FINE STRUCTURE OF THE ZEROS OF ORTHOGONAL POLYNOMIALS, III. PERIODIC RECURSION COEFFICIENTS

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ABSTRACT. We discuss asymptotics of the zeros of orthogonal polynomials on the real line and on the unit circle when the recursion coefficients are periodic. The zeros on or near the absolutely continuous spectrum have a clock structure with spacings inverse to the density of zeros. Zeros away from the a.c. spectrum have limit points mod p and only finitely many of them.

1. Introduction

This paper is the third in a series [17, 18] that discusses detailed asymptotics of the zeros of orthogonal polynomials with special emphasis on distances between nearby zeros. We discuss both orthogonal polynomials on the real line (OPRL) where the basic recursion for the orthonormal polynomials, $p_n(x)$, is

$$xp_n(x) = a_{n+1}p_{n+1}(x) + b_{n+1}p_n(x) + a_np_{n-1}(x) \tag{1.1}$$

($a_n > 0$ for $n = 1, 2, \dots$, b_n real, and $p_{-1}(x) \equiv 0$), and orthogonal polynomials on the unit circle (OPUC) where the basic recursion is

$$\varphi_{n+1}(z) = \rho_n^{-1}(z\varphi_n(z) - \bar{\alpha}_n\varphi_n^*(z)) \tag{1.2}$$

Here α_n are complex coefficients lying in the unit disk $\mathbb{D}$ and

$$\varphi_n^*(z) = z^n\,\overline{\varphi_n(1/\bar{z})} \tag{1.3}$$

and

$$\rho_n = (1 - |\alpha_n|^2)^{1/2} \tag{1.4}$$

In this paper, we focus on the case where the Jacobi coefficients $\{a_n\}_{n=1}^\infty, \{b_n\}_{n=1}^\infty$ or the Verblunsky coefficients $\{\alpha_n\}_{n=0}^\infty$ are periodic, that is, for some p,

$$a_{n+p} = a_n \qquad b_{n+p} = b_n \tag{1.5}$$

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or

$$\alpha_{n+p} = \alpha_n \tag{1.6}$$

It should be possible to say something about perturbations of a periodic sequence, say $\alpha_n^{(0)}$, which obeys (1.6) and $\alpha_n = \alpha_n^{(0)} + \delta\alpha_n$ with $|\delta\alpha_n| \to 0$ sufficiently fast. We leave the details to be worked out elsewhere.

To describe our results, we begin by summarizing some of the basics of the structure of the measures and recursion relations when (1.5) or (1.6) holds. We will say more about this underlying structure in the sections below. In this introduction, we will assume that all gaps are open, although we don't need and won't use that assumption in the detailed discussion.

When (1.5) holds, the continuous part of the underlying measure, $d\rho$, on $\mathbb{R}$ is supported on p closed intervals $[\alpha_j, \beta_j]$, $j = 1, \dots, p$, called bands, with gaps (β_j, α_{j+1}) in between. Each gap has zero or one mass point. The m-function of the measure $d\rho$,

$$m(z) = \int \frac{d\rho(x)}{x - z} \tag{1.7}$$

has a meromorphic continuation to the genus $p-1$ hyperelliptic Riemann surface, $\mathcal{S}$, associated to $[\prod_{j=1}^p (x-\alpha_j)(x-\beta_j)]^{1/2}$. This surface has a natural projection $\pi : \mathcal{S} \to \mathbb{C}$, a twofold cover except at the branch points $\{\alpha_j\}_{j=1}^p \cup \{\beta_j\}_{j=1}^p$. $\pi^{-1}[\beta_j, \alpha_{j+1}]$ is a circle and $m(z)$ has exactly one pole $\gamma_1, \dots, \gamma_{p-1}$ on each circle.

It has been known for many years (see Faber [2]) that the density of zeros dk is supported on $\cup_{j=1}^p [\alpha_j, \beta_j] \equiv B$ and is the equilibrium measure for B in potential theory. We define $k(E) = \int_{\alpha_1}^E dk$. Then $k(\beta_j) = j/p$. Our main results about OPRL are:

(1) We can describe the zeros of $p_{np-1}(x)$ exactly (not just asymptotically) in terms of $\pi(\gamma_j)$ and $k(E)$.
(2) Asymptotically, as $n \to \infty$, the number of zeros of p_n in each band $[\alpha_j, \beta_j]$, $N^{(n,j)}$, obeys $\sup_n |\frac{n}{p} - N^{(n,j)}| < \infty$, and the zeros $\{x_\ell^{(n,j)}\}_{\ell=1}^{N(n,j)}$ obey

$$\sup_{\substack{j \\ \ell=1,2,\dots,N^{(n,j)}-1}} n\left| k(x_{\ell+1}^{(n,j)}) - k(x_\ell^{(n,j)}) - \frac{1}{n} \right| \to 0 \tag{1.8}$$

as $n \to \infty$.
(3) $z \in \mathbb{C}$ is a limit of zeros of p_n if and only if z lies in $\operatorname{supp}(d\rho)$.
(4) Outside the bands, there are at most $2p + 2b - 3$ points which are limits of zeros of p_{mp+b-1} for each $b = 1, \dots, p$ and, except for these limits, zeros have no accumulation points in $\mathbb{C}\setminus$bands.

For OPUC, the continuous part of the measure, $d\mu$, is supported on p disjoint intervals $\{e^{i\theta} \mid x_j \leq \theta \leq y_j\}$, $j = 1, \dots, p$, in $\partial\mathbb{D}$ with p gaps in between $\{e^{i\theta} \mid y_j \leq \theta \leq x_{j+1}\}$ with $x_{p+1} \equiv 2\pi + x_1$. Each gap has zero or one mass point. The Carathéodory function of the measure $d\mu$,

$$F(z) = \int \frac{e^{i\theta} + z}{e^{i\theta} - z}\, d\mu(\theta) \tag{1.9}$$

has a meromorphic continuation from $\mathbb{D}$ to the genus $p-1$ hyperelliptic Riemann surface, $\mathcal{S}$, associated to $[\prod_{j=1}^{p}(z - e^{ix_j})(z - e^{iy_j})]^{1/2}$. The surface has a natural projection $\pi : \mathcal{S} \to \mathbb{C}$, and the closure of each gap has a circle as the inverse image. F has a single pole in each such circle, so p in all at $\gamma_1, \dots, \gamma_p$.

Again, the density of zeros is the equilibrium measure for the bands and each band has mass $1/p$ in this measure. See [16], especially Chapter 11, for a discussion of periodic OPUC. Our main results for OPUC are:

(1′) We can describe the zeros of $\varphi_{np}^* - \varphi_{np}$ exactly (note, not zeros of φ_{np}).
(2′) Asymptotically, as $n \to \infty$, the number of zeros of φ_n near each band, $N^{(n,j)}$, obeys $\sup_n |\frac{n}{p} - N^{(n,j)}| < \infty$, and the points on the bands closest to the zeros obey an estimate like (1.8).
(3′) $z \in \mathbb{C}$ is a limit of zeros of φ_n if and only if z lies in $\operatorname{supp}(d\mu)$.
(4′) There are at most $2p + 2b - 1$ points which are limits of zeros of φ_{mp+b} for each $b = 1, \dots, p$ and, except for these limits, zeros have no accumulation points in $\mathbb{C}\backslash$bands.

In Section 2, we discuss OPRL when (1.5) holds, and in Section 3, OPUC when (1.6) holds. Each section begins with a summary of transfer matrix techniques for periodic recursion coefficients (Floquet theory).

While I am unaware of any previous work on the precise subject of Sections 2 and 3, the results are closely related to prior work of Peherstorfer [6, 7], who discusses zeros in terms of measures supported on a union of bands with a particular structure that overlaps our class of measures. For a discussion of zeros for OPUC with two bands, see [5].

These papers also consider situations where the recursion coefficients are only almost periodic. For any finite collection of closed intervals on $\mathbb{R}$ or closed arcs on $\partial\mathbb{D}$, there is a natural isospectral torus of OPRL or OPUC where the corresponding m- or F-function has minimal degree on the Riemann surface (see, e.g., [16, Section 11.8]). It would be interesting to extend the results of the current paper to that case.

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2. OPRL With Periodic Jacobi Coefficients

In this section, we analyze the zeros of OPRL with Jacobi coefficients obeying (1.5). We begin with a summary of the theory of transfer matrices, discriminants, and Abelian functions associated to this situation. A reference for much of this theory is von Moerbeke [20]; a discussion of the discriminant can be found in Hochstadt [3], von Moerbeke [20], Toda [19], and Last [4]. The theory is close to the OPUC theory developed in Chapter 11 of [16].

Define the 2×2 matrix,

$$A_k(z) = \frac{1}{a_{k+1}} \begin{pmatrix} z - b_{k+1} & -a_k \\ a_{k+1} & 0 \end{pmatrix} \tag{2.1}$$

where

$$a_0 \equiv a_p \tag{2.2}$$

Thus

$$\det(A_k) = \frac{a_k}{a_{k+1}} \tag{2.3}$$

and the abstract form of (1.1)

$$zu_n = a_{n+1}u_{n+1} + b_{n+1}u_n + a_n u_{n-1} \tag{2.4}$$

is equivalent to

$$\begin{pmatrix} u_{n+1} \\ u_n \end{pmatrix} = A_n \begin{pmatrix} u_n \\ u_{n-1} \end{pmatrix} \tag{2.5}$$

So, in particular,

$$\begin{pmatrix} p_{n+1}(z) \\ p_n(z) \end{pmatrix} = A_n A_{n-1} \dots A_0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.6}$$

This motivates the definition of the transfer matrix,

$$T_n(z) = A_{n-1}(z) \dots A_0(z) \tag{2.7}$$

for $n = 1, 2, \dots$. We have, by (1.5), that

$$T_{mp+b} = T_b(T_p)^m \tag{2.8}$$

suggesting that T_p plays a basic role. By (2.3) and (2.2),

$$\det(T_p) = 1 \tag{2.9}$$

A fundamental quantity is the discriminant

$$\Delta(z) = \operatorname{Tr}(T_p(z)) \tag{2.10}$$

By (2.6), we have

$$T_n(z) = \begin{pmatrix} p_n(z) & q_{n-1}(z) \\ p_{n-1}(z) & q_{n-2}(z) \end{pmatrix} \tag{2.11}$$

where $q_n(z)$ is a polynomial of degree n that is essentially the polynomial of the second kind (the normalization is not the standard one but involves an extra a_p).

By (2.9) and (2.10), $T_p(z)$ has eigenvalues

$$\Gamma_\pm(z) = \frac{\Delta(z)}{2} \pm \sqrt{\left(\tfrac{\Delta(z)}{2}\right)^2 - 1} \tag{2.12}$$

In a moment, we will define branch cuts in such a way that on all of $\mathbb{C}\backslash$cuts,

$$|\Gamma_+(z)| > |\Gamma_-(z)| \tag{2.13}$$

so (2.8) implies the Lyapunov exponent is given by

$$\lim_{n\to\infty} \frac{1}{n} \log \|T_n(z)\| = \frac{1}{p} \log|\Gamma_+(z)| \equiv \gamma(z) \tag{2.14}$$

(2.12) means $|\Gamma_+| = |\Gamma_-|$ if and only if $\Delta(z) \in [-2,2]$, and one shows that this only happens if z is real. Moreover, if $\Delta(z) \in (-2,2)$, then $\Delta'(x) \neq 0$. Thus, for x very negative, $(-1)^p\Delta(x) > 0$ and solutions of $(-1)^p\Delta(x) = \pm 2$ alternate as $+2, -2, -2, +2, +2, -2, -2, \dots$, which we label as

$$\alpha_1 < \beta_1 \leq \alpha_2 < \beta_2 \leq \alpha_3 < \cdots < \beta_p \tag{2.15}$$

Since $\Delta(x)$ is a polynomial of degree p, there are p solutions of $\Delta(x) = 2$ and of $\Delta(x) = -2$, so $2p$ points $\{\alpha_j\}_{j=1}^p \cup \{\beta_j\}_{j=1}^p$.

The bands are $[\alpha_1, \beta_1], [\alpha_2, \beta_2], \dots, [\alpha_p, \beta_p]$ and the gaps are (β_1, α_2), $(\beta_2, \alpha_3), \dots, (\beta_{p-1}, \alpha_{p+1})$. If some $\beta_j = \alpha_{j+1}$, we say the j-th gap is closed. Otherwise we say the gap is open.

If we remove the bands from $\mathbb{C}$, $\Gamma_\pm(z)$ are single-valued analytic functions and (2.13) holds. Moreover, Γ_+ has an analytic continuation to the Riemann surface, $\mathcal{S}$, of genus $\ell \leq p-1$ where ℓ is the number of open gaps. $\mathcal{S}$ is defined by the function $[(z-\alpha_1)(z-\beta_p)\prod_{\text{open gaps}}(z-\beta_j)(z-\alpha_{j+1})]^{1/2}$. Γ_- is precisely the analytic continuation of Γ_+ to the second sheet.

The Dirichlet data are partially those x's where

$$T_p(x)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = c_x \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.16}$$

that is, points where the 21 matrix element of T_p vanishes. It can be seen that the Dirichlet data x's occur, one to each gap, that is, $x_1, \dots, x_{p-1}$ with $\beta_j \leq x_j \leq \alpha_{j+1}$. If x is at an edge of a gap, then

$c_j \equiv c_{x_j}$ is ± 1. Otherwise $|c_j| \neq 1$. If $|c_j| > 1$, we add the sign $\sigma_j = -1$ to x_j, and if $|c_j| < 1$, we add the sign $\sigma_j = +1$ to x_j. Thus the values of Dirichlet data for each open gap are two copies of $[\alpha_j, \beta_j]$ glued at the ends, that is, a circle. The set of Dirichlet data is thus an ℓ-dimensional torus. It is a fundamental result [20] that the map from a's and b's to Dirichlet data sets up a one-one correspondence to all a's and b's with a given Δ, that is, the set of a's and b's with a given Δ is an ℓ-dimensional torus.

The m-function (1.7) associated to $d\rho$ has a meromorphic continuation to the Riemann surface, $\mathcal{S}$, with poles precisely at the points x_j on the principal sheet if $\sigma_j = +1$ and on the bottom sheet if $\sigma_j = -1$. ρ has point mass precisely at those $x_j \in (\beta_j, \alpha_{j+1})$ with $\sigma_j = +1$. It has absolutely continuous support exactly the union of the bands, and has no singular part other than the possible point masses in the gaps.

Finally, in the review, we note that the potential theoretic equilibrium measure dk for the set of bands has several critical properties:

(1) If $k(x) = \int_{\alpha_1}^x dk$, then
$$k(\beta_j) = k(\alpha_{j+1}) = \frac{j}{p} \tag{2.17}$$
(2) The Thouless formula holds:
$$\gamma(z) = \int \log|z-x|\, dk(x) + \log C_B \tag{2.18}$$
where γ is given by (2.14) and C_B is the (logarithmic) capacity of B.

(3) The (logarithmic) capacity of the bands is given by
$$C_B = \biggl(\prod_{j=1}^p a_j\biggr)^{-1} \tag{2.19}$$
(4)
$$\Gamma_+(z) = C_B \exp\biggl(p\int \log(z-x)\, dk(x)\biggr) \tag{2.20}$$

That completes the review of periodic OPRL. We now turn to the study of the zeros. We begin by describing exactly (not just asymptotically!) the zeros of P_{mp-1}:

Theorem 2.1. *The zeros of $P_{mp-1}(x)$ are exactly*

(i) *The $p-1$ Dirichlet data points $\{x_j\}_{j=1}^{p-1}$.*

(ii) *The* $(m-1)p$ *points* $\{x_{k,q}^{(m)}\}_{\substack{k=1,\dots,p\\ q=1,\dots,m-1}}$ *where*

$$k(x_{k,q}^{(m)}) = \frac{k-1}{p} + \frac{q}{mp} \tag{2.21}$$

Remarks. 1. The points of (2.21) can be described as follows. Break each band $[\alpha_j, \beta_j]$ into m pieces of equal size in equilibrium measure. The $x_{k,q}^{(m)}$ are the interior break points.

2. If a gap is closed, we include its position in the "Dirichlet points" of (i).

3. Generically, there are not zeros at the band edges, that is, (2.21) has $q = 1, \dots, m-1$ but not $q = 0$ or $q = m$. But it can happen that one or more of the Dirichlet data points is at an α_{j+1} or a β_j.

4. This immediately implies that once one proves that the density of zeros exists, that it is given by dk.

5. It is remarkable that this result is new, given that it is so elegant and its proof so simple! I think this is because the OP community most often focuses on measures and doesn't think so much about the recursion parameters and the Schrödinger operator community doesn't usually think of zeros of P_n.

Example 2.2. Let $b_n \equiv 0$, $a_n \equiv \frac{1}{2}$ which has period $p = 1$. It is well-known in this case that the P_n are essentially Chebyshev polynomials of the second kind, that is,

$$P_n(\cos\theta) = \frac{1}{2^n} \frac{\sin(n+1)\theta}{\sin\theta} \tag{2.22}$$

Thus P_{m-1} has zeros at points where

$$\theta = \frac{j\pi}{m} \qquad j = 1, \dots, m-1 \tag{2.23}$$

(the zeros at $\theta = 0$ and $\theta = \pi$ are cancelled by the $\sin(\theta)$). $k(x) = \pi - \arccos(x)$ and (2.23) is (2.21). We see that Theorem 2.1 generalizes the obvious result on the zeros of the Chebyshev polynomials of the second kind. □

First Proof of Theorem 2.1. By (2.11), zeros of P_{mp-1} are precisely points where the 12 matrix element of T_{mp} vanishes, that is, points where $\binom{1}{0}$ is an eigenvector of T_{mp}. That is, zeros of P_{mp-1} are Dirichlet points for this period mp problem.

When (1.5) holds, we can view the a's and b's as periodic of period mp. There are closed gaps where $T_{mp}(z) = \pm\mathbf{1}$, that is, interior points to the original bands where $(\Gamma_\pm)^m = 1$, that is, points where (2.21) holds. Thus, the Dirichlet data for T_{mp} are exactly the points claimed. $\square$

Theorem 2.1 immediately implies point (2) from the introduction.

Theorem 2.3. *Let $P_n(x)$ be a family of OPRL associated to a set of Jacobi parameters obeying* (1.5)*. Let (α_j, β_j) be a single band and let $N^{(n,j)}$ be the number of zeros of P_n in that band. Then*

$$|N^{(mp+b,j)} - (m-1)| \leq \min(b+1, p-b) \tag{2.24}$$

for $-1 \leq b \leq p-1$. In particular,

$$\left| N^{(n,j)} - \frac{n}{p} \right| \leq 1 + \frac{p}{2} \tag{2.25}$$

Proof. By a variational principle for any n, n',

$$|N^{(n,j)} - N^{(n',j)}| \leq |n - n'| \tag{2.26}$$

(2.24) is immediate from Theorem 2.1 if we take $n' = mp - 1$ and $n' = mp + (p-1)$. (2.25) follows from (2.24) given that $\min(b+1, p-b) \leq p/2$. $\square$

Remark. Because of possibilities of Dirichlet data zeros at α_j and/or β_j, we need (α_j, β_j) in defining $N^{(n,j)}$. It is more natural to use $[\alpha_j, \beta_j]$. If one does that, (2.24) becomes $2 + \min(b+1, p-b)$ and (2.25), $3 + \frac{p}{2}$.

To go beyond these results and prove clock behavior for the zeros of p_{mp+b} ($b \not\equiv -1 \bmod p$), we need to analyze the structure of p_n in terms of Γ_+, Γ_-. For z not a branch point (or closed gap), $\Gamma_+ \neq \Gamma_-$. Γ_+ is well-defined on $\mathbb{C}\backslash$bands since $|\Gamma_+| > |\Gamma_-|$. On the bands, $|\Gamma_+| = |\Gamma_-|$ and, indeed, the boundary values on the two sides of a band are distinct. But Γ_+ is analytic on $\mathbb{C}\backslash$bands, so for such z, we can define $P_\pm$ by

$$T_p(z) = \Gamma_+ P_+ + \Gamma_- P_- \tag{2.27}$$

where P_+, P_- are 2×2 rank one projections obeying

$$P_+^2 = P_+ \qquad P_-^2 = P_- \qquad P_+ P_- = P_- P_+ = 0 \tag{2.28}$$

and

$$P_+ + P_- = \mathbf{1} \tag{2.29}$$

It follows from (2.27) and (2.29) that

$$P_+ = \frac{T_p(z) - \Gamma_-(z)\mathbf{1}}{\Gamma_+ - \Gamma_-} \tag{2.30}$$

$$P_- = \frac{T_p(z) - \Gamma_+(z)\mathbf{1}}{\Gamma_- - \Gamma_+} \tag{2.31}$$

which, in particular, shows that P_+ is a meromorphic function on $\mathcal{S}$ whose second-sheet values are just P_-.

Define

$$a(z) = \left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix}, P_+(z) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle \tag{2.32}$$

$$b(z) = \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, P_+(z) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle \tag{2.33}$$

so (2.29) implies

$$\left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix}, P_-(z) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle = -a(z) \tag{2.34}$$

$$\left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, P_-(z) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle = 1 - b(z) \tag{2.35}$$

Under most circumstances, $a(z)$ has a pole at band edges where $\Gamma_+ - \Gamma_- \to 0$. For later purpose, we note that $\langle \binom{0}{1}, (T_p(z) - \Gamma_- \mathbf{1}) \binom{1}{0} \rangle = \langle \binom{0}{1}, T_p(z) \binom{1}{0} \rangle$ has a finite limit at such points. Later we will be looking at

$$\begin{aligned} a(z)(\Gamma_+^m - \Gamma_-^m) &= \left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix}, T_p(z) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle \frac{\Gamma_+^m - \Gamma_-^m}{\Gamma_+ - \Gamma_-} \\ &\to \left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix}, T_p(z) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle m \Gamma_+^{m-1} \end{aligned}$$

if $\Gamma_+ - \Gamma_- \to 0$. This is zero if and only if $\langle \binom{0}{1}, T_p(z) \binom{1}{0} \rangle = 0$, that is, if and only if the edge of the band is a Dirichlet data point.

(2.27) and (2.28) imply

$$T_{mp}(z) = T_p(z)^m = \Gamma_+^m P_+ + \Gamma_-^m P_- \tag{2.36}$$

so

$$T_{mp}(z) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = [a(z)(\Gamma_+^m - \Gamma_-^m)] \begin{pmatrix} 0 \\ 1 \end{pmatrix} + [b(z)\Gamma_+^m + (1 - b(z))\Gamma_-^m] \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.37}$$

Thus, by (2.25) for $b \geq 0$,

$$P_{mp+b-1} = \left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix}, T_b T_{mp} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle \tag{2.38}$$

$$= [(\Gamma_+^m - \Gamma_-^m) a(z)] q_{b-2}(z) + [b(z)\Gamma_+^m + (1 - b(z))\Gamma_-^m] p_{b-1}(z) \tag{2.39}$$

where

$$q_{-2}(z) \equiv 1 \qquad q_{-1}(z) \equiv 0 \tag{2.40}$$

Second Proof of Theorem 2.1. For $b=0$, $p_{b-1}\equiv 0$ and $q_{b-2}=1$, so

$$p_{mp-1}(z)=(\Gamma_+^m-\Gamma_-^m)a(z) \tag{2.41}$$

Its zeros are thus points where $a(z)=0$ or where $\Gamma_+^m=\Gamma_-^m$, except that at branch points, $a(z)$ can have a pole which can cancel a zero of $\Gamma_+^m-\Gamma_-^m$.

$a(z)=0$ if and only if $\binom{1}{0}$ is an eigenvector of $T_p(z)$, that is, exactly at the Dirichlet data points.

$\Gamma_+^m=\Gamma_-^m$ is equivalent to $\Gamma_+^{2m}=1$ since $\Gamma_-=\Gamma_+^{-1}$. This implies $|\Gamma_+|=|\Gamma_-|$, so can only happen on the bands. On the bands, by (2.24),

$$\Gamma_+(x)=\exp(\pi ip\,k(x)) \tag{2.42}$$

and $\Gamma_+^{2m}=1$ if and only if

$$mp\,k(x)\in\mathbb{Z} \tag{2.43}$$

that is, if (2.21) holds for some $q=0,\dots,m$. But at $q=0$ or $q=m$, $a(z)$ has a pole that cancels the zero of $\Gamma_+^m-\Gamma_-^m$, so the zeros of p_{mp-1} are precisely given by (i) and (ii) of Theorem 2.1. □

We can use (2.39) to analyze zeros of p_{mp+b-1} for large m. We begin with the region away from the bands:

Theorem 2.4. *Let $z\in\mathbb{C}\backslash$bands and let b be fixed. Then*

$$\lim_{m\to\infty}\Gamma_+(z)^{-m}p_{mp+b-1}(z)=a(z)q_{b-2}(z)+b(z)p_{b-1}(z) \tag{2.44}$$

In particular, if the right side of (2.44) is called $j_b(z)$, then

(1) *If $j_b(z_0)\neq 0$, then $p_{mp+b-1}(z)$ is nonvanishing near z_0 for m large.*
(2) *If $j_b(z_0)=0$, then $p_{mp+b-1}(z)$ has a zero (k zeros if z has a k-th order zero at z_0) near z_0 for m large.*
(3) *There are at most $2p+2b-3$ points in $\mathbb{C}\backslash$bands where $j_b(z_0)$ is zero.*

Proof. (2.44) is immediate from (2.39) and $|\Gamma_-/\Gamma_+|<1$. (1) and (2) then follow by Hurwitz's theorem if we show that $j_b(z)$ is not identically zero.

By (2.1) and (2.7) near $z=\infty$,

$$T_p(z)=\left(\prod_{j=1}^p a_j\right)^{-1}z^p\begin{pmatrix}1&0\\0&0\end{pmatrix}+O(z^{p-1})$$

which implies $\Gamma_+=(\prod_{j=1}^p a_j)^{-1}z^p+O(z^{p-1})$ and $\Gamma_-(z)=O(z^{-p})$. It follows that $a(z)\to 0$ as $z\to\infty$ and $b(z)\to 1$. Thus, since p_{b-1} has degree $b-1$, (2.39) shows that as $z\to\infty$ on the main sheet, $f(z_0)$ has a pole of order $b-1$.

On the other sheet, P_+ changes to P_-, so $a(z) \to 0$ and $b(z) \to 0$ on the other sheet. It follows that $j(z)$ has a pole at ∞ of degree at most $b-2$. j also has poles of degree at most 1 at each branch point. Thus, $j_b(z)$ as a function on $\mathcal{S}$ has total degree at most $2p+(b-1)+(b-2) = 2p+2b-3$ which bounds the number of zeros. $\square$

Finally, we turn to zeros on the bands. A major role will be played by the function on the right side of (2.44) (j is for "Jost" since this acts in many ways like a Jost function):

$$j_b(z) = a(z)q_{b-2}(z) + b(z)p_{b-1}(z) \tag{2.45}$$

Lemma 2.5. *j_b is nonvanishing on the interior of the bands.*

Remark. By $j_b(x)$ for x real, we mean (2.45) with a defined via $\lim_{\varepsilon\downarrow 0} a(x+i\varepsilon)$ since $P_\pm$ are only defined off $\mathbb{C}\backslash$bands.

Proof. As already mentioned, the boundary values obey

$$\lim_{\varepsilon\downarrow 0} P_+(x+i\varepsilon) = \lim_{\varepsilon\downarrow 0} P_-(x-i\varepsilon) \tag{2.46}$$

(by the two-sheeted nature of P_+ and P_-). Thus, by (2.30) and (2.31),

$$a(x+i0) = -a(x-i0) \tag{2.47}$$

$$b(x+i0) = 1 - b(x-i0) \tag{2.48}$$

Moreover, since T_p and $\Gamma_\pm$ are real on $\mathbb{R}\backslash$bands, $a(z)$ and $b(z)$ are real on $\mathbb{R}\backslash$bands (by (2.26)). Thus

$$a(x+i0) = \overline{a(x-i0)} \tag{2.49}$$

$$b(x+i0) = \overline{b(x-i0)} \tag{2.50}$$

The last four equations imply for x in the bands

$$\mathrm{Re}(a(x+i0)) = 0 \tag{2.51}$$

$$\mathrm{Re}(b(x+i0)) = \tfrac{1}{2} \tag{2.52}$$

p and q are real on $\mathbb{R}$, so

$$\mathrm{Re}(j_b(x)) = \tfrac{1}{2}\, p_{b-1}(x) \tag{2.53}$$

Thus, if $j_b(x_0) = 0$ on the bands, $p_{b-1}(x_0) = 0$.

As we have seen, $a(z) = 0$ only at the Dirichlet points and so not in the bands. If $p_{b-1}(x_0) = 0 = j_b(x_0)$, then since $a(x_0) \neq 0$, we also have $q_{b-2}(x_0) = 0$. By (2.11), if $p_{b-1}(x_0) = q_{b-2}(x_0)$, then $\det(T_b(x_0)) = 0$, which is false. We conclude via proof by contradiction that $j_b(x)$ has no zeros. $\square$

Theorem 2.6. *For each b and each band j, there is an integer $D_{b,j}$ so the number of zeros $N_{b,j}(m)$ of p_{mp+b-1} is either $m-D_{b,j}$ or $m-D_{b,j}+1$. In particular,*

$$\sup_{n,j}\left|\frac{n}{p}-N^{(n,j)}\right|<\infty \tag{2.54}$$

Moreover, (1.8) *holds.*

Proof. By (2.39), (2.46), (2.47), and (2.48), we have

$$p_{mp+b-1}(x)=j_b(x)\Gamma_+(x)^m+\overline{j_b(x)}\;\overline{\Gamma_+(x)}^m \tag{2.55}$$

on the bands. By the lemma, $j_b(x)$ is nonvanishing inside band j, so

$$j_b(x)=|j_b(x)|e^{i\gamma_b(x)} \tag{2.56}$$

where γ_b is continuous — indeed, real analytic — and by a simple argument, γ_b and γ_b' have limits as $x\downarrow a_j$ or $x\uparrow b_j$.

By (2.42), (2.55) becomes

$$p_{mp+b-1}(x)=2|j_b(x)|\cos(\pi mp\,k(x)+\gamma_b(x)) \tag{2.57}$$

Define $D_{b,j}$ to be the negative of the integral part of $[\gamma_b(b_j)-\gamma_b(a_j)]/\pi$. Since $\sup_{\text{bands}}|\gamma_b'(x)|<\infty$, there is, for large m, at most one solution of $\pi mp\,k(x)+\gamma_b(x)=\pi\ell$ for each ℓ. Given this, it is immediate that the number of zeros is $m-D_{b,j}$ or $m-D_{b,j}+1$.

Finally, (1.8) is immediate from (2.57). Given that γ is C^1, we even get that

$$k(x_{\ell+1}^{(n,j)})-k(x_\ell^{(n,j)})=\frac{1}{n}+O\left(\frac{1}{n^2}\right) \tag{2.58}$$

□

As for point (3) from the introduction, the proof of Theorem 2.4 shows that if z_0 is not in the bands and is a limit of zeros of $p_{mp+b-1}(z)$, then $p_{mp+b-1}(z_0)$ goes to zero exponentially (as Γ_-^m). If this is true for each b, then $\sum_{n=0}^\infty|p_n(z)|^2<\infty$, which means z_0 is in the pure point spectrum of $d\mu$. Since the bands are also in the spectrum, we have

Proposition 2.7. *$z_0\in\mathbb{C}$ is a limit of zeros of $p_n(z)$ (all n) if and only if $z_0\in\operatorname{supp}(d\mu)$.*

Remark. This also follows from a result of Denisov-Simon [1], but their argument, which applies more generally, is more subtle.

3. OPUC With Periodic Verblunsky Coefficients

In this section, we analyze the zeros of OPUC with Verblunsky coefficients obeying (1.6). We begin with a summary of the transfer matrices, discriminants, and Abelian functions in this situation. These ideas, while an obvious analog of the OPRL situation, seem not to have been studied before their appearance in [16], which is the reference for more details. Many of the consequences of these ideas were found earlier in work of Peherstorfer and Steinbauer [8, 9, 10, 11, 12, 13, 14].

Throughout, we will suppose that p is even. If $(\alpha_0, \dots, \alpha_{p-1}, \alpha_p, \dots)$ is a sequence with odd period, $(\beta_0, \beta_1, \dots) = (\alpha_0, 0, \alpha_1, 0, \alpha_2, \dots)$ has even period and

$$\Phi_{2n}(z, \{\beta_j\}) = \Phi_n(z^2, \{\alpha_j\}) \tag{3.1}$$

so results for the even p case immediately imply results for the odd p.

Define the 2×2 matrix

$$A_k(z) = \frac{1}{\rho_k}\begin{pmatrix} z & -\bar{\alpha}_k \\ -z\alpha_k & 1 \end{pmatrix} \tag{3.2}$$

where ρ_k is given by (1.4). Then

$$\det(A_k(\alpha)) = z$$

(1.2) and its * are equivalent to

$$\begin{pmatrix} \varphi_{n+1} \\ \varphi_{n+1}^* \end{pmatrix} = A_n(z)\begin{pmatrix} \varphi_n \\ \varphi_n^* \end{pmatrix} \tag{3.3}$$

The second kind polynomials, $\psi_n(z)$, are the OPUC with Verblunsky coefficients $\{-\alpha_j\}_{j=0}^\infty$. Then it is easy to see that

$$\begin{pmatrix} \psi_{n+1} \\ -\psi_{n+1}^* \end{pmatrix} = A_n(z)\begin{pmatrix} \psi_n \\ -\psi_n^* \end{pmatrix} \tag{3.4}$$

with A given by (3.2).

We thus define

$$T_n(z) = A_{n-1}(z)\dots A_0(z) \tag{3.5}$$

By (1.6), we have

$$T_{mp+b} = T_b(T_p)^m \tag{3.6}$$

(3.3) and (3.4) imply that

$$\begin{pmatrix} \varphi_n \\ \varphi_n^* \end{pmatrix} = T_n\begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{3.7}$$

$$\begin{pmatrix} \psi_n \\ -\psi_n^* \end{pmatrix} = T_n\begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{3.8}$$

so that

$$T_n(z) = \tfrac{1}{2} \begin{pmatrix} \varphi_n(z) + \psi_n(z) & \varphi_n(z) - \psi_n(z) \\ \varphi_n^*(z) - \psi_n^*(z) & \varphi_n^*(z) + \psi_n^*(z) \end{pmatrix} \tag{3.9}$$

The discriminant is defined by

$$\Delta(z) = z^{-p/2} \operatorname{Tr}(T_p(z)) \tag{3.10}$$

The $z^{-p/2}$ factor (recall p is even) is there because $\det(z^{-p/2}T_p(z)) = 1$, so $z^{-p/2}T_p(z)$ has eigenvalues $\Gamma_\pm(z)$ given by (2.12). $\Delta(z)$ is real on $\partial\mathbb{D}$ so

$$\Delta(z) = \overline{\Delta(1/\bar z)} \tag{3.11}$$

$\Delta(z) \in (-2, 2)$ only if $z = e^{i\theta}$ and there are p roots, each of $\operatorname{Tr}(T_p(z)) \mp 2z^{p/2} = 0$, that is, p solutions of $\Delta(z) = \pm 2$. These alternate on the circle at points $+2, -2, -2, +2, +2, -2, -2, \dots$, so we pick

$$0 \le x_1 < y_1 \le x_2 < y_2 \le \cdots < y_p \le 2\pi \tag{3.12}$$

where e^{ix_j}, e^{iy_j} are solutions of $\Delta(z) = \pm 2$.

The bands

$$B_j = \{e^{i\theta} \mid x_j \le \theta \le y_j\} \tag{3.13}$$

are precisely the points where $\Delta(z) \in [-2, 2]$. In between are the gaps

$$G_j\{e^{i\theta} \mid y_j < \theta < x_{j+1}\} \tag{3.14}$$

where $x_{p+1} = x_1 + 2\pi$. Some gaps can be closed, that is, G_j is empty (i.e., $y_j = x_{j+1}$).

We also see that on $\mathbb{C}\backslash$bands, $|\Gamma_+| > |\Gamma_-|$, so the Lyapunov exponent is given by

$$\lim_{n\to\infty} \frac{1}{n} \log \|T_n(z)\| = \frac{1}{2} \log|z| + \frac{1}{p} \log|\Gamma_+(z)| \equiv \gamma(z) \tag{3.15}$$

If we remove the bands from $\mathbb{C}$, (2.13) holds. Moreover, $\Gamma_+(z)$ has an analytic continuation to the Riemann surface, $\mathcal{S}$, of $[\prod_{\text{open gaps}}(z - e^{iy_{j+1}})(z - e^{ix_j})]^{1/2}$. The genus of $\mathcal{S}$, $\ell \le p-1$, where $\ell+1$ is the number of open gaps. (In some sense, the OPRL case, where the genus ℓ is the number of gaps, has $\ell + 1$ gaps also, but one gap is $\mathbb{R}\backslash[\alpha_1, \beta_p]$ which includes infinity.) Γ_- is the analytic continuation of Γ_+ to the second sheet.

The Dirichlet data are partly these points in $\partial\mathbb{D}$, z_j,

$$T_p(z)\begin{pmatrix} 1 \\ 1 \end{pmatrix} = c_z \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{3.16}$$

It can be shown there is one such z_j in each gap (including closed gaps) for the p roots of $\varphi_p(z) - \varphi_p^*(z)$. We let $c_j = c_{z_j}$. If z_j is at a gap edge, $|c_j| = 1$; otherwise $|c_j| \ne 1$. If $|c_j| > 1$, we add sign -1 to z_j and place the Dirichlet point on the lower sheet of $\mathcal{S}$ at point z_j. If $|c_j| < 1$, we add sign $+1$ and put the Dirichlet point on the initial sheet. $+1$ points correspond to pure points in $d\mu$.

As in the OPRL case, the set of possible Dirichlet data points is a torus, but now of dimension $\ell + 1$. This torus parametrizes those μ with periodic α's and discriminant Δ.

The F-function, (1.9), has a meromorphic contribution to $\mathcal{S}$ with poles precisely at the Dirichlet data points.

The potential theoretic equilibrium measures dk for the bands have several critical properties:

(1) If $k(e^{i\theta_0}) = k(\{e^{i\theta} \mid x_1 < \theta < \theta_0\})$, then

$$k(e^{iy_j}) = k(e^{ix_{j+1}}) = \frac{j}{p} \tag{3.17}$$

(2) The Thouless formula holds:

$$\gamma(z) = \int \log|z - e^{i\theta}|\, dk(e^{i\theta}) + \log C_B \tag{3.18}$$

where γ is given by (3.15) and C_B is the capacity of the bands.

(3) We have

$$C_B = \prod_{j=0}^{p-1} (1 - |\alpha_j|^2)^{1/2} \tag{3.19}$$

(4)

$$\Gamma_+(z) = C_B z^{-p/2} \exp\biggl(p \int \log(z - e^{i\theta})\, dk(e^{i\theta})\biggr) \tag{3.20}$$

This completes the review of periodic OPUC. The analog of Theorem 2.1 does not involve Φ_n but $\Phi_n - \Phi_n^*$:

Theorem 3.1. *The zeros of* $\Phi_{mp}(z) - \Phi_{mp}^*(z)$ *are at the following points:*

(i) *the* p *Dirichlet data* z_j*'s in each gap of the period* p *problem.*

(ii) *the* $(m-1)p$ *points where*

$$k(e^{i\theta}) = \frac{k-1}{p} + \frac{q}{mp} \tag{3.21}$$

$k = 1, \dots, p$; $q = 1, \dots, m-1$.

Proof. As noted (and proven several ways in [16, Chapter 11]), for a period mp problem, $\Phi_{mp} - \Phi_{mp}^*$ has its zeros, one in each gap. The gaps of the mp problem are the gaps of the original problem plus a closed gap at each point where (3.21) holds. There is a zero in each closed gap and at each point where (3.16) holds since then $T_{mp}(z)\binom{1}{1} = c_j^m \binom{1}{1}$. $\square$

We now turn to the analysis of zeros of $\varphi_{mp+b}(z)$, $b = 0, 1, \dots, p-1$; $m = 0, 1, 2, \dots$. The analog of (2.38) is, by (3.7),

$$\varphi_{mp+b} = \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, T_b(T_p)^m \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\rangle \tag{3.22}$$

As in Section 2, we write, for $z \in \mathbb{C}\backslash\text{bands}$:

$$z^{-p/2} T_p(z) = \Gamma_+(z) P_+(z) + \Gamma_-(z) P_-(z) \tag{3.23}$$

where $P_\pm$ are 2×2 matrices which are complementary projections, that is, (2.28)/(2.29) hold. (2.30)/(2.31) are replaced by

$$P_+ = \frac{z^{-p/2} T_p(z) - \Gamma_-(z)\mathbf{1}}{\Gamma_+ - \Gamma_-} \tag{3.24}$$

$$P_- = \frac{z^{-p/2} T_p(z) - \Gamma_+(z)\mathbf{1}}{\Gamma_- - \Gamma_+} \tag{3.25}$$

So, in particular, $P_\pm$ have meromorphic continuations to $\mathcal{S}$, and P_+ continued to the other sheet is P_-.

Define

$$a(z) = \frac{1}{2} \left\langle \begin{pmatrix} 1 \\ 1 \end{pmatrix}, P_+ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\rangle \tag{3.26}$$

$$b(z) = \frac{1}{2} \left\langle \begin{pmatrix} 1 \\ -1 \end{pmatrix}, P_+ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\rangle \tag{3.27}$$

so that, by (2.29),

$$\frac{1}{2} \left\langle \begin{pmatrix} 1 \\ 1 \end{pmatrix}, P_- \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\rangle = 1 - a(z) \tag{3.28}$$

$$\frac{1}{2} \left\langle \begin{pmatrix} 1 \\ -1 \end{pmatrix}, P_- \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\rangle = -b(z) \tag{3.29}$$

Thus, since $\frac{1}{\sqrt{2}}\binom{1}{1}, \frac{1}{\sqrt{2}}\binom{1}{-1}$ are an orthonormal basis,

$$\begin{aligned} z^{-mp/2} T_{mp} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \Gamma_+^m &\left[a(z) \begin{pmatrix} 1 \\ 1 \end{pmatrix} + b(z) \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right] \\ &+ \Gamma_-^m \left[(1 - a(z)) \begin{pmatrix} 1 \\ 1 \end{pmatrix} - b(z) \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right] \end{aligned} \tag{3.30}$$

Therefore, by (3.7), (3.8), and (3.22),

$$\begin{aligned} \varphi_{mp+b}(z) = \varphi_b(z) &[a(z) z^{mp/2} \Gamma_+^m + (1-a) z^{mp/2} \Gamma_-^m] \\ &+ \psi_b(z) [b(z) z^{mp/2} \Gamma_+^m - b(z) z^{mp/2} \Gamma_-^m] \end{aligned} \tag{3.31}$$

We thus define

$$j_b(z) = a(z)\varphi_b(z) + b(z)\psi_b(z) \tag{3.32}$$

and we have, since $|\Gamma_+| > |\Gamma_-|$ on $\mathbb{C}\backslash$bands:

Theorem 3.2. *For $z \in \mathbb{C}\backslash$bands,*

$$\lim_{m\to\infty} z^{-mp/2}\Gamma_+^{-m}\varphi_{mp+b}(z) = j_b(z) \tag{3.33}$$

In addition, j_b is nonvanishing near $z = \infty$.

In particular, if $z_0 \notin$ bands and $j_b(z_0) \neq 0$, then for some $\varepsilon > 0$ and M, we have $\varphi_{mp+b}(z_0) \neq 0$ if $|z - z_0| < \varepsilon$ and $m \geq M$. If $z_0 \notin$ bands and $j_b(z_0)$ has a zero of order k, then for some $\varepsilon > 0$ and all m large, $\varphi_{mp+b}(z)$ has precisely k zeros (counting multiplicity). The number of z_0 in $\mathbb{C}\backslash$bands with $j_b(z_0) = 0$ is at most $2p + 2b - 1$.

Proof. As noted, (3.31) and $|\Gamma_+| > |\Gamma_-|$ imply (3.33). To analyze $j_b(z)$ near $z = \infty$, we proceed as follows: We have, by (3.2) and (3.5), that as $|z| \to \infty$,

$$T_p(z) = z^p\left(\prod_{j=0}^{p-1}\rho_j^{-1}\right)\left[\begin{pmatrix}1 & 0\\ -\alpha_{p-1} & 0\end{pmatrix}\cdots\begin{pmatrix}1 & 0\\ -\alpha_0 & 0\end{pmatrix}\right] + O(z^{p-1}) \tag{3.34}$$

$$= z^p\left(\prod_{j=1}^{p-1}\rho_j^{-1}\right)\begin{pmatrix}1 & 0\\ -\alpha_{p-1} & 0\end{pmatrix} + O(z^{p-1}) \tag{3.35}$$

from which it follows that

$$P_+ = \begin{pmatrix}1 & 0\\ -\alpha_{p-1} & 0\end{pmatrix} + O(z^{-1}) \tag{3.36}$$

$$P_- = \begin{pmatrix}0 & 0\\ \alpha_{p-1} & 1\end{pmatrix} + O(z^{-1}) \tag{3.37}$$

and

$$a(z) = \tfrac{1}{2}\,(1 - \alpha_{p-1}) + O(z^{-1}) \tag{3.38}$$

$$b(z) = \tfrac{1}{2}\,(1 + \alpha_{p-1}) + O(z^{-1}) \tag{3.39}$$

We have

$$\varphi_b(z) = \left(\prod_{j=0}^{p-1}\rho_j^{-1}\right)z^b + O(z^{b-1}) \tag{3.40}$$

$$\psi_b(z) = \left(\prod_{j=0}^{p-1}\rho_j^{-1}\right)z^b + O(z^{b-1}) \tag{3.41}$$

from which we see that

$$j_b(z) = \left(\prod_{j=0}^{p-1}\rho_j^{-1}\right)z^b + O(z^{b-1}) \tag{3.42}$$

since (3.38)/(3.39) imply $a(z)+b(z) = 1+O(z^{-b})$. In particular, $j_b(z)$ is not zero near ∞, so j_b is not identically zero, and the assertion about locations of zeros of $\varphi_{mp+b}(z)$ follows from Hurwitz's theorem.

Since $1 - a(z) - b(z) = O(z^{-1})$, (3.28)/(3.29) imply that, on the second sheet, the analytic continuation of $j_b(z)$ near ∞ is $O(z^{b-1})$. It follows that j_b has a pole of order b at ∞ on the main sheet (regular if $b = 0$) and a pole of order at most $b-1$ (a zero if $b = 0$ and is regular if $b = 1$) at ∞ on the second sheet. j_b also can have at most $2p$ simple poles at the $2p$ branch points.

It follows that the degree of j_b as a meromorphic function on $\mathcal{S}$ is at most $2p + 2b - 1$ (if $b = 0$, $2p$). Thus the number of zeros is at most $2p+2b-1$ if $b \neq 0$. If $b = 0$, there are at most $2p+2b$ zeros. But since then one is at ∞ on the second sheet, the number of zeros on finite points is at most $2p + 2b - 1$. □

Next, we note that

Theorem 3.3. *Let $\{\alpha_n\}$ be periodic and not at all 0. z_0 is a limit of zeros of $\varphi_n(z)$ (i.e., there exist z_n with $\varphi_n(z_n) = 0$ and $z_n \to z_0$ if and only if z_0 lies in the support of $d\mu$).*

Remark. $\alpha_n = 0$ has 0 as a limit point of zeros at $\varphi_n(z) = z^n$, so one needs some additional condition on the α's to assure this result.

Proof. By Theorems 8.1.11 and 8.1.12 of [15], if $z_0 \in \text{supp}(d\mu)$, then it is a limit point of zeros. For the other direction, suppose $z_0 \notin$ bands and is a limit point of zeros. By Theorem 3.2, $j_b(z_0) = 0$ for each $b = 0, 1, \dots, p-1$, so by (3.31), $\varphi_{mp+b}(z) \sim C(\Gamma_- z_0^{p/2})^m$ which, since $|z_0| \le 1$ and $|\Gamma_-| < 1$, implies that $\varphi_n(z_0)$ goes to zero exponentially.

Since α_n is not identically zero, some α_j, $j \in \{0, 1, \dots, p-1\}$ is nonzero. Thus, by Szegő recursion for φ_j,

$$\varphi^*_{mp+j}(z_0) = \alpha_j^{-1}[z_0\varphi_{mp}(z_0) - \rho_j\varphi_{mp+1}(z_0)]$$

goes to zero exponentially in m.

Since α_n is periodic, $\sup_n|\alpha_n| < 1$, and so, $\sup_n \rho_n^{-1} < \infty$. Since

$$\varphi^*_{mp+j+1}(z_0) = \rho_{j+1}^{-1}(\varphi^*_{mp+j}(z_0) - \alpha_j\varphi_{mp+j}(z_0))$$

we see $\varphi^*_{mp+j+1}(z_0)$ decays exponentially and so, by induction, $\varphi^*_n(z_0)$ decays exponentially. By the Christoffel-Darboux formula (see [15, eqn. (2.2.70)]), $|\varphi^*_n(z_0)|^2 \ge 1 - |z_0|^2$, so the decay implies $|z_0| = 1$. But if $z_0 \in \partial\mathbb{D}$ and $\sum_n|\varphi_n(z_0)|^2 < \infty$, then $\mu(\{z_0\}) > 0$ (see [15, Theorem 2.7.3]).

Thus if z_0 is a limit of zeros, either $z_0 \in$ bands or $\mu(\{z_0\}) > 0$, that is, $z_0 \in \text{supp}(d\mu)$. □

Finally, in our analysis of periodic OPUC, we turn to zeros near to the bands. We define $\tilde{j}_b$ on $\mathbb{C}\backslash$bands so that (3.31) becomes

$$\varphi_{mp+b}(z) = j_b(z) z^{mp/2} \Gamma_+^m + \tilde{j}_b(z) z^{mp/2} \Gamma_-^m \tag{3.43}$$

While $\varphi_{mp+b}(z)$ is continuous across the bands, j_b, $\tilde{j}_b$, and $\Gamma_\pm$ are not. In fact, Γ_+ (resp. j_b) continued across a band becomes Γ_- (resp. $\tilde{j}_b$). We define all four objects at $e^{i\theta} \in \partial\mathbb{D}$ as limits as $r \uparrow 1$ of the values at $re^{i\theta}$.

Proposition 3.4. (i) *In the bands,*

$$e^{i\theta p/2} \Gamma_+(e^{i\theta}) = \exp(-i\pi p\, k(\theta)) \tag{3.44}$$

(ii) *At no point in the bands do both $j_b(e^{i\theta})$ and $\tilde{j}_b(e^{i\theta})$ vanish.*
(iii) *$\tilde{j}_b$ is everywhere nonvanishing on the interiors of the bands.*

Proof. (i) This follows from (3.20). There is an issue of checking that it is $\exp(-i\pi p\, k(\theta))$, not $\exp(i\pi p\, k(\theta))$. To confirm this, note that $\frac{\partial}{\partial\theta} \operatorname{Im} \log(\exp(-i\pi p\, k(\theta))) \le 0$ and mainly < 0. Since $\partial|\Gamma_+|/\partial r \le 0$ at $r = 1$, this is consistent with (3.44) and the Cauchy-Riemann equations.

(ii) follows from (3.43) and the fact that $\varphi_n(z)$ is nonvanishing on $\partial\mathbb{D}$.

(iii) Continue (3.43) through the cut. Since φ_m is entire, the continuation onto the "second sheet" is also φ_m. $\Gamma_\pm$ get interchanged by crossing the cut. Let us use $j_{b,2}, \tilde{j}_{b,2}$ for the continuation to the second sheet (of course, $j_{b,2}$ is $\tilde{j}_b$ on the second sheet, but that will not concern us).

By this (3.43) continued, $\varphi_{mp+b}(z) = 0$ if and only if

$$\left(\frac{\Gamma_-(z)}{\Gamma_+(z)} \right)^m = -\frac{\tilde{j}_{b,2}(z)}{j_{b,2}(z)} \tag{3.45}$$

If $\tilde{j}_b(z_0) = 0$ for $z_0 \in \partial\mathbb{D}$, then $|\tilde{j}_{b,2}(rz_0)/j_{b,2}(rz_0)|$ goes from 0 to a nonzero value as r increases. On the other hand, since $|\Gamma_-/\Gamma_+| < 1$ on $\mathbb{C}\backslash$bands, for m large, $|\Gamma_-(rz_0)/\Gamma_+(rz_0)|^m$ goes from 1 to a very small value as r increases. It follows that for m large,

$$\left| \frac{\Gamma_-(z)}{\Gamma_+(z)} \right|^m = \left| \frac{\tilde{j}_{b,2}(z)}{j_{b,2}(z)} \right|$$

has a solution $r_m z_0$ with $r_m > 1$ and $r_m \to 1$. As in [17], we can change the phase slightly to ensure (3.45) holds for some point, z_m, near $r_m z_0$ with $|z_m| > 1$. Since φ has no zero in $\mathbb{C}\backslash\mathbb{D}$, this is a contradiction. □

Remark. This proof shows that in the bands $|\tilde{j}_b(e^{i\theta})| > |j_b(e^{i\theta})|$.

(3.43) says we want to solve

$$\left(\frac{\Gamma_-(z)}{\Gamma_+(z)}\right)^m = g_b(z) \tag{3.46}$$

to find zeros of $\varphi_{mp+b}(z)$. We have, by the remark, that $|g(\theta)| < 1$.

Definition. We call $z_0 \in$ bands a singular point of order k if $j_b(z_0) = 0$ and the zero is of order k.

We do not know if there are singular points in any example! If so, they should be nongeneric. We define the functions

$$\tilde g_b(\theta) = -\frac{j_b(e^{i\theta})}{\tilde j_b(e^{i\theta})} \tag{3.47}$$

and

$$g_b(z) = -\frac{j_b(z)}{\tilde j_b(z)} \tag{3.48}$$

For $e^{i\theta}$ in the interior of a band minus the singular points, let $A(\theta)$ be given by

$$\frac{\tilde g_b(\theta)}{\overline{\tilde g_b(\theta)}} = \exp(2iA(\theta)) \tag{3.49}$$

with A continuous away from the singular points.

The analysis of a similar equation to (3.46) in [17] shows that:

(a) The solutions of (3.46) near $|z| = 1$ lie in sectors where

$$2\pi p\, k(\theta) = A(\theta) + \frac{2\pi j}{m} + O\left(\frac{1}{m \log m}\right) \tag{3.50}$$

with exactly one solution in each such sector.

(b) The magnitudes of the solutions obey

$$|z| = 1 - O\left(\frac{\log m}{m}\right) \tag{3.51}$$

(c) Successive zeros z_{k+1}, z_k obey

$$k(\arg(z_{k+1})) - k(\arg z_k) = \frac{1}{mp} + O\left(\frac{1}{m \log m}\right) \tag{3.52}$$

and

$$\left|\frac{z_{k+1}}{z_k}\right| = 1 + O\left(\frac{1}{m \log m}\right) \tag{3.53}$$

(d) All estimates in (a)–(c) are uniform on a band.

(e) Away from singular points, all $O(1/m \log m)$ errors can be replaced by $O(1/m^2)$ and $O(\log m/m)$ in (3.51) by $O(1/m)$. If there are no singular points, these are uniform over a band.

It is easy to see that the total variation of A in each interval between singular points (or band endpoints) is finite, so (3.50) and the fact that k varies by $1/p$ over a band say that the number of solutions in a band differs from m by a finite amount. This implies

Theorem 3.5. *Let $N^{(n,j)}$ be the number of zeros, z_0, of $\varphi_n(z)$ that obey*
(a) $\arg z_0 \in$ *band j*
(b)
$$(1 - |z_0|) \leq n^{-1/2} \tag{3.54}$$
Then
(a) $\sup_{n,j} |N^{(n,j)} - \frac{n}{p}| < \infty$
(b) *For n large, all such zeros have*
$$(1 - |z_0|) \leq C\,\frac{\log n}{n} \tag{3.55}$$
and if there are no singular points, we can replace $\log n/n$ in (3.55) *by $1/n$.*

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