Finite-Amplitude Baroclinic Wave Packets

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ABSTRACT

A theory is presented for the propagation of wave packets in a slightly unstable baroclinic shear flow in a quasi-geostrophic two-layer model on the beta plane. The theory for inviscid motions is considered and packet solutions resembling solitary waves are found. It is shown that the propagation speed of the packet, which is a function of its amplitude, *exceeds* the most naturally defined group velocity. A physical explanation is presented and it is suggested that the enhancement of the signal velocity above the group velocity is a general property of systems possessing linear instability and nonlinear stability.

1. Introduction

In a recent paper [Pedlosky (1970) hereafter referred to as FAB] I discussed the finite-amplitude dynamics of a single wave in a baroclinic two-layer model on the beta plane. The analysis showed that for a slightly supercritical inviscid wave the finite-amplitude state consisted of a long-period oscillation. The oscillation is described by a second-order nonlinear ordinary differential equation whose dependent variable is the wave amplitude and whose independent variable is the "long" time variable T related to the dimensional time t_* by

$$T=\frac{U}{L}|\Delta|^{\frac{1}{2}}t_*, \tag{1.1}$$

where U is a characteristic horizontal velocity of the flow and L a characteristic horizontal length. The vertical shear of the flow exceeds its critical value by a

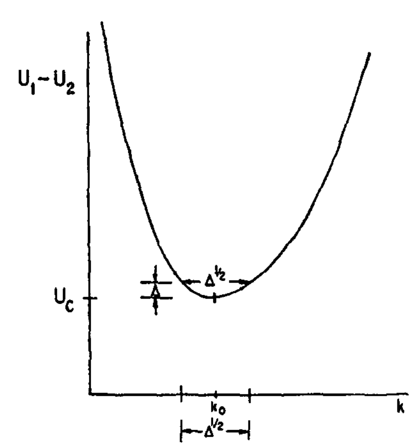

FIG. 1. The stability diagram showing the $\Delta^{\frac{1}{2}}$ bandwidth of unstable waves excited by an O(Δ) increment of the vertical shear above its critical value.

small amount Δ (in nondimensional units) and the factor $|\Delta|^{\frac{1}{2}}$ in (1.1) reflects the fact that for $\Delta\ll1$ the linear growth rate is O($\Delta^{\frac{1}{2}}$).

The purpose of the present paper is to discuss the finite-amplitude dynamics of the *group* of waves which becomes unstable when the critical value of the shear is exceeded by an amount Δ. Fig. 1 is a schematic picture of the linear stability diagram for the two-layer model. We see that if the minimum critical shear U_c is exceeded by an amount Δ all wavenumbers in a band of width $\Delta^{\frac{1}{2}}$ around the most unstable wavenumber k_0 are destabilized. It may happen that geometrical constraints prevent some or even all of the waves in that wavenumber band from being present. For example, large-scales waves in the atmosphere have their zonal wavenumber quantized so that only integral multiples of a fundamental are allowed. On the other hand, if the wavelength is sufficiently small compared to the scale of the domain of the wave, it is a very good approximation to allow the wave spectrum to be continuous in which case all wavenumbers in the unstable band are anticipated. Such a situation may well be relevant to baroclinic waves in the ocean. Assuming that the preferred wavelength of instability is on the order of the Rossby radius of deformation, we anticipate for the oceanic case wavelengths of the order 50–100 km embedded in the general oceanic circulation whose scale is O(5000 km).

A wavenumber band of thickness $\Delta^{\frac{1}{2}}$ about the central wavenumber k_0 corresponds, if $\Delta\ll1$, to a nearly periodic wave with wavelength $2\pi/k_0$ whose amplitude slowly varies in the longitudinal direction on the scale $L/|\Delta|^{\frac{1}{2}}$. We are led, therefore, to consider the dynamics of finite-amplitude wave packets in a baroclinic current possessing both multiple time and multiple space scales. In particular, we seek to discover equations governing the propagation of disturbances (i.e., the propagation of wave-like or meander patterns) which are of finite extent longitudinally. Although the model is highly simplified, an important result which emerges from the analysis is that the signal velocity with which the amplitude of the packet propagates may considerably exceed the group velocity of the waves. This enhancement of the signal speed above the group speed is a result of both the instability of the basic state and the nonlinear dynamics of the wave.

2. The model

As in FAB the model used in this paper consists of two layers of homogeneous fluids, each with a different constant density, on a plane rotating with angular velocity Ω. The lighter fluid lies above the heavier fluid. The fluid is bounded above by a rigid horizontal plane a distance D above the lower plane. The interface between the two fluids is considered for purposes of simplicity to lie equidistant between the two horizontal planes. The system is confined laterally to a channel infinite in length whose width is L. In order to include the effects of the earth's sphericity, Ω is assumed to vary linearly with the latitude coordinate y', *viz*:

$$2\Omega = f_0 + \beta' y'.$$

If the scales $[L, D, L/U, U, (D/L)U]$ are chosen for the horizontal coordinates, the vertical coordinate, time and the horizontal and vertical velocity, respectively, the governing equations of motion for small Rossby number (U/f_0L) are the quasi-geostrophic potential vorticity equations

$$\left.\begin{aligned}\left(\frac{\partial}{\partial t}+\frac{\partial\psi_1}{\partial x}\frac{\partial}{\partial y}-\frac{\partial\psi_1}{\partial y}\frac{\partial}{\partial x}\right)(\nabla^2\psi_1+F(\psi_2-\psi_1)+\beta y)=0\\ \left(\frac{\partial}{\partial t}+\frac{\partial\psi_2}{\partial x}\frac{\partial}{\partial y}-\frac{\partial\psi_2}{\partial y}\frac{\partial}{\partial x}\right)(\nabla^2\psi_2+F(\psi_1-\psi_2)+\beta y)=0\end{aligned}\right\}, \tag{2.1}$$

where x and y are the longitude and latitude coordinates and ψ_1 and ψ_2 are the nondimensional geostrophic streamfunctions (scaled with UL) in the upper and lower layer, respectively. The dimensionless parameters which have appeared are

$\beta = \beta' L^2/U$, the ratio of the planetary vorticity gradient to the relative vorticity gradient

$F = f_0^2 L^2 \bigg/ \left[\dfrac{\rho_2-\rho_1}{\rho_2} g \dfrac{D}{2}\right]$, the ratio of the square of the characteristic geometric length L to the square of the Rossby radius of deformation (2.2)

and the operator $\nabla^2 = (\partial^2/\partial x^2)+(\partial^2/\partial y^2)$. The reader is referred to FAB for the details of the derivation of (2.1).

3. Derivation of the packet propagation equation

We choose for the basic state upon which the packet perturbation is placed the simple zonal shear flow

$$\psi_1^{(0)} = -U_1 y, \quad \psi_2^{(0)} = -U_2 y, \tag{3.1}$$

upon which we place the packet perturbations φ_1 and φ_2. In line with our remarks in Section 1, we consider φ_1 and φ_2 to be functions of x, y, t and the long space and time scales

$$\left.\begin{aligned} T &= |\Delta|^{\frac{1}{2}} t\\ X &= |\Delta|^{\frac{1}{2}} x\end{aligned}\right\}, \tag{3.2}$$

where Δ is the increment above the minimum critical shear β/F (Pedlosky, 1964). The absolute values of $|\Delta|$ are used in (3.2) for we may also wish to consider subcritical states where $\Delta<0$. Since we are considering states which differ from the minimum critical shear by an amount Δ, we may write

$$U_1 = U_2 + \beta/F + \Delta, \tag{3.3}$$

while t and x derivatives in (2.1) transform according to the rule

$$\left.\begin{aligned}\frac{\partial}{\partial t} &\rightarrow \frac{\partial}{\partial t}+|\Delta|^{\frac{1}{2}}\frac{\partial}{\partial T}\\ \frac{\partial}{\partial x} &\rightarrow \frac{\partial}{\partial x}+|\Delta|^{\frac{1}{2}}\frac{\partial}{\partial X}\end{aligned}\right\}. \tag{3.4}$$

Furthermore, we anticipate that the equilibrated amplitude of the wave will be $O(\Delta^{\frac{1}{2}})$ as in FAB. The nonlinear equations governing the perturbations φ_1 and φ_2 for the upper layer are then

$$\begin{aligned}&\left[\frac{\partial}{\partial t}+\left(U_2+\frac{\beta}{F}\right)\frac{\partial}{\partial x}\right][\nabla^2\varphi_1+F(\varphi_2-\varphi_1)]+2\beta\frac{\partial\varphi_1}{\partial x}\\ &+|\Delta|^{\frac{1}{2}}\left\{\left[\frac{\partial}{\partial T}+\left(U_2+\frac{\beta}{F}\right)\frac{\partial}{\partial X}\right][\nabla^2\varphi_1+F(\varphi_2-\varphi_1)]+2\beta\frac{\partial\varphi_1}{\partial X}+\left[\frac{\partial}{\partial t}+\left(U_2+\frac{\beta}{F}\right)\frac{\partial}{\partial x}\right]\frac{2\partial^2\varphi_1}{\partial x\partial X}\right\}\\ &+|\Delta|\left\{\frac{\Delta}{|\Delta|}\frac{\partial}{\partial x}[\nabla^2\varphi_1+F(\varphi_2-\varphi_1)]+\frac{\Delta}{|\Delta|}F\frac{\partial\varphi_1}{\partial x}+\left[\frac{\partial}{\partial t}+\left(U_2+\frac{\beta}{F}\right)\frac{\partial}{\partial x}\right]\frac{\partial^2\varphi_1}{\partial X^2}+\left[\frac{\partial}{\partial T}+\left(U_2+\frac{\beta}{F}\right)\frac{\partial}{\partial X}\right]\frac{2\partial^2\varphi_1}{\partial x\partial X}\right\}\\ &+\frac{\partial\varphi_1}{\partial x}\frac{\partial}{\partial y}[\nabla^2\varphi_1+F(\varphi_2-\varphi_1)]-\frac{\partial\varphi_1}{\partial y}\frac{\partial}{\partial X}[\nabla^2\varphi_1+F(\varphi_2-\varphi_1)]\\ &+|\Delta|^{\frac{1}{2}}\left\{\left[\frac{\partial\varphi_1}{\partial X}\frac{\partial}{\partial y}-\frac{\partial\varphi_1}{\partial y}\frac{\partial}{\partial X}\right][\nabla^2\varphi_1+F(\varphi_2-\varphi_1)]+2\left[\frac{\partial\varphi_1}{\partial x}\frac{\partial}{\partial y}-\frac{\partial\varphi_1}{\partial y}\frac{\partial}{\partial x}\right]\frac{\partial^2\varphi_1}{\partial x\partial X}\right\}=0,\end{aligned} \tag{3.5a}$$

where terms of $O(\Delta^2)$ have been ignored as being inconsequential to the subsequent development. Similarly, for the lower layer, we have

$$\left(\frac{\partial}{\partial t}+U_2\frac{\partial}{\partial x}\right)[\nabla^2\varphi_2+F(\varphi_1-\varphi_2)]+|\Delta|^{\frac{1}{2}}\left[\left(\frac{\partial}{\partial T}+U_2\frac{\partial}{\partial X}\right)(\nabla^2\varphi_2+F(\varphi_1-\varphi_2))+\left(\frac{\partial}{\partial t}+U_2\frac{\partial}{\partial x}\right)\frac{2\partial^2\varphi_2}{\partial x\partial X}\right]$$
$$+|\Delta|\left[\left(\frac{\partial}{\partial t}+U_2\frac{\partial}{\partial x}\right)\frac{\partial^2\varphi_2}{\partial X^2}+\left(\frac{\partial}{\partial T}+U_2\frac{\partial}{\partial X}\right)\frac{2\partial^2\varphi_2}{\partial x\partial X}-F\frac{\Delta}{|\Delta|}\frac{\partial\varphi_2}{\partial x}\right]+\left(\frac{\partial\varphi_2}{\partial x}\frac{\partial}{\partial y}-\frac{\partial\varphi_2}{\partial y}\frac{\partial}{\partial x}\right)[\nabla^2\varphi_2-F(\varphi_2-\varphi_1)]$$
$$+|\Delta|^{\frac{1}{2}}\left\{\left[\frac{\partial\varphi_2}{\partial X}\frac{\partial}{\partial y}-\frac{\partial\varphi_2}{\partial y}\frac{\partial}{\partial X}\right][\nabla^2\varphi_2-F(\varphi_2-\varphi_1)]+2\left[\frac{\partial\varphi_2}{\partial x}\frac{\partial}{\partial y}-\frac{\partial\varphi_2}{\partial y}\frac{\partial}{\partial x}\right]\frac{\partial^2\varphi_2}{\partial x\partial X}\right\}=0. \quad (3.5b)$$

The appropriate boundary conditions are

$$\left[\frac{\partial}{\partial x}+|\Delta|^{\frac{1}{2}}\frac{\partial}{\partial X}\right]\varphi_n=0,\quad y=0,\,1,\quad n=1,\,2. \quad (3.6)$$

The perturbation streamfunction φ_n $(n=1,2)$ are expanded in the asymptotic series

$$\varphi_n=|\Delta|^{\frac{1}{2}}\varphi_n^{(1)}+|\Delta|\varphi_n^{(2)}+|\Delta|^{\frac{3}{2}}\varphi_n^{(3)}. \quad (3.7)$$

When (3.7) is inserted into (3.5a) and (3.5b) and like powers of $|\Delta|^{\frac{1}{2}}$ are set equal to zero, a sequence of linear problems results, the lowest order $(\Delta^{\frac{1}{2}})$ being

$$\left.\begin{aligned}&\left[\frac{\partial}{\partial t}+(U_2+\beta/F)\frac{\partial}{\partial x}\right][\nabla^2\varphi_1^{(1)}+F(\varphi_2^{(1)}-\varphi_1^{(1)}]+2\beta\frac{\partial\varphi_1^{(1)}}{\partial x}=0\\&\left[\frac{\partial}{\partial t}+U_2\frac{\partial}{\partial x}\right][\nabla^2\varphi_2^{(1)}+F(\varphi_1^{(1)}-\varphi_2^{(1)}]=0\end{aligned}\right\}, \quad (3.8)$$

which possesses wave packet solutions of the form

$$\left.\begin{aligned}\varphi_1&=\mathrm{Re}A(X,T)e^{ik(x-ct)}\sin m\pi y\\\varphi_2&=\mathrm{Re}\gamma A(X,T)e^{ik(x-ct)}\sin m\pi y\end{aligned}\right\}, \quad (3.9)$$

$$\left.\begin{aligned}c&=U_2+\frac{\beta(a^4-2F^2)}{2a^2F(a^2+2F)}(1\pm1)\\\gamma&=\frac{a^2+F}{F}-[2\beta/F/(U_2+\beta/F-c)]\end{aligned}\right\}, \quad (3.10)$$

where $a^2=k^2+m^2\pi^2$. The solutions (3.9) and (3.10) hold for all values of k. In particular, there are two modes corresponding to the choice of the plus or minus sign in (3.10). However, the wavenumber of the marginally stable wave (FAB)

$$a^2=\sqrt{2}F \quad (3.11)$$

corresponds to a coalescence of the two modes, both of which possess the phase speed

$$c=U_2, \quad (3.12)$$

and

$$\gamma=\sqrt{2}-1.$$

We note in passing that although the phase speeds coalesce, the group velocities of the two modes at marginal instability do not, i.e.,

$$\frac{\partial kc}{\partial k}\equiv C_{g2}=U_2 \quad (3.13)$$

if the minus sign in (3.10) is chosen, or

$$\frac{\partial kc}{\partial k}\equiv C_{g1}=U_2+\beta k^2(1+\gamma^2)/F^2 \quad (3.14)$$

if the plus sign is chosen. Thus, at the point of neutral stability there are two group speeds, each of which corresponds to the limit of a distinct but coalescing mode. At this point we attribute no particular significance to C_{g1} or C_{g2} beyond their definitions given by (3.13) and (3.14). Their significance will only become apparent when the packet amplitude equation is determined.

We note that at this order the amplitude $A(X,T)$ is still undetermined. We must consider the higher order problems to find an equation for A. The method of determining that equation is so similar to that found in FAB that only the results need be displayed here. The details of the analysis are essentially unaltered.

The order $|\Delta|$ problem quite simply yields

$$\left.\begin{aligned}\varphi_1^{(2)}&=\Phi_1^{(2)}(y,X,T)\\\varphi_2^{(2)}&=\Phi_2^{(2)}(y,X,T)-\mathrm{Re}\frac{i2F}{k\beta}\\&\times\left[\frac{\partial}{\partial T}+\left(U_2+\frac{\beta}{F}\right)\frac{\partial}{\partial X}\right]Ae^{ik(x-ct)}\sin m\pi y\end{aligned}\right\}. \quad (3.15)$$

The functions $\Phi_1^{(2)}$ and $\Phi_2^{(2)}$ are as yet unknown corrections to the zonal flow. That is, they are independent of x and t and non-wavelike, but they are dependent, slowly, on longitude. The second term in

(3.15) yields the phase shift between the waves in the upper and lower layers.

At this order both the mean flow corrections and the packet amplitude are still undetermined. When we consider the $O(|\Delta|^{\frac{3}{2}})$ problem we find, as in FAB, that unless certain restrictions are placed on the wave amplitude and the mean flow corrections the perturbation expansion (3.7) will contain secular terms and will not be valid for times of the $O(\Delta^{-\frac{1}{2}})$. In order to maintain our expansion valid for that time scale (which is the time scale over which the wave amplitude evolves to finite amplitude), we must eliminate those secular terms. This condition on the validity of our expansion yields, as in FAB, equations governing the functions $\Phi_1^{(2)}$, $\Phi_2^{(2)}$ and A. The analysis is lengthy, tedious but completely straightforward; the results are

$$\left[\frac{\partial}{\partial T}+\left(U_2+\frac{\beta}{F}\right)\frac{\partial}{\partial X}\right]\times\left[\frac{\partial^2\Phi_1^{(2)}}{\partial y^2}+F(\Phi_2^{(2)}-\Phi_1^{(2)})\right]+2\beta\frac{\partial\Phi_1^{(2)}}{\partial X}=\frac{F^2}{2\beta}\left[\left(\frac{\partial}{\partial T}+U_2\frac{\partial}{\partial X}\right)|A|^2 m\pi\sin 2m\pi y\right],\quad(3.16)$$

$$\left[\frac{\partial}{\partial T}+U_2\frac{\partial}{\partial X}\right]\left[\frac{\partial^2\Phi_2^{(2)}}{\partial y^2}+F(\Phi_1^{(2)}-\Phi_2^{(2)})\right]=-\frac{F^2}{2\beta}\left\{\frac{\partial}{\partial T}+\left[U_2+\frac{\beta k^2(1+\gamma^2)}{F^2}\right]\frac{\partial}{\partial x}\right\}\times|A|^2 m\pi\,\mathrm{sim}2m\pi y,\quad(3.17)$$

$$\left[\frac{\partial}{\partial T}+U_2\frac{\partial}{\partial X}\right]\left[\frac{\partial}{\partial T}+\left(U_2+\frac{\beta k^2(1+\gamma^2)}{F^2}\right)\frac{\partial}{\partial X}\right]A=\sigma^2 A+\frac{k^2\gamma^2\beta A}{F^2}\int_0^1\left[\frac{\partial^2\Phi_2^{(2)}}{\partial y^2}+F(\Phi_1^{(2)}-\Phi_2^{(2)})\right]\times m\pi\sin 2m\pi y dy,\quad(3.18)$$

where

$$\sigma^2=\frac{\Delta}{2|\Delta|}k^2\gamma^2\frac{\beta}{F}.\quad(3.19)$$

Although the analysis leading to (3.16)–(3.18) is lengthy, the resulting equations have a simple interpretation. Eqs. (3.16) and (3.17) are simply the perturbation forms for each layer of the conservation of potential vorticity for a flow which varies only slowly with longitude and time. The forcing terms on the right-hand side of (3.16) and (3.17) represent the advection of potential vorticity on the long space and time scales produced by the nonlinear self-interaction of the wave packet. This interaction acts like a source term for the slowly evolving, *nearly* zonal correction to the potential vorticity of the basic state.

The interpretation of (3.18) is almost equally straightforward. The left-hand side is an expression for the rate of change of A and consists of the application of two convective operators on A. The first is the time rate of change moving with C_{g1} and the second with C_{g2}. The first term on the right-hand side of (3.18) yields the rate of change of A due to the linear destabilization process, i.e., the increment Δ in vertical shear above critical. Indeed, σ^2 is the square of the growth rate, in non-dimensional units, given by linear theory. The last term on the right-hand side of (3.18) represents the equilibration mechanism of the interaction of the primary wave with the continuously altering basic flow. It is fundamentally a nonlinear term and it couples the amplitude equation to the equation for the evolution of the mean flow.

If we define the function

$$B(X,T)=-\frac{4\beta}{F^2 m\pi}\int_0^1 dy\sin 2m\pi y\times\left[\frac{\partial^2\Phi_2^{(2)}}{\partial y^2}+F(\Phi_1^{(2)}-\Phi_2^{(2)})\right],\quad(3.20)$$

we can rewrite the problem for the propagation of the wave packet amplitude $A(X,T)$ in the simple form

$$\left(\frac{\partial}{\partial T}+C_{g1}\frac{\partial}{\partial X}\right)\left(\frac{\partial}{\partial T}+C_{g2}\frac{\partial}{\partial X}\right)A=\sigma^2 A-NAB,\quad(3.21)$$

$$\left(\frac{\partial}{\partial T}+C_{g2}\frac{\partial}{\partial X}\right)B=\left(\frac{\partial}{\partial T}+C_{g1}\frac{\partial}{\partial X}\right)|A|^2,\quad(3.22)$$

where

$$N=\frac{k^2\gamma^2 m^2\pi^2}{4}.\quad(3.23)$$

Eqs. (3.21) and (3.22) are the fundamental equations governing the propagation of the wave packet whose central wavenumber is given by (3.11). Should we arbitrarily consider disturbances independent of X (i.e., an infinite plane wave), Eqs. (3.21) and (3.22) will reduce to the amplitude equations derived in FAB. We note that, in general, the packet propagation equations consist of two coupled partial differential equations and the system is nonlinear to boot.

4. The propagation of signals

The packet propagation equations (3.21) and (3.22) are quite difficult to discuss in general; thus, in this paper we shall content ourselves with a discussion of several special cases. A more general analysis will require, for arbitrary initial conditions, the numerical integration of the equations. Some aspects of the fundamental nature of the phenomena, however, can be deduced quite easily.

First, let us examine the case where the wave amplitude is small enough so that we may linearize (3.21). In that case the equation for A decouples from that for the mean flow. Further, let us examine the special case where $\sigma^2=0$, i.e., where the flow state is just marginally stable. Now $\sigma^2=0$ from (3.19) really means letting $\Delta\rightarrow 0$ while keeping $|\Delta|\neq 0$. This is not as absurd as it looks for it is equivalent to setting the increment of the vertical shear above critical to zero (which is measured by Δ) while maintaining that the packet length scale is $L|\Delta|^{-\frac{1}{2}}$, where now $|\Delta|^{-\frac{1}{2}}$ is unrelated to the shear increment. In this simple case (3.21) reduces to

$$\left(\frac{\partial}{\partial T}+C_{g1}\frac{\partial}{\partial X}\right)\left(\frac{\partial}{\partial T}+C_{g2}\frac{\partial}{\partial X}\right)A=0, \tag{4.1}$$

whose general solution is

$$A=R(X-C_{g1}T)+S(X-C_{g2}T), \tag{4.2}$$

where R and S are arbitrary functions.

The solution for arbitrary initial conditions consists of *two* packets each of which moves with the speed C_{g1} or C_{g2}. This is the real justification for interpreting (3.13) and (3.14) as proper defining relations for the group speeds of the two coalescing modes at the minimum critical shear. Although each of the modes has the same phase speed they are still really distinguishable because they possess different group speeds. The energy present in each of the modes will spatially separate if $C_{g1}\neq C_{g2}$ which will always be true if $\beta\neq 0$. For example, if

$$\left.\begin{aligned} A(X,0)&=A_0(X)\\ \frac{\partial A}{\partial T}(X,0)&=0\end{aligned}\right\}, \tag{4.3}$$

then for all $T>0$

$$A(X,T)=-\frac{C_{g2}}{C_{g1}-C_{g2}}A_0(X-C_{g1}T)+\frac{C_{g1}}{C_{g1}-C_{g2}}A_0(X-C_{g2}T). \tag{4.4}$$

Thus, if $C_{g1}>C_{g2}>0$, the faster mode will have the smaller amplitude and its amplitude will be opposite in sign to that of both the initial condition and the larger, slower packet as both propagate to the right (eastward). In any event the signal velocity is given by the appropriate group velocity.

If we still suppress the effects of nonlinearity but allow σ^2 to be greater than zero (the linearly unstable case), Eq. (3.21) becomes

$$\left(\frac{\partial}{\partial T}+C_{g1}\frac{\partial}{\partial X}\right)\left(\frac{\partial}{\partial T}+C_{g2}\frac{\partial}{\partial X}\right)A=\sigma^2 A. \tag{4.5}$$

By a trivial Galilean transformation we can consider the case where, without the loss of generality, $C_{g1}=-C_{g2}=C_G$. Then, using Laplace transforms, the solution to (4.5), again subject to (4.3), is

$$A(X,T)=\frac{1}{2\pi i}\int_\Gamma \frac{dp}{2C_G}\frac{e^{pt}}{(p^2-\sigma^2)^{\frac{1}{2}}}\times\left[\int_{-\infty}^{X}d\xi\exp\left(-(p^2-\sigma^2)^{\frac{1}{2}}\frac{X-\xi}{C_G}\right)A_0(\xi)\times\int_{X}^{\infty}d\xi\exp\left(-(p^2-\sigma^2)^{\frac{1}{2}}\frac{X-\xi}{C_G}\right)A_0(\xi)\right], \tag{4.6}$$

where Γ is a contour stretching from $p=-i\infty$ to $p=+i\infty$ to the right of the branch point at $p=\sigma$. The details of (4.6) do not concern us here. However, it is important to note that, by standard methods, we can show that the signal velocity implied by (4.6) is $\pm C_G$ or C_{g1} and C_{g2}. For example, if $A_0(X)$ is a delta function centered on $x=x_0$, A will be zero if either

$$\left.\begin{aligned} X-x_0&>C_GT\\ \text{or}\quad X&<C_GT+x_0\end{aligned}\right\}. \tag{4.7}$$

Thus, for the linear disturbance, the domain of influence of an arbitrary disturbance is limited in a given time by the distance a disturbance moves when travelling with the group velocities C_{g1} and C_{g2}.

As we mentioned above it is difficult to make general statements about the nonlinear case. However, a particular example can easily be found which confounds the notion that the signal speed is always the group velocity. A particularly interesting solution to (3.12) and (3.22) is a travelling wave packet, limited in spatial extent, which maintains its form as it propagates. Such a solution represents a wave group or meander pattern in a mature finite-amplitude state which is limited in longitude. Far upstream and downstream of the pattern the flow is strictly zonal, while within the domain of the

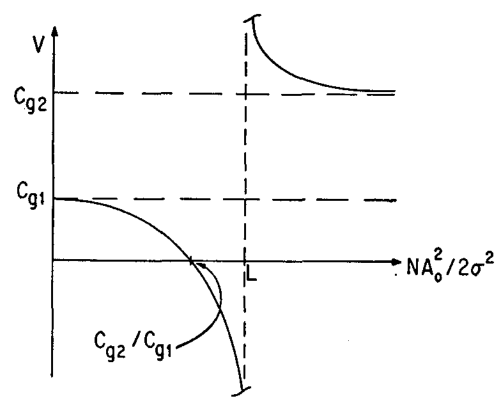

FIG. 2. The speed of the baroclinic wave packet, V, as a function of the packet amplitude A_0.

perturbation equilibrated finite-amplitude baroclinic waves exist.

Let

$$\left.\begin{aligned} A(X,T)&=A(X-VT)\\ B(X,T)&=B(X-VT)\end{aligned}\right\}, \tag{4.8}$$

where V is the propagation speed of the packet and is to be determined. Eq. (3.22) becomes

$$(C_{g2}-V)\frac{dB}{d\theta}=(C_{g1}-V)\frac{d|A|^2}{d\theta}, \tag{4.9}$$

where

$$\theta=X-VT,$$

which can immediately be integrated to give

$$B=\frac{(C_{g1}-V)}{(C_{g2}-V)}|A|^2, \tag{4.10}$$

where the constant of integration is set to zero since both A and B are assumed to vanish for large X. Eq. (3.21) then becomes

$$(C_{g1}-V)(C_{g2}-V)\frac{d^2A}{d\theta^2}-\sigma^2A+\frac{N(C_{g1}-V)}{(C_{g2}-V)}A|A|^2=0. \tag{4.11}$$

A solution of (4.11) which satisfies the condition that $A(X-VT)$ vanish for large X is

$$A(X,T)=A_0\operatorname{sech}\kappa(X-VT), \tag{4.12}$$

where

$$V=C_{g1}+(C_{g1}-C_{g2})\Big/\left(\frac{NA_0^2}{2\sigma^2}-1\right), \tag{4.13}$$

$$\kappa=\sigma[NA_0^2/2\sigma^2-1\Big/\left[\left(\frac{NA_0^2}{2\sigma^2}\right)^{\frac{1}{2}}(C_{g1}-C_{g2})\right]. \tag{4.14}$$

The propagation speed of the packet depends on the packet amplitude A_0. Fig. 2 displays the most striking

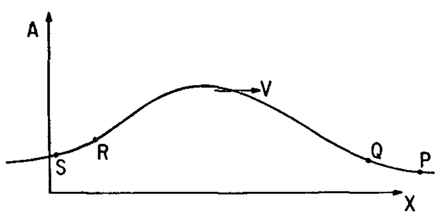

FIG. 3. A schematic diagram of the packet amplitude. Because of the nonlinear stabilization, the wave at P has a greater effective growth rate than that at the point Q. Similarly, the point at S decays more rapidly than the point at R.

characteristic of the solution. For all values of A_0 the speed of propagation of the packet lies *outside* the velocity interval bounded by the group speeds C_{g1} and C_{g2}. There is a critical amplitude ($NA_0^2/2\sigma^2=1$) for which $|V|$ becomes infinite; however, in that case $\kappa\to 0$ and the solution tends to the infinitely long plane wave for which the notion of propagation loses its significance.

The physical explanation of this strange behavior is really rather simple. Fig. 3 shows a schematic view of the packet envelope. The fluid at point P experiences a disturbance and will start to grow in size. The growth is due to two causes. The first is just the arrival of larger packet amplitude as would occur in any wave propagation of a finite packet. The fluid, however, is unstable and the amplitude at both points P and Q will grow by extracting energy from the mean flow. However, the fluid is stabilized by finite-amplitude effects ($N>0$) so that the amplitude at P will grow more rapidly than at Q where the larger amplitude leads to slower growth. Thus, due to the amplitude-dependent extraction of energy from the environment, the propagation of the region of maximum amplitude is enhanced by the nonlinear stability of the flow. At the points S and R the reverse process is taking place (the flow is inviscid and hence reversible), energy is being pumped back into the mean flow more rapidly at S than R, again causing the pattern to more more rapidly than the group velocity. *The enhancement of the signal velocity above the group velocity is due to the combination of linear instability and nonlinear stability.*

This can be seen by examining the solutions of (4.11) in the case

$$\sigma^2=-\omega^2<0 \quad (\text{i.e. } \Delta<0), \tag{4.15}$$

where we again have

$$A=A_0\operatorname{sech}\kappa(X-VT). \tag{4.16}$$

However, now

$$V=\frac{(C_{g1}NA_0^2/2\omega)^2+C_{g2}}{(NA_0^2/2\omega)^2+1}, \tag{4.17}$$

$$\kappa=\omega\left(\frac{2\omega^2}{NA_0^2}\right)^{\frac{1}{2}}\left(1+\frac{NA_0^2}{2\omega^2}\right)\Big/(C_{g1}-C_{g2}), \tag{4.18}$$

so that

$$C_{g1}\geqslant V\geqslant C_{g2}. \tag{4.19}$$

In the absence, therefore, of either instability or nonlinearity the signal speed is the group speed. The physical explanation given above does indeed require both.

Naturally, (4.12) is only a special solution of (3.12) and (3.22). To discuss the case for arbitrary initial conditions will require numerical calculations. Indeed the "solitary wave" solution presented here may itself be unstable to slight alterations in the initial conditions. It nevertheless demonstrates what I think is a general feature of wave propagation in unstable fluids, namely, the enhancement of the signal speed above the group speeds.

5. Steady solutions

If at $X=0$ the fluid is continuously perturbed by an oscillation with frequency kU_2 corresponding to the frequency of the marginally unstable wave, spatially modulated solutions which are independent of T are possible if the *amplitude* of the perturbation at $X=0$ is constant over the long time T, for then (3.21) and (3.22) reduce to

$$\frac{d^2A}{dX^2}-\frac{\sigma^2 A}{C_{g1}C_{g2}}+\frac{NA}{C_{g2}^2}[|A|^2-|A(0)|^2]=0, \quad (5.1)$$

where

$$A(0)\equiv A(X=0), \quad (5.2)$$

and where we have insisted that

$$B(0)=0. \quad (5.3)$$

For all "initial" conditions on A at $X=0$ the solution will consist of a long wavelength oscillation downstream of the disturbance. Since the form of the amplitude disturbance in X is identical to the form of the temporal oscillation of the single plane wave found in FAB, the details will not be presented here. It is, perhaps, from a geophysical point of view a rather artificial example for in the more typical case the applied perturbation would also be changing on the time scale T. On the other hand, it is perhaps the simplest model of a steady meander envelope for travelling meanders.

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