 

# FINITE-AMPLITUDE SURFACE WAVES IN ELECTROHYDRODYNAMICS*

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**Abstract.** The stability of weakly nonlinear waves on the surface of a fluid layer in the presence of an applied electric field is investigated by using the derivative expansion method. A nonlinear Schrödinger equation for the complex amplitude of quasi-monochromatic traveling wave is derived. The wave train of constant amplitude is unstable against modulation. The equation governing the amplitude modulation of the standing wave is also obtained which yields the nonlinear cut-off wave number.

**1. Introduction.** The effect of the electric field on the motion of fluids has been studied by a number of workers since the pioneering work of Rayleigh [1] and Stokes [2]. In his investigation on the stability of an incompressible, inviscid, perfectly conducting fluid layer in the presence of the electrostatic forces, Michael [3] found that the electrostatic forces can have a destabilizing effect on the fluid motions. Shivamoggi [4] has also examined the stability for such a problem in the neighborhood of the linear cut-off wave number.

Nonlinear dispersive waves have been a subject of intense study (see Lighthill [5], Whitham [6], Karpman [7]). In this paper, we examine the asymptotic behavior of weakly nonlinear dispersive waves on the surface of a fluid layer when an externally applied electric field is present. To that purpose we employ the derivative expansion method [8]. In Sec. 2, we give the basic equations and outline the procedure for obtaining the linear and the successive nonlinear partial differential equations of the various orders. Using the linear theory, we have obtained the uniformly valid solution for the second-order problem. It is also shown that, to the lowest order in the expansion parameter $\varepsilon$, the amplitude $A$ remains constant in a frame of reference moving with the group velocity of the waves. In Sec. 3, it is demonstrated that the complex amplitude of the quasi-monochromatic waves can be described by the nonlinear Schrödinger equation. The wave train solution of constant amplitude is modulationally unstable for $k > 1.388$. The result also indicates that although the Schrödinger equation so obtained is valid for a wide range of wave numbers, it does not hold near the cut-off wave number. A similar but slightly modified analysis is presented in Sec. 4 for the standing waves. It is shown that the amplitude modulation of the standing wave is governed by the nonlinear Schrödinger equation with the role of the time and the space variables interchanged. The

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nonlinear cut-off wave number which separates the stable from the unstable motions depends sensitively on the initial conditions.

**2. Formulation.** We study the two-dimensional wave propagation of finite amplitude on the surface of a perfectly conducting fluid layer with thickness $a_0$ supported by a conducting electrode at $z = -a_0$. The fluid is assumed to be inviscid and incompressible with the unperturbed free surface at $z = 0$. Another conducting plate is maintained above the surface of the fluid at $z = b_0$ with the fixed potential $v_0$. The motion ensues from rest and the flow field so generated due to wave motion is assumed to be irrotational. The basic equations relevant to our problem are:

$$\nabla^2\Omega(x, z, t) = 0, \qquad -a_0 < z < \eta, \tag{1}$$

$$\nabla^2\phi(x, z, t) = 0, \qquad \eta < z < b_0, \tag{2}$$

where $t$ denotes the time and $\eta$ the elevation of the free surface measured from the unperturbed level. Here, $\Omega(x, z, t)$ and $\phi(x, z, t)$ represent the velocity and electrostatic potentials, respectively. The various physical quantities are normalized with respect to the characteristic length $l = (T/\rho g)^{1/2}$ and the time $t_c = (l/g)^{1/2}$, where $g$ is the acceleration due to gravity and $T$ is the surface tension of the fluid. The nondimensional potential functions are taken to be $g^{-1/2}l^{3/2}\Omega(x, z, t)$ and $\phi_c\phi(x, z, t)$, where $\phi_c = (l^3 g\rho)^{-1/2}v_0$. The boundary conditions are:

$$\frac{\partial\eta}{\partial t} + \frac{\partial\Omega}{\partial x}\frac{\partial\eta}{\partial x} = \frac{\partial\Omega}{\partial z} \quad \text{at} \quad z = \eta, \tag{3}$$

$$(\partial/\partial z)\Omega = 0 \quad \text{at} \quad z = -a, \tag{4}$$

$$\phi = 1 \quad \text{at} \quad z = b, \qquad \phi = 0 \quad \text{at} \quad z = \eta, \tag{5, 6}$$

$$\frac{\phi_c^2}{8\pi}\left(\frac{\partial\phi}{\partial n}\right)^2 - \frac{\partial\Omega}{\partial t} - \frac{1}{2}\left[\left(\frac{\partial\Omega}{\partial x}\right)^2 + \left(\frac{\partial\Omega}{\partial z}\right)^2\right] + \frac{\partial^2\eta}{\partial x^2}\left(1 + \left(\frac{\partial\eta}{\partial x}\right)\right)^{-3/2} - \eta = \text{constant}, \tag{7}$$

where $a$, $b$ are the dimensionless quantities and $n$ represents the outward normal.

**3. Perturbation analysis.** In investigating the modulation of a weakly nonlinear quasi-monochromatic wave with narrow band spectrum. we employ the multiple-scale method. We introduce the variables

$$x_n = \varepsilon^n x, \qquad t_n = \varepsilon^n t \qquad (n = 0, 1, 2, \ldots, N),$$

where the small parameter $\varepsilon$ represents the weakness of the nonlinearity. To describe the nonlinear interactions of small but finite amplitude waves, we first expand $\eta$, $\phi$, and $\Omega$ in the following asymptotic series:

$$\eta(x, t) = \sum_{n=1}^{N+1} \varepsilon^n\eta_n(x_0, x_1, \ldots, x_n; t_0, t_1, \ldots, t_n) + O(\varepsilon^{N+2}), \tag{8}$$

$$\phi(x, t) = \sum_{n=0}^{N+1} \varepsilon^n\phi_n(x_0, x_1, \ldots, x_n; t_0, t_1, \ldots, t_n) + O(\varepsilon^{N+2}), \tag{9}$$

$$\Omega(x, t) = \sum_{n=1}^{N+1} \varepsilon^n\Omega_n(x_0, x_1, \ldots, x_n; t_0, t_1, \ldots, t_n) + O(\varepsilon^{N+2}), \tag{10}$$

$$\frac{\partial}{\partial t} = \sum_{n=0}^{N+1} \varepsilon^n \frac{\partial}{\partial t_n} + O(\varepsilon^{N+2}), \tag{11}$$

$$\frac{\partial}{\partial x} = \sum_{n=0}^{N+1} \varepsilon^n \frac{\partial}{\partial x_n} + O(\varepsilon^{N+2}). \tag{12}$$

It turns out that for the problem under investigation it is sufficient to take $N = 2$ as far as the lowest significant order is concerned. On using Taylor's series expansion around $z = 0$, the conditions (3), (6) and (7) lead to the linear and the successive nonlinear partial differential equations of various orders (see Appendix).

*a. Linear theory.* The progressive wave solution of the first-order problem governed by the equations (A4) to (A10) with respect to the lower scales $x_0$ and $t_0$ yields

$$\eta_1 = i(A \exp(i\theta) - \bar{A} \exp(-i\theta)), \tag{13}$$

$$\Omega_1 = \frac{\omega}{k} \frac{\cosh k(z + a)}{\sinh ka} A \exp(i\theta) + c \cdot c + B_1, \tag{14}$$

$$\phi_1 = \frac{i \sinh k(z - b)}{b \sinh kb} (A \exp(i\theta) - \bar{A} \exp(-i\theta)), \tag{15}$$

where

$$\theta = kx_0 - \omega t_0. \tag{16}$$

The constant $B_1$ is assumed to be real and independent of the lower scales $x_0$, $t_0$, and the amplitude $A$ is a function of the fast variables $x_1$, $t_1$; $x_2$, $t_2$. For the above solutions to be nontrivial, the frequency $\omega$ and the wave number $k$ must satisfy the following dispersion relation:

$$\omega^2 = k(k^2 + 1 - \alpha k \coth kb)\tanh ka, \tag{17}$$

where

$$\alpha = \phi_c^2/4\pi b^2. \tag{18}$$

From Eq. (17), we conclude that the fluid layer is unstable for all deformations which have wave numbers less than $k_c$, where $k_c$ is given by the transcendental equation

$$1 + k^2 - \alpha k \coth kb = 0. \tag{19}$$

On solving (19) when $\alpha = 1$ and $b = 0.5$, we get $k_c = 1.093$. If $kb \gg 1$ (i.e., long wave approximation), the solution of (19) takes the form

$$k_c = \left(\frac{\alpha}{b} - 1\right)^{1/2}. \tag{20}$$

On the other hand, for small $kb$, we obtain

$$k_{c_{1,2}} = 0.5\alpha \pm (0.25\alpha^2 - 1)^{1/2}. \tag{21}$$

Consequently, the progressive wave solutions are possible only for $k > k_c$. Since our aim is to study the amplitude modulation of the traveling waves, we now proceed to the second- and third-order problems furnished by Eqs. (A11) to (A25).

*b. Second-order solutions.* The nonsecularity condition for the second-order perturbation is

$$\frac{\partial A}{\partial t_1} + v_g \frac{\partial A}{\partial x_1} = 0 \tag{22}$$

together with its complex conjugate relation. Here $v_g = d\omega/dk$ represents the group velocity. Eq. (22) implies that the complex amplitude $A$ is constant in the frame of reference moving with the group velocity of the wave train. With this solvability condition, the particular solution of the second-order problem is

$$\eta_2 = \eta_{20}(x_1, x_2; t_1, t_2) + i\omega^{-1}p_1 \exp(i\theta) + A^2 p_2 \exp(2i\theta) + c.c., \tag{23}$$

$$\Omega_2 = -i\omega k^{-1}(z+a)\frac{\sinh k(z+a)}{\sinh ka}\frac{\partial A}{\partial x_1}\exp(i\theta) + \omega k^{-1}p_3\frac{\cosh 2k(z+a)}{\sinh 2ka}A^2\exp(2i\theta) + c.c. + B_2, \tag{24}$$

$$\phi_2 = b^{-1}(z-b)\frac{\cosh k(z-b)}{\sinh kb}\frac{\partial A}{\partial x_1}\exp(i\theta) + b^{-2}(z-b)q\left(\frac{\partial B_1}{\partial t_1} + 2p_5 A\bar{A}\right) + ib^{-1}p_4 A^2\frac{\sinh 2k(z-b)}{\sinh 2kb}\exp(2i\theta) + c.c. + B_3, \tag{25}$$

$$\eta_{20} = q\left(\frac{\partial B_1}{\partial t_1} + 2p_6 A\bar{A}\right), \tag{26}$$

$$p_1 = \frac{\partial A}{\partial t_1} + \omega k^{-1}(1 + ak\coth ka)\frac{\partial A}{\partial x_1}, \tag{27}$$

$$p_2 = ip_3 - k\coth ka, \tag{28}$$

$$p_3 = \frac{-2i}{D(2k, 2\omega)}\left[\frac{\omega^2}{2}(\coth^2 ka - 3) + (4k^2+1)k\coth ka + \frac{\alpha}{2}k^2(3 - \coth^2 kb - 4\coth 2kb(\coth ka + \coth kb))\right], \tag{29}$$

$$p_4 = p_3 + ik(\coth ka + \coth kb), \tag{30}$$

$$p_5 = \tfrac{1}{2}\omega^2 \operatorname{cosech}^2 ka + \frac{\alpha}{2}k^2(3 - \coth^2 kb) - k\coth kb, \tag{31}$$

$$p_6 = \tfrac{1}{2}\omega^2 \operatorname{cosech}^2 ka + \frac{\alpha}{2}k^2(3 - \coth^2 kb) - \alpha b^{-1}k\coth kb, \tag{32}$$

$$q = (\alpha b^{-1} - 1)^{-1}, \tag{33}$$

$$D(2\omega, 2k) = 2k(4k^2 + 1 - 2\alpha k\coth 2kb) - 4\omega^2\coth 2ka. \tag{34}$$

Here $B_2$ and $B_3$ are arbitrary constants to be determined by considering the equations of higher orders. We shall assume that $D(2\omega, 2k) \neq 0$. The case when $D(2\omega, 2k) = 0$ corresponds to the second harmonic resonance.

**3. Nonlinear self-modulation.** We now proceed to the third problem to get the equation for the amplitude modulation. The nonsecularity condition for $\eta_3$ is given by

$$a\frac{\partial^2 B_1}{\partial x_1^2} + \frac{\partial \eta_{20}}{\partial t_1} + 2\omega \coth ka \frac{\partial}{\partial x_1}(A\bar{A}) = 0. \tag{35}$$

With $B_1$ and $\eta_{20}$ assumed to depend upon the slower scales through $A$, Eqs. (22), (26), and (35) yield

$$\partial B_1/\partial x_1 = 2(a + qv_g)^{-1}[qv_g p_6 - \omega \coth ka]A\bar{A} + c(x_2, t_2), \tag{36}$$

where $c(x_2, t_2)$ is a constant of integration to be evaluated from the initial or the boundary conditions. We impose the condition that there is no flow at infinity. This condition gives

$$c(x_2, t_2) = 2(a + qv_g)^{-1}[\omega \coth ka - qv_g p_6]|A_0|^2, \tag{37}$$

where $A = A_0$ when $x \to \infty$. On substituting (36), (37) into (23), we get the mean elevation of the free surface due to nonlinearity.

We now introduce the following transformations:

$$\xi = \varepsilon^{-1}(x_2 - v_g t_2) = x_1 - v_g t_1; \qquad \tau = \varepsilon^2 t_2 = \varepsilon t_1.$$

On substituting Eqs. (13)–(17) and (23)–(34) into the third-order perturbation equations (A18) to (A25), we get the nonsecularity condition

$$i\frac{\partial A}{\partial \tau} + P\frac{\partial^2 A}{\partial \xi^2} = QA^2\bar{A} + RA, \tag{38}$$

where

$$P = \tfrac{1}{2}(dv_g/dk), \tag{39}$$

$$Q = \left(\frac{\partial D}{\partial \omega}\right)^{-1} \sum_{m=1}^{5} \left(\frac{\partial D}{\partial \omega}\right)^{-1} q_m, \tag{40}$$

$$R = -[1 - v_g q(1 + b^{-1})]i, \tag{41}$$

$$q_1 = p_3[\alpha k^2 \coth kb\{2(\coth 2ka + \coth 2kb) + (\coth ka + \coth kb) + 2\coth 2kb\} - k(k^2+1)(\coth ka + 2\coth 2ka) - 3\alpha k^2 - 2\omega^2 \coth ka \coth 2ka], \tag{42}$$

$$q_2 = ik[\alpha k^2\{(\coth ka + \coth kb)(4\coth kb \coth 2kb + \coth ka \coth kb - 2) - \tfrac{5}{2}\coth kb - \coth ka\} + \tfrac{3}{2}(k^3 - \omega^2 \coth ka) - k(k^2+1)(\coth^2 ka + \tfrac{3}{2})], \tag{43}$$

$$q_3 = -2i\omega\, d \coth ka, \tag{44}$$

$$q_4 = 2i\alpha k^2 b^{-1} q \coth kb(2p_5 - dv_g), \tag{45}$$

$$q_5 = iq(k^2\alpha \operatorname{cosech}^2 kb - \omega^2 \operatorname{cosech}^2 ka)(2p_6 - dv_g), \tag{46}$$

$$d = 2(a + qv_g)^{-1}[qv_g p_6 - \omega \coth ka]. \tag{47}$$

Eq. (38) describes the nonlinear self-modulation of the capillary-gravity waves on a liquid layer in the presence of an externally applied electric field. From the known solutions of the nonlinear Schrödinger equation (38), it is interesting to note that the plane wave solution is unstable against modulation if $PQ < 0$ (see Lighthill [4], Hasimoto and

Ono [9] and Kakatuni *et al.* [10]). Typically, for values of $\alpha = 1$, $a = b = 0.5$, this condition is satisfied for the wave number $k > 1.388$.

The term $P(\partial^2 A/\partial x_1^2)$ in (38) is essential to describe the long-time asymptotic behavior of the wave modulation. Physically, this plays the role of checking the steepening of the wave form giving rise to the equilibrium solution. This solution can be expressed in terms of the Jacobian elliptic functions, which include a depressive solitary wave, a phase jump, and a wave train of constant amplitude as special cases. The nonstationary solution of (38) has been obtained by Zakharov and Shabat [11] through the inverse scattering technique. They found that the finite-amplitude solution is modulationally unstable in such a way that an arbitrary initial motion ultimately breaks up into a series of solitons.

**4. Amplitude modulation of standing waves.** Since we are interested in the waves near $k = k_c$ and $\omega = 0$, the carrier wave is not a progressive wave but instead a standing wave. For this case the coefficients $P$ and $Q$ in (38) become unbounded as $k$ approaches $k_c$. This necessitates modification of the previous analysis. The starting solutions of the first-order problem we take here are:

$$\eta_1 = i(A(x_1 x_2;\, t_1,\, t_2)\exp(ik_c x_0) - \bar{A}(x_1,\, x_2;\, t_1,\, t_2)\exp(-ik_c x_0)), \tag{48}$$

$$\Omega_1 = B_1(x_1,\, x_2;\, t_1,\, t_2), \tag{49}$$

$$\phi_1 = \frac{i \sinh k(z-b)}{b \sinh kb}[A \exp(ik_c x_0) - \bar{A} \exp(-ik_c x_0)]. \tag{50}$$

Proceeding as before, we find the nonsecularity condition for the second-order perturbation

$$\partial A/\partial x_1 = 0, \tag{51}$$

which implies that the amplitude is independent of the faster variable $x_1$. The uniformly valid solution of the second-order problem now becomes

$$\Omega_2 = ik^{-1}\frac{\partial A}{\partial t_1}\frac{\cosh k(z+a)}{\sinh ka}\exp(ik_c x_0) + c.c. + B_2, \tag{52}$$

$$\phi_2 = b^{-2}(z-b)q\left(\frac{\partial B_1}{\partial t_1} + s_1 \bar{A}A\right) + ib^{-1}s_2\frac{\sinh 2k(z-b)}{\sinh 2kb}A^2\exp(2ik_c x_0) + c.c., \tag{53}$$

$$\eta_2 = q\frac{\partial B_1}{\partial t_1} + (qs_1 - 2k\coth kb)A\bar{A} + (is_2 + k\coth kb)A^2\exp(2ik_c x_0) + c.c., \tag{54}$$

where

$$s_1 = \alpha k^2(1 - \coth^2 kb) - 2k\coth kb, \tag{55}$$

$$s_2 = i\left[\alpha\frac{k^2}{2}(\coth^2 kb - 1) + k\coth kb(1 + 4k^2)\right]\left[1 + 4k^2 - 2\alpha k\coth kb\right]^{-1}. \tag{56}$$

Considering only the constant terms in the third-order problem, we obtain the nonsecularity condition

$$\partial B_1/\partial t_1 = [2k(\coth ka + \coth kb)q - s_1]A\bar{A} + E(x_2,\, t_2), \tag{57}$$

where $E(x_2, t_2)$ is an integration constant to be determined by the initial or the boundary conditions. Invoking the fact that there is no flow at infinity gives

$$E(x_2, t_2) = [s_1 - 2k(\coth ka + \coth kb)q]A_0 \bar{A}_0. \tag{58}$$

The requirement that the third-order perturbation be nonsecular yields

$$i\frac{\partial A}{\partial x_2} + P\frac{\partial^2 A}{\partial t_1^2} = QA^2\bar{A} + RA \tag{59}$$

where

$$P = -\frac{1}{2}\frac{\partial^2 k}{\partial \omega^2}\bigg|_{\omega=0}, \tag{60}$$

$$\begin{aligned} Q = -i\left(\frac{\partial^2 k}{\partial \omega^2}\right)^{-1}_{\omega=0} & [\{k(1+k^2)(2\coth 2kb + \coth kb) + \alpha k^2(2\coth kb \coth 2kb - 5)\}s_2 \\ & + i\{k^2(1+k^2)(\coth^2 kb + \tfrac{3}{2}) - \tfrac{3}{2}k^4 + \tfrac{1}{2}\alpha k^3 \coth kb\} \\ & + i\{(1+k^2) + \alpha k \coth kb\}2kb^{-1}(\coth ka + \coth kb) \\ & + q\{(1+k^2)\coth kb - \alpha k^2\}\{2k\coth kb(1-q) + 2k\coth ka\}], \end{aligned} \tag{61}$$

$$R = -\left(\frac{\partial^2 k}{\partial \omega^2}\right)^{-1}_{\omega=0}[b^{-1}q^{-1}(\alpha k \coth kb + 1 + k^2) + q^{-1}\{(1+k^2)\coth kb - \alpha k^2\}]E. \tag{62}$$

The above equation is the nonlinear Schrödinger equation with the role of time and the space variables interchanged. The interaction term $RA$ can be absorbed by the following transformation:

$$A(x_2, t_1) = A\exp\left(-i\int^{x_2} R(x_2')\,dx_2'\right). \tag{63}$$

We now consider the wave-train solution of the form

$$A(x_2, t_1) = a_0 \exp(i(Kx_2 - \Omega t_1)), \tag{64}$$

where $A_0$ is a constant. On substituting (64) into (59), we get the dispersion relation

$$\Omega^2 = -\frac{1}{P}(K + Q|a_0|^2). \tag{65}$$

Here $k_c = 1.093$, $Q = -8.256$, and $P = -0.186$ with $\alpha = 1$, and $a = b = 0.5$. For $\Omega$ to be imaginary, we require $K < Q|a_0|^2$. The nonlinear cut-off wave number therefore is

$$k_n = k_c + Q|a_0|^2\varepsilon^2 - \frac{\varepsilon^2}{x_2}\int^{x_2} R(x_2')\,dx_2. \tag{66}$$

This shows that $k_n$ depends sensitively upon the initial condition with respect to $t_1$. Moreover, the nonlinearity changes slightly the range of unstable wave numbers. The band width of spectrum is of $O(\varepsilon^2)$ in the wave number space for the standing waves. The equilibrium solution of (59) can also be expressed in terms of the Jacobian elliptic functions. A solitary wave, the phase jump and the wave train of constant amplitude are then just the special cases.

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**Appendix.** With the introduction of the following linear operators:

$$\mathscr{D}[\Omega_n] = \frac{\partial}{\partial z}\Omega_n, \qquad L[\eta_n, \Omega_n] = \frac{\partial \eta_n}{\partial t_0} - \mathscr{D}\Omega_n, \tag{A1, A2}$$

$$M[\eta_n, \Omega_n, \phi_n] = \left(\alpha b \mathscr{D}\phi_n - \frac{\partial \Omega_n}{\partial t_0} - \eta_n + \frac{\partial^2 \eta_n}{\partial x_0^2}\right), \tag{A3}$$

the first-order problem for $O(\varepsilon)$ is given by

$$\nabla^2 \Omega_1 = 0, \qquad \nabla^2 \phi_1 = 0 \tag{A4, A5}$$

with the boundary conditions

$$L[\eta_1, \Omega_1] = 0 \quad \text{at} \quad z = 0, \tag{A6}$$

$$\phi_1 = 0 \quad \text{at} \quad z = b, \tag{A7}$$

$$\phi_1 = -\eta_1 b^{-1} \quad \text{at} \quad z = 0, \tag{A8}$$

$$\mathscr{D}[\Omega_1] = 0 \quad \text{at} \quad z = -a, \tag{A9}$$

$$M[\eta_1, \Omega_1, \phi_1] = 0 \quad \text{at} \quad z = 0. \tag{A10}$$

The second-order problem for $O(\varepsilon^2)$ is

$$\nabla^2 \Omega_2 = -2(\partial^2 \Omega_1 / \partial x_0\, \partial x_1), \tag{A11}$$

$$\nabla^2 \phi_2 = -2(\partial^2 \phi_1 / \partial x_0\, \partial x_1), \tag{A12}$$

with

$$\mathscr{D}[\Omega_2] = 0 \quad \text{at} \quad z = -a, \tag{A13}$$

$$L[\eta_2, \Omega_2] = -\left(\frac{\partial \Omega_1}{\partial x_0}\right)\left(\frac{\partial \eta_1}{\partial x_0}\right) - \frac{\partial \eta_1}{\partial t_1} + \eta_1 \mathscr{D}^2 \Omega_1 \quad \text{at} \quad z = 0, \tag{A14}$$

$$\phi_2 = 0 \quad \text{at} \quad z = b, \tag{A15}$$

$$\phi_2 = -\eta_1 \mathscr{D}\phi_1 - \eta_2 b^{-1} \quad \text{at} \quad z = 0, \tag{A16}$$

$$\begin{aligned} M[\eta_2, \Omega_2, \phi_2] = {} & -\frac{\phi_c^2}{4\pi}\left[-\tfrac{1}{2}(\mathscr{D}\phi_1)^2 - \eta_1 b^{-1}(\mathscr{D}^2\phi_1) + b^{-2}\left(\frac{\partial \eta_1}{\partial x_0}\right)^2 \right. \\ & \left. + b^{-1}\left(\frac{\partial \eta_1}{\partial x_0}\right)\left(\frac{\partial \phi_1}{\partial x_0}\right)\right] + \frac{\partial \Omega_1}{\partial t_1} + \frac{1}{2}\left[(\mathscr{D}\Omega_1)^2 + \left(\frac{\partial \Omega_1}{\partial x_0}\right)^2\right] \\ & + \eta_1 \mathscr{D}\left(\frac{\partial \Omega_1}{\partial t_0}\right) - 2\frac{\partial^2 \eta_1}{\partial x_0\, \partial x_1} \quad \text{at} \quad z = 0. \end{aligned} \tag{A17}$$

The third-order problem for $O(\varepsilon^3)$ now becomes

$$\nabla^2 \Omega_3 = -2\frac{\partial^2 \Omega_2}{\partial x_0\, \partial x_1} - \frac{\partial^2 \Omega_1}{\partial x_1^2} - 2\frac{\partial^2 \Omega_1}{\partial x_0\, \partial x_2}, \tag{A18}$$

$$\nabla^2 \phi_3 = -2\frac{\partial^2 \phi_2}{\partial x_0\, \partial x_1} - \frac{\partial^2 \phi_1}{\partial x_1^2} - 2\frac{\partial^2 \phi_1}{\partial x_0\, \partial x_2}, \tag{A19}$$

with

$$\mathscr{D}[\Omega_3] = 0 \quad \text{at} \quad z = -a, \qquad \mathscr{D}[\Omega_3] = 0 \quad \text{at} \quad z = -a, \tag{A20, A21}$$

$$\begin{aligned} L[\eta_3, \Omega_3] = &-\frac{\partial \eta_2}{\partial t_1} - \frac{\partial \eta_1}{\partial t_2} + \eta_1 \mathscr{D}^2 \Omega_2 + \eta_2 \mathscr{D}^2 \Omega_1 + \tfrac{1}{2}\eta_1^2 \mathscr{D}^3 \Omega_1 \\ &- \frac{\partial \Omega_1}{\partial x_0}\left(\frac{\partial \eta_1}{\partial x_1} + \frac{\partial \eta_2}{\partial x_0}\right) - \frac{\partial \eta_1}{\partial x_0}\left(\frac{\partial \Omega_2}{\partial x_0} + \frac{\partial \Omega_1}{\partial x_1} + \eta_1 \frac{\partial^2 \Omega_1}{\partial z\, \partial x_0}\right) \quad \text{at} \quad z = 0, \end{aligned} \tag{A22}$$

$$\phi_3 = 0 \quad \text{at} \quad z = b, \tag{A23}$$

$$\phi_3 = -\eta_1 \mathscr{D}\phi_2 - \eta_2 \mathscr{D}\phi_1 - \tfrac{1}{2}\eta_1^2 \mathscr{D}^2 \phi_1 - \eta_3 b^{-1} \quad \text{at} \quad z = 0, \tag{A24}$$

$$\begin{aligned} M[\eta_3, \Omega_3, \phi_3] = &\frac{\partial \Omega_2}{\partial t_1} + \frac{\partial \Omega_1}{\partial t_2} + \eta_1 \mathscr{D}\left[\frac{\partial \Omega_2}{\partial t_0} + \frac{\partial \Omega_1}{\partial t_1}\right] + \tfrac{1}{2}\eta_1^2 \mathscr{D}^2\left(\frac{\partial \Omega_1}{\partial t_0}\right) \\ &+ \eta_2 \mathscr{D}\left(\frac{\partial \Omega_1}{\partial t_0}\right) + (\mathscr{D}\Omega_1)[\mathscr{D}\Omega_2 + \eta_1 \mathscr{D}^2 \Omega_1] \\ &+ \left(\frac{\partial \Omega_1}{\partial x_0}\right)\left[\frac{\partial \Omega_2}{\partial x_0} + \frac{\partial \Omega_1}{\partial x_1} + \eta_1 \frac{\partial^2 \Omega_1}{\partial z\, \partial x_0}\right] \\ &- \frac{\partial^2 \eta_1}{\partial x_1^2} - 2\frac{\partial^2 \eta_2}{\partial x_1\, \partial x_0} - 2\frac{\partial^2 \eta_1}{\partial x_0\, \partial x_2} + \frac{3}{2}\left(\frac{\partial \eta_1}{\partial x_0}\right)^2 \frac{\partial^2 \eta_1}{\partial x_0^2} \\ &- \frac{\phi_c^2}{4\pi}\Bigg[(\mathscr{D}\phi_2)(\mathscr{D}\phi_1) + \eta_1\{b^{-1}\mathscr{D}^2\phi_2 + (\mathscr{D}\phi_1)(\mathscr{D}^2\phi_1)\} \\ &+ \eta_2 b^{-1}\mathscr{D}^2\phi_1 - b^{-1}\frac{\partial \eta_2}{\partial x_0}\frac{\partial \phi_1}{\partial x_0} + \tfrac{1}{2}\eta_1^2 b^{-1}\mathscr{D}^3\phi_1 \\ &- b^{-1}\left\{\frac{\partial \eta_1}{\partial x_1}\frac{\partial \phi_1}{\partial x_0} + \frac{\partial \eta_1}{\partial x_0}\frac{\partial \phi_2}{\partial x_0} + \frac{\partial \eta_1}{\partial x_0}\frac{\partial \phi_1}{\partial x_1} + \left(\frac{\partial \eta_1}{\partial x_0}\right)^2 (\mathscr{D}\phi_1)\right\} \\ &- (\mathscr{D}\phi_1)\left(\frac{\partial \eta_1}{\partial x_0}\right)\left(\frac{\partial \phi_1}{\partial x_0}\right) - b^{-1}\eta_1\left(\frac{\partial \eta_1}{\partial x_0}\right)\left(\frac{\partial^2 \phi_1}{\partial x_0\, \partial z}\right) \\ &- b^{-2}\left(\frac{\partial \eta_1}{\partial x_0}\right)\left(\frac{\partial \eta_1}{\partial x_1} + \frac{\partial \eta_2}{\partial x_0}\right)\Bigg] \quad \text{at} \quad z = 0. \end{aligned} \tag{A25}$$

In writing Eqs. (A8) to (A25), we have used the following zeroth-order solutions of Eqs. (1) to (7):

$$\Omega_0 = 0, \tag{A26}$$

$$\phi_0 = z/b. \tag{A27}$$

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