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FINITE ABELIAN SEMIGROUPS REPRESENTED INTO THE POWER SET OF FINITE GROUPS

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Finite abelian groups have very well-defined structures and are direct sums of cyclic groups. If 2^G is the collection of nonempty subsets of a semigroup G, then $AB = \{ab \mid a \in A, b \in B\}$ defines a semigroup for 2^G. Although finite abelian groups have been investigated, 2^G is a relatively new object for research. BYRD, LLOYD, PEDERSON, and STEPP studied the automorphisms of 2^G (see [2]) and have made contributions to the understanding of 2^G.

If one allows G to be any abelian group and not just finite then TRNKOVÁ in [5] proved that every abelian semigroup is embeddable (one-to-one homomorphism) into 2^G for some abelian group G. But 2^G for an arbitrary abelian group is rather untractable. So further restriction was needed. In [1], BILYEU and LAU studied the collection (hyperspace) of compact subsets of a compact group and certain topological embeddings were derived.

But underlying all the general studies, a very basic question has not been settled:

Problem. If S is a finite abelian semigroup, then is S embeddable in 2^G for some finite abelian group G?

A finite abelian semigroup is said to be *representable* (in this paper) if it is embeddable in 2^G for some finite abelian group G. A z-semigroup is a semigroup having a unique idempotent which is a zero for the semigroup (see YAMADA [6] and [7]). If S is a finite semigroup, then it has a minimal ideal denoted by $M(S)$ and $S/M(S)$ is the Rees quotient. If S has an identity 1, then $H(1)$ is the group of units.

We were not able to solve the general problem but were able to prove that if finite abelian z-semigroups are representable, then finite abelian semigroups are representable. The following lemmas are helpful to establish this fact.

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Lemma 1. *If $G_1, \ldots, G_n$ are finite groups, then* $\prod_{i=1}^{n} 2^{G_i}$ *is embeddable in* $2^{\Pi G_i}$.

Proof. Use the function which sends $(A_1, \ldots, A_n)$ to $A_1 \times \ldots \times A_n$.

Lemma 2. *If S is a finite abelian semigroup and for each pair $x \neq y$ in S, there is a homomorphism f from S into 2^G for some finite abelian group G so that $f(x) \neq f(y)$, then S is representable.*

Proof. Since there are finitely many homomorphisms from S into $2^{G_1}, \ldots, 2^{G_n}$ to separate points, then S is embeddable in $\prod 2^{G_i}$, hence in $2^{\Pi G_i}$ by Lemma 1.

Lemma 3. *If S, T are semigroups and $i : S \to T$ is a one-to-one homomorphism, then $i^* : 2^S \to 2^T$ is a one-to-one homomorphism where $i^*(A) = i(A)$.*

Lemma 4. *If S is a semigroup and $\sigma : 2^{2^S} \to 2^S$ is defined by $\sigma(\mathscr{A}) = \bigcup\{A \mid A \in \mathscr{A}\}$, then σ is a homomorphism.*

Theorem. *If each finite abelian z-semigroup is representable, then every finite abelian semigroup is representable.*

Proof. Induct on the order of S where S is a finite abelian semigroup. Suppose $M(S)$ has more than one element. Let $e = e^2 \in M(S)$. Note that $M(S)$ is a group since S is abelian. Then $f : S \to M(S)$ by $f(x) = xe$ and $p : S \to S/M(S)$ would separate points. But $S/M(S)$ has an order less than that of S. By induction, $S/M(S)$ is representable.

We can now assume that S has a zero. Choose $e = e^2 \neq 0$ so that it is minimal with respect to the idempotent ordering of all nonzero idempotents. Again $f : S \to Se$ by $f(x) = xe$ and $S \to S/Se$ separate points. Hence we can assume that $S = Se$, i.e., S has an identity 1 and has only two idempotents 0 and 1.

Suppose $H(1) = \{1\}$. Then $I = S \setminus H(1)$ is a finite abelian z-semigroup. Let j be an embedding of I into 2^G for some finite abelian group G. Let H be a finite abelian group having more than one element. Then $J : S \to 2^{G \times H}$ defined by:

$$J(x) = \begin{cases} j(x) \times H & \text{if} \quad x \neq 1, \\ \{(1, 1)\} & \text{if} \quad x = 1, \end{cases}$$

is an embedding.

Assume that the set of idempotents of S is $\{0, 1\}$ and $H(1) \neq \{1\}$.

Let $H = H(1)$. Since $|I \cup \{1\}| < |S|$, then by induction, we have $j : I \cup \{1\} \to 2^G$ an embedding for some finite abelian group G. Let

1. $J : H \times (I \cup \{1\}) \to H \times 2^G$ be defined by $J(h, x) = (h, j(x))$,
2. $K : H \times 2^G \to 2^{H \times G}$ be defined by $K(h, A) = \{h\} \times A$,
3. $m : H \times (I \cup \{1\}) \to S$ be defined by $m(h, x) = hx$.

Then

$$m^{-1}(x) = \begin{cases} \{(h, h^{-1}x) \mid h \in H\} & \text{if} \quad x \in I, \\ \{(x, 1)\} & \text{if} \quad x \in H(1). \end{cases}$$

Claim. $M : S \to 2^{H \times (I \cup \{1\})}$ is a homomorphism where $M(x) = m^{-1}(x)$.

Let $x, y \in S$. Then $M(x) M(y) \subseteq M(xy)$ since m is a homomorphism.

Case A. Suppose $x \in H$ and $y \in I$. Then $xy \in I$. Let $(h, z) \in M(xy)$. Then $hz = xy$, $m^{-1}(x) = (x, 1)$ and $(h, z) = (x, 1)(hx^{-1}, z) \in M(x) M(y)$.

Case B. Suppose $x \in H$ and $y \in H$. Then $M(xy) = (xy, 1) = (x, 1)(y, 1) = M(x) \cdot M(y)$.

Case C. Suppose $x, y \in I$. Let $(h, z) \in M(xy)$. Then $hz = xy$. Hence $(h, z) = (h, h^{-1}x)(1, y) \in M(x) M(y)$.

Consider $i : S \to 2^{H \times G}$ defined by composing these four functions:

$$S \to^{M} 2^{H \times (I \cup \{1\})} \to^{J^*} 2^{H \times 2^G} \to^{K^*} 2^{2^{H \times G}} \to^{\sigma} 2^{H \times G}.$$

We shall prove that $i = \sigma K^* J^* M$ is an embedding. It is clear that it is a homomorphism.

Case 1. Let $x, y \in I$.

$$i(x) = \sigma K^* J^* M(x) = \sigma K^* J^* \{(h, h^{-1}x) \mid h \in H\} = \sigma K^* \{(h, j(h^{-1}x)) \mid h \in H\} =$$
$$= \sigma\{\{h\} \times j(h^{-1}x) \mid h \in H\} = \bigcup_{h \in H} \{h\} \times j(h^{-1}x).$$
$$i(y) = \bigcup_{h \in H} \{h\} \times j(h^{-1}y).$$

Suppose $i(x) = i(y)$. Then $\{1\} \times j(x) \subseteq \bigcup_{h \in H} \{h\} \times j(h^{-1}y)$. Hence $\{1\} \times j(x) \subseteq \{1\} \times j(y)$. Conversely, $\{1\} \times j(y) \subseteq \{1\} \times j(x)$. But $j(x) = j(y)$ implies $x = y$.

Case 2. Let $x, y \in H$.

$$i(x) = \sigma K^* J^* M(x) = \sigma K^* J^* \{(x, 1)\} = \sigma K^* \{(x, j(1))\} =$$
$$= \sigma\{\{x\} \times j(1)\} = \{x\} \times j(1).$$
$$i(y) = \{y\} \times j(1).$$

Hence $i(x) = i(y)$ implies $x = y$.

Case 3. Let $x \in H$, $y \in I$. Then

$$i(x) = \{x\} \times j(1)$$

and

$$i(y) = \bigcup_{h \in H} \{h\} \times j(h^{-1}y).$$

Hence $i(x) \neq i(y)$ since H has more than one element.

Remark. Left zero semigroups ($xy = x$ for all x, y) are not embeddable in 2^G for any finite group G. Hence the commutative property of the semigroup is important to the problem.

Remark. The structure of finite abelian z-semigroups was thoroughly discussed in [6] and [7] but we are still unable to solve the general problem.

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