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FINGERPRINTS THEOREMS FOR ZERO-CROSSINGS

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Abstract.

We prove that the scale map of the zero-crossings of almost all signals filtered by the second derivative of a gaussian of variable size determines the signal uniquely, up to a constant scaling and a harmonic function. Our proof provides a method for reconstructing almost all signals from knowledge of how the zero-crossing contours of the signal, filtered by a gaussian filter, change with the size of the filter. The proof assumes that the filtered signal can be represented as a polynomial of finite, albeit possibly very high, order. An argument suggests that this restriction is not essential. Stability of the reconstruction scheme is briefly discussed. The result applies to zero- and level-crossings of linear differential operators of gaussian filters. The theorem is extended to two dimensions, that is to images. These results are reminiscent of Logan's theorem. They imply that extrema of derivatives at different scales are a complete representation of a signal.

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1. Introduction

Images are often described in terms of "edges", that are usually associated with the zeros of some differential operator. For instance, zero-crossings in images convolved with the laplacian of a gaussian have been extensively used as the basis representation for later processes such as stereopsis and motion (Marr, 1982). In a similar way, sophisticated processing of 1-D signals requires that a symbolic description must first be obtained, in terms of *changes* in the signal. These descriptions must be concise and, at the same time, they must capture the meaningful information contained in the signal.

It is clearly important, therefore, to characterize in which sense the information in an image or a signal is captured by extrema of derivatives.

Ideally, one would like to establish a unique correspondence between the changes of intensity in the image and the physical surfaces and edges which generate them through the imaging process. This goal is extremely difficult to achieve in general, although it remains one of the primary objectives of a comprehensive theory of early visual processing.

A more restricted class of results, that does not exploit the constraints dictated by the signal or image generation process, has been suggested by work on zero-crossings of images filtered with the laplacian of a gaussian. Logan (1977) had shown that the zero-crossings of a 1-D signal ideally bandpass with a bandwidth of less than an octave determine uniquely the filtered signal (up to scaling). The theorem has been extended—only in the special case of oriented bandpass filters—to 2-D images (Poggio, et al., 1982; Marr, et al., 1979) but it cannot be used for gaussian filtered signals or images, since they are not ideally bandpass. Nevertheless, Marr et al. (1979) conjectured that the zero-crossings maps, obtained by filtering the image with the second derivative of gaussians of variable size, are very rich in information about the signal itself (see also Grimson, 1981; Marr and Hildreth, 1980; Marr, 1982; for multiscale representations see also Crowley, 1982 and Rosenfeld, 1982 also for more references).

More recently, Witkin (1983) (see also Stansfield, 1980) introduced a scale-space description of zero-crossings, which gives the position of the zero-crossing across a continuum of scales, i.e., sizes of the gaussian filter (parametrized by the σ of the gaussian). The signal—or the result of applying to the signal a linear (differential) operator—is convolved with a gaussian filter over a continuum of sizes of the filter. Zero- or level- crossings of the filtered signal are contours on the $x-\sigma$ plane (and surfaces in the x, y, σ space). The appearance of the *scale map* of the zero-crossing—an example is shown in Figure 1—is suggestive of a *fingerprint*. Witkin has proposed that this concise map can be effectively used to obtain a rich and qualitative description of the signal. Furthermore, it has been proved in 1-D (Babaud et al, 1983; Yuille and Poggio, 1983) and 2-D (Yuille and Poggio, 1983) that the gaussian filter is the only filter with a "nice" scaling behavior, i.e., a simple behavior of zero-crossing across scales, with several attractive properties for further processing. In this paper, we prove a stronger *completeness* property: the map of the zero-crossing across scales determines the signal uniquely for almost all signals (in the absence of noise). The scale maps obtained by gaussian filters are true *fingerprints* of the signal. Our proof is constructive. It shows how the original signal can be reconstructed by information from the zero-crossing contours across scales. It is important to emphasize that our result applies to level-crossings of any arbitrary linear (differential) operator of the gaussian, since it applies to functions that obey the diffusion equation.

Our *fingerprints theorems* can be regarded as an extension of Logan's result to gaussian filtered, nonbandpass signals *and* 2-D images. There are, however, some important differences between Logan's Theorem and the *fingerprints theorems*. Logan uses a bandpass filter, at one scale only, and shows that the zero-crossings determine the filtered signal. His proof is non-constructive and only applies in 1-D (2-D generalizations exist

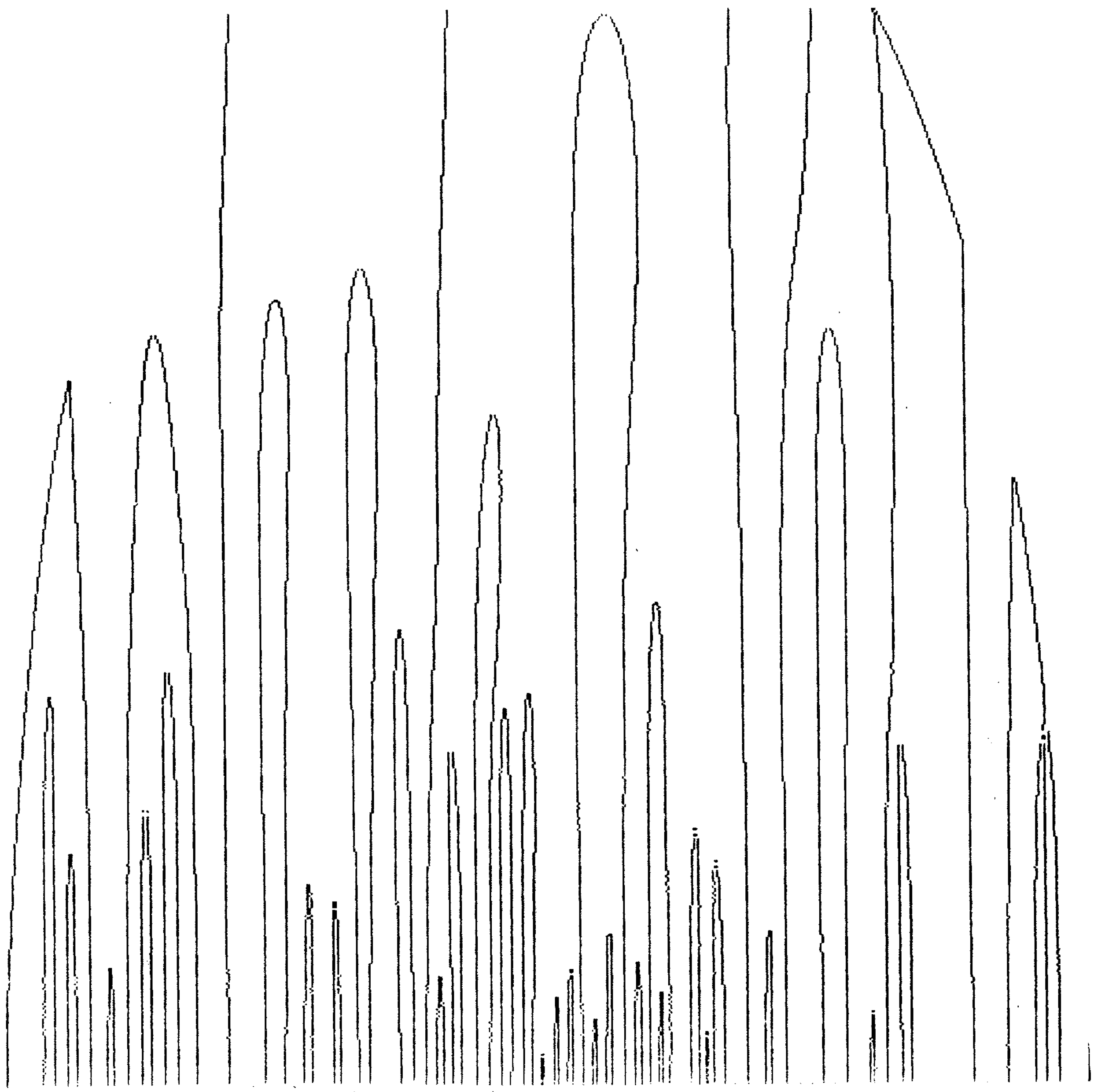

Figure 1 *The scale map of the zero-crossings of the second derivative of a signal (a 1-D slice of a natural image). The x axis is the abscissa; the scale, i.e. σ, increases from the bottom to the top. Our theorem states that this map is a true fingerprint since it determines uniquely the signal (modulus the null space of the operator).*

[Poggio et al,1982] but none are fully satisfactory). The *fingerprints theorems* determine the *original* image from the zero-crossings of the image filtered at different scales. The proof is constructive and applies in both 1-D and 2-D. Reconstruction of the signal is of course not the goal of early signal processing: Symbolic primitives must be extracted from the signals and used for later processing. Our results imply that scale-space fingerprints are complete primitives, that capture the whole information in the signal and characterize it uniquely. Subsequent processes can therefore work on this more compact representation instead of the original signal.

Our results have theoretical interest in that they answer the question as to what information is conveyed by the zero- and level-crossings of multiscale gaussian filtered signals. From a point of view of applications, the results in themselves do not justify the use of the fingerprint representation. *Completeness* of a representation (connected with Nishihara's *sensitivity*) is not sufficient (Nishihara, 1981). A good representation must, in addition, be *robust* (i.e. *stable* in Nishihara's terms) against photometric and geometric distortions (the general point of view argument). It should also possibly be *compact* for the given class of signals. Most importantly it should make *explicit* the information that is required by later processes. Fingerprints of images may have these additional properties. Their compactness property, for instance, can be defended with the same type of heuristic arguments used to justify edge detection.[1]

2. Assumptions and results

We consider the zero-crossings of a signal $I(x)$, space-scale filtered with the second derivative of a gaussian, as a function of x, σ. Let F and E be defined by

$$F(x) = \frac{d^2}{dx^2} I$$

$$E(x,\sigma) = \frac{d^2}{dx^2} I * G$$

$$E(x,\sigma) = I(x) * \frac{d^2}{dx^2}[G(x,\sigma)] = \int I(\varsigma) \frac{1}{\sigma} \frac{d^2}{dx^2} exp^{-\frac{(x-\varsigma)^2}{2\sigma^2}} d\varsigma. \qquad [2.1]$$

Notice that $E(x,\sigma)$ obeys the diffusion equation in x and σ:

$$\frac{\partial^2 E}{\partial x^2} = \frac{1}{\sigma}\frac{\partial E}{\partial \sigma}. \qquad [2.2]$$

We restrict ourselves to images, or signals, P such that E can be expressed as a finite Taylor series of arbitrarily high order. Observe that any filtered image can be approximated arbitrarily well in this way.

We will show that the local behavior of the zero-crossing curves (defined by $E(x,\sigma) = 0$) on the $x - \sigma$ plane determines the image up to an harmonic[2] function $\varphi(x)$, such that $\frac{d^2}{dx^2}\varphi = 0$. The proof of this result will then be generalized to 2-D. We will also discuss its (obvious) extension to zero- and level-crossings of linear (differential) operators. More precisely we will prove the following theorem:

Theorem 1: *The derivatives (including the zero-order derivative) of the zero-crossings contours defined by $E(x,\sigma) = 0$, at two distinct points at the same scale, determine uniquely*

[1] Clearly, the scale map fingerprint cannot always be a more concise description of the signal than the signal itself, unless the signal is redundant in precisely the way that the fingerprint representation can exploit. We expect this to be the case for images, if an appropriate differential operator is used, because images are not a purely random array of numbers. Usually images consist of rather homogeneous regions that do not change much over significant scale intervals.

[2] This indeterminancy is not a problem. It has long been known that the human visual system is rather insensitive to linear illumination gradients. Our reconstruction scheme provides the Laplacian of the image I in terms of Hermite polynomials. It is easy to integrate a function of this type to obtain I.

a signal of class P up to an harmonic function of x and constant scaling (except on a set of measure zero).[3]

Note that the theorem does not apply to signals that do not have at least two distinct zero-crossings contours. Another remark is relevant here: the gaussian filter seems critical for our proof, but we cannot show that it is the only filter with this property. In section 4 we will extend Theorem 1 to the two dimensional case:

Theorem 2: *Derivatives of the zero-crossings contours, defined by $E(x, y, \sigma) = 0$, at two distinct points at the same scale, uniquely determine an image of class P up to an harmonic function of x, y and a scaling factor (except on a set of measure zero).*

If the signal is not a polynomial, a similar weaker result can be proved.[4] A best solution can always be found but it may not be unique. These theorems break down when all the zero-crossing contours are independent of scale (i.e. the contours go straight up in the scale-space fingerprint). This is a rare, though interesting, special case and is discussed in detail in a future paper [Yuille and Poggio 1983, in preparation]. It can only occur for functions which cannot be represented as finite polynomials.

The theorems do not directly address the *stability* of this reconstruction scheme. The first question concerns stability of the reconstruction of the *filtered* function $E(x, \sigma)$ at the σ where the derivatives are taken. Note that our result relies only on two points on the zero-crossing contours. Exploitation of the whole zero-crossings contours should make the reconstruction considerably robust. The second question is about the stability of the recovery of the unfiltered signal $\nabla^2 I(x)$ from $E(x, \sigma)$. This is equivalent to inverting the diffusion equation, which is numerically unstable. Reconstruction is, however, possible with an error depending on the signal to noise behaviour (see later).[5]

2.1. Outline of the 1-D Proof

We summarize here the 1-D proof from a slightly different point of view that clarifies its bare structure.

The proof starts by taking derivatives along the zero crossing contours at a certain point. Such derivatives split into combinations of x and t derivatives (where $t = \sigma^2/2$). Because the filter is assumed to be gaussian, however, derivatives can be expressed in terms of x derivatives. This is a key point: since the filtered signal $E(x, t)$ satisfies the diffusion equation, the t derivatives can be expressed in terms of the x derivatives simply by $E_t = E_{xx}$. The next stage is to find the x derivatives of $E(x, t)$ up to an arbitrary degree n from such derivatives along the zero crossing contours in the $x - t$ plane. We show that this can be done by using 2 points on 2 contours. (It is possible that one point is sufficient, but we are as yet unable to prove this.) Since $E(x, t)$ is entire analytic, because of the gaussian filtering (see Appendix 2), it can be represented by a Taylor series expansion in x. Since we know the values of the n derivatives of $E(x, t)$ with respect to x, we know its Taylor series expansion and hence $E(x, t)$. The unfiltered signal $F(x)$, ($E(x, t) = F(x) * G(x, t)$) can then be recovered

[3] For a general operator the reconstruction is modulus the null space of the operator. Harmonic functions are the null space of the Laplacian.

[4] In this case, the signal is determined by the zero-crossing contours in the L^2 sense only. This means that the signal may not be determined correctly on a set of measure zero. However, if the image is assumed to be an analytic entire function (see Appendix 2), section 3.4 implies that it can be determined *exactly* everywhere. Since images - like any physical signal - are effectively bandlimited by the measurement (or imaging) process, they can be considered as restrictions to the reals of analytic entire functions (see Appendix 2).

[5] If $E(x, \sigma)$ is obtained at all σ – this can be done by applying the reconstruction scheme at two points at each σ – robust reconstruction of $I(x)$ can be achieved in the following way (Hummel and Zucker, in prep.). Since $\int E(x, t)dt = \int (I * G)_{xx} dt = \int (I * G)_t dt = E(x, 0)$ with the integration between 0 and ∞, $I(x)$ can be reconstructed by integrating $E(x, \sigma)$ across σ (with $t = \sigma^2/2$ the diffusion equation is $E_x x = E_t$). In practice, the limits of integration will be finite, originating small errors in the reconstruction.

in the ideal noiseless case by deblurring the gaussian. A particularly simple way of doing this is provided by a property of the function ϕ_n in which we will expand the function F: the coefficients of an expansion of $F(x)$ in terms of ϕ_n are equal to the coefficients of the Taylor series expansion of $E(x,t)$. In the presence of noise, the recovery of $F(x)$ from $E(x,t)$ is obviously unstable. It is limited by S/N ratio since high spatial frequencies in the signal are masked by the noise for increasing t. (For instance, if $F(x) = \sum a_\mu e^{i\mu x}$, the filtered signal is $E(x,t) = \sum a_\mu e^{i\mu x} e^{-\mu^2 t}$.) Note that since the zero-crossing contours are available at all scales a reconstruction scheme that exploits more than 2 points will be significantly more robust. As one would expect, the reconstruction of the unfiltered signal is therefore affected by noise. The reconstruction of the filtered signal $E(x,t)$ is likely to be considerably more robust. We plan to study theoretically and with computer simulations the noise sensitivity of the reconstruction scheme.

3. Proof of the Theorem in 1-D

We divide our proof into three main steps. In the first we show that derivatives at a point on a zero-crossing contour put strong constraints on the "moments" of the Fourier tranform of $E(x,\sigma)$ (see eq. 3.1.4). The second section relates the "moments" to the coefficients of the expansion of $F(x) = E(x,0)$ in functions related to the Hermite polynomial. In the third section we show that the "moments" can be uniquely determined by the derivatives on a second point of a different zero-crossing contour.

3.1. The "moments" of the signal are constrained by the zero-crossing contours

Let the Fourier tranform of the signal $I(x)$ be $\tilde{I}(\omega)$ and the gaussian filter be $G(x,\sigma) = \frac{1}{\sigma} e^{\frac{-x^2}{2\sigma^2}}$ with Fourier transform $\tilde{G}(\omega) = e^{\frac{-\sigma^2\omega^2}{2}}$.

The zero crossings are given by solutions of $E(x,t) = 0$. Using the convolution theorem we can express $E(x,t)$ as

$$E(x,t) = \int e^{-\omega^2 t} e^{i\omega x} \omega^2 \tilde{I}(\omega) d\omega. \qquad [3.1.1]$$

and $t = \sigma^2/2$. The Implicit Function theorem gives curves $x(t)$ which are C^∞ (this is a property of the gaussian filter and of the diffusion equation, see Appendix 2 and Yuille and Poggio, 1983). Let ς be a parameter of the zero crossing curve. Then

$$\frac{d}{d\varsigma} = \frac{dx}{d\varsigma}\frac{\partial}{\partial x} + \frac{dt}{d\varsigma}\frac{\partial}{\partial t}. \qquad [3.1.2]$$

On the zero-crossing surface, $E = 0$ and $\frac{d^n}{d\varsigma^n} E = 0$ for all integers n. Knowledge of the zero crossing curve is equivalent to knowledge of all the derivatives of x and t with respect to ς.

We compute the derivatives of E with respect to ς at (x_o, t_o). The first derivative is :

$$\frac{d}{d\varsigma}E(x,t) = \frac{dx}{d\varsigma}\int e^{-\omega^2 t} e^{i\omega x}(i\omega)\omega^2 \tilde{I}(\omega) d\omega + \frac{dt}{d\varsigma}\int e^{-\omega^2 t}(-\omega^2) e^{i\omega x}\omega^2 \tilde{I}(\omega) d\omega \qquad [3.1.3]$$

and is expressed in terms of the first and second moments of the function $e^{-\omega^2 t} e^{i\omega x}\omega^2 I(\omega)$. The moment of order n is defined by:

$$M_n = \int_{-\infty}^{\infty} (i\omega)^n e^{-\omega^2 t} e^{i\omega x} \omega^2 \tilde{I}(\omega) d\omega. \quad [3.1.4]$$

The second derivative is

$$\begin{aligned} \frac{d^2}{d\varsigma^2} E(x,t) = & \frac{d^2 x}{d\varsigma^2} \int e^{-\omega^2 t} e^{i\omega x} (i\omega) \omega^2 \tilde{I}(\omega) d\omega \\ & + \frac{d^2 t}{d\varsigma^2} \int e^{-\omega^2 t} (-\omega^2) e^{i\omega x} \omega^2 \tilde{I}(\omega) d\omega \\ & + \left(\frac{dx}{d\varsigma}\right)^2 \int e^{-\omega^2 t} e^{i\omega x} (-\omega^2) \omega^2 \tilde{I}(\omega) d\omega \\ & + 2 \frac{dx}{d\varsigma} \frac{dt}{d\varsigma} \int e^{-\omega^2 t} (-\omega^2) e^{i\omega x} (i\omega) \omega^2 \tilde{I}(\omega) d\omega \\ & + \left(\frac{dt}{d\varsigma}\right)^2 \int e^{-\omega^2 t} (\omega^4) e^{-i\omega x} \omega^2 \tilde{I}(\omega) d\omega. \end{aligned} \quad [3.1.5]$$

Since the parametric derivatives along the zero crossing curve are zero, equation [3.1.3] is a homogeneous linear equation in the first two moments. Similarly, [3.1.5] is a homogeneous linear equation in the first four moments. In general, the n^{th} equation, $\frac{d^n}{d\varsigma^n} E(x,t) = 0$, is a homogeneous equation in the first $2n$ moments. We choose our axes such that $x_o = 0$. The next section shows that the moments of $e^{-\omega^2 t} \omega^2 I(\omega)$ are the coefficients a_n in the expression of the function $F(x)$ in Hermite polynomials. So we have n equations in the first $2n$ coefficients a_n. To determine the a_n uniquely, we need n additional and independent equations which, as we will show in section 3.3, can be provided by considering a neighboring zero crossing curve at (x_1, t_o).

3.2. The "moments" are the coefficients of the expansion of $F(x)$

In this section we show that the moments defined by [3.1.4] can be related to the coefficients of the expansion of $F(x)$ in functions related to the Hermite polynomials. We expand

$$F(x) = \frac{d^2}{dx^2} I(x) \quad [3.2.1]$$

in terms of the functions $\varphi_n(x,\sigma)$ related to the Hermite polynomials $H_n(x)$ (see Appendix 1) by

$$\varphi_n(x,\sigma) = (-1)^n \frac{\sigma^{n-1}}{(\sqrt{2})^{n+1} \sqrt{\pi}} H_n\left(\frac{x}{\sqrt{2}\sigma}\right) \quad [3.2.2]$$

$$F(x) = \sum_{n=0} a_n(\sigma) \varphi_n(x,\sigma) \quad [3.2.3]$$

The coefficients $a_n(\sigma)$ of the expansion are given by

$$a_n(\sigma) = \langle w_n(x,\sigma), F(x) \rangle \quad [3.2.4]$$

where $\langle , \rangle$ denotes inner product in L^2 and $\{w_n(x,\sigma)\}$ is the set of functions biorthogonal to $\{\varphi_n(x,\sigma)\}$. The $\{\varphi_n(x,\sigma)\}$ are given explicitly by

$$\varphi_n(x,\sigma) = \frac{\sigma^{2n-1}}{n! \sqrt{2} \sqrt{\pi}} e^{\frac{x^2}{2\sigma^2}} \frac{d^n}{dx^n} e^{\frac{-x^2}{2\sigma^2}} \quad [3.2.5]$$

and the $w_n(x,\sigma)$ by

$$w_n(x,\sigma) = (-1)^n \frac{d^n}{dx^n} e^{\frac{-x^2}{2\sigma^2}} \quad [3.2.6]$$

Since

$$F(x) = \frac{1}{\sqrt{2\pi}} \int e^{i\omega x} \omega^2 \tilde{I}(\omega) d\omega \quad [3.2.7]$$

the a_n are given by

$$a_n(\sigma) = \frac{1}{\sqrt{2\pi}} (-1)^n \int \langle \frac{d^n}{dx^n} e^{\frac{-x^2}{2\sigma^2}}, e^{i\omega x} \rangle \omega^2 \tilde{I}(\omega) d\omega \quad [3.2.8]$$

The inner product in [3.2.8] is just the inverse Fourier transform of $w_n(x)$. Therefore,

$$a_n(\sigma) = \int (i\omega)^n e^{\frac{-\omega^2\sigma^2}{2}} \omega^2 \tilde{I}(\omega) d\omega \quad [3.2.9]$$

which is equal to M_n modulus a factor $e^{i\omega x}$. We will need to consider the derivatives along the zero-crossing contours at two points. We can choose coordinates so that these points are $(0,\sigma_0)$ and (x_1,σ_0).

Therefore knowledge of the image is equivalent to knowing the a_n.

3.3. Combining information from two contours

The derivatives at (x_1, t_o) give us n equations in the first $2n$ moments of $e^{-\omega^2 t} e^{i\omega x_1} \omega^2 \tilde{I}(\omega)$. We can relate them to the expansion coefficients of the function

$$F(x + x_1) = \int e^{i\omega x} e^{i\omega x_1} \omega^2 \tilde{I}(\omega) d\omega \quad [3.3.1]$$

in terms of the φ_n functions.

We write

$$F(x + x_1) = \sum_0^{2n} b_n \varphi_n(x) \quad [3.3.2]$$

We have n equations for the $2n$ unknowns b_n. Now observe that

$$\sum_0^{2n} b_n \varphi_n(x) = \sum_0^{2n} a_n \varphi_n(x + x_1) \quad [3.3.3]$$

Any $\varphi_n(x + x_1)$ can be expressed as a linear combination of $\varphi_m(x)$ with $m \leq n$, as we will show in section 3.3.1.

Thus we express each b_n's in terms of a_n's and then we combine the equations from two points to obtain $2n$ equations for the $2n$ coefficients a_n. Thus with the results of the next two sections the proof will be complete.

3.3.1. Change of basis

For a given σ, the functions φ_n are given in terms of Hermite polynomials (see eq. 3.2.2). We show how we can express Hermite polynomials with origin at $x = 0$ in terms of Hermite polynomials with origin at $x = -x_1$ and hence perform a change of basis.

We show that $H_n(x - x_1)$ can be expressed as a linear combination of $H_n(x)$ with $m \leq n$. Let us consider the generating function e^{-p^2+2px} which defines the Hermite polynomials as

$$e^{-p^2+2px} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} p^n \qquad [3.3.1.1]$$

Equation [3.3.1.1] gives at $x + x_1$

$$e^{-p^2+2p(x-x_1)} = \sum_{n=0}^{\infty} \frac{H_n(x - x_1)}{n!} p^n \qquad [3.3.1.2]$$

The left hand side of equation [3.3.1.2] can be expanded as

$$e^{-p^2+2px} e^{-2px_1} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} p^n \sum_{m=0}^{\infty} \frac{(-2x)^m p^m}{m!} \qquad [3.3.1.3]$$

Term by term comparison of equation [3.3.1.2] and [3.3.1.3] gives

$$H_n(x - x_1) = \sum_{m=0}^{n} \binom{n}{m} H_m(x)(-2x_1)^{n-m} \qquad [3.3.1.4]$$

The series obtained by substituting equation [3.3.1.4] into

$$f(x) = \sum_{n=0}^{\infty} b_n H_n(x - x_1) \qquad [3.3.1.5]$$

is a series of the form

$$\sum_{n=0}^{\infty} d_n H_n(x) \qquad [3.3.1.6]$$

that converges to $f(x)$ as the following argument demonstrates. If $f(x)$ is in L^2 it can be expanded in terms of the H_n. Similarly, $g(x) = f(x - x_1)$ is also in L^2 and can be expanded as

$$g(x) = \sum_{n=0}^{\infty} c_n H_n(x) \qquad [3.3.1.7]$$

Thus, $f(x - x_1)$ can be expressed as a linear combination of $H_n(x)$ and this series coincides with eq.(3.3.1.6) because of the uniqueness of the expansion. In particular, we obtain for the φ_n

$$\varphi_n(x - x_1, t) = \varphi_n(x, t) + x_1 \varphi_{n-1}(x, t) + \ldots \qquad [3.3.1.8]$$

where the remaining terms are functions $\varphi_m(x, t)$, with $m \leq n - 1$, multiplied by polynomials in x_1.

Observe that if we restrict ourselves to polynomial functions of x a change of basis will correspond to transformating an n-th order polynomial in x into an n-th order polynomial in $x - x_1$. This argument suffices for our theorems which are restricted to finite polynomial functions of class P. For the sake of completeness, however, Appendix 3 deals with the general case.

3.4. Independence of the equations

We have to show that information from 2 points yield a unique solution. The first n equations in the $2n$ first moments from a point can be written as

$$\begin{pmatrix} \frac{dx}{d\varsigma} & \frac{dt}{d\varsigma} & 0 & 0 & 0 & \dots & 0 \\ \frac{d^2x}{d\varsigma^2} & \frac{d^2t}{d\varsigma^2} + \left(\frac{dx}{d\varsigma}\right)^2 & 2\frac{dx}{d\sigma}\frac{dt}{d\varsigma} & \left(\frac{dt}{d\varsigma}\right)^2 & 0 & \dots & 0 \\ \dots & \dots \end{pmatrix} \begin{pmatrix} M_1(x) \\ M_2(x) \\ \vdots \\ M_{2n}(x) \end{pmatrix} = 0 \tag{3.4.1}$$

The matrix of the coefficients is a nx$2n$ matrix. Note that its rows are linearly independent (since the coefficients of the r^{th} row vector are zero after the $2r^{th}$ component).

The next n equations are given by the matrix of the derivatives at a second point, x_i, that have the same form as eq. (3.4.1), multiplied by the moments at $(x + x_1)$.

$$\begin{pmatrix} \frac{d\bar{x}}{d\varsigma} & \frac{d\bar{t}}{d\varsigma} & 0 & 0 & 0 & \dots & 0 \\ \frac{d^2\bar{x}}{d\varsigma^2} & \frac{d^2\bar{t}}{d\varsigma^2} + \left(\frac{d\bar{x}}{d\varsigma}\right)^2 & 2\frac{d\bar{x}}{d\varsigma}\frac{d\bar{t}}{d\varsigma} & \left(\frac{d\bar{t}}{d\varsigma}\right)^2 & 0 & \dots & 0 \\ \dots & \dots \end{pmatrix} \begin{pmatrix} M_1(x + x_1) \\ M_2(x + k_1) \\ \vdots \\ M_{2n}(x + x_1) \end{pmatrix} = 0. \tag{3.4.2}$$

The moments at $(x + x_1)$ can be expressed in terms of the moments at x by the following transformation (see section 3.3).

$$\begin{pmatrix} 1 & x_1 & \frac{x_1^2}{2} & \frac{x_1^3}{3!} & \dots & \frac{x_1^{2n}}{2n!} \\ 0 & 1 & x_1 & \dots \\ 0 & 0 & 1 & x_1 & \dots \\ & & & & & 1 \end{pmatrix} \begin{pmatrix} M_1(x) \\ M_2(x) \\ \vdots \\ M_{2n}(x) \end{pmatrix} = \begin{pmatrix} M_1(x + x_1) \\ M_2(x + x_1) \\ \vdots \\ M_{2n}(x + x_1) \end{pmatrix} \tag{3.4.3}$$

Equation (3.4.3) substituted into (3.4.2) gives, together with equation 3.4.1, the full set of $2n$ equations in the $2n$ unknowns $M_i(x)$. The $2n$ x $2n$ matrix of the coefficients can be thought of as originating from the first point (the top half) and from the second point (the bottom half) on the zero-crossing curves.

In general, the determinant of this matrix is non-zero. Intuitively, if the filtered signal has nonzero moments of order higher than $2n$, the system of $2n$ equations would not have a solution. A proof for this claim is given in Appendix 4. The argument is based on the fact that the determinant of the coefficients is a polynomial in x_1. If this vanishes, then x_1 can be expressed in terms of the first n derivatives at the two points. We show, however, that in general it is possible to change x continuously without altering the first n derivatives. This implies that the determinant is almost always different from zero. The argument breaks down if the filtered signal is a polynomial of degree $2n$ or less.

In this case, the determinant must be zero, since the homogeneous set of equations has at least one solution. At this point, we have to show that the solution is unique. We first observe that the determinant of the coefficients of the $2n$ x $2n$ system of equations is a polynomial in x_1. This polynomial is nontrivial[6] since the first n and the second n equations

[6] This argument cannot be applied when all zero-crossing contours are vertical straight lines: in this case it is impossible to reconstruct the signal [Yuille and Poggio 1983, in preparation]

separately are independent. It follows that the determinant vanishes at a finite number (at most $2n$) of values of x_1.

Suppose the determinant is zero. Observe that x_1 is known from the position of the zero-crossing curves (x_1 is the distance between the two points at which derivatives are taken). Typically the roots of the polynomial in x_1 will be distinct and there will be a unique zero eigenvector of the matrix. Thus we have proved that n derivatives at two points determine uniquely (modulus a common scaling factor) the $2n$ moments of a polynomial of degree $2n$. The case of multiple zero eigenvectors is nongeneric, i.e., an arbitrarily small perturbation in the "image" would annihilate eventual multiple zero eigenvectors. Furthermore, multiple zero eigenvectors of the matrix of degree $2n$ must also be multiple zero eigenvectors of all higher order matrices which is even more unlikely (except on a set of measure zero).

Our proof is limited to filtered function of the polynomial type (albeit of very high degree). We now stretch an argument suggesting that the result holds also for most filtered functions $E(x, y)$ which are not polynomials.

Consider the homogeneous system of equations obtained from two points up to the moment M_{2n}. Denote with A' the matrix of the coefficients. Let A be the matrix of the coefficients of the inhomogeneous system of equations obtained by dividing all unknowns M_1 to M_{2N} by the first moment. The system, $AM' = Z$, where Z is the first column of A' divided by M_1, does not in general have solutions as we have shown (see Appendix 4). Furthermore, A has no null vector (if it has, then A' must also have a null vector, which is impossible since $det A' \neq 0$.). Then there is a unique least square solution of the equation $\|AM' - Z\| = 0$ given by $M' = A^{+}Z$, where A^{+} is the pseudoinverse of A [see Albert, 1972]. Thus for every finite M there is a unique least square solution to the system of equations $AM' = A$ but no exact solution. As n goes to infinity, however, at least one exact solution must appear.

To summarize, in section 3.1 we show that the moments of the signal are constrained by the derivatives of the zero-crossing contours at one point. Section 3.2 shows that the moments are equal to the coefficients of the expansion of the unfiltered signal $F(x)$ in our Hermite-like expansion (and also equal the coefficients of the Taylor expansion of the filtered signal $E(x, t)$). In section 3.3 we show how we can combine constraints from two different points on the zero-crossing contours at the same scale. Finally, section 3.4 demonstrates that the equations obtained in this way from two points determine a unique solution. The stability of the solution was briefly discussed in section 2.

4. The Extension to Two Dimensions

The function

$$F(\underline{x}) = E(\underline{x}, 0) = \int e^{i\underline{\omega}\cdot\underline{x}}\underline{\omega}^2 I(\underline{\omega})d\underline{\omega} \tag{4.1}$$

can be expanded in terms of the $\varphi_n(x, t)$ and $\varphi_n(y, t)$ as

$$F(\underline{x}) = \sum_{n,m} a_{nm}(t)\varphi_n(x, t)\varphi_m(y, t) \tag{4.2}$$

with the coefficients given by

$$a_{nm}(t) = \langle F(\underline{x}), w_n(x, t)w_m(y, t)\rangle = \int (i\omega_x)^n (i\omega_y)^m e^{-\underline{\omega}^2 t} I(\underline{\omega})d\underline{\omega}. \tag{4.3}$$

We define $T(\omega)$ as

$$T(\omega) = e^{i\underline{\omega}\cdot\underline{x}}\underline{\omega}^2 I(\underline{\omega})e^{-\underline{\omega}^2 t} \tag{4.4}$$

and

$$E(\underline{x}, t) = \int T(\underline{\omega})d\underline{\omega}. \tag{4.5}$$

We now take derivatives of $E(\underline{x}, t)$ on the zero-crossing surface. Thus with

$$\frac{d}{d\varsigma} = \frac{dx}{d\varsigma}\frac{\partial}{\partial x} + \frac{dy}{d\varsigma}\frac{\partial}{\partial y} + \frac{dt}{d\varsigma}\frac{\partial}{\partial t} \tag{4.6}$$

the first equation

$$\frac{dE}{d\varsigma} = 0 \tag{4.7}$$

gives, where $\underline{\omega} = (\omega_x, \omega_y) = (\omega_1, \omega_2)$ and $\omega^2 = {\omega_x}^2 + {\omega_y}^2$,

$$\frac{dx}{d\varsigma}\int \omega_x T(\underline{\omega})d\underline{\omega} + \frac{dy}{d\varsigma}\int \omega_y T(\underline{\omega})d\underline{\omega} + \frac{dt}{d\varsigma}\int \omega^2 T(\underline{\omega})d\underline{\omega} = 0 \tag{4.8}$$

and the second equation

$$\frac{d^2E}{d\varsigma^2} = 0 \tag{4.9}$$

gives

$$\begin{aligned} \frac{d^2x_i}{d\varsigma^2}\int \omega_i T(\underline{\omega})d\underline{\omega} &+ \frac{d^2t}{d\varsigma^2}\int \omega^2 T(\underline{\omega})d\underline{\omega} \\ &+ \frac{dx_i}{d\varsigma}\frac{dx_j}{d\varsigma}\int \omega_i\omega_j T(\underline{\omega})d\underline{\omega} \\ &+ 2\frac{dt}{d\varsigma}\frac{dx_i}{d\varsigma}\int \omega^2\omega_i T(\underline{\omega})d\underline{\omega} \\ &+ \left(\frac{dt}{d\varsigma}\right)^2\int \omega^4 T(\underline{\omega})d\underline{\omega} = 0 \end{aligned} \tag{4.10}$$

where we use the summation convention over i, j.

These equations, and the higher order ones that can be obtained in the same way, are equations in the moments S_{mpq} where

$$S_{mpq} = \int \omega^{2m}\omega_x^p\omega_y^q T(\underline{\omega})d\underline{\omega}. \tag{4.11}$$

The n-th order equation will consist of terms of the type S_{mpq} with $m+p+q \leq n$. We will show that, using different curves on the zero-crossing surface through (x_0, y_0, t_0), we can produce one equation for each pair of moments with $m+p+q = n$ in terms of the moments with $m+p+q < n$. As in the 1-D case we have half the equations we need to solve for the moments. Again we can consider a second point on another zero-crossing contour (at the same scale). Combining the equations from two points, after a change of basis, yields enough information to determine the image.

There are an infinite number of curves on the zero-crossing surface which pass through any given point (x_0, y_0, t_0). Since the surface is two-dimensional, the tangents to these curves form a two-dimensional vector space. Each curve will give different equations for the moments. We will show that by taking different curves through the same point (x_0, y_0, t_0) we can nearly obtain enough linearly independent equations to determine the moments. As in the one-dimensional, case we show that we can obtain the remaining information by considering the behaviour at a second point on the zero-crossing surfaces.

We first consider the lowest order equation (4.8). As we mentioned above there are two linearly independent tangents to the zero-crossing surface at (x_0, y_0, t_0). Thus we have two linearly independent vectors $(\frac{dx}{d\varsigma_1}, \frac{dy}{d\varsigma_1}, \frac{dt}{d\varsigma_1})$ and $(\frac{dx}{d\varsigma_2}, \frac{dy}{d\varsigma_2}, \frac{dt}{d\varsigma_2})$ where we use ς_1 and ς_2 to denote the parameters on the two curves. We can substitute these vectors into equation(4.8) and obtain two equations for three unknowns (the three moments with $n = 1$). These are sufficient to determine the moments up to a scaling factor. This factor corresponds to scaling the image I and cannot be determined from the zero-crossings.

We now consider the n-th order equations and show that there is one equation for each pair of moments with $p + q + m = n$ in terms of the moments with $p + q + m < n$. The moments we need to solve for are the S_{mpq} with $m + p + q = n$. Fix m and consider the moments as p and q vary. There are $n - m + 1$ possible moments, however we will show that only 2 of them are independent. The moments are given in equation (4.11). Adding the moments $S_{m,p,q}$ and $S_{m,p+2,q-2}$ gives us the moment $S_{m+1,p,q-2}$. Now, since $m + p + q = n$, we find $(m + 1) + p + (q - 2) = n - 1$ and so the moment $S_{m+1,p,q-2}$ is of order $n - 1$. Thus if we know $S_{m,p,q}$ we also know $S_{m,p+2,q-2}$. We can repeat this argument adding 2 to p and subtracting 2 from q or vice versa. Thus if we know the moments $S_{m,0,n-m}$ and $S_{m,1,n-m-1}$ we can use this argument to find the other moments. Hence for each value of m there are only 2 independent moments. The case $m = n$ is special and only has one term. m can vary from 0 to n and so there are a total of $2n + 1$ independent moments with $m + p + q = n$. We show that for each m it is possible to get one equation for the two moments.

The coefficients of the unknowns will be A_{pqm} where

$$A_{pqm} = \frac{n!}{m!p!q!}\left(\frac{dt}{d\varsigma}\right)^m \left(\frac{dx}{d\varsigma}\right)^p \left(\frac{dy}{d\varsigma}\right)^q \tag{4.12}$$

evaluated at x_0, y_0, t_0 and thus the expansion containing the unknowns is of the form

$$\sum_{m,p,q} A_{mpq} S_{mpq} \tag{4.13}$$

We consider now the terms for a fixed n. Since we can take the derivatives along arbitrary directions on the surface, the terms $dx/d\varsigma$ and $dy/d\varsigma$ take independent values (while $dt/d\varsigma$ is constrained since the curve must lie on the zero-crossing surface). We will show that it is possible to eliminate all the new moments (those with $m + p + q = n$) with $m > 0$ and obtain one equation for the two independent moments with $m = 0$. In a similar way we can get one equation for the two independent moments with $m = 1$ and so on. First we eliminate the moment with $m = n$. We consider N curves with parameters $(\varsigma_1, \varsigma_2, \ldots \varsigma_N)$. We normalize the curves by requiring

$$\frac{dt}{d\varsigma_i} = 1, i = 1, 2, \ldots N \tag{4.14}$$

This is always possible unless we are at an extrema of the zero-crossing surface. Suppose we write down the n-th order equations corresponding to two different curves. From equations (4.12) and (4.13) the coefficients of $S_{n,0,0}$ are unity in both cases. Thus we can subtract one equation from the other and obtain a linear equation for the new moments with $m < n$.

Now we show we can eliminate the $m = n$ and $m = n - 1$ moments simultaneously. We take three curves with parameters $\varsigma_1, \varsigma_2, \varsigma_3$ and add the three resulting n-th order equations multiplying the first by λ the second by μ and the third by ν. Comparing coefficients of $S_{n,0,0}$, $S_{n-1,1,0}$ and $S_{n-1,0,1}$ gives us three simultaneous equations

$$\lambda + \mu + \nu = 0, \tag{4.15}$$

$$\lambda \frac{dx}{d\varsigma_1} + \mu \frac{dx}{d\varsigma_2} + \nu \frac{dx}{d\varsigma_3} = 0, \tag{4.16}$$

$$\lambda \frac{dy}{d\varsigma_1} + \mu \frac{dy}{d\varsigma_2} + \nu \frac{dy}{d\varsigma_3} = 0, \tag{4.17}$$

These can always be solved since the tangent vectors (i.e the vectors $(1, \frac{dx}{d\varsigma_i}, \frac{dy}{d\varsigma_i})$) form a two dimensional vector space and hence any three vectors in this space are linearly dependent and satisfy equations (4.15), (4.16) and (4.17). Note that if two curves are the same we can satisfy these equations but the resulting equation for the moments will vanish. Let the normal to the zero-crossing surface be (n_1, n_2, n_3). Then the curves must satisfy the equation

$$n_1 + n_2 \frac{dx}{d\varsigma_i} + n_3 \frac{dy}{d\varsigma_i} = 0 \tag{4.18}$$

So we can only vary one of $\frac{dx}{d\varsigma_i}$ and $\frac{dy}{d\varsigma_i}$ independently.

Now we try to eliminate the moments with $m = n$, $m = n - 1$ and $m = n - 2$. We combine the n-th order equations from curves with parameters ς_i multiplying the equations by λ_i and taking the sum. The coefficients of the moments $S_{n,0,0}$, $S_{n-1,1,0}$, $S_{n-1,0,1}$, $S_{n-2,2,0}$ and $S_{n-2,1,1}$ (using $S_{n-2,0,2} = S_{n-1,0,0} - S_{n-2,2,0}$, where $S_{n-1,0,0}$ is a lower order moment) form a vector l_i

$$\begin{pmatrix} 1 \\ \frac{dx}{d\varsigma_i} \\ \frac{dy}{d\varsigma_i} \\ \frac{dx}{d\varsigma_i}^2 - \frac{dy}{d\varsigma_i}^2 \\ \frac{dx}{d\varsigma_i}\frac{dy}{d\varsigma_i} \end{pmatrix} \tag{4.19}$$

where, for simplicity, we have incorporated the factors of n into the moments. To eliminate the moments with $m \geq n - 2$ we must solve

$$\sum_{i=1}^{N} \lambda_i l_i = 0. \tag{4.20}$$

Because of equation (4.18) there are only three linearly independent vectors which can be formed by varying the $\frac{dx}{d\varsigma_i}$ and taking linear combinations. Thus if $N = 4$ we can solve the equations (4.20) and eliminate moments with $m \geq n - 2$. By increasing the number of curves by 1 each time we decrease m by 1 we can eliminate all the moments with $m > 1$. We are left with one equation in the two unknown moments with $m = 0$. In a similar way we can eliminate all the moments except the two with $m = 1$ and obtain one equation for these two moments. We repeat this process for the higher order moments.

Thus the n-th order equations give us one equation for each pair of independent moments with $p+q+m=n$. If we consider all the equations as n varies we find that, as in the 1-D case, at each point on a zero-crossing contour we have only half the number of equations needed to solve for the moments. As in the one-dimensional case we now consider a second zero-crossing surface at the same value of t and repeat the argument. We change the basis of the Hermite polynomials as in section 3.3 and obtain enough equations to solve for the moments. Substituting these into the equation (4.2) for $F(x)$ completes the proof of Theorem 2.

5. Examples

We now illustrate the theorems by considering some special cases. If the signal is a low order polynomial in X, it is possible to obtain the zero crossing curves explicitly. We then use the derivatives of these curves to reconstruct the image, as in the theorem. These examples also suggest that the derivatives of the curves *at a single point* will usually give sufficient information to reconstruct the signal.

Suppose the signal $F(X)$ is a second order polynomial in X.

$$F(X) = 1 + AX + BX^2 \tag{5.1}$$

where A and B are arbitrary coefficients. All the moments of the signal are zero except for the first two. We convolve this signal with a gaussian at scale σ and obtain

$$E(X,\sigma) = 1 + AX + BX^2 + B\sigma^2 \tag{5.2}$$

We consider the curves given by

$$E(X,\sigma) = 0 \tag{5.3}$$

We write these in the form

$$\sigma^2 + \{X^2 + (A/B)X + 1/B\} = 0 \tag{5.4}$$

and see that they correspond to circles in the (X,σ) plane. Define X_1 and X_2 by

$$\begin{aligned} X_1X_2 &= 1/B \\ -(X_1+X_2) &= A/B \end{aligned} \tag{5.5}$$

Then we can rewrite the equations as

$$\sigma^2 + \left\{X - \left(\frac{X_1+X_2}{2}\right)\right\}^2 = \left(\frac{X_2-X_1}{2}\right)^2 \tag{5.6}$$

Thus the zero crossing curve corresponds to a semi-circle which intersects the X-axis at X_1 and X_2 (see figure 2).

We now parameterize the curve by an angle θ so that

$$\begin{aligned} X(\theta) &= \left(\frac{X_1+X_2}{2}\right) + \left(\frac{X_2-X_1}{2}\right)cos\theta \\ \sigma(\theta) &= \left(\frac{X_2-X_1}{2}\right)sin\theta \end{aligned} \tag{5.7}$$

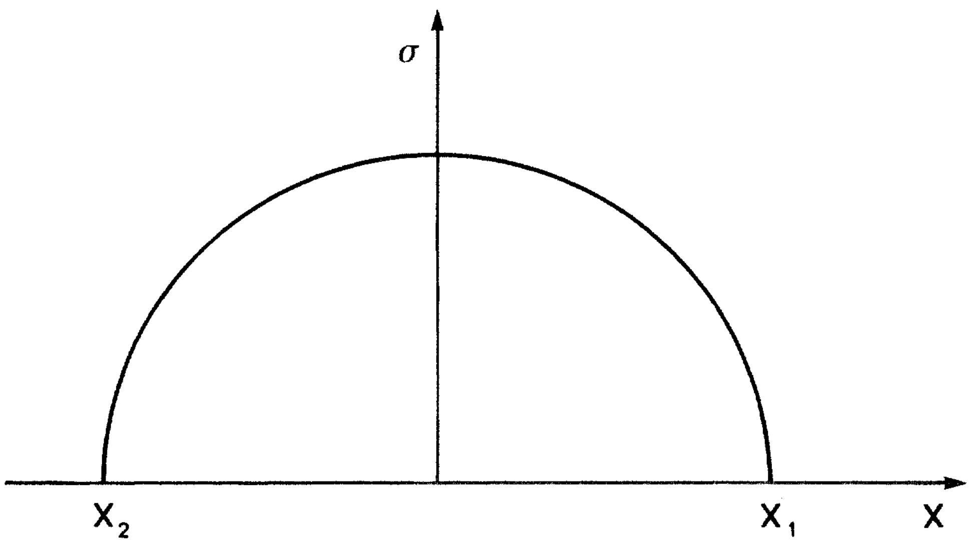

Figure 2 *See text.*

We calculate the derivatives

$$\frac{dX}{d\theta} = -\left(\frac{X_2 - X_1}{2}\right) sin\theta$$
$$\frac{d\sigma}{d\theta} = \left(\frac{X_2 - X_1}{2}\right) cos\theta \tag{5.8}$$

Recalling that $t = \sigma^2/2$, we combine (5.7) and (5.8) to obtain

$$\frac{dt}{d\theta} = \left(\frac{X_2 - X_1}{2}\right)^2 sin\theta cos\theta \tag{5.9}$$

We differentiate again to obtain

$$\frac{d^2X}{d\theta^2} = -\left(\frac{X_2 - X_1}{2}\right) cos\theta$$
$$\frac{d^2t}{d\theta^2} = \left(\frac{X_2 - X_1}{2}\right)^2 \{cos^2\theta - sin^2\theta\} \tag{5.10}$$

We set $\frac{X_2 - X_1}{2} = b$. Then we write the first two equations at $\theta = \theta_1$ as

$$\begin{pmatrix} -bsin\theta_1 & b^2 sin\theta_1 cos\theta_1 & 0 & 0 \\ -bcos\theta_1 & b^2 cos^2\theta_1 & -2b^3 sin^2\theta_1 cos\theta_1 & b^4 sin^2\theta_1 cos^2\theta_1 \end{pmatrix} \begin{pmatrix} M_1 \\ M_2 \\ M_3 \\ M_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{5.11}$$

We pick another point on the curve with the same value of σ. This point has parameter $\theta_2 = \pi - \theta_1$ (with $0 < \theta_2 < \pi/2$). This gives us a second equation

$$\begin{pmatrix} -bsin\theta_1 & -b^2 sin\theta_1 cos\theta_1 & 0 & 0 \\ bcos\theta_1 & 2cos^2\theta_1 & 2b^3 sin^2\theta_1 cos^2\theta_1 & b^4 sin^2\theta_1 cos^2\theta_1 \end{pmatrix} \begin{pmatrix} 1 & X_1 & \frac{X_1^2}{2!} & X_1^3/3! \\ 0 & 1 & X_1 & X_1^2/2! \\ 0 & 0 & 1 & X_1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} M_1 \\ M_2 \\ M_3 \\ M_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{5.12}$$

Now we consider the equation for the first two moments obtained by taking the first derivative at both points. From (5.11) and (5.12), this becomes

$$\begin{pmatrix} -bsin\theta_1 & b^2 sin\theta_1 cos\theta_1 \\ -bsin\theta_1 & -b^2 sin\theta_1 cos\theta_1 - X_1 bsin\theta_1 \end{pmatrix} \begin{pmatrix} M_1 \\ M_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{5.13}$$

The condition for there to be a solution of (5.13) is that the determinant of the matrix vanishes. This occurs at

$$X_1 = -2bcos\theta_1 \tag{5.14}$$

From (5.7), we see that this is indeed the distance between the two points and so we can solve for M_1 and M_2. We obtain:

$$M_1 = bcos\theta_1 M_2 \tag{5.15}$$

Substituting for $bcos\theta_1$ from (5.7) yields

$$M_1 = \left(X_o - \left\{\frac{X_1 + X_2}{2}\right\}\right) M_2 \tag{5.16}$$

where X_o is the position of the first point. The reconstructed function is

$$F(X) = -M_1 \varphi_1(X - X_o, \sigma_1) + M_2 \varphi_2(X - X_o, \sigma) \tag{5.17}$$

Without loss of generality set $X_o = 0$. Then up to a scale factor

$$F(X) = \frac{\{X_1 + X_2\}}{2} \frac{X}{\sqrt{2\pi}\sigma_1} + \frac{1}{2\sqrt{2\pi}} \left\{-\sigma_1 + \frac{X^2}{\sigma_1}\right\} \tag{5.18}$$

Now σ_1 lies on the circle

$$\sigma^2 + \left(X - \left\{\frac{X_1 + X_2}{2}\right\}\right)^2 = \left(\frac{X_2 - X_1}{2}\right)^2 \tag{5.19}$$

at the point where $X = 0$. Hence

$$\sigma_1^2 = -X_1 X_2 \tag{5.20}$$

Note that X_1 and X_2 have opposite signs if $X = 0$ lies on the circle. Substituting (5.20) into (5.18) gives

$$F(X) = \frac{1}{2\sqrt{2\pi}\sqrt{-X_1X_2}}\{X_1X_2 - (X_1 + X_2)X + X^2\} \tag{5.21}$$

From (5.1) and (5.5), we see that this is indeed the original function up to a scaling factor. Thus we have demonstrated how to reconstruct the signal.

We should check that $X_1 = -2b\cos\theta_1$ remains a root of the determinant for the higher order determinants. We will calculate the result for the case $n = 2$. From (5.11) and (5.12), the determinant equation becomes (in unconventional notation)

$$\begin{pmatrix} -b\sin\theta_1 & b^2\sin\theta_1\cos\theta_1 & 0 & 0 \\ -b\cos\theta_1 & b^2\cos^2\theta_1 & -2b^3\sin^2\theta_1\cos\theta_1 & b^4\sin^2\theta_1\cos^2\theta_1 \\ -b\sin\theta_1 & -X_1b\sin\theta_1 - b^2\sin^2\theta_1\cos\theta_1 & \frac{-X_1^2}{2!}b^2\sin\theta_1\cos\theta_1 - X_1b^2\sin\theta_1\cos\theta_1 & C \\ b\cos\theta_1 & X_1b\cos\theta_1 + b^2\cos^2\theta_1 & X_1^2/2!b\cos\theta_1 + X_1b^2\cos^2\theta_1 + 2b^3\sin^2\theta_1\cos^2\theta_1 & B \end{pmatrix} = 0 \tag{5.22}$$

where $C = \frac{-X_1^3}{3!}b\sin\theta_1 - \frac{X_1^2}{2!}b^2\sin\theta_1\cos\theta_1$ and $B = \frac{X_1^3}{3!}b\cos\theta_1 + \frac{X_1^2}{2!}\cos^2\theta_1 + X_1 2b^3\sin^2\theta_1\cos^2\theta_1 + b^4\sin^2\theta_1\cos^2\theta_1$. Dividing the matrix by factors common to rows, this becomes

$$\begin{pmatrix} -1 & b\cos\theta_1 & 0 & 0 \\ -1 & b\cos\theta_1 & -2b^2\sin^2\theta_1 & b^3\sin^2\theta_1\cos\theta_1 \\ -1 & -X_1 - b\cos\theta_1 & -X_1^2/2! - X_1b\cos\theta_1 & -X_1^3/3! - X_1^2/2!b\cos\theta_1 \\ 1 & X_1 + b\cos\theta_1 & X_1^2/2! + X_1b\cos\theta_1 + 2b^2\sin^2\theta_1\cos\theta_1 & V \end{pmatrix} = 0 \tag{5.23}$$

where $V = \frac{X_1^3}{3!} + \frac{X_1^2}{2!}b\cos\theta_1 + X_1 2b^2\sin^2\theta_1\cos\theta_1 + b^3\sin^2\theta_1\cos\theta_1$. By adding and subtracting rows, this reduces to

$$\begin{pmatrix} -1 & b\cos\theta_1 & 0 & 0 \\ 0 & 0 & -2b^2\sin^2\theta_1 & b^3\sin^2\theta_1\cos\theta_1 \\ 0 & -X_1 - 2b\cos\theta_1 & -X_1^2/2! - X_1b\cos\theta_1 & -X_1^3/3! - X_1^2/2!b\cos\theta_1 \\ 0 & 0 & 2b^2\sin^2\theta_1\cos\theta_1 & S \end{pmatrix} = 0 \tag{5.24}$$

,

where $S = X_1 2b^2\sin^2\theta_1\cos\theta_1 + b^3\sin^2\theta_1\cos\theta_1$. Thus, the equation becomes (removing common factors)

$$(X_1 + 2b\cos\theta_1)\begin{vmatrix} -1 & b \\ \cos\theta_1 & 2X_1 + b \end{vmatrix} = 0 \tag{5.25}$$

and can be expressed as

$$(X_1 + 2b\cos\theta_1)(2X_1 + b + b\cos\theta_1) = 0 \tag{5.26}$$

Hence, $X_1 = -2b\cos\theta_1$ remains a root for the $n = 2$ case.

We turn now to another example and an alternative approach to the problem of determining the image from the zero-crossings across scales. Let the signal be a third order polynomial.

$$F(X) = A + BX + CX^2 + X^3 \tag{5.27}$$

If we know that the signal is a third order polynomial we can determine its coefficients by derivatives of the zero-crossing contours at a single point. It is straightforward to show that this gives

$$E(X,t) = A + BX + CX^2 + X^3 + 2(C + 3X)t \tag{5.28}$$

The zero crossings curves are given by

$$A + BX + CX^2 + X^3 + 2(C + 3X)t = 0 \tag{5.29}$$

We will show that by taking a sufficient number of derivatives *at one point* it is possible to reconstruct the signal. From (5.29), we calculate

$$B + 2CX + 3X^2 + 6t + 2(C + 3X)\frac{dt}{dx} = 0 \tag{5.30}$$

$$2C + 6X + 2(C + 3X)\frac{d^2t}{dx^2} + 12\frac{dt}{dx} = 0 \tag{5.31}$$

$$6 + 18\frac{d^2t}{dx^2} + 2(C + 3X)\frac{d^3t}{dx^3} = 0 \tag{5.32}$$

At a point X t, $\frac{dt}{dx}$, $\frac{d^2t}{dx^2}$, $\frac{d^3t}{dx^3}$ are known. We write (5.29), (5.30), (5.31) as equations in the unknowns A, B, C.

$$\begin{pmatrix} 1 & X & X^2 + 2t \\ 0 & 1 & 2X + 2\frac{dt}{dx} \\ 0 & 0 & 2 + 2\frac{d^2t}{dx^2} \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} -X^3 - 6Xt \\ -3X^2 - 6t \\ -6X - 12\frac{dt}{dx} \end{pmatrix} \tag{5.33}$$

It will be possible to solve these equations uniquely for A, B, and C, provided the determinant of the matrix on the left hand size is nonzero. But the matrix is the Wronskian of the functions 1, X, $X^2 + 2t$. Except for a set of measure zero, which we discard, it will only vanish if the functions 1, X, $X^2 + 2t$ are linearly dependent. But from (5.29) we see this can only happen if the function $(C + 3X)$ divides the function $(A + BX + CX^2 + X^3)$. Apart from this special case, the determinant of the matrix will be non-zero and it is possible to solve for A, B and C.

We now consider the special case. The condition that $(C+3X)$ divides $(A+BX+CX^2+X^3)$ is

$$B = \frac{3A}{C} + \frac{2C^2}{9} \tag{5.34}$$

and the result is

$$\frac{X^2}{3} + \frac{2CX}{9} + \frac{A}{C} + \sigma^2 = 0 \tag{5.35}$$

Note firstly that although for the general third order polynomial (5.29) there are usually three zero crossing curves, there are now only two. Secondly, this equation is of similar form to equation (5.4) for the second order polynomial but the relative coefficients of σ^2 and X^2 are different, so the two cases can be distinguished.

We can differentiate (5.35) and show that the coefficients A and C can be determined *at a single point.*

It seems likely that this result will apply to all polynomials functions $F(X)$. Hence knowledge of the degree of the polynomial signal allows reconstruction from derivatives at a single point. It is furthermore likely that the degree of the image could be determined by the shape of the zero-crossing contours. If this conjecture is true this would represent an alternative constructive proof of Theorem 1.

6. Conclusions

We conclude with a brief discussion of a few issues that are raised by this paper and that will require further work.

a)*Stability of the reconstruction.* Although we have not yet rigorously addressed the question of numerical stability of the whole reconstruction scheme, there seem to be various ways for designing a robust reconstruction scheme. The first step to consider is the reconstruction of the *filtered* signal $E(x,t)$. One could exploit the derivatives at n points – at the given σ – and then solve the resulting highly constrained linear equations with least squares methods. Alternatively, it may be possible to fit a smooth curve through several points on one contour, and then obtain the derivatives there in terms of this interpolated curve. The same process must be performed on a second separate zero-crossing contour. This scheme provides a rigorous way of proving that instead of derivatives at two points, the location of the whole zero-crossing contour across scales can be used directly to reconstruct the signal (since the Implicit Function theorem shows that the zero-crossing curve is C^{∞}).

The second step involves the reconstruction of the unfiltered signal $I(x)$. We have constructed $\nabla^2 I$ explicitly. The construction is in terms of Hermite polynomials which can be integrated up straightforwardly to give us I (up to a function ϕ such that $\nabla^2\phi = 0$). Alternatively we can consider $F(x)$ to be the second moment of $I(x)$ (see equation (3.2.7)) and then use the moment equations to determine the second and higher order moments of $I(x)$ leaving the first two moments undetermined. This reconstruction step is unstable, as we discussed earlier, if only one scale is used. $I(x)$ can of course be reconstructed only modulus the null space of the (differential) operator. When the differential operator is the Laplacian and $E(x,t)$ is available for all t, then the representation is invertible and $I(x)$ can be recovered.[7 8]

b) *Degenerate fingerprints.* Our uniqueness result applies to *almost* all signal: a restricted but well known class of signals, with vertical zero-crossings in the scale-space diagram, correspond to nonunique fingerprints. These signals, which will be discussed in a forthcoming paper [Yuille and Poggio 1983, in preparation], do not belong to the class P introduced in Theorem 1 and 2. Interestingly, level-crossings (with a level different from zero) can distinguish between elements of this class. Note that there is a further, obvious constraint on the reconstruction: the original signal $I(x)$ can be reconstructed modulus the null space of the (differential) filter.

c) *Extensions.* Our main results are not restricted to second derivatives. They apply to zero- and level- crossings of a signal filtered by a gaussian filter of variable size. They also apply

[7] This reconstruction scheme may play a role in the computation of lightness in the vertebrate visual system, (Poggio, possibly in preparation).

[8]Notice that it may often be possible to assume that the image is an analytic entire function, i.e., a distribution already "diffused" by the imaging process. In this case, the Hermite expansion converges analytically everywhere and so does the associated Taylor expansion. If the convergence, however, can only be ensured in L^2 (i.e., the image is not analytic) there may be some parts of the image where the series expansion is not a faithful representation. We conjecture that it should usually be possible to infer the presence of such an anomalous region from the behavior of the derivatives of the image.

to transformations of a signal under a linear space-invariant operator – in particular they apply to the linear derivatives of a signal and to linear combinations of them. In both 1-D and 2-D, local information at just two points is sufficient. In practice, since many derivatives are needed at each point, information about the whole contour, to which the point belongs, is in fact exploited.

d) *Are the fingerprints redundant?* The proof of our theorem implies that two points on the fingerprint contours are sufficient. As we mentioned earlier, several points are probably required to make the reconstruction robust. We conjecture, however, that the fingerprints are redundant and that appropriate constraints derived from the process underlying signal generation (the imaging process in the case of images) should be used to characterize how to collapse the fingerprints into more compact representations. Witkin made already this point and discussed various heuristic ways to achieve this goal.

e) *Implications of the results.* As we discussed in the introduction, our results imply that the fingerprint representation is a *complete* representation of a signal or an image. Zero- and level-crossings across scales of a filtered signal capture full information about it. These results also suggest a central role for the gaussian in multiscale filtering that assure that zero- and level-crossing indeed contain full information. Note, however, that the fingerprint theorems do not constrain or characterize in any way the differential filter that has to be used. The filter may be just the identity operator, provided of course that enough zero-crossings contours exist. Independent arguments, based on the constraints of the signal formation process, must be exploited to characterize a suitable filter for each class of signals. For images, second derivative operators such as the Laplacian are suggested by work that takes into account the physical properties of objects and of the imaging process (Grimson, 1982; Poggio and Torre,in preparation; Yuille, 1983). We plan to explore this approach in the near future.

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Appendix 1

Properties of Hermite polynomials and truncation of the expansion

The set of Hermite functions is defined by

$$\psi_n(x) = \frac{e^{\frac{-x^2}{2}} H_n(x)}{\sqrt{2^n n! \sqrt{\pi}}} \quad [1]$$

where H_n are the Hermite polynomials.

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} \quad [2]$$

The Hermite functions are an orthonormal basis of functions which is *complete* for L^2 functions. The completeness is expressed by

$$\sum_n \psi_n(x)\psi_n(\varsigma) = \delta(x - \varsigma) \quad [3]$$

and the orthonormality by

$$\int \psi_n(x)\psi_m(x)dx = \delta_{nm} \quad [4]$$

In general, the Hermite expansion of a L^2 function does not converge uniformly, but only in the L^2 norm. The series will converge to the function except at a set of point of measure zero. At any point, the series can be truncated at a term of order N such that the remainder of the series is arbitrarily small. If we only consider a finite number of points where the series converges, the series can be truncated and the function approximated arbitrarily well by a finite number of Hermite components.

The Hermite polynomials defined in equation [2] and the set of function $w_n(x)$ defined as (see equation 3.2.6):

$$w_n(x) = \frac{1}{2^n n! \sqrt{\pi}} \frac{d^n}{dx^n} e^{-x^2} \quad [5]$$

are biorthogonal sets of functions, i.e.,

$$\int H_n(x) w_m(x) dx = \delta_{nm} \quad [6]$$

They also obey a completeness property

$$\sum_{nm} H_n(x) w_n(\varsigma) = \delta(x - \varsigma) \quad [7]$$

and therefore a L^2 function $f(x)$ can be expanded in either set of functions as

$$f(x) = \sum_n a_n H_n(x)$$
$$f(x) = \sum_n b_n w_n(x) \quad [8]$$

with

$$a_n = \langle f, w_n \rangle$$
$$b_n = \langle f, H_n \rangle \qquad [9]$$

Appendix 2

Analytic properties of solutions of the diffusion equation

For completeness we provide results about the analytic properties of functions convolved with the gaussian, i.e. solutions of the heat equation with the Huygens property (see later). Notice that images can be considered to have undergone already a "diffusion", since the imaging process has the effect of convolving the light distribution with a point spread function, usually very close to a gaussian.

Lemma: The solutions of the diffusion equations $u(x,t)$ as functions of x for a given t are restrictions to reals of functions which are entire functions of the complex variable (Widder, 1975, page 64).[9]

This result is noteworthy. The condition that $u_{xx} = u_t$ automatically brings with it C^∞ for $u(x,t)$ and even analyticity for the space variable x. Considered as a function of t, $u(x,t)$ can be extended analytically into the complex plane, although it will not generally be analytic (see Widder, 1975, page 65).

$$F(x,\sigma) = \sum_n a_n(x,\sigma) v_n(x,\sigma)$$

The heat polynomials v_n are defined as the coefficients of $\frac{z^n}{n!}$ in the expansion of

$$e^{xz+tz^2} = \sum_{n=0}^{\infty} v_n(x,t)\frac{z^n}{n!} \qquad [1]$$

where $t = \frac{\sigma^2}{2}$. v_n is a polynomial of degree n and is given by

$$v_n(x,t) = n! \sum_{k=0}^{[n/2]} \frac{t^k}{k!} \frac{x^{n-2k}}{(n-2k)!} \qquad [2]$$

They are related to the Hermite polynomials H_n by

$$v_n(x,t) = (-t)^{n/2} H_n\left(\frac{x}{\sqrt{-4t}}\right) \qquad [3]$$

The heat polynomials are solutions of the diffusion equation. Among other nice properties, they obey the following theorem [Widder, 1975].

Theorem: Any function $u(x,t)$ which obeys the diffusion equation and has the Huygens property can be expanded in a series of heat polynomials which *converges uniformly* in (x,t).

[9] Entire functions are functions which are analytic everywhere.

A solution $u(x,t)$ of the diffusion equation is said to have the Huygens property if it can be expressed as the convolution of the Gaussian with its initial value at $t = 0$. Filtered images satisfy this requirement by definition.

This theorem is important since it ensures that the expansion converges not only in the square integrable norm, as is usually the case, but also uniformally (i.e., at each point). This ensures that undesirable behavior, like the Gibbs phenomenon, does not end.

Appendix 3

Convergence of change of basis

We can write $F(x)$ in terms of the b_n's as

$$F(x) = \sum_{n=0}^{\infty} b_n(t)\varphi_n(x + x_1, t). \tag{1}$$

This series converges in the L^2 sense: given an $\epsilon > 0$, however small, it is possible to find an N_1 such that

$$\int \left(F(x) - \sum_{n=0}^{N_1} b_n(t)\varphi_n(x + x_1, t) \right)^2 dx < \epsilon. \tag{2}$$

Using (1) we can write equation (2) as

$$\int \left(\sum_{n=N_1+1}^{\infty} b_n(t)\varphi_n(x + x_1, t) \right)^2 dx < \epsilon. \tag{3}$$

Similarly for the expansion of $F(x)$ in terms of a_n's we can find an N_2 such that

$$\int \left(F(x) - \sum_{n=0}^{N_2} b_n(t)\varphi_n(x, t) \right)^2 dx < \epsilon. \tag{4}$$

Set $N = \max(N_1, N_2)$. If we can cut off the series for $F(x)$ at N then we can change the basis, as in subsection 3.3.1., equate coefficients of the $\varphi_n(x,t)$'s and hence relate the a_n,s to the b_n's. We would then have N equations for the first N moments. We now show that cutting off the series is permissible: if we choose ϵ sufficiently small we can make the error involved as small as we like.

The a_n's are related to the b_n's by

$$a_n(t) = \left\langle \sum_{n=0}^{\infty} b_n(t)\varphi(x + x_1, t), w_n(x,t) \right\rangle. \tag{5}$$

This can be written as

$$a_n(t) = \left\langle \sum_{n=0}^{N} b_n(t)\varphi(x + x_1, t), w_n(x,t) \right\rangle + \left\langle \sum_{n=N+1}^{\infty} b_n(t)\varphi(x + x_1, t), w_n(x,t) \right\rangle. \tag{6}$$

The second term is called $\bar{a}_n$ and is neglected when we cut off the series at N. We must now show that this is justified. We can write its square as

$$\bar{a}_n^2(t) = \left(\int \left(\sum_{n=N+1}^{\infty} b_n(t)\varphi(x+x_1,t)\right) w_n(x,t)dx\right)^2. \tag{7}$$

Using the Cauchy-Schwarz inequality we obtain

$$\left(\int \left(\sum_{n=N+1}^{\infty} b_n(t)\varphi(x+x_1,t)\right) w_n(x,t)dx\right)^2 \leq \int w_n(x,t)^2 dx \int \left(\sum_{n=N+1}^{\infty} b_n(t)\varphi(x+x_1,t)\right)^2 dx. \tag{8}$$

The first term on the right hand side of (6) is

$$a_n - \bar{a}_n = \left\langle \sum_{n=0}^{N} b_n(t)\varphi(x+x_1,t), w_n(x,t)\right\rangle. \tag{9}$$

Hence, using (8) and (3), we have

$$\bar{a}_n{}^2 \leq \epsilon \int \left(w_n(x,t)\right)^2 dx. \tag{10}$$

Thus if we make ϵ very small the errors $\bar{a}_n$'s will be negligible (we can scale $w_n(x,t)$ and $\varphi_n(x,t)$ by functions of n so that $\int \left(w_n(x,t)\right)^2 dx$ tends to a finite limit as n tends to ∞; alternatively, note that both $\bar{a}_n(x,t)$ and $a_n - \bar{a}_n$ depend linearly on $w_n(x,t)$ and so scaling it will not alter their relative sizes).

The errors involved in terminating the series can be made arbitrarily small by making the cutoff N sufficiently large. Thus we can change the basis and obtain N equations for the first N moments. We solve these equations and obtain the first N terms in the expansion of $F(x)$ in terms of the $\varphi_n(x,t)$. Taking the limit as N tends to ∞ we reconstruct the image (in the L^2 sense).

Appendix 4

We will show that the 2nth order determinant is generally non-zero. Recall that the determinant is a polynomial in x_1 (of degree at most $2n$) with the coefficients being functions of the first n derivation of the curves at the two points. If this determinant always vanished, it would mean that the distance between any two curves with prescribed values of their first n derivation could only take a finite set of values (at most $2n$) whatever the values of the higher order derivation of the curves. We will show that, by changing the values of the higher order derivatives, it is possible to alter the value of X_1 continuously while keeping the first n derivation of the curves constant.

We take two points $(0,\ t_1)$ and $(x_1,\ t_1)$ lying on zero-crossing curves. At these points, we assume we know the derivatives $\frac{\partial E}{\partial x}$, $\frac{\partial E}{\partial t}$, $\frac{\partial^2 E}{\partial x \partial t}$, ... up to order n. (This means we can reconstruct $\frac{dx}{d\sigma}, \ldots, \frac{d^n x}{d\sigma^n}$ from the implicit function theorem.) We can use the diffusion equation to write these as $\frac{\partial E}{\partial x}, \frac{\partial^2 E}{\partial x^2}, \ldots, \frac{\partial^{2n} E}{\partial x^{2n}}$.

So we have

$$\begin{aligned} E(0,t_1) &= 0 \\ \frac{\partial E}{\partial x}(0,t_1) &= K_1 \\ &\cdots \\ \frac{\partial^{2n} E}{\partial^{2n} x}(0,t_1) &= K_{2n} \end{aligned} \tag{1}$$

and

$$\begin{aligned} E(x_1,t_1) &= 0 \\ \frac{\partial E}{\partial x}(x_1,t_1) &= C_1 \\ &\cdots \\ \frac{\partial^{2n} E}{\partial x^{2n}}(x_1,t_1) &= C_{2n} \end{aligned} \tag{2}$$

where $K_1,\ldots,K_{2n}$ and $C_1,\ldots,C_{2n}$ are specified. Now we will try to alter the value of x_1 while keeping $K_1,\ldots,K_{2n}$ and $C_1,\ldots,C_{2n}$ constant.
We have

$$E(x,t) = \int e^{i\omega x} e^{-\omega^2 t} \omega^2 I(\omega) d\omega \tag{3}$$

Introduce a "deformation" parameter λ and a function $Y(\omega,\lambda)$ where

$$Y(\omega,0) = \omega^2 I(\omega) \tag{4}$$

and $x_1 = x_1(\lambda)$.
Let

$$E(x,t,\lambda) = \int e^{i\omega x} e^{-\omega^2 t} Y(\omega,\lambda) d\omega \tag{5}$$

Allow $x_1(\lambda)$ to vary while maintaining equations (1) and (2). For the first point this gives

$$\begin{aligned} \int e^{-\omega^2 t} \frac{\partial}{\partial\lambda} Y(\omega,\lambda) d\omega &= 0 \\ &\cdots \\ \int e^{-\omega^2 t} \omega^{2n} \frac{\partial Y}{\partial\lambda}(\omega,\lambda) d\omega &= 0 \end{aligned} \tag{6}$$

For the second point we obtain

$$\begin{aligned} \frac{dx_1}{d\lambda} \int e^{i\omega x_1} e^{-\omega^2 t} (i\omega) Y(\omega,\lambda) d\omega + \int e^{i\omega x_1} e^{-\omega^2 t} \frac{\partial}{\partial\lambda} Y(\omega,\lambda) d\omega &= 0 \\ \cdots\cdots \\ \frac{dx_1}{d\lambda} \int e^{i\omega x_1} e^{-\omega^2 t} (i\omega) \omega^{2n} Y(\omega,\lambda) d\omega + \int e^{i\omega x_1} e^{-\omega^2 t} \omega^{2n} \frac{\partial}{\partial\lambda} Y(\omega,\lambda) d\omega &= 0 \end{aligned} \tag{7}$$

We want to solve equations (6) and (7) for $\frac{\partial Y}{\partial\lambda}(\omega,\lambda)$ in terms of $\frac{dx_1}{d\lambda}$. Then the result follows. Equation (6) implies that the first $2n$ moments of $\frac{\partial Y}{\partial\lambda}(\omega,\lambda)$ are zero. Equation (7) means that the first $2n$ moments of $e^{i\omega x_1}\frac{\partial Y}{\partial\lambda}(\omega,\lambda)$ take prescribed values. (We assume $Y(\omega,\lambda)$ is known but not $\frac{\partial}{\partial\lambda}Y(\omega,\lambda)$.)

Expanding $e^{i\omega x_1}$ as a Taylor series (and using equation (6)), we write equation (7) as

$$\int \left\{ \sum_{m=2n+1}^{\infty} (i\omega)^m \frac{x_1^m}{m!} \right\} e^{-\omega^2 t} \frac{\partial}{\partial \lambda} Y(\omega,\lambda) d\omega = -\frac{dx_1}{d\lambda} \int e^{i\omega x_1} e^{-\omega^2 t} (i\omega) Y(\omega,\lambda) d\omega$$

$$\cdots \tag{8}$$

$$\int \left\{ \sum_{m=1}^{\infty} (i\omega)^m \frac{{x_1}^m}{m!} \right\} \omega^{2n} e^{-\omega^2 t} \frac{\partial}{\partial \lambda} Y(\omega,\lambda) d\omega = -\frac{dx_1}{d\lambda} \int e^{i\omega x_1} e^{-\omega^2 t} (i\omega) \omega^{2n} Y(\omega,\lambda) d\omega$$

The moments of $\frac{\partial}{\partial\lambda} Y(\omega,\lambda)$ are

$$W_m = \int e^{-\omega^2 t} \frac{\partial}{\partial \lambda} Y(\omega,\lambda) \omega^m d\omega \tag{9}$$

and define

$$A_p = \frac{-dx_1}{d\lambda} \int e^{i\omega x_1} e^{-\omega^2 t} (i\omega) \omega^p Y(\omega,\lambda) d\omega \tag{10}$$

Using (9) and (10) we rewrite equations (8) as

$$\sum_{m=2n+1}^{\infty} \frac{(ix_1)^m}{m!} W_m = A_1$$

$$\cdots\cdots \tag{11}$$

$$\sum_{m=2n+1}^{\infty} \frac{(ix_1)^{m-2n}}{(m-2n)!} W_m = A_{2n+1}$$

It will always be possible to solve these equations for W_m and there will be infinitely many solutions. To see this, we set

$$W_m = 0 \qquad , m > 4n+1 \tag{12}$$

and write equation (11) as

$$\begin{pmatrix} \frac{(ix_1)}{(2n+1)!} & \cdots & \frac{(ix_1)^{4n+1}}{(4n+1)!} \\ \vdots & & \\ \frac{(ix_1)}{1!} & \cdots & \frac{(ix_1)^{2n+1}}{2n!} \end{pmatrix} \begin{pmatrix} W_{2n+1} \\ \vdots \\ W_{4n+1} \end{pmatrix} = \begin{pmatrix} A_1 \\ \vdots \\ A_{2n+1} \end{pmatrix} \tag{13}$$

It is possible to solve (13) if the determinant is non-zero. The determinant is of form $\lambda(x_1)^{(2n+i)^2}$. (This follows directly from the form of the matrix) and so is either zero for all x_1 or else never zero. The determinant is also the Wronskian of the function $\frac{(ix_1)^{2n+1}}{2n!}, \ldots, \frac{(ix_1)^{4n+1}}{(4n+1)!}$ and as these functions are linearly independent, it cannot vanish everywhere. Hence the determinant never vanishes and we can solve for the W_m's in terms of the A_p's. Relaxing the condition (12) gives us infinitely many solutions.

Thus, we have shown that it is possible to alter x_1 continuously without changing the values of the first n derivatives at both points. This means that the determinant of the $2n$-order matrix in the moments will in general be non-zero; it can only be zero for a finite set of x_1 and there are an infinite set of possible values for x_1 compatible with the first n derivatives at the points.

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