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FINE TOPOLOGY ON FUNCTION SPACES

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ABSTRACT. This paper studies the topological properties of two kinds of "fine topologies" on the space $C(X,Y)$ of all continuous functions from X into Y.

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1. INTRODUCTION.

The topology of pointwise convergence and the compact-open topology are two of the most commonly used topologies on the set $C(X,Y)$ of continuous functions from a space X into a space Y. These spaces will be denoted by $C_p(X,Y)$ and $C_k(X,Y)$, respectively. If Y is a metric space, the supremum metric topology on $C(X,Y)$ is also commonly used. However, sometimes none of these topologies is strong enough to apply a function space to a given situation, in which case a finer topology may be needed. A good example of this is the use of a "fine topology" on a function space in [4], in which the Baire space property of the function space is used to obtain certain kinds of embeddings into infinite-dimensional manifolds.

This paper studies the topological properties of two kinds of "fine topologies" on $C(X,Y)$. In order to avoid pathologies, all spaces will be Tychonoff spaces. The symbol $\mathbb{R}$ will denote the real line with the usual topology, and $\mathbb{R}^+$ will denote the positive real line. Also $C(X,\mathbb{R})$ and $C(X,\mathbb{R}^+)$ will be abbreviated as $C(X)$ and $C^+(X)$. Finally let ω denote the set of natural numbers.

1. UNIFORM TOPOLOGIES.

Whenever a space Y has a compatible uniform structure on it, this induces a uniform structure on $C(X,Y)$. If δ is a diagonal uniformity on Y, then for each $D\in\delta$, define

$$\hat{D} = \{(f,g)\in C(X,Y)^2 : \text{for every } x\in X,\ (f(x),g(x))\in D\}. \tag{1.1}$$

The family $\{\hat{D} : D\in\delta\}$ is a base for a diagonal uniformity $\hat{\delta}$ on $C(X,Y)$. Denote the resulting topological space by $C_\delta(X,Y)$. On the other hand, if μ is a covering uniformity on Y, then for each $\mathcal{U}\in\mu$, let

$$\hat{\mathcal{U}} = \{(f,g)\in C(X,Y)^2 : \text{for every } x\in X, \text{ there exists a } U\in\mathcal{U} \text{ with } (f(x),g(x))\in U^2\}. \tag{1.2}$$

The family $\{\hat{\mathcal{U}} : \mathcal{U}\in\mu\}$ is also a base for a diagonal uniformity $\hat{\mu}$ on $C(X,Y)$. Denote this space by $C_\mu(X,Y)$.

There is a natural way of passing from a diagonal uniformity δ to a covering uniformity μ_δ, so that $\mu_{\delta_\mu} = \mu$ and $\delta_{\mu_\delta} = \delta$ (cf. Willard [5], section 36). It can be easily verified that $C_{\mu_\delta}(X,Y) = C_\delta(X,Y)$ and $C_{\delta_\mu}(X,Y) = C_\mu(X,Y)$. Therefore a uniform structure on Y may be considered either as a diagonal uniformity or a covering uniformity, and the resulting uniform structures on C(X,Y) will generate the same topology, called the uniform topology.

Let ν stand for the fine (covering) uniformity on Y. Whenever Y is paracompact, ν has as a base the family of all open covers of Y. The topology on $C_\nu(X,Y)$ will be called the fine uniform topology.

If μ is any compatible uniformity on Y, then the relationships between the various topologies discussed above are given by

$$C_p(X,Y) \leq C_k(X,Y) \leq C_\mu(X,Y) \leq C_\nu(X,Y), \tag{1.3}$$

where the inequality means the space on the right is finer than that on the left.

Each compatible bounded metric ρ on Y induces the supremum metric $\hat{\rho}$ on C(X,Y), defined by $\hat{\rho}(f,g) = \sup\{\rho(f(x),g(x)) : x\in X\}$. The resulting topological space will be denoted by $C_\rho(X,Y)$. A base for $C_\rho(X,Y)$ consists of the metric balls $\{B_\rho(f,e) : f\in C(X,Y) \text{ and } e\in \mathbb{R}^+\}$. If μ_ρ is the uniformity on Y generated by ρ, then $C_{\mu_\rho}(X,Y) = C_\rho(X,Y)$.

For a metrizable space Y, let M(Y) be the family of all compatible bounded metrics on Y. The following fact gives a useful tool for working with the fine uniform topology.

PROPOSITION 1.1. If Y is metrizable, then $C_\nu(X,Y)$ has as a base $\{B_\rho(f,e) : \rho\in M(Y),\ f\in C(X,Y) \text{ and } e\in \mathbb{R}^+\}$.

PROOF. To see that $B_\rho(f,e)$ is a neighborhood of f in $C_\nu(X,Y)$, define the open cover $\mathcal{U} = \{B_\rho(y,e/3) : y\in Y\}$ of Y, and let $g\in\mathcal{U}[f]$. Then for each $x\in X$, there is a $y\in Y$ with $(f(x), g(x)) \in B_\rho(y, e/3)$. Therefore each $\rho(f(x),g(x))<2e/3$, so that $\hat{\rho}(f,g)\leq 2e/3<e$. This establishes that $g\in B_\rho(f,e)$, and it follows that $\hat{\mathcal{U}}[f]\subset B_\rho(f,e)$.

On the hand, let $\mathcal{U}\in\mu$ and $f\in C(X,Y)$. Let $\mathcal{U}_1 >^* \mathcal{U}_2 >^* \ldots$ be a normal sequence of open covers of Y so that $\mathcal{U}_1$ refines $\mathcal{U}$, and let ρ be the metric defined by this sequence (see Willard [5], p. 167, for this construction). It follows from the construction of ρ that there is an $e\in\mathbb{R}^+$ so that $B_\rho(f,e)\subset\hat{\mathcal{U}}[f]$.

Let μ be any compatible uniformity on Y. If X is compact then $\hat{\mu}$ on C(X,Y) is the same as the uniformity of uniform convergence on compact sets, which is known to generate the compact-open topology (cf. Willard [5], section 43). The converse is in fact also true for most Y.

PROPOSITION 1.2. If Y contains a nontrivial path, then for any compatible uniformity μ on Y, $C_\mu(X,Y) = C_k(X,Y)$ if and only if X is compact.

PROOF. Let $\phi : I \to Y$ be a continuous function from the closed unit interval into Y such that $\phi(0) \neq \phi(1)$. Let $y = \phi(0)$, let $z = \phi(1)$, and let f be the constant map from X to y. Define $\mathcal{U}=\{Y\backslash\{y\}, Y\backslash\{z\}\}$, which is an open cover of Y. To see that $\mathcal{U}\in\mu$, let $D\in\delta_\mu$ with $z\notin D[y]$. Choose a symmetric $E\in\delta_\mu$ with $E\circ E\subset D$. It follows that $\{E[p] : p\in Y\}$ refines $\mathcal{U}$, so that $\mathcal{U}\in\mu$.

Suppose that X is not compact. To see that $\hat{\mathcal{U}}[f]$ is not a neighborhood of f in $C_k(X,Y)$, let $W = [A_1,V_1] \cap \ldots \cap [A_n,V_n]$ be any basic open subset of $C_k(X,Y)$ which contains f, where each $[A_i,V_i] = \{f\varepsilon C(X,Y) : f(A_i) \subset V_i\}$ and the A_i are compact and the V_i are open. If $A = A_1 \cup \ldots \cup A_n$, then there is some $x\varepsilon X\setminus A$. Define $g: A\cup\{x\} \to Y$ by $g(a) = y$ for $a\varepsilon A$ and $g(x) = z$. Since $\phi(I)$ is arcwise connected, then g extends to some $\hat{g}\varepsilon C(X,Y)$. Then $\hat{g}\varepsilon W\setminus\hat{\mathcal{U}}[f]$, which establishes that $C_\mu(X,Y)$ is finer than $C_k(X,Y)$.

It follows from Proposition 1.2 that whenever X is compact, all compatible uniformities on Y generate the same topology on C(X,Y) - the compact-open topology. Also when Y is compact, a compatible uniformity on Y generates a unique topology on C(X,Y) since there is only one compatible uniformity on Y - but in this case the topology on C(X,Y) is not the compact-open topology, unless X is also compact.

PROPOSITION 1.3. Let Y be a metrizable space. If X is psuedocompact, then all compatible uniformities on Y generate the same topology on C(X,Y).

PROOF. Let $\mathcal{U}$ be an open cover of $\mathbb{R}$, and let $f\varepsilon C(X,Y)$. For each $x\varepsilon X$, let $V_x \varepsilon \mathcal{U}$ with $f(x) \varepsilon V_x$. Then for each such x there exists a $\mathcal{U}_x \varepsilon \mu$ with $(D_x \circ D_x)[f(x)] \subset V_x$, where $D_x = \cup\{U^2 : U\varepsilon\mathcal{U}_x\}$. Since X is psuedocompact and Y is metrizable, then f(X) is compact. So there exist $x_1, \ldots, x_n \varepsilon X$ such that $f(X) \subset D_{x_1}[f(x_1)] \cup \ldots \cup D_{x_n}[f(x_n)]$. Now there exists an $\mathcal{U}_0 \varepsilon \mu$ which refines each of $\mathcal{U}_{x_1}, \ldots, \mathcal{U}_{x_n}$.

In order to show that $\hat{\mathcal{U}}_0[f] \subset \hat{\mathcal{U}}[f]$, let $g\varepsilon\hat{\mathcal{U}}_0[f]$ and $x\varepsilon X$. Then there exists an i such that $f(x)\varepsilon D_{x_1}[f(x_i)]$. Let $U\varepsilon\mathcal{U}_0$ with $(f(x), g(x)) \varepsilon U^2$. There is some $U_i \varepsilon \mathcal{U}_{x_1}$ such that $U \subset U_i$, so that $(f(x),g(x)) \varepsilon U_i^2 \subset D_{x_1}$. Therefore $f(x)\varepsilon D_{x_1}[f(x_i)] \subset (D_{x_1} \circ D_{x_1})[f(x_i)] \subset V_{x_1}$ and $g(x) \varepsilon (D_{x_1} \circ D_{x_1})[f(x_i)] \subset V_{x_1}$. Since x is arbitrary, $g\varepsilon\hat{\mathcal{U}}[f]$.

The topology generated in Proposition 1.3 is in general not the compact-open topology.

The "completeness" of a function space can be a useful property for obtaining the existence of certain kinds of functions. If ρ is a complete metric on Y, then $\hat{\rho}$ is a complete metric on C(X,Y). On the other hand, $C_\nu(X,Y)$ may not be metrizable. So complete metrizability of $C_\nu(X,Y)$ is too much to expect in general. But $C_\nu(X,Y)$ does have "completeness" to the following extent.

THEOREM 1.4. If Y is completely metrizable, then $C_\nu(X,Y)$ is a Baire space.

PROOF. Let ρ be a compatible bounded complete metric on Y. Also let $\{W_n : n \varepsilon\omega\}$ be a sequence of dense open subsets of $C_\nu(X,Y)$, and let W be a nonempty open subset of $C_\nu(X,Y)$ Choose $d_1 \varepsilon M(Y)$, $f_1 \varepsilon C(X,Y)$, and $0<\varepsilon_1< 1/2$ so that $B_{d_1}(f_1,\varepsilon_1) \subset W_1 \cap W$. Define $\rho_1 = \max\{\rho, d_1\}$, so that $B_{\rho_1}(f_1,\varepsilon_1) \subset B_{d_1}(f_1,\varepsilon_1)$. Continue by induction so that at the n+1 step, choose $d_{n+1}\varepsilon M(Y)$, $f_{n+1}\varepsilon C(X,Y)$, and $0<\varepsilon_{n+1}<1/2^{n+1}$ such that $B_{d_{n+1}}(f_{n+1},\varepsilon_{n+1}) \subset W_{n+1} \cap B_{\rho_n}(f_n,\varepsilon_n/2)$; and define $\rho_{n+1} = \max\{\rho_n, d_{n+1}\}$. Now $\{f_n : n\varepsilon\omega\}$ is a Cauchy sequence in $C_\rho(X,Y)$, and therefore converges to some $f\varepsilon C_\rho(X,Y)$. Also for each $i\varepsilon\omega$, $\{f_n : n\varepsilon\omega\}$ will converge to f in $C_{\rho_i}(X,Y)$, so that f is in the closure of $B_{\rho_i}(f_i,\varepsilon_i/2)$ in $C_{\rho_i}(X,Y)$. Therefore $f\varepsilon B_{\rho_i}(f_i,\varepsilon_i) \subset B_{d_i}(f_i,\varepsilon_i) \subset W_i \cap W$ for every $i\varepsilon\omega$.

The conclusion of Theorem 1.4 cannot be strengthened to Cech-completeness, which can be seen from Theorem 4.1. Also the hypothesis cannot be changed to Y being compact. For example, if X is the closed unit interval I with the usual topology, and $Y = I^2$

with the order topology with respect to lexicographic ordering, then some functions in $C_\nu(X,Y)$ have open neighborhoods which can be written as countable unions of nowhere dense sets.

2. FINE TOPOLOGIES.

Throughout this section, (Y,ρ) will be a metric space. In this case there is a natural way to generate a topology on $C(X,Y)$ which may be even finer than the fine uniformity topology.

For each $f\varepsilon C(X,Y)$ and $\phi\varepsilon C^+(X)$, define $B_\rho^+(f,\phi) = \{g\varepsilon C(X,Y)$: for every $x\varepsilon X$, $\rho(f(x),g(x))<\phi(x)\}$. Since for each $\phi,\psi\varepsilon C^+(X)$, $\max\{\phi,\psi\} \varepsilon C^+(X)$, then $\{B_\rho^+(f,\phi) : f\varepsilon C(X,Y)$ and $\phi\varepsilon C^+(X)\}$ is a base for a topology on $C(X,Y)$. This topology is called the fine topology with respect to ρ (Munkres [3], p.285), and will be denoted by $C_{f_\rho}(X,Y)$. Certainly $C_{f_\rho}(X,Y)$ is finer than $C_\rho(X,Y)$, and is in general strictly finer. However, for psuedocompact X, they are the same.

PROPOSITION 2.1. If X is pseudocompact, then $C_{f_\rho}(X,Y) = C_\rho(X,Y)$.

PROOF. Let $f\varepsilon C_{f_\rho}(X,Y)$, and let $\phi\varepsilon C^+(X)$. Since $1/\phi\varepsilon C^+(X)$ and since X is psuedocompact, then $1/\phi$ is bounded. So there is a number M with $1/\phi(x)< M$ for all $x\varepsilon X$. Therefore $B_\rho(f,1/M)\subset B_\rho^+(f,\phi)$.

From Propositions 1.3 and 2.1 it follows that if X is psuedocompact then $C_{f_\rho}(X,Y) = C_\nu(X,Y)$. But even if X is not psuedocompact, $C_{f_\rho}(X,Y)$ can still be related to $C_\nu(X,Y)$ as follows.

PROPOSITION 2.2. If X is paracompact, then $C_\nu(X,Y)\leq C_{f_\rho}(X,Y)$.

PROOF. Let $d\varepsilon M(Y)$, let $f\varepsilon C(X,Y)$, and let $e\varepsilon \mathbb{R}^+$. For each $x\varepsilon X$, there exists a $\psi(x)\varepsilon \mathbb{R}^+$ such that $B_\rho(f(x),\psi(x))\subset B_d(f(x),e/2)$. Let $\mathcal{U} =\{U_\alpha : \alpha \varepsilon A\}$ be a star-refinement of $\{f^{-1}(B_\rho(f(x),\psi(x)/2)) : x\varepsilon X\}$. Let $\{\phi_\alpha : \alpha \varepsilon A\}$ be a partition of unity subordinated to $\mathcal{U}$. For each $\alpha\varepsilon A$, let n_α be the cardinality of $\{\beta\varepsilon A : U_\alpha\cap U_\beta\neq\emptyset\}$, let $x_\alpha\varepsilon X$ be such that $U_\alpha\subset f^{-1}(B_\rho(f(x_\alpha),\psi(x_\alpha)))$, and let $m_\alpha = \min\{\psi(x_\beta)/2 : U_\alpha\cap U_\beta\neq \emptyset\}$. Then define $\phi = \Sigma\{(m_\alpha/n_\alpha)\phi_\alpha : \alpha\varepsilon A\}$, which is a member of $C^+(X)$.

To establish that $B_\rho^+(f,\phi) \subset B_d(f,e)$, let $g\varepsilon B_\rho^+(f,\phi)$ and let $x\varepsilon X$. Also let $U_{\alpha_1}, \ldots, U_{\alpha_k}$ be the members of $\mathcal{U}$ containing x. Take i to be such that $m_{\alpha_1}\phi_{\alpha_1}(x) = \min\{m_{\alpha_1}\phi_{\alpha_1}(x),\ldots,m_{\alpha_k}\phi_{\alpha_k}(x)\}$. Then $\phi(x) \leq m_{\alpha_1}\phi_{\alpha_1}(x)(1/n_{\alpha_1}+\ldots+1/n_{\alpha_k})$. $\leq m_{\alpha_i}\phi_{\alpha_i}(x) \leq \psi(x_{\alpha_i})/2$. It follows that $\rho(g(x), f(x)) < \psi(x_{\alpha_i})/2$. Since $f(x)\varepsilon B_\rho(f(x_{\alpha_i}),\psi(x_{\alpha_i})/2)$, then $g(x)\varepsilon B_\rho(f(x_{\alpha_i}),\psi(x_{\alpha_i}))\subset B_d(f(x_{\alpha_i}),e/2)$. Also $f(x)\varepsilon B_d(f(x_{\alpha_i}),e/2)$, so that $g(x)\varepsilon B_d(f(x),e)$.

The inequality in Proposition 2.2 is in general not an equality as indicated in the comment after Corollary 3.5.

Whenever ρ is complete then $C_{f_\rho}(X , Y)$ is a Baire space ([4], p.297). Instead of giving a proof of this here, a proof will be given in the next section that $C_{f_\rho}(X)$ is psuedo-complete, which is a property stronger than being a Baire space.

3. REAL-VALUED FUNCTIONS.

For the rest of the paper, ρ will denote the usual metric on $\mathbb{R}$ bounded by 1; that is, $\rho(s,t) = \min\{1, |s-t|\}$.

LEMMA 3.1. Let $f,g\varepsilon C(X)$ and let $\phi,\psi\varepsilon C^+(X)$. Then the closure of $B_\rho^+(f,\phi)$ in $C_{f_\rho}(X)$ is contained in $B_\rho^+(g,\psi)$ if and only if for each $x\varepsilon X$ the closure of $B_\rho(f(x),\phi(x))$ in $\mathbb{R}$ is contained in $B_\rho(g(x),\psi(x))$.

PROOF. For the sufficiency, let $h\varepsilon B^+_\rho(f,\phi)$ and let $x\varepsilon X$. Suppose $h(x)$ were contained in the complement of the closure of $B_\rho(f(x),\phi(x))$; call this set V. Then the set of functions taking x into V would be a neighborhood of h in $C_{f_\rho}(X)$ which misses $B^+_\rho(f,\phi)$. This contradiction shows that $h(x)\ \varepsilon\ cl(B_\rho(f(x),\phi(x)))\subset B_\rho(g(x),\phi(x))$.

For necessity, let $x_0\ \varepsilon\ X$ and let $t\ \varepsilon\ cl(B_\rho(f(x_0),\phi(x_0)))$ in $\mathbb{R}$. Define $h(x) = f(x) + (t-f(x_0))\cdot\phi(x)/\phi(x_0)$ for all $x\ \varepsilon\ X$. This defines an $h\ \varepsilon\ C(X)$ such that $h(x_0)=t$. Also $|h(x)-f(x)|\le\phi(x)$ for each x.

To see that $h\varepsilon cl(B^+_\rho(f,\phi))$ in $C_{f_\rho}(X)$, let $B^+_\rho(h,\xi)$ be a basic neighborhood of h. Define $k(x) = h(x) + \operatorname{sign}(f(x) - h(x))\cdot\min\{\xi(x)/2,|h(x)-f(x)|\}$, which is an element of C(X). Now $|k(x)-h(x)| = \min\{\xi(x)/2,|h(x)-f(x)|\}\le \xi(x)/2 <\xi(x)$. Also if $h(x)>f(x)$, then $|k(x)-f(x)| = |h(x)-f(x)-\min\{\xi(x)/2, h(x)-f(x)\}|$. If this is positive, then it is equal to $\xi(x)/2$, and $\xi(x)/2\le|h(x)-f(x)|<\phi(x)$. The same argument shows that if $h(x) < f(x)$, then also $|k(x)-f(x)| < \phi(x)$. Therefore $k\varepsilon B^+_\rho(f,\phi)$, so that $B^+_\rho(h,\xi)\cap B^+_\rho(f,\phi)\neq\emptyset$. From this it follows that $h\varepsilon cl(B^+_\rho(f,\phi))\subset B^+_\rho(g,\psi)$. But then $|t-g(x_0)| = |h(x_0) - g(x_0)|<\psi(x_0)$, so that $t\varepsilon B_\rho(g(x_0),\psi(x_0))$.

The proof of Lemma 3.1 depends on the metric (and algebraic) structure of the range space. In fact the lemma is not true in general. For example, let $X=I$ and $Y = \{(s,\sin(2\pi/s))\varepsilon\ \mathbb{R}^2 : 0<s\le2\}\cup\{(0,0)\}$ with metric d on Y defined by $d((s_1,t_1),(s_2,t_2)) = \max\{|s_1-s_2|,|t_1-t_2|\}$. Also let $y=(2,0)$, let $z=(0,0)$, let f be the constant map taking X to y, and let g be the constant map taking X to z. Then $B_d(f,2) = C(X,Y)\backslash\{g\}$. Since g is isolated, $B_d(f,2)$ is closed, so that $cl(B_d(f,2))\subset B_d(f,2)$. On the other hand, for any $x\varepsilon X$, $z\varepsilon cl(B_d(f(x),2))\backslash B_d(f(x),2)$ in Y.

THEOREM 3.2. The space $C_{f_\rho}(X)$ is psuedo-complete.

PROOF. For each $n\varepsilon\omega$, let $C^+_n(X) = \{\phi\varepsilon C^+(X) : \text{for all } x\varepsilon X,\ \phi(x)<1/2^n\}$, and define $\mathcal{B}_n = \{B^+_\rho(f,\phi) : f\varepsilon C(X) \text{ and } \phi\varepsilon C^+_n(X)\}$. Each $\mathcal{B}_n$ is a base for $C_{f_\rho}(X)$. It remains to show that if $B_n\varepsilon\mathcal{B}_n$ for each n with $cl(B_n)\subset B_n$ then $\cap\{B_n:n\varepsilon\omega\}\neq\emptyset$. If each $B_n=B^+_\rho(f_n,\phi_n)$, then by Lemma 3.1, for each $n\varepsilon\omega$ and each $x\varepsilon X$, $cl(B_\rho(f_{n+1}(x),\phi_{n+1}(x)))\subset B_\rho(f_n(x),\phi_n(x))$. Since each $\phi_n(x)<1/2^n$, then $\cap\{B_\rho(f_n(x),\phi_n(x)) : n\varepsilon\omega\} = \{f(x)\}$ for some $f(x)\varepsilon\ \mathbb{R}$. This defines the function f, which is the uniform limit of $\{f_n : n\varepsilon\omega\}$, and is hence continuous. Clearly $f\varepsilon\cap\{B_n : n\varepsilon\omega\}$, as desired.

The algebraic structure on $\mathbb{R}$ induces an algebraic structure on C(X). This structure interacts well with some topologies on C(X). For example, $C_k(X)$ and $C_p(X)$ are always locally convex linear topological spaces. On the other hand, $C_\rho(X)$ is only a topological group under addition while the scalar multiplication operation is not continuous for non-compact X. The space $C_{f_\rho}(X)$ behaves much like $C_\rho(X)$ in this regard. It is straightforward to show that $C_{f_\rho}(X)$ is a topological group under addition. As a result, $C_{f_\rho}(X)$ is homogeneous; and for many arguments it suffices to consider only basic neighborhoods of the zero function, f_0.

The next result establishes when f_0 has a countable base. It is stated for $(\mathbb{R},\rho)$, but it is also true for any metric space containing a nontrivial path.

PROPOSITION 3.3. If X is normal and f_0 has a countable base in $C_{f_\rho}(X)$, then X is countably compact.

PROOF. Suppose X is not countably compact. Then X contains a countable closed discrete set, $\{x_n : n\epsilon\omega\}$. Let $\{\phi_n : n\epsilon\omega\}$ be any sequence in $C^+(X)$. The goal is to show that $\{B^+_\rho(f_0,\phi_n) : n\epsilon\omega\}$ cannot be a base at f_0. For each n, let $e_n \epsilon \mathbb{R}^+$ be such that interval $[0,e_n]$ is contained in $B_\rho(0,\phi_n(x_n)/2)$. Since X is normal, there exists a $\phi\epsilon C^+(X)$ such that $\phi(x_n) = e_n$ for every n.

It remains to show that $B^+_\rho(f_0,\phi_n) \not\subset B^+_\rho(f,\phi)$ for each n. Fix $n\epsilon\omega$, and let U be a neighborhood of x_n such that $\phi_n(U) \subset (\phi_n(x_n)/2,\infty)$. Because X is a Tychonoff space, there exists an $f\epsilon C(X)$ such that $f(x_n) = e_n$, $f(X\backslash U)=\{0\}$, and $f(U) \subset [0,e_n]$. Clearly $f \notin B^+_\rho(f_0,\phi)$ since $f(x_n) = e_n = \phi(x_n)$. To see that $f\epsilon B^+_\rho(f_0,\phi_n)$, let $x\epsilon X$. If $x\notin U$, then $\rho(f(x), f_0(x))=0$. If $x\epsilon U$, then $f(x) \epsilon [0,e_n] \subset B_\rho(0,\phi_n(x_n)/2)$. Also $\phi_n(x) > \phi_n(x_n)/2$, so that $\rho(f(x),f_0(x)) < \phi_n(x)$.

Therefore for a normal space X, $C_{f_\rho}(X)$ is first countable if and only if it is already equal to the metrizable space $C_\rho(X)$.

The situation is little different for $C_\nu(X)$. To begin with, $C_\nu(X)$ is in general not homogeneous, as the next proposition shows.

PROPOSITION 3.4. An element of $C_\nu(X)$ has a countable base if and only if it is a bounded function.

PROOF. First suppose f is an unbounded function in C(X). Without loss of generality, suppose there is a sequence $\{x_n : n\epsilon\omega\}$ in X such that $f(x_n) = n$ for each $n\epsilon\omega$. Let $\{\mathcal{U}_n[f] : n\epsilon\omega\}$ be any sequence of basic neighborhoods of f in $C_\nu(X)$.

For each $n\epsilon\omega$, let $V_n\epsilon\mathcal{U}_n$ with $n\epsilon V_n$, and let I_n be a closed interval containing n in its interior and contained in $V_n\cap(n-1/2,n+1/2)$. Also let t_n be a point of the interior of I_n different than n, and W_n be an open subset of the interior of I_n which contains n but not t_n. Then let $W = \mathbb{R}\backslash\omega$, and define $\mathcal{U} = \{W\}\cap\{W_n : n\epsilon\omega\}$.

To see that each $\hat{\mathcal{U}}_n[f]$ is not contained in $\hat{\mathcal{U}}[f]$, let $\phi_n:I_n\to I_n$ be a homeomorphism which fixes the endpoints of I_n and moves n to t_n. Then define $\phi\epsilon C(\mathbb{R})$ by $\phi(s)=\phi_n(s)$ if $s\epsilon I_n$ and $\phi(s)=s$ otherwise. It follows that $\phi\circ f\epsilon\hat{\mathcal{U}}_n[f]\backslash\hat{\mathcal{U}}[f]$. Therefore $\{\hat{\mathcal{U}}_n[f] : n\epsilon\omega\}$ cannot be a base at f.

For the converse, suppose that f is a bounded function in C(X). The goal will be to show that $\{B_\rho(f,1/n) : n\epsilon\omega\}$ is a base at f in $C_\nu(X)$. Let $d\epsilon M(\mathbb{R})$ and let $e>0$. Also let $M\epsilon\omega$ such that f(X) is contained in the interval $[-M,M]$. For each $t\epsilon[-M,M]$, there exists an $e_t\epsilon\mathbb{R}^+$ such that $B_\rho(t,e_t) \subset B_d(t,e/3)$. There exist $t_1,\ldots,t_m\epsilon[-M,M]$ such that $[-M,M] \subset B_\rho(t_1,e_{t_1}/2)\cap\ldots\cap B_\rho(t_m,e_{t_m}/2)$. Take $n\epsilon\omega$ with $1/n \leq\min\{e_{t_1}/2,\ldots, e_{t_m}/2\}$.

To see that $B_\rho(f, 1/n)\subset B_d(f,e)$, let $g\epsilon B_\rho(f, 1/n)$ and let $x\epsilon X$. There is some k with $f(x)\epsilon B_\rho(t_k,e_{t_k}/2)$. Now $\rho(g(x), f(x)) < 1/n\leq e_{t_k}/2$ and $\rho(f(x), t_k) < e_{t_k}/2$, so that $\rho(g(x),t_k)<e_{t_k}$. Therefore $g(x)\epsilon B_\rho(t_k,e_{t_k})\subset B_d(t_k,e/3)$, and similarly $f(x)\epsilon B_d(t_k,e/3)$. Hence $d(g(x),f(x))<2e/3$, so that $d(g,f)\leq 2e/3<e$.

COROLLARY 3.5. If $C_\nu(X)$ is first countable, then X is psuedocompact.

It follows from Proposition 3.4 that whenever X is not psuedocompact then $C_\nu(X)$ is not homogeneous, and is thus not a topological group under addition. Therefore $C_\nu(X)$ and $C_{f_\rho}(X)$ are different whenever X is not psuedocompact.

4. COUNTABILITY PROPERTIES

The spaces $C_\nu(X)$ and $C_{f_\rho}(X)$ have a property which is useful for studying countability properties. That property is submetrizability; i.e., these topologies contain weaker metrizable topologies. Certain results follow immediately. For example, singleton sets in $C_\nu(X)$ and $C_{f_\rho}(X)$ are G_δ-sets. Also the concepts of compactness, countable compactness, and sequential compactness are equivalent for subsets of these spaces.

There is a concept which is weaker than first countability that will be useful to consider. A space is of point countable type if every point is contained in a compact set which has a countable base. Every Cech-complete space has this property. Also a space which is of point countable type and in which singleton sets are G_δ-sets is first countable. As a result, a number of properties are equivalent for the fine and fine uniform topologies. The proof of the following theorem then follows from Proposition 1.3 and Corollary 3.5.

THEOREM 4.1. If C(X) has the fine uniform topology (or the fine topology for normal X), then the following are equivalent.

(a) C(X) is first countable.

(b) C(X) is of point countable type.

(c) C(X) is Cech-complete

(d) C(X) is metrizable.

(e) C(X) is completely metrizable.

(f) $C(X) = C_\rho(X)$.

(g) All compatible uniformities on $\mathbb{R}$ induce the same topology on C(X).

(h) X is pseudocompact.

Theorem 4.1 is also true with $\mathbb{R}$ replaced by any complete metric space which contains a closed ray; i.e., a closed copy of the interval $[0,\infty)$.

THEOREM 4.2. If C(X) has the fine uniform topology (or the fine topology), then the following are equivalent.

(a) C(X) is separable.

(b) C(X) has the countable chain condition.

(c) C(X) is Lindelof.

(d) C(X) has a countable network.

(e) C(X) is second countable.

(f) C(X) is separable and completely metrizable.

(g) $C_\rho(X)$ is separable.

(h) X is compact and metrizable.

PROOF. Since $C_\rho(X) \le C(X)$, each of (a) through (f) implies (g). That (g) implies (h) is well-known. Finally, if (h) is true, then $C(X) = C_\rho(X)$.

These same arguments can be extended to generalize Theorem 4.2. In particular, this theorem will be true if $\mathbb{R}$ is replaced by a separable complete metric space which contains a nontrivial path.

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