

NASA TECHNICAL NOTE



NASA TN D-4803

61

NASA TN D-4803



FINITE AMPLITUDE WAVES IN
FLUID-FILLED ELASTIC TUBES:
WAVE DISTORTION, SHOCK WAVES,
AND KOROTKOFF SOUNDS

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NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

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By Richard M. Beam

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SUMMARY

The Lagrangian form of the one-dimensional equations of motion for a fluid-filled elastic tube is developed. The resulting nonlinear equations are combined with a general nonlinear internal pressure cross-sectional area relation for the tube to obtain the finite amplitude "wave equation." After determination of the tube pressure-area relation which leads to a linear (non-distorting) wave equation, a solution for the general (distorting) wave equation is presented. The solution for the general wave equation is applicable for simple (nonreflecting) waves in tubes with general pressure-area relations. The solution is then used to develop criteria for the steepening and nonsteepening of finite amplitude waves.

The theory is further developed to include the calculation of wave distortion. The criterion for a sharp wave is derived and the critical length of tube required for a sharp wave to develop is determined. The propagation of the resulting sharp wave or shock wave is considered. The velocity and the energy loss formulas for the shock wave are developed.

Finally, the analysis and experimental results are used to support new hypotheses for the mechanism and source of the Korotkoff sounds which have a major role in the indirect determination of blood pressure. A limited amount of experimental data (from Bramwell) is presented to correlate quantitatively the analysis and the new hypotheses.

INTRODUCTION

The study of the dynamics of fluid-filled elastic tubes, or cylinders, has received extensive interest over the past two centuries. The greatest bulk of papers and books on the subject has appeared within the last 10 years in the fields of biomechanics¹ and aeromechanics (see the bibliography of ref. 2). Studies in aeromechanics are, in general, concerned with the interaction of liquid propellants and their elastic containers while the studies in biomechanics are concerned with the interaction of blood and arteries or veins. Although the physical analogies in this report refer to biomechanics problems, the application of the theory is not restricted to one field.

¹An excellent historical and theoretical review including an extensive bibliography was given by Shalak in 1966, reference 1.

For mathematical simplicity, most investigators have restricted their analyses to linear theory. The solutions to the nonlinear equations have been limited either to numerical integration utilizing the method of characteristics (ref. 3) or to the perturbation methods (ref. 1). Although both methods of solution provide useful information, neither provides the general information available from a closed-form solution.

The purpose of this report is twofold; first, to present an analysis for the propagation of a simple (no reflection) wave of finite amplitude in a fluid-filled elastic tube. Of particular interest are the distortion of a smooth wave until a sharp wave develops and the propagation of the resulting shock wave. Although several writers have speculated on the development of shock waves (refs. 1, 4, and 5) in fluid-filled elastic tubes, this is, to the knowledge of this author, the first presentation of a theory for the development and propagation of the shock waves. The experimental verification remains to be done.

The second purpose of this report is to present new hypotheses for the mechanism and source of the Korotkoff sounds which were the motivation for this study. The hypotheses concern the source of the Korotkoff sounds that occur when blood pressure is measured with a pressure cuff. Briefly, the mechanism hypothesis states that the sounds result from the development of a sharp, or possibly, a shock pulse wave in the compressed portion of the artery under the inflated cuff. The source hypothesis states that the source of the "sounds" is the response of the "listening" system to a sharp pressure wave.

SYMBOLS

$A(x,t)$	internal cross-sectional area of tube
A_e	equilibrium internal cross-sectional area of tube
A_1, A_2	discrete values of cross-sectional area of tube on each side of shock wave
C	wave velocity of finite amplitude nondistorting wave
$c(A)$	spatial wave velocity as function of area A (the argument may be other than A)
c_s	spatial velocity of shock wave
c_s^*	shock-wave velocity in compressed coordinate system
c_1	a constant (eq. (59))
E_L	energy loss coefficient for shock wave
$F\left(\frac{\partial u}{\partial x}\right)$	function defined by equation (27)

$f(a)$	pressure cross-sectional area relation for tube
$f_7(A)$	pressure cross-sectional area relation for tube which leads to a linear wave equation
$g\left(\frac{\partial u}{\partial x}\right)$	function defined by equation (25)
$h\left[t - \frac{x}{c(p)}\right]$	pressure spatial time relation for simple wave
$L_c(p^*)$	length of tube required for a sharp wave to develop at pressure p^*
L_{cm}	minimum length of tube required for a sharp wave to develop
$p(x,t)$	pressure differential across tube wall (Subscripts i and o denote inside and outside tube $p = p_i - p_o$.)
p_e	differential pressure across tube wall at equilibrium condition (Subscripts ie and oe denote inside and outside tube pressure $p_e = p_{ie} - p_{oe}$.)
p^*	pressure at which pressure gradient becomes infinite
p_2, p_1	discrete pressures on each side of shock (p_1 is also used as constant in eqs. (58) through (63))
t	time parameter
t_0	discrete time at which pressure equals p^* (eq. (52))
$u(x,t)$	displacement of particle which had coordinate x at time $t = 0$
$v(x,t)$	particle velocity (eq. (34))
v_2, v_1	discrete particle velocities on each side of shock
x	spatial coordinate (fig. 1)
y	spatial coordinate (fig. 3)
$\alpha(x,t)$	dimensionless relation for tube cross-sectional area (eq. (32))
$\epsilon(x,t)$	dimensionless relation for tube cross-sectional area (eq. (20))
ϵ_1, ϵ_2	discrete values of ϵ on each side of shock

- μ particle-to-particle propagation rate (eq. (36))
- ρ fluid density
- ω frequency constant
- ($\bar{}$) dummy variable of integration

I. LAGRANGIAN EQUATIONS OF MOTION

Derivation of Equations of Motion for Large Displacements

The purpose of this section is (1) to develop the large displacement equations of motion (Lagrangian form) for a fluid-filled elastic tube, (2) to establish the conditions for a nondistorting propagating wave, and (3) to obtain the finite amplitude wave propagation properties. In order to simplify the mathematics and to emphasize the physical phenomena relevant to this report, some gross assumptions regarding the physical system have been made, namely:

- (a) The fluid is incompressible and inviscid.
- (b) The fluid flow is one-dimensional and the radial inertia is negligible.
- (c) The mass of the tube wall is negligible (i.e., the kinetic energy of the wall is small when compared to the kinetic energy of the fluid).
- (d) The flexural rigidity and prestress of the tube in the axial direction do not affect the stiffness of the tube. This is equivalent to the assumption that each element along the length of the tube behaves as an isolated element disconnected from its neighbors.
- (e) Under equilibrium conditions the tube has uniform (not necessarily circular) cross section along the length of the tube. The elastic properties are also uniform along the tube.

The validity of these assumptions in the analysis of Korotkoff sounds will be reviewed in section IV of this report.

Euler (ref. 6) developed two forms for the equations of motion of a fluid. One considers the velocity, pressure, and density of all points in the space occupied by the fluid for all time, while the other considers the history of every element of the fluid for all time. The two forms are generally designated as the Eulerian and Lagrangian forms, respectively. For most fluid flow problems, the Eulerian form is more tractable than the Lagrangian, and the Lagrangian form becomes quite awkward for two- and three-dimensional flow. For some special one-dimensional flows, however, the Lagrangian form has some advantages which will be evident in the following analysis.

Consider the equilibrium of an element of fluid which had original length δx . Let $u(x,t)$ be the displacement of the particle which had coordinate x at time $t = 0$ (see fig. 1). The x coordinate system is then an inertial

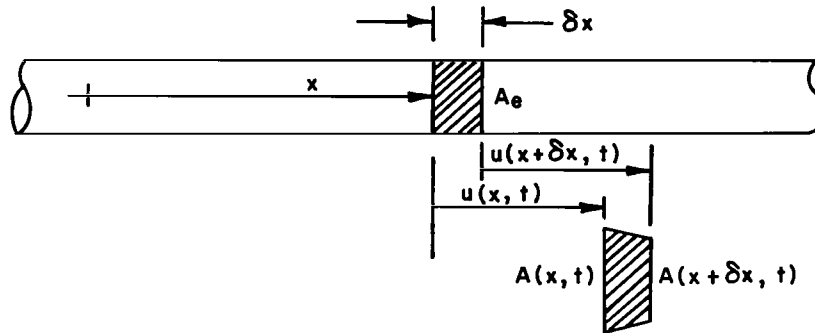


Figure 1.- Coordinate system for development of Lagrangian equation.

coordinate system along the axis of the tube. In addition, define

$A(x,t)$ cross-sectional area at time t of the fluid element which had coordinate x at $t=0$ (i.e., the cross-sectional area of the element at $x + u$)

A_e original ($t = 0$) cross-sectional area of the fluid element

$p(x,t)$ pressure at time t in the fluid element which had coordinate x at $t = 0$ (i.e., the pressure in the element at $x + u$)

ρ fluid density (a constant for incompressible fluid)

The force on the left side of the deformed element is

$$p(x,t)A(x,t) \quad (1)$$

while the force on the right side is

$$-p(x + \delta x,t)A(x + \delta x,t) \quad (2)$$

In addition, the forces acting on the sides of the element have an axial component

$$-\frac{p(x,t) + p(x + \delta x,t)}{2} [A(x,t) - A(x + \delta x,t)] \quad (3)$$

The forces acting on the surfaces of the element (eqs. (1), (2), and (3)) must be offset by the inertial force of the accelerating element

$$\rho \delta x A_e \frac{1}{2} \left[\frac{\partial^2 u(x,t)}{\partial t^2} + \frac{\partial^2 u(x + \delta x,t)}{\partial t^2} \right] \quad (4)$$

Note that expressions (3) and (4) are only approximate for finite δx but are exact in the limit as δx approaches zero. Equating the external forces to the inertial forces acting on the element one obtains

$$\begin{aligned}
 p(x,t)A(x,t) - p(x+\delta x,t)A(x+\delta x,t) - \frac{p(x,t) + p(x+\delta x,t)}{2} [A(x,t) - A(x+\delta x,t)] \\
 = \rho \delta x A_e \frac{1}{2} \left[\frac{\partial^2 u(x,t)}{\partial t^2} + \frac{\partial^2 u(x+\delta x,t)}{\partial t^2} \right]
 \end{aligned} \tag{5}$$

If the terms of equation (5) with argument $x + \delta x$ are expanded about x

$$\begin{aligned}
 p(x,t)A(x,t) - \left[p(x,t) + \delta x \frac{\partial p(x,t)}{\partial x} + O(\delta x^2) \right] \left[A(x,t) + \delta x \frac{\partial A(x,t)}{\partial x} + O(\delta x^2) \right] \\
 - \frac{1}{2} \left[p(x,t) + p(x,t) + \delta x \frac{\partial p(x,t)}{\partial x} + O(\delta x^2) \right] \left[A(x,t) - A(x,t) - \frac{\partial A(x,t)}{\partial x} \delta x - O(\delta x^2) \right] \\
 = \rho \delta x A_e \frac{1}{2} \left[\frac{\partial^2 u(x,t)}{\partial t^2} + \frac{\partial^2 u(x,t)}{\partial t^2} + \delta x \frac{\partial^2}{\partial t^2} \frac{\partial u(x,t)}{\partial x} + O(\delta x^2) \right]
 \end{aligned} \tag{6}$$

where $O(\delta x^2)$ represents terms involving terms of second and higher order in δx and, after terms of $O(\delta x^2)$ are regrouped, equation (6) may be written

$$- \frac{\partial p(x,t)}{\partial x} A(x,t) \delta x + O(\delta x^2) = \rho \delta x A_e \frac{\partial^2 u(x,t)}{\partial t^2} \tag{7}$$

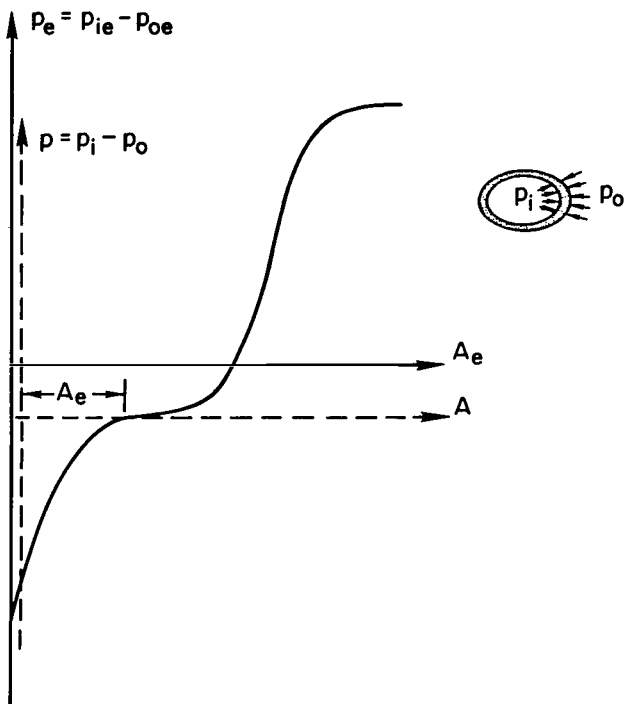
Since δx may be taken as small as one chooses, in the limit as $\delta x \rightarrow 0$ equation (7) becomes

$$- \frac{\partial p(x,t)}{\partial x} A(x,t) = \rho A_e \frac{\partial^2 u(x,t)}{\partial t^2} \tag{8}$$

In addition to equilibrium of the deformed element, the mass, or in this analysis (incompressible fluid) the volume, must be conserved. Equating the volume of the undeformed element to the volume of the deformed element produces

$$\delta x A_e = [\delta x + u(x+\delta x,t) - u(x,t)] \frac{1}{2} [A(x,t) + A(x+\delta x,t)] \tag{9}$$

which is approximate for finite δx but exact in the limit as δx approaches zero.



- p_{ie} pressure inside the tube when tube is in equilibrium with outside pressure p_{oe} and has inside cross-sectional area A_e .
- p_{oe} pressure outside the tube when tube is in equilibrium.
- p_e differential pressure ($p_i - p_o$) across the tube wall when tube is in equilibrium with internal cross-sectional area A_e .
- A_e equilibrium internal cross-sectional area of tube.
- p change in differential pressure across tube wall from p_e (subscripts i and o refer to inside and outside tube).
- A cross-sectional area of the tube wall corresponding to pressure p .

Figure 2.- Pressure-area relation for typical elastic tube. Dashed axes indicate pressure-area relation applicable for perturbation about equilibrium conditions (p_e, A_e).

After expansion of equation (9), the limit as $\delta x \rightarrow 0$ is

$$A_e = \left[1 + \frac{\partial u(x,t)}{\partial x} \right] A(x,t) \quad (10)$$

and substitution from the mass conservation equation (10) into the equilibrium equation (8) leads to an equation relating the pressure and the displacement

$$-\frac{\partial p(x,t)}{\partial x} = \rho \left[1 + \frac{\partial u(x,t)}{\partial x} \right] \frac{\partial^2 u(x,t)}{\partial t^2} \quad (11)$$

The preceding equations are independent of the properties of the tube wall. The following analysis will be restricted to the special case where the internal cross-sectional area of the tube and the pressure difference across the tube wall are related by a single-valued function f , in particular

$$p = f(A) \quad (12)$$

The qualitative shape for a typical elastic tube pressure-area curve is illustrated in figure 2. Note that p of figure 2 represents the differential pressure across the tube wall but the $p(x,t)$ in equations (1) through (11) is the pressure in the fluid. This apparent inconsistency in notation is rectified by letting the outside pressure change be zero ($p_o = \text{constant}$) throughout the remaining analysis. Note also that the pressure-area curve depends on the initial cross-sectional area, A_e , and pressure, p_e .

The pressure-area curve of figure 2 includes the full range of cross-sectional area. With large negative values of p_e ($p_{ie} \ll p_{oe}$) the tube is completely collapsed ($A_e = 0$). As the external pressure is decreased (or internal pressure is increased) the area gradually increases until the pressure difference approaches the classical buckling pressure for the tube (ref. 7). In practice, the tubes do not

exhibit instability characteristics since they are not perfect cylinders.² Rather, they exhibit regions of large change in area for small changes in pressure. As p_e becomes positive the cross section of the tube becomes quite circular and remains so until the pressure becomes sufficiently high at which time the tube reaches an unstable equilibrium configuration. The pressures at which this later type of instability occurs are outside the range of interest of this report.

With the pressure-area relation defined by equation (12) the derivation of the equation for $u(x,t)$ may be completed. Since the pressure is known as a function of A then

$$\frac{\partial p(x,t)}{\partial x} = \frac{df(A)}{dA} \frac{\partial A(x,t)}{\partial x} = f'[A(x,t)] \frac{\partial A(x,t)}{\partial x} \quad (13)$$

but from differentiation of both sides of equation (10) with respect to x

$$\frac{\partial A(x,t)}{\partial x} = - \frac{A_e \frac{\partial^2 u(x,t)}{\partial x^2}}{\left(1 + \frac{\partial u}{\partial x}\right)^2} \quad (14)$$

or from (13) and (14)

$$\frac{\partial p(x,t)}{\partial x} = -f'[A(x,t)]A_e \frac{\frac{\partial^2 u(x,t)}{\partial x^2}}{\left[1 + \frac{\partial u(x,t)}{\partial x}\right]^2} \quad (15)$$

Equation of $\frac{\partial p(x,t)}{\partial x}$ from equations (11) and (15) leads to

$$f'[A(x,t)]A_e \frac{\frac{\partial^2 u(x,t)}{\partial x^2}}{\left[1 + \frac{\partial u(x,t)}{\partial x}\right]^3} = \frac{\partial^2 u(x,t)}{\partial t^2} \quad (16)$$

Equation (16) is the general nonlinear wave equation for large-amplitude waves and a general nonlinear tube pressure-area relation. In order to obtain an equation for $u(x,t)$ alone, A can be eliminated from equation (16) by

²Although the collapse of an artery differs from that of a rubber tube (see ref. 8), the essential features of figure 2 are the same for both.

substitution of A from equation (10). The resulting equation will, in general, be nonlinear (the complexity depending on the form of $f'(A)$) but susceptible to analysis.

Condition for Nondistortion of Large-Amplitude Wave

It will prove advantageous to eliminate the bracketed term in equation (16) by substitution from equation (10)

$$\frac{1}{A_e^2 \rho} f'[A(x,t)] A^3(x,t) \frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial^2 u(x,t)}{\partial t^2} \quad (17)$$

Note that equation (17) has the form of the linear wave equation if

$$f'_\gamma[A(x,t)] A^3(x,t) = \text{constant}$$

and that the corresponding wave velocity is C provided

$$\frac{1}{A_e^2 \rho} f'_\gamma[A(x,t)] A^3(x,t) = C^2 \quad (18)$$

The required $f(A)$, denoted $f_\gamma(A)$, is found from equation (18), in the form,

$$df_\gamma(A) = C^2 \rho A_e^2 \frac{dA}{A^3}$$

therefore

$$f_\gamma(A) = C^2 \rho A_e^2 \left(-\frac{1}{2A^2} \right) + \text{const}$$

The constant of integration is obtained from $f_\gamma(A_e) = 0$, therefore,

$$f_\gamma(A) = \frac{C^2 \rho}{2} \left[1 - \left(\frac{A_e}{A} \right)^2 \right] \quad (19)$$

If the pressure-area relation has the particular form prescribed by equation (19), the linear wave equation will prevail and simple waves with finite amplitude will propagate without distortion and with velocity C . It is interesting to note that a longitudinally constrained rubber tube with circular cross section has precisely the pressure-area relation of equation (19) (finite amplitude displacements with cross section remaining circular).

The nondistortion of large-amplitude waves in rubber tubes with circular cross section was pointed out previously by Olsen and Shapiro (ref. 3). Using the Eulerian form of equations, they introduced the pressure-area curve for a

rubber tube (circular cross section). After developing the equations for the characteristic curves, they considered a finite amplitude wave traveling in one direction and with the appropriate argument concluded that "every part of the wave moves with the same speed and that the wave neither steepens or flattens. This extraordinary result depends on the particular pressure-area relation for the tube."

The above analysis (eqs. (17) through (19)) proves that the only form of pressure-area curve which yields a linear wave equation (nondistorting waves) for finite amplitude waves is given by equation (19).

Finite Amplitude Wave Distortion

The primary concern of the present investigation is the distortion of finite amplitude waves as they propagate. Return now to the general nonlinear equation of motion (eq. (16)). From equation (10)

$$\epsilon(x,t) = \frac{A_e}{A(x,t)} - 1 \quad \text{or} \quad A(x,t) = \frac{A_e}{1 + \epsilon(x,t)} \quad (20)$$

where

$$\epsilon(x,t) = \frac{\partial u(x,t)}{\partial x} \quad (21)$$

then

$$f'(A) = \frac{df(A)}{dA} = \frac{df(\epsilon)}{d\epsilon} \frac{d\epsilon}{dA} = f'(\epsilon) \frac{d\epsilon}{dA}$$

but

$$\frac{d\epsilon}{dA} = -\frac{A_e}{A^2} = -\frac{(1 + \epsilon)^2}{A_e}$$

therefore

$$f'(A) = -f'(\epsilon) \frac{(1 + \epsilon)^2}{A_e} \quad (22)$$

Substitution of $f'(A)$ from equation (22) into the general equation of motion (16) leads to

$$\left[\frac{f'(\epsilon)}{(1 + \epsilon)} \right] \frac{\partial^2 u(x,t)}{\partial t^2} = \frac{\partial^2 u(x,t)}{\partial t^2} \quad (23)$$

where ϵ is, of course, a function of x and t defined by equation (21). Equation (23) could, of course, be written

$$\left\{ \frac{f' \left[\frac{\partial u(x,t)}{\partial x} \right]}{\rho \left[1 + \frac{\partial u(x,t)}{\partial x} \right]} \right\} \frac{\partial^2 u(x,t)}{\partial t^2} = \frac{\partial^2 u(x,t)}{\partial t^2} \quad (23a)$$

or, more generally,

$$g \left[\frac{\partial u(x,t)}{\partial x} \right] \frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial^2 u(x,t)}{\partial t^2} \quad (24)$$

where

$$g \left[\frac{\partial u(x,t)}{\partial x} \right] = - \frac{f' \left[\frac{\partial u(x,t)}{\partial x} \right]}{\rho \left[1 + \frac{\partial u(x,t)}{\partial x} \right]} \quad (25)$$

Consider the propagation of a wave to the right (x positive) into the undisturbed tube. The solution must satisfy the differential equation (24) in addition to the conditions of the undisturbed tube ahead of the wave, namely

$$\frac{\partial u(x,t)}{\partial x} = \epsilon(x,t) = 0, \quad \frac{\partial u(x,t)}{\partial t} = 0 \quad (26)$$

In accordance with the development of Earnshaw (refs. 6 and 9), we seek solutions of the form

$$\frac{\partial u}{\partial t} = F \left(\frac{\partial u}{\partial x} \right) \quad (27)$$

where F is a function to be determined. One may deduce from differentiation of equation (27)

$$\frac{\partial}{\partial t} \left(\frac{\partial u}{\partial t} \right) = \frac{\partial}{\partial t} F \left(\frac{\partial u}{\partial x} \right) = F' \left(\frac{\partial u}{\partial x} \right) \frac{\partial}{\partial t} \left(\frac{\partial u}{\partial x} \right) = F' \left(\frac{\partial u}{\partial x} \right) \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial t} \right) = \left[F' \left(\frac{\partial u}{\partial x} \right) \right]^2 \frac{\partial^2 u}{\partial x^2}$$

Therefore

$$\frac{\partial^2 u}{\partial t^2} = \left[F' \left(\frac{\partial u}{\partial x} \right) \right]^2 \frac{\partial^2 u}{\partial x^2} \quad (28)$$

Equation (27) provides a solution to equation (24) if

$$g \left(\frac{\partial u}{\partial x} \right) = \left[F' \left(\frac{\partial u}{\partial x} \right) \right]^2 \quad (29)$$

Therefore equation (29) is the equation for $F \left(\frac{\partial u}{\partial x} \right)$ or $F(\epsilon)$.

Integration of equation (29) leads to

$$\int_{F_1}^F d\bar{F}(\epsilon) = \pm \int_{\epsilon_1}^{\epsilon} \sqrt{g(\bar{\epsilon})} d\bar{\epsilon}$$

However, if the integration is begun in the undisturbed region, one obtains from the conditions (26) and equation (27)

$$F_1 = \frac{\partial u}{\partial t} = 0, \quad \epsilon_1 = \frac{\partial u}{\partial x} = 0$$

therefore

$$F(\epsilon) = \pm \int_0^{\epsilon} \sqrt{g(\bar{\epsilon})} d\bar{\epsilon} \quad (30)$$

The particle velocity as a function of area can thus be obtained with the aid of equations (20), (22), (25), and (30)

$$\frac{\partial u(x,t)}{\partial t} = \pm \int_0^{\epsilon} - \left[\frac{f'(\bar{\epsilon})}{\rho(1 + \bar{\epsilon})} \right]^{1/2} d\bar{\epsilon} = \pm \int_{A_e}^A \left[\frac{A_e}{\rho} \frac{f'(\bar{A}) \bar{A}^3}{A_e^3} \right]^{1/2} \left[- \frac{\left(\frac{A_e}{\bar{A}} \right)^2}{A_e} \right] d\bar{A}$$

or simply

$$\frac{\partial u(x,t)}{\partial t} = \mp \int_{A_e}^A \left[\frac{f'(\bar{A})}{\bar{A}\rho} \right]^{1/2} d\bar{A} \quad (31)$$

A compression wave ($A > A_e$) traveling to the right (x positive) produces positive particle velocity and a decompression wave ($A < A_e$) traveling to the right produces negative particle velocity. The converse is true for waves traveling to the left ($A > A_e$, $\partial u/\partial t < 0$; $A < A_e$, $\partial u/\partial t > 0$); therefore the positive sign preceding the integral of equation (31) corresponds to waves traveling to the right and the negative sign corresponds to waves traveling to the left.

For future reference in the evaluation of the wave distortion, it will be useful to introduce yet another parameter α defined as

$$\alpha = \frac{1}{2} \left[1 - \left(\frac{A_e}{A} \right)^2 \right] = -\frac{1}{2} (2\epsilon + \epsilon^2) \quad (32)$$

Note that α has been chosen such that the linear wave equation prevails if the pressure is a linear function of α (i.e., if $f(\alpha) \sim \alpha$, see eq. (19)). The particle velocity as a function of α is obtained from equations (30), (31), and (32)

$$\frac{\partial u(x,t)}{\partial t} = \pm \int_0^\epsilon \left[-\frac{f'(\bar{\epsilon})}{\rho(1+\bar{\epsilon})} \right]^{1/2} d\bar{\epsilon} = \mp \int_0^\alpha \left[+\frac{f'(\bar{\alpha})}{\rho(1-2\bar{\alpha})} \right]^{1/2} d\bar{\alpha} \quad (33)$$

The sign of the integral is, in accordance with the discussion of equation (31), positive for waves traveling to the right and negative for waves traveling to the left.

The final objective of this section is to evaluate the spatial velocity of propagation of a disturbance and the distortion of the disturbance. The procedure will be to evaluate the rate of propagation from particle to particle and with this result to consider the absolute velocity of propagation, or the wave velocity, of the disturbance.

Let $v(x,t)$ denote the particle velocity, that is,

$$v(x,t) = \frac{\partial u(x,t)}{\partial t} \quad (34)$$

The velocity of a particle at x at time t will be the same as the velocity of the particle at $x + \delta x$ at time $t + \delta t$, provided

$$v(x,t) = v(x + \delta x, t + \delta t)$$

or

$$\frac{\partial v(x,t)}{\partial x} \delta x + \frac{\partial v(x,t)}{\partial t} \delta t = 0 \quad (35)$$

but from equations (27) and (34)

$$\frac{\partial u(x,t)}{\partial t} = v(x,t) = F \left(\frac{\partial u}{\partial x} \right)$$

therefore

$$F' \left(\frac{\partial u}{\partial x} \right) \frac{\partial^2 u(x,t)}{\partial x^2} \delta x + \frac{\partial^2 u(x,t)}{\partial t^2} \delta t = 0$$

Substitution of $\frac{\partial^2 u}{\partial t^2}$ from equation (28) produces

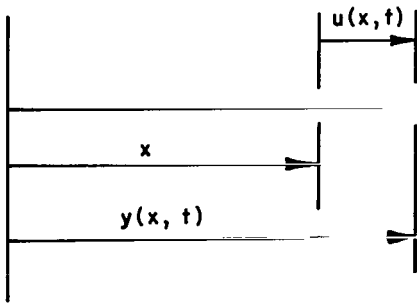
$$F' \left(\frac{\partial u}{\partial x} \right) \frac{\partial^2 u(x,t)}{\partial x^2} \left[\delta x + F' \left(\frac{\partial u}{\partial x} \right) \delta t \right] = 0$$

which is true for all values of x and t if

$$\delta x + F' \left(\frac{\partial u}{\partial x} \right) \delta t = 0$$

Therefore if the rate of propagation from particle to particle $\left(\frac{\delta x}{\delta t} \right)$ of the velocity v is denoted μ , then

$$\mu = \frac{\delta x}{\delta t} = -F' \left(\frac{\partial u}{\partial x} \right) \quad (36)$$



Let the absolute location of a particle with velocity $v(x,t)$ be $y(x,t)$ (see fig. 3), that is,

$$y(x,t) = x + u(x,t) \quad (37)$$

then the absolute location of a particle with velocity v at time $t + \delta t$

Figure 3.- Definition of the spatial coordinate y . will be

$$y(x + \mu \delta t, t + \delta t) = y(x,t) + \frac{\partial y}{\partial x} \mu \delta t + \frac{\partial y}{\partial t} \delta t$$

Thus the absolute spatial velocity of particle velocity v (denoted Dy/Dt) is

$$\frac{Dy}{Dt} = \frac{\partial y}{\partial x} \mu + \frac{\partial y}{\partial t} \quad (38)$$

Note that this represents the propagation rate of particle velocity v as opposed to the propagation rate of a particle. Substitution from equations (36) and (37) reduces this to

$$\frac{Dy}{Dt} = \left(1 + \frac{\partial u}{\partial x} \right) \left[-F' \left(\frac{\partial u}{\partial x} \right) \right] + \frac{\partial u}{\partial t}$$

or with the aid of equations (26), (27), (29), and (30)

$$\frac{Dy}{Dt} = \mp(1 + \epsilon) \sqrt{g(\epsilon)} \pm \int_0^\epsilon \sqrt{g(\bar{\epsilon})} d\bar{\epsilon} \quad (39)$$

Note that Dy/Dt may be interpreted as the "wave velocity" of the portion of a disturbance with velocity v . Denoting this wave velocity by c and substituting from equations (20) and (31), one obtains

$$c(A) = \mp \left[\frac{f'(A)A}{\rho} \right]^{1/2} \mp \int_{A_e}^A \left[\frac{f'(\bar{A})}{\bar{A}\rho} \right]^{1/2} d\bar{A} \quad (40)$$

Not only does v propagate with velocity $c(A)$ but so does A ;³ therefore equation (40) is the relation required to calculate the distortion of a large-amplitude wave as it propagates into the undisturbed tube. The only quantities required are (1) the pressure-area curve for the tube $f(A)$, (2) the density of the fluid, ρ , (3) the undisturbed cross-sectional area of the tube, A_e , and (4) the spatial distribution of the cross-sectional area (or pressure since there is a one-to-one correspondence by equation (12)) at some initial time t_0 , $A(x, t_0)$. The actual calculation of the distortion will be considered in a later section.

Criteria for Steepening

Of primary concern in considerations which follow will be the waves which steepen as they propagate. A steepening wave is defined as one in which the absolute value of the slope ($|\partial p/\partial x|$ or $|\partial A/\partial x|$) for some value of pressure,

³The derivation of the propagation velocity of A differs from that of v only in the determination of the particle-to-particle propagation rate. Instead of equation (35), one obtains

$$\frac{\partial A(x, t)}{\partial x} \delta x + \frac{\partial A(x, t)}{\partial t} \delta t = 0$$

and with equations (20), (21), and (27)

$$\frac{\delta x}{\delta t} = - \frac{\frac{\partial A}{\partial t}}{\frac{\partial A}{\partial x}} = - \frac{\frac{\partial \epsilon(x, t)}{\partial t}}{\frac{\partial \epsilon(x, t)}{\partial x}} = - \frac{\frac{\partial}{\partial x} \left(\frac{\partial u}{\partial t} \right)}{\frac{\partial^2 u}{\partial x^2}} = -F' \left(\frac{\partial u}{\partial x} \right)$$

which is identical to equation (36). Since the pressure p is assumed to have a one-to-one correspondence with the area A , then the function $c(p)$ can be calculated from $c(A)$.

or cross-sectional area, increases as the wave propagates. If a steepening wave propagates for a sufficiently long time a sharp wave (infinite slope) will develop (at least mathematically).

One could consider the steepening phenomenon by evaluating dc/dA , that is, from equation (40)

$$\frac{dc(A)}{dA} = \mp \frac{1}{2} \left[\frac{f'(A)A}{\rho} \right]^{-1/2} \rho [f'(A) + Af''(A)] \mp \left[\frac{f'(A)}{A\rho} \right]^{1/2} \quad (41)$$

However, equation (41) does not yield a simple set of criteria for steepening waves due to the complexity of the right-hand side of the equation. However, if c is written as a function of the parameter α (eq. (32)) one obtains with the aid of equations (33) and (39)

$$c(\alpha) = \mp \left[\frac{(1 - 2\alpha)f'(\alpha)}{\rho} \right]^{1/2} \mp \int_0^\alpha \left[\frac{f'(\bar{\alpha})}{\rho(1 - 2\bar{\alpha})} \right]^{1/2} d\bar{\alpha} \quad (42)$$

Differentiation of $c(\alpha)$ in equation (42) produces

$$\frac{dc(\alpha)}{d\alpha} = \mp \frac{1}{2} \left[\frac{1 - 2\alpha}{\rho f'(\alpha)} \right]^{1/2} f''(\alpha) \quad (43)$$

Recall that the positive sign in equation (43) is applicable to waves traveling to the right (positive x) and the negative sign applicable to waves traveling to the left (negative x). Therefore, for waves traveling to the right

$$\left. \begin{aligned} \frac{dc}{d\alpha} &> 0 && \text{if } f''(\alpha) > 0 \\ \frac{dc}{d\alpha} &= 0 && \text{if } f''(\alpha) = 0 \\ \frac{dc}{d\alpha} &< 0 && \text{if } f''(\alpha) < 0 \end{aligned} \right\} \quad (44)$$

Note also that for compressive waves

$$\frac{A_e}{A} \text{ decreases, } \alpha \text{ increases} \quad (45)$$

and for decompression waves

$$\frac{A_e}{A} \text{ increases, } \alpha \text{ decreases} \quad (46)$$

The criteria for steepening waves may now be stated in terms of the pressure as a function of the parameter α . The most useful criteria are probably the ones in terms of a plot $p(\alpha)$.

If one plots the curve $p[1 - (A_e/A)^2]$ or $p(\alpha)$, the criteria for steepening waves may be stated as:

(a) Compression waves will steepen for ranges of p in which the slope of the $p(\alpha)$ curve is increasing ($f''(\alpha) > 0$). Conversely, they will flatten otherwise.

(b) Decompression waves will steepen for ranges of p in which the slope of the $p(\alpha)$ curve is decreasing ($f''(\alpha) < 0$). Conversely, they will flatten otherwise.

(c) Compression and decompression waves will propagate undistorted for ranges of p in which the slope of the $p(\alpha)$ curve is constant ($f''(\alpha) = 0$).

The last of the criteria is in accordance with the condition of nondistorting waves (eq. (19)). It seems appropriate to reemphasize that the criteria for steepening are stated entirely in terms of a function relating the pressure to the area of the tube.

II. CALCULATION OF WAVE DISTORTION

General Equation of Distortion

The purpose of this section is to establish the formulas for calculating the distortion of a large-amplitude wave propagating into an undisturbed tube. In addition, later parts of this section will be devoted to determining the criteria for a sharp wave (infinite slope) and the critical distance required for a sharp wave to develop. Finally, an example calculation will be presented to demonstrate an application of the theory.

Section I provides the theory necessary to determine the spatial wave velocity as a function of the pressure (or area) amplitude. If this quantity is denoted by $c(p)$, then it follows from the discussion of the first section that the pressure, p , may be expressed as a function of $t - [x/c(p)]$, namely

$$p(x,t) = h \left[t - \frac{x}{c(p)} \right] \quad (47)$$

where $c(p)$ is a known function. Equation (47) is applicable to a wave traveling to the right (positive x).

Sharp-Wave Criterion

The pressure gradient will become infinite when

$$\frac{\partial p(x,t)}{\partial x} \rightarrow \infty \quad (48)$$

Evaluating $\partial p/\partial x$ from equation (47) yields

$$\frac{\partial p(x,t)}{\partial x} = h' \left[t - \frac{x}{c(p)} \right] \frac{-c(p) + x \frac{\partial c(p)}{\partial x}}{[c(p)]^2}$$

but

$$\frac{\partial c(p)}{\partial x} = \frac{dc(p)}{dp} \frac{\partial p}{\partial x}$$

therefore,

$$\frac{\partial p(x,t)}{\partial x} = h' \left[t - \frac{x}{c(p)} \right] \left[-\frac{1}{c(p)} + \frac{x}{c^2(p)} \frac{dc(p)}{dp} \frac{\partial p(x,t)}{\partial x} \right]$$

or, solving for $\partial p/\partial x$,

$$\frac{\partial p(x,t)}{\partial x} = \frac{h' \left[t - \frac{x}{c(p)} \right]}{c(p) - x \frac{dc(p)}{dp} \frac{1}{c(p)} h' \left[t - \frac{x}{c(p)} \right]} \quad (49)$$

The function $h[t - x/c(p)]$ is assumed to be known for some tube position, namely, $x = 0$, so that $h(t)$ is a prescribed function. Since this study is concerned with the transition from "smooth" waves to sharp waves, $h(t)$ is assumed to be continuous and $\partial h(t)/\partial t$ (or $h'(t)$) is assumed to be finite. The pressure gradient $\partial p(x,t)/\partial x$ becomes infinite if

$$1 - x \frac{dc(p)}{dp} \frac{1}{c^2(p)} h' \left[t - \frac{x}{c(p)} \right] = 0 \quad (50)$$

The criterion for a sharp wave to develop (eq. (50)) depends on (a) the prescribed form of the pressure, $h(t)$, (b) the wave velocity pressure curve, $c(p)$, and (c) the location on the tube, x .

Critical Length

As the pressure wave passes some position $x = L_c$ along the tube, the slope of the wave $\partial p(L,t)/\partial x$ will be infinite for pressure level p^* if (from eq. (50))

$$L_c = \frac{\frac{c^2(p^*)}{dc(p^*)/dp}}{h' \left[t_0 - \frac{L_c}{c(p^*)} \right]} \quad (51)$$

Note that t_0 is the time at which $p(L_c,t) = p^*$ which is (from eq. (47))

$$p^*(L_c, t_0) = h \left[t_0 - \frac{L_c}{c(p^*)} \right]$$

or

$$t_0 = \frac{L_c}{c(p^*)} + h^{-1}[p^*] \quad (52)$$

where h^{-1} denotes the inverse function and should not be confused with h' , which denotes a derivative with respect to the argument of the function, nor with the negative one power. The denominator of equation (51) can be transformed by substitution from equation (52) to obtain

$$h' \left[t_0 - \frac{L_c}{c(p^*)} \right] = h' [h^{-1}(p^*)] \quad (53)$$

Differentiation of both sides of equation (47) with respect to t yields

$$h' \left[t - \frac{x}{c(p)} \right] = \frac{\frac{\partial p(x,t)}{\partial t}}{1 + \frac{x[dc(p)/dp][\partial p(x,t)/\partial t]}{c(p)^2}} \quad (54)$$

The right side of equation (53), $h' [h^{-1}(p^*)]$, is the value of h' evaluated for the value of the argument at which $h = p^*$. This may be evaluated from equation (54) by choosing $x = 0$ and evaluating the right side of the equation for $p = p^*$ thus

$$h' [h^{-1}(p^*)] = h'(t) \Big|_{h=p^*} = \frac{\partial p(0,t)}{\partial t} \Big|_{p=p^*} = \frac{\partial h(t)}{\partial t} \Big|_{h=p^*} \quad (55)$$

where $[\partial h(t)/\partial t] \Big|_{p=p^*}$ can be readily calculated from the known function $h(t)$.

The distance along the tube, L_c , or the "critical length" at which the slope of the wave, $\partial p / \partial x$, becomes infinite at $p = p^*$ is (from eqs. (51), (53), and (55))

$$L_c(p^*) = \frac{c^2(p^*)}{\left. \frac{dc(p^*)}{dp} \right|_{h=p^*}} \quad (56)$$

Thus $L_c(p^*)$ can be calculated if the wave velocity pressure function, $c(p)$, and the pressure-time curve at $x = 0$, $h(t)$, are prescribed.

Since the critical length depends on the pressure amplitude p^* , one must examine each amplitude of pressure to obtain the "minimum critical length" L_{cm} . This latter quantity may be obtained more formally by taking the derivative of $L_c(p^*)$ and equating the result to zero, that is,

$$\frac{dL_c(p^*)}{dp^*} = 0 \quad (57)$$

One must, of course, assure himself that the result is a minimum and not a maximum. In addition, obtaining a minimum from equation (57) does not insure that the minimum value of L_c has been obtained (because of the finite range of p^*). In fact, it is not necessary for a minimum (from the calculus) value of L_c to exist in the allowable range of p^* . However, there is always a value of L_{cm} .

Illustrative Example

The application of the formulas for critical length and minimum critical length is best illustrated by an example calculation. Let the prescribed pressure-time curve at $x = 0$ be

$$h(t) = p_1 \sin \omega t \quad (58)$$

and the prescribed wave velocity-pressure curve be

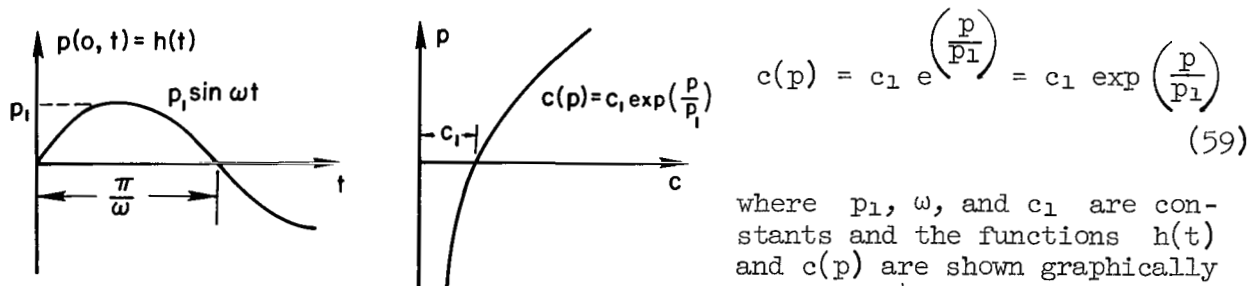


Figure 4. - Pressure-time and pressure-wave velocity functions for illustrative example.

where p_1 , ω , and c_1 are constants and the functions $h(t)$ and $c(p)$ are shown graphically in figure 4. The traveling wave is a series of compression and decompression waves. As the wave propagates along the tube,

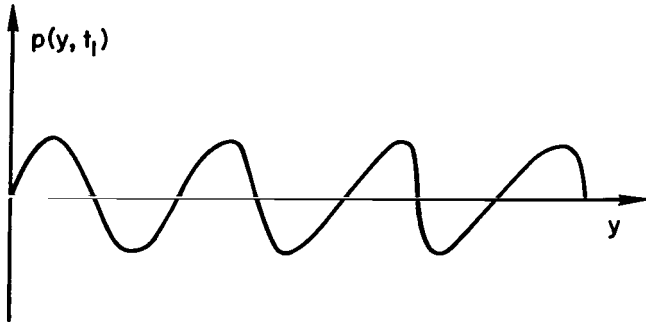


Figure 5.- Spatial distortion of pressure wave for illustrative example.

the compression waves sharpen while the decompression waves become less steep so that at some time t_1 the wave has the shape shown in figure 5. The pressure in the tube is

$$p(x,t) = p_1 \sin \omega \left[t - \frac{x}{c(p)} \right] \quad (60)$$

The computation of the critical length $L_c(p^*)$ requires the evaluation of $\frac{\partial h}{\partial t} \Big|_{h=p^*}$. From equation (58),

$$\frac{\partial h}{\partial t} = p_1 \omega \cos \omega t$$

and the time at which $h = p^*$ is

$$t \Big|_{h=p^*} = \frac{1}{\omega} \sin^{-1} \frac{p^*}{p_1}$$

Therefore

$$\frac{\partial h}{\partial t} \Big|_{h=p^*} = p_1 \omega \cos \left[\sin^{-1} \frac{p^*}{p_1} \right]$$

or simply

$$\frac{\partial h}{\partial t} \Big|_{h=p^*} = \pm \omega \sqrt{p_1^2 - p^{*2}} \quad (61)$$

where the choice of \pm sign depends on the position on the function $h(t)$ and, therefore, the position on the wave.

The numerator term in the calculation of $L_c(p^*)$ is, from equations (56) and (59),

$$\frac{c^{*2}(p^*)}{dc(p^*)/dp} = c_1 p_1 \exp \left(\frac{p^*}{p_1} \right) \quad (62)$$

Substitution from equations (61) and (62) into the critical length equation yields

$$L_c(p^*) = \frac{c_1 p_1 \exp(p^*/p_1)}{\pm \omega \sqrt{p_1^2 - p^{*2}}}$$

or

$$L_c(p^*) = \frac{c_1 \exp(p^*/p_1)}{\pm \omega \sqrt{1 - (p^*/p_1)^2}} \quad (63)$$

In order to obtain the minimum critical distance, differentiate the right side of equation (63) with respect to p^* and equate the result to zero

$$\frac{dL_c(p^*)}{dp^*} = \frac{c_1 \exp(p^*/p_1)}{p_1 \omega} \frac{1}{\pm [1 - (p^*/p_1)^2]^{3/2}} \left[1 + \frac{p^*}{p_1} - \left(\frac{p^*}{p_1} \right)^2 \right] = 0 \quad (64)$$

thus a relative minimum or maximum of the function $L_c(p^*)$ occurs when

$$\frac{p^*}{p_1} = \frac{1}{2} \mp \frac{\sqrt{5}}{2}$$

but the range of interest of p^*/p_1 is $-1 \leq p^*/p_1 \leq 1$. Therefore, the minimum of interest is when

$$\frac{p^*}{p_1} = \frac{1}{2} - \frac{\sqrt{5}}{2} \approx -0.618$$

or from equation (63)

$$L_{cm} = \frac{c_1}{\omega} 0.686$$

The negative pressure is a differential pressure in the previous relations. The absolute pressure must, of course, be positive. The computation of the minimum length of tube required for the development of a steep wave is extremely simple for the example problem. Although this simplicity will not prevail for the more general problem (i.e., more complex $c(p)$ and $h(t)$ functions), a numerical calculation should always be practical.

III. SHOCK WAVES

The formulas, developed for calculating the length of tube required for a sharp wave ($\partial p/\partial x = \infty$), strictly speaking, do not apply in the limit ($\partial p/\partial x = \infty$) but should be applied only as $\partial p/\partial x$ approaches infinity, for when $\partial p/\partial x = \infty$ the particle acceleration also becomes infinite. Also, for

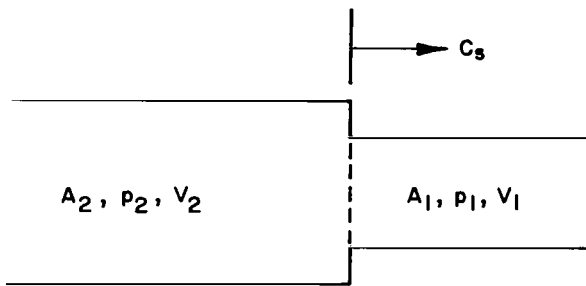
very sharp pressure gradients the axial bending rigidity of the tube wall should be included in the analysis.⁴

The final transition from the nearly sharp to a truly sharp wave presents a formidable mathematical problem. However, the propagation of a sharp wave, once the transition phase has been completed, can be treated quite simply provided one again assumes that the axial bending rigidity of the tube is negligible.

Associated with the sharp waves are discontinuities in the particle velocity and pressure and (except for the linear wave equation) energy losses. For the remainder of this report the sharp waves will be called "shock waves."

Shock-Wave Velocity

Consider the propagation of a compression shock wave through a fluid-filled elastic tube (fig. 6). Let the cross-section area, the pressure, and



the particle velocity (A , p , and v , respectively) be distinguished by subscripts 1 and 2 referring to the right and left side of the shock wave, respectively. The spatial velocity of the shock wave is denoted by c_s . Conservation of fluid mass across the shock front requires that

Figure 6.- Shock-wave notation.

$$(c_s - v_1)A_1 = (c_s - v_2)A_2 \quad (65)$$

A useful parameter in further computations is c_s^* which is defined as the shock-wave velocity in the compressed (or lengthened) coordinate system. The relation of c_s to c_s^* may be written

$$c_s = v_1 + c_s^*(\epsilon_1 + 1) \quad (66)$$

where $\epsilon_1 = A_2/A_1 - 1$ (see eq. (20)). Or if one had chosen to write the relation in terms of the left side of the shock

$$c_s = v_2 + c_s^*(\epsilon_2 + 1) \quad (67)$$

⁴The axial bending rigidity in the development of a shock requires further investigation. This effect may or may not be important. Certainly, however, it would be more attractive from the physical viewpoint if the tube wall displacement discontinuity were eliminated. Indeed it is not certain that a shock wave will develop if the bending rigidity is included.

Note that c_s of equations (66) and (67) must be equal, therefore

$$(v_2 - v_1) = c_s^*(\epsilon_1 - \epsilon_2) \quad (68)$$

which is equivalent to equation (65).

Besides mass, the momentum of the fluid must also be conserved. The impulse of the forces on the fluid acting along the tube is

$$\left[p_2 A_2 - p_1 A_1 - \int_{A_1}^{A_2} f(A) dA \right] \frac{\Delta x}{c_s - v_1}$$

which must equal the change of momentum of the fluid

$$\rho \Delta x A_1 (v_2 - v_1)$$

therefore

$$\left[p_2 A_2 - p_1 A_1 - \int_{A_1}^{A_2} f(A) dA \right] \frac{1}{c_s - v_1} = \rho A_1 (v_2 - v_1) \quad (69)$$

Elimination of c_s and v_2 from equation (69) with the aid of equations (66) and (68) produces

$$(c_s^*)^2 = \frac{1}{\rho A_1} \frac{1}{(1 + \epsilon_1)} \frac{1}{(\epsilon_1 - \epsilon_2)} \left[p_2 A_2 - p_1 A_1 - \int_{A_1}^{A_2} f(A) dA \right]$$

or

$$(c_s^*)^2 = \frac{p_2 A_2 - p_1 A_1 - \int_{A_1}^{A_2} f(A) dA}{\rho A_e \left(\frac{A_e}{A_1} - \frac{A_e}{A_2} \right)} \quad (70)$$

Integration by parts of the numerator of the right-hand side of equation (70) simplifies the expression for c_s^*

$$(c_s^*)^2 = \frac{\int_{A_1}^{A_2} A \frac{df(A)}{dA} dA}{\rho A_e \left(\frac{A_e}{A_1} - \frac{A_e}{A_2} \right)} \quad (71)$$

and the shock wave velocity in the undeformed coordinate system is (from eqs. (66) and (71))

$$(c_s - v_1)^2 = (c_s^*)^2 \left(\frac{A_e}{A_1}\right)^2 = \frac{1}{\rho A_1^2} \frac{\int_{A_1}^{A_2} A \frac{df(A)}{dA} dA}{\left(\frac{1}{A_1} - \frac{1}{A_2}\right)} \quad (72)$$

The velocity of the shock wave can be determined if the pressure-area curve of the tube, the conditions ahead of the shock (A_1, v_1), and the amplitude of the shock $A_2 - A_1$ (or $p_2 - p_1$) are prescribed.

The next step is to estimate the shock-wave velocity in terms of the small-amplitude wave velocities of disturbances at the foot ($A = A_1$) or peak ($A = A_2$) of the original wave (fig. 7). This is accomplished by introducing

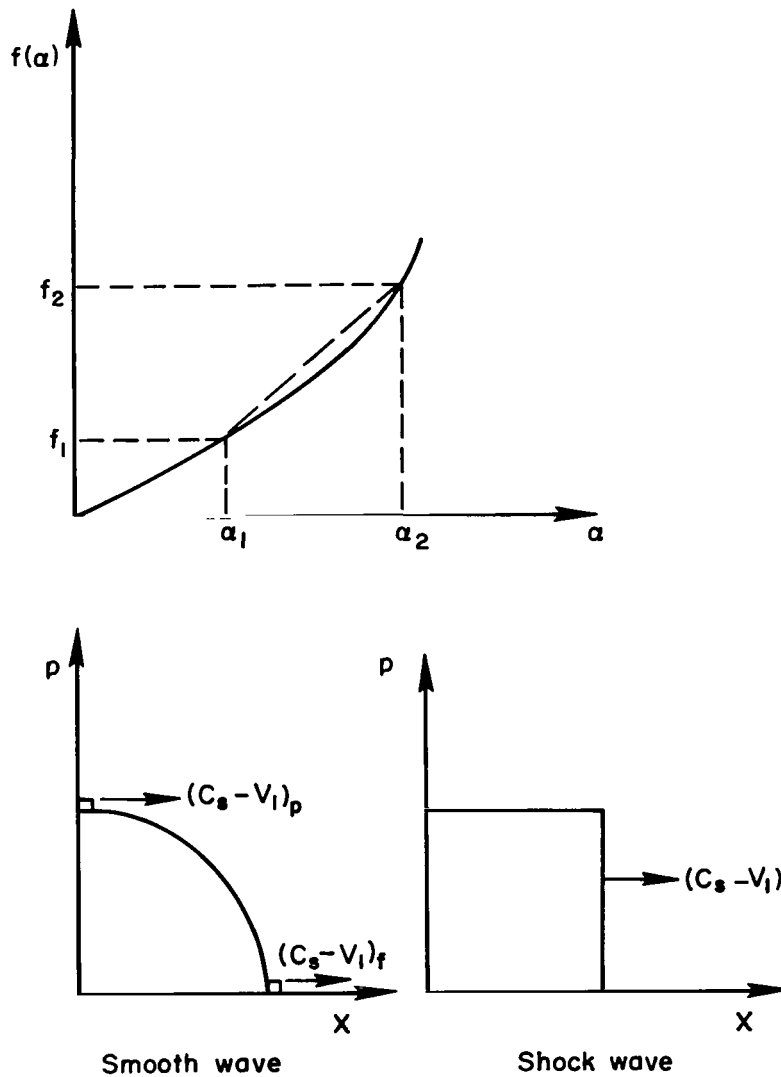


Figure 7.- The pressure as a function of a for use in shock-wave description.

the α notation (eq. (32)) into equation (71)

$$(c_s^*)^2 = \frac{\int_{\alpha_1}^{\alpha_2} \frac{df(\alpha)}{d\alpha} \frac{1}{\sqrt{1-2\alpha}} d\alpha}{\rho(\sqrt{1-2\alpha_1} - \sqrt{1-2\alpha_2})} \quad (73)$$

or from equation (72)

$$(c_s - v_1)^2 = \frac{(1-2\alpha_1) \int_{\alpha_1}^{\alpha_2} \frac{df(\alpha)}{d\alpha} \frac{1}{\sqrt{1-2\alpha}} d\alpha}{\rho(\sqrt{1-2\alpha_1} - \sqrt{1-2\alpha_2})} \quad (74)$$

The equations derived thus far in this section are valid for infinitesimal as well as finite wave amplitudes. The wave velocity of the foot of the wave which developed into the shock wave was (from eq. (74)), taking the limit as $\alpha_2 \rightarrow \alpha_1$,

$$(c_s - v_1)_f^2 = \frac{1-2\alpha_1}{\rho} \frac{df(\alpha_1)}{d\alpha} \quad (75)$$

and the velocity at the peak of the original wave⁵ was

$$(c_s - v_1)_p^2 = \frac{1-2\alpha_2}{\rho} \frac{df(\alpha_2)}{d\alpha} \quad (76)$$

Note that if $df/d\alpha$ is positive and $d^2f/d\alpha^2$ is positive for $\alpha_1 \leq \alpha \leq \alpha_2$ (i.e., $df(\alpha_1)/d\alpha < df(\alpha_2)/d\alpha$) as in figure 7 (which is the criterion for sharp waves, section I), then from equations (74), (75) and (76)

$$(c_s - v_1)_f^2 < (c_s - v_1)^2 < (c_s - v_1)_p^2 \quad (77)$$

The velocity of the developed shock wave is greater than the velocity of the foot of the original wave but less than the velocity of the peak of the original wave. In the limiting case in which $df/d\alpha = \text{constant}$ (nondistorting wave, see section I) then

$$(c_s - v_1)_f^2 = (c_s - v_1)^2 = (c_s - v_1)_p^2 \quad (78)$$

⁵Strictly speaking, the "original" wave is a smooth wave with amplitude equal to the amplitude of the shock wave. It has not been proved that the smooth wave will develop a shock wave of equal amplitude.

Energy Loss

Since the shock-wave velocity is determinable from considerations of conservation of mass and momentum, the energy loss across the shock can be calculated directly.

The external work rate on the fluid is

$$p_2 A_2 v_2 - p_1 A_1 v_1 \quad (79)$$

the rate of change of kinetic energy of the fluid is

$$\frac{1}{2} \rho A_1 (v_2^2 - v_1^2) (c_s - v_1) \quad (80)$$

the rate of change in potential energy of the tube is

$$\int_{A_1}^{A_2} f(A) dA c_s \quad (81)$$

and the energy loss rate is defined as

$$(c_s - v_1) E_L \quad (82)$$

Equating the work rate to the energy change rate yields

$$p_2 A_2 v_2 - p_1 A_1 v_1 = \frac{1}{2} \rho A_1 (v_2^2 - v_1^2) (c_s - v_1) + \int_{A_1}^{A_2} f(A) dA c_s + E_L (c_s - v_1) \quad (83)$$

If v_2 and c_s are eliminated from equation (83) with the aid of equations (65) and (72) then, after considerable manipulation of terms, one obtains

$$E_L = \frac{1}{2} \left[p_2 (A_2 - A_1) + p_1 A_1 \left(1 - \frac{A_1}{A_2} \right) - \left(1 + \frac{A_1}{A_2} \right) \int_{A_1}^{A_2} f(A) dA \right] \quad (84)$$

Thus the energy loss coefficient E_L can be calculated if the conditions ahead of the shock wave, the amplitude of the shock wave $A_2 - A_1$ (or $p_2 - p_1$), and the pressure-area curve for the tube are known.

The case considered here is one of a particular class of shocks which require only the mechanical shock conditions (ref. 10) for the determination of the shock. The energy loss is generally attributed to the generation of an equivalent amount of heat.

IV. KOROTKOFF SOUNDS

Sphygmomanometry

Although blood pressure can be measured directly by inserting pressure measuring devices into the arteries or veins, indirect methods of measurement have been devised which provide less discomfort to the patient, eliminate the risk of infection, and provide greater convenience to the medical practitioner.

Several indirect methods (ref. 11) have been devised and used successfully; however, one method, sphygmomanometry, has come to be accepted as a standard in the medical field. Fortunately, the method is simpler than its name and is familiar to almost everyone as part of a physical examination. The details of the method may not be so familiar to those outside the medical profession and, at the risk of being superfluous, will be presented.

The required equipment consists of a compression cuff, an inflation bulb with pressure control valve, and a mercury manometer or other pressure measuring device. When the cuff is snugly applied to the arm just above the elbow (fig. 8), inflation of the cuff with the bulb compresses the tissues which in

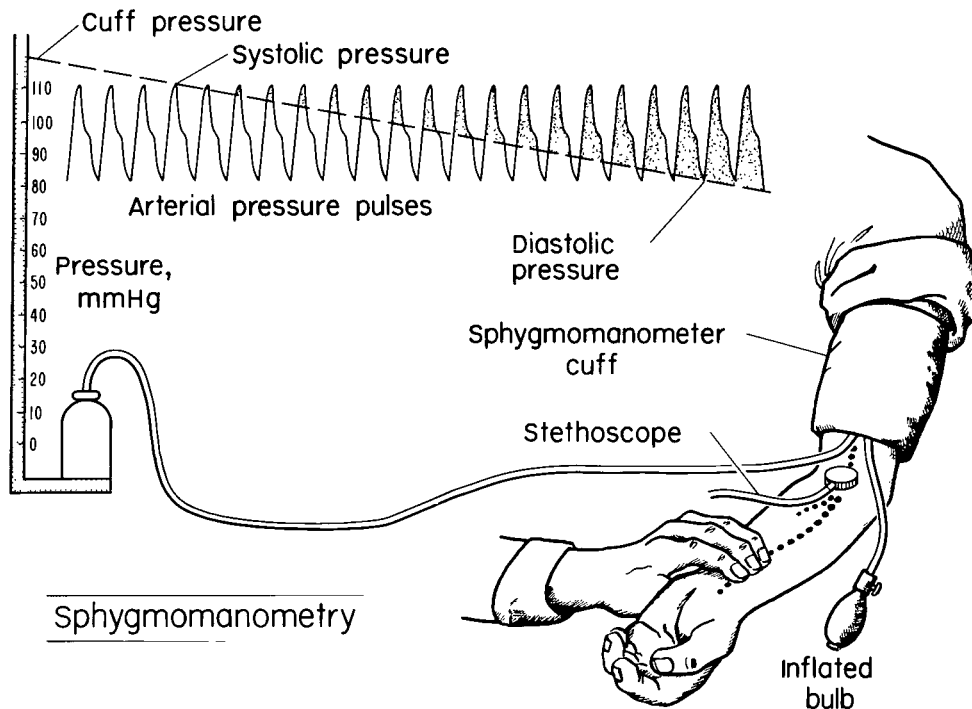


Figure 8.- The method of sphygmomanometry.

turn compress the brachial artery. The pressure in the cuff is increased until no pulse wave can be felt in the artery downstream from the cuff.

A stethoscope is positioned just downstream from the pressure cuff and the pressure in the cuff is slowly reduced by the pressure control valve. As

the cuff pressure decreases, certain distinctive sounds may be heard until the cuff pressure is reduced to a sufficiently low level at which the sounds disappear. The sounds are called Korotkoff sounds and the pressures indicated on the manometer (cuff pressures) at which the sounds appear and disappear are generally accepted as a measure of maximum blood pressure (systolic) and minimum blood pressure (diastolic) in the brachial artery.

Direct and Indirect Method Correlation

The simplest and most common correlation of the sounds and the actual pressure pulse is shown graphically in figure 8. This correlation assumes that the highest pressure at which sounds first occur corresponds to the maximum value of the pulse pressure and that the lowest pressure at which sounds disappear corresponds to the minimum value of the actual pulse pressure.

Several medical researchers have attempted to establish the validity of the indirect method by comparison with blood pressure measurements obtained from intra-arterial pressure transducers (refs. 12-15). Although the degree of correlation varies between investigations (the medical history of the subjects varies also), it is not unusual to obtain discrepancies between the two methods as great as 20 percent. There are, of course, cases when the indirect method produces zero diastolic readings - a grave condition if the method were correct.

It seems unreasonable to expect that the discrepancies between the two methods will be understood until the conditions required for the generation of the Korotkoff sounds and the source of the sounds are known.

Various Hypotheses

Hypotheses explaining the Korotkoff sounds have been plentiful.⁶ No attempt will be made to be all inclusive but four of the more popular hypotheses from the literature will be briefly reviewed.

Water hammer. - Early in his investigation of blood pressure estimation by indirect methods, Erlanger (ref. 17) (1916) came to the conclusion that the main mechanism of Korotkoff sounds is as follows:

"Under compressions which permit the pulse to determine relatively wide excursions of the arterial wall in the compression chamber, that is, under compressing pressures ranging from systolic arterial pressure to, and even a variable distance below, diastolic pressure, the volume of the compressed artery increases abruptly with each pulse. This permits a considerable volume of blood to enter the opening artery with a high velocity. The motion of this column of blood is, however, suddenly checked where it comes into contact with the stationary, or practically stationary, column of blood filling the uncompressed artery below. The water hammer that is

⁶The introductions to references 16 and 8 provide historical reviews.

thus set into play distends the arterial wall at the point of impact with unusual violence. This distention sets the arterial wall into vibration and the sound is produced."

After further experiments, however, Erlanger (ref. 18) changed his opinion and supported the hypothesis of Bramwell.

"Breaker" phenomenon.- While working in Erlanger's laboratory, Bramwell (refs. 19 and 20) (1923) experimentally studied the change in the form of the pulse wave in the course of transmission. He correlated the propagation of the pulse wave with the propagation of ocean waves toward a beach and concluded that for the pulse wave:

"Hence, the more rapidly moving elements will tend to catch up to those which are moving more slowly; in other words the wave will tend to become more vertical, and, under suitable conditions, parts of it may actually tend to topple over and form 'breakers.' Even if the wave as a whole does not 'break,' any individual portion of it which does so will set up irregular vibrations in the arteries, which, if they be of sufficient amplitude, will be transmitted to the periphery"

Later Bramwell related the "breaker" to the sound production in arteries and specifically to the Korotkoff sounds.

Bernoulli effect.- Rodbard (ref. 21) experimentally studied the flow of fluid through collapsible tubes. Due to the coupling of the flow (through the Bernoulli or flow velocity pressure effect) and the flexibility of the tube, he was able to produce instabilities and vibrations of the fluid-filled tube. He termed this phenomenon "flutter" and one of his conclusions was

"The Korotkoff murmurs heard when arteries are compressed during the measurement of blood pressure are probably also due to flutter of the walls"

It should be noted that the "flutter" of which Rodbard speaks requires relatively high flow velocities and it is known that the Korotkoff sounds can be reproduced with zero mean flow (ref. 22).

Dynamic instability.- Anliker and Raman formulated a hypothesis for the Korotkoff sounds at diastole (ref. 16) and Raman later extended the hypothesis to include systole (ref. 23). The hypotheses interpret the sounds as "a phenomenon of dynamic instability (oscillations with increasing amplitude), the instability being induced by the application of a pressure cuff." They explain their hypothesis as follows:

"Like any other sound perceived by a human ear, the Korotkoff sounds heard with the aid of the stethoscope are aggregates of vibrations whose frequencies and intensity place them within the audibility range. It is conceivable that the Korotkoff sounds are due to disturbances in the flow that are induced locally, that is, in the segment of the brachial artery which is compressed by the cuff.

However, it is also possible that they are a part of the noise associated with pulsative flow through the circulatory system that is selectively amplified to a level above the audibility threshold in the compressed section of the brachial. It is well known that pulsating flow through the complex system of blood vessels generates a multitude of wave motions; the associated vibrations are not audible because their amplitudes and in many cases their frequencies are far below the audibility threshold. Irrespective of the origin of the disturbances leading to Korotkoff sounds, it will be shown that the application of a pressure cuff can change the system locally from one that is dynamically stable to one that is intermittently unstable and thus capable of mechanically amplifying certain low intensity vibrations (disturbances) to such an extent that they become audible"

Theoretical work to support the hypotheses is practically nonexistent except for that of Anliker and Raman. They have done the linear analysis for a fluid-filled circular cylinder (simulating diastole) and Raman did a similar analysis for systole except that the shell geometry was changed to simulate the "flattened" artery. Their analyses do not provide a means of computing the frequencies or amplitudes of the sounds (which are assumed to be amplifications of disturbances already present) but establishes the differential pressure for instability of the system (buckling of shell) and amplification of the disturbances.

Bramwell and Hill (ref. 19) did make some calculations of the distortion of the arterial pulse wave assuming that the small displacement wave velocity of the pulse wave was known. Their calculations were, however, incorrect because they, in effect, included only one term of the right side of equation (40). In particular, they used only the small displacement velocity (measured) which corresponds to

$$\left[\frac{f'(A)A}{\rho} \right]^{1/2}$$

They have therefore neglected the term

$$\int_{A_e}^A \left[\frac{f'(\bar{A})}{\bar{A}\rho} \right]^{1/2} d\bar{A}$$

or the "integrated particle velocity."

Applicability of Present Theory

The intent of this section is to investigate the fundamental physical phenomena associated with the production of Korotkoff sounds. Parameters that influence the character of the sounds, but are nonessential for production of the sounds, will be neglected. This is done to isolate the physical features essential to the production of the sounds. Future analysis should provide a

method for determining the effect of those parameters which influence but are not essential to the production of the sounds.

The applicability of the analysis of sections I through III to the investigation of the Korotkoff sounds depends on the ability of the mathematical model to represent the important properties of the physical system which produces the sounds. This analysis will be applicable if it can be demonstrated that:

(1) The phenomenon of Korotkoff sounds can be simulated by pulsatile flow in a compressed isolated artery (that is, an artery detached from its surrounding tissue but still intact at its ends), and

(2) The assumptions of section I are applicable in formulating the mathematical model for the pulsatile flow.

The validity of the first condition has been demonstrated in experiments on anesthetized dogs by Erlanger (ref. 24) and in experiments on simulated arteries by Sacks et al. (ref. 25). The results of these experiments demonstrate that the Korotkoff sounds occur without the surrounding tissue. This should not be taken to imply that the surrounding tissue will not change the quality of the sounds. Anliker and Raman (ref. 16), in their experiments on a simulated artery, have shown that simulated tissue surrounding the artery muffled the sounds. The quality of the sounds is affected by the tissue but the tissue is not essential to produce the sounds.

The second condition to applying the analysis will be satisfied by considering each assumption of section I explicitly.

(a) The effect of compressibility of the blood in the study of pulsatile flow in the arteries has been shown to be small (ref. 1). The effect of viscosity on the Korotkoff sound phenomenon was considered experimentally by Sacks et al. (ref. 26). They concluded that the effects of viscosity are relatively small since it was demonstrated that Korotkoff sounds could be produced over a range of viscosities from 1 to 30 cm^2/sec . The kinematic viscosity of blood is of the order 5 cm^2/sec .

(b) The assumption of one-dimensional flow and the neglect of radial inertia have been shown to be reasonable approximations as a result of experiments performed by the author to supplement the available experimental data. These experiments are discussed below.

In the author's experiments the simulated artery was either a gum rubber tube (1-inch diameter and 1/16-inch wall thickness) or a transparent flexible plastic tube of similar dimensions. The fluid was either pure water or a water-glycerin mixture (90-percent water, 10-percent glycerin). The pulsatile flow was provided by a positive displacement pump (fig. 9). The mean flow was zero in accordance with the findings of Sacks et al. (ref. 25) that the mean flow effect was negligible. The compression (simulated cuff) of the tube was provided by decreasing the mean internal pressure and thus allowing the atmospheric pressure to provide the cuff pressure. This method of compressing

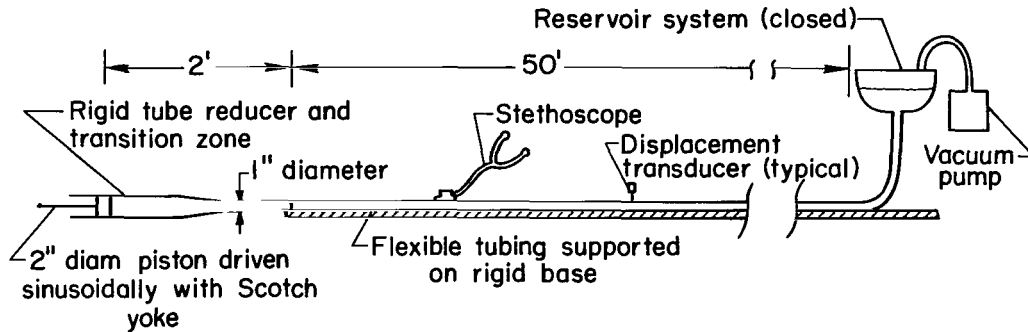


Figure 9.- Diagram of test fixture for author's experiments.

the tube eliminates any influence a mechanical cuff may have on the sounds.⁷ In addition, this method allows almost unlimited access to the tube for instrumentation.

Flow visualization experiments were performed using the clear plastic tube and a solution of water (90 percent) and glycerin (10 percent) of the appropriate density to suspend small plastic beads (approximately 0.05-inch diameter). With the pump providing a sinusoidal displacement input at the end of the tube the mean pressure differential across the tube wall was adjusted to produce sounds at the stethoscope location. High-speed motion pictures were then taken of the area surrounding the stethoscope so that the tube and fluid motion could be studied in detail. The pictures showed that the velocity of the suspended particles and thus the velocity of the fluid was uniform across the cross section of the tube. In addition, the maximum lateral motion of the particles was about 7 percent of the longitudinal motion. On the basis of these observations the assumptions of a one-dimensional flow and negligible energy associated with the lateral motion are justified.

(c) Since the density of the tube wall is approximately the same as the density of the fluid, the mass of the tube can be neglected for the same reason the lateral inertia of the fluid can be neglected.

(d) Although prestress of the tube in the axial direction affects the quality of the Korotkoff sounds (ref. 18), prestress is not essential to the production of sounds. This was demonstrated experimentally by Bramwell and confirmed by the author in his experiments.

The flexural rigidity in the axial direction of the tube has no effect for long wave lengths (ref. 3) but the effect must be accounted for as the pressure gradient becomes sharp. The transition region (pressure gradient) at which the flexural rigidity must be included requires further investigation. The justification for omitting axial flexural rigidity in the present

⁷The "cuff" in the author's experiments was actually the same length as the elastic tube (approximately 50 feet). The "effective" cuff length was, however, determined by the location of the stethoscope. If the pulse wave form were measured at station x_1 and the stethoscope were located at station x_2 (downstream from x_1) then the effective length of the cuff was $x_2 - x_1$.

investigation of Korotkoff sounds is based on the good quantitative agreement of the author's analysis and the experimental data of Bramwell (presented in a later paragraph).

(e) The assumption of uniformity of the tube under equilibrium conditions is certainly justified for the simulated artery experiments. The application to the brachial artery is validated by the work of Tickner and Sacks who made extensive measurements on excised brachial and other arteries (ref. 27).

Hypothesis for Mechanism

The present hypothesis for the Korotkoff sounds is divided into two parts. The first considers the development of necessary conditions (mechanism) for the production of sounds and the second part considers the sounds. The proposed mechanism should already be apparent from the development of sections I through III. The hypothesis is:

As the pulse wave traverses the compressed portion of the artery, it may, with favorable system parameters, steepen and, in fact, become sharp (possibly a shock wave). The system parameters include (1) the shape of the arterial pulse as it enters the compressed section of the cuff, (2) the physical properties of the tube wall, (3) the cuff pressure, and (4) the length of the cuff. The development of a shock wave is not essential to the production of sounds but the wave must become relatively steep (see the sound hypothesis).

It should be noted that the cuff pressure could be zero and the other parameters favorable to produce steep waves. This could account for the zero diastolic reading sometimes obtained in practice.

This hypothesis does not differ significantly from that of Bramwell except for the final stage of the steepening wave which is assumed here to be a shock wave rather than the less probable "breaking wave." The major contributions to this hypothesis are (i) the theoretical support of sections I to III which make the hypothesis both plausible and useful, and (ii) the explanation for the developed sharp wave.

The Sounds

The mechanism hypothesis offers no explanation for the sounds heard through the stethoscope. Generally, one associates sound with oscillatory pressures. However, no oscillatory components of pressure are predicted by the analysis and in the author's experiments sounds could be obtained when the displacement of the tube wall contained no oscillatory components. High-speed motion pictures (2000 frames per second) and tube wall displacement transducers revealed only the sharply rising and slowly decaying wall displacement as each "pulse" wave passed. In addition, the flow visualization studies showed no turbulence in the flow when the sounds were heard or during any other part of the cycle. This last observation is in agreement with the

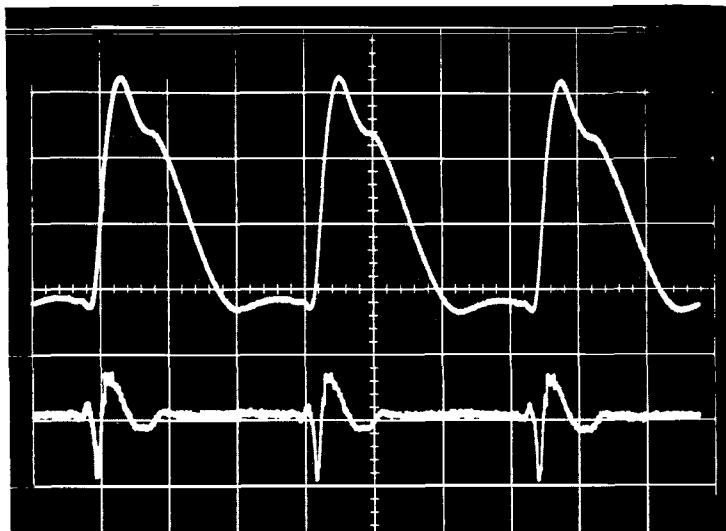


Figure 10.- Typical pulse wave and simulated Korotkoff sounds. Upper trace is pressure recorded from pressure transducer inside tube under stethoscope (see fig. 11). Lower trace is signal from microphone mounted inside stethoscope earpiece. Time base 0.2 sec per division.

turbulence studies of Anliker and Raman (ref. 16) who used dye for flow visualization. Measurements from pressure transducers inside the tube (fig. 10) showed no oscillatory pressure components, only the rapid pressure rise as the "pulse" wave passed the transducer.

The author's experimental results and the analysis indicate that the predominant source of the sounds is the sharp pressure gradient. In order to determine if this alone were a plausible explanation, a series of additional experiments were conducted.

The signal from a pressure transducer inside the tube and flush with the inner

wall of the tube was used as the "driving" signal for an audio amplifier speaker system (fig. 11). A stethoscope was placed on the tube wall directly over the pressure transducer. The sounds produced by the speaker system were then compared with those obtained by listening through the stethoscope. The comparison of sounds by the human ear is at best qualitative; however, the sounds from the speaker and those from the stethoscope were essentially but not indistinguishably the same. In order to insure that the speaker sounds were not affected by artifacts of the pulse pressure wave which were not observed on an oscilloscope display, a function generator was used to simulate the pressure transducer waveform. The function chosen was a ramp with the slope approximating the pulse pressure gradient. When this signal from the function generator was introduced into the speaker system the sounds were indistinguishable from those obtained when the pulse pressure signal was used. It was concluded that the sharp pulse wave alone could account for the predominant sounds.

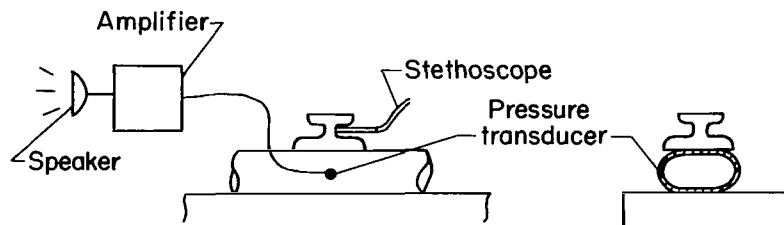
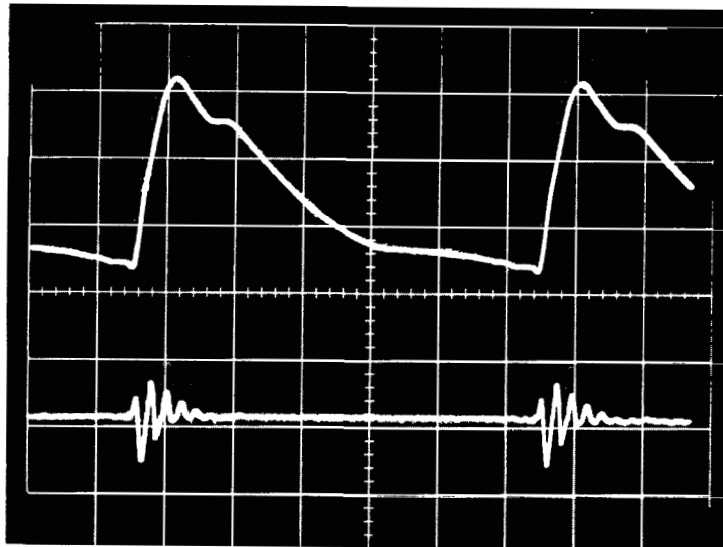


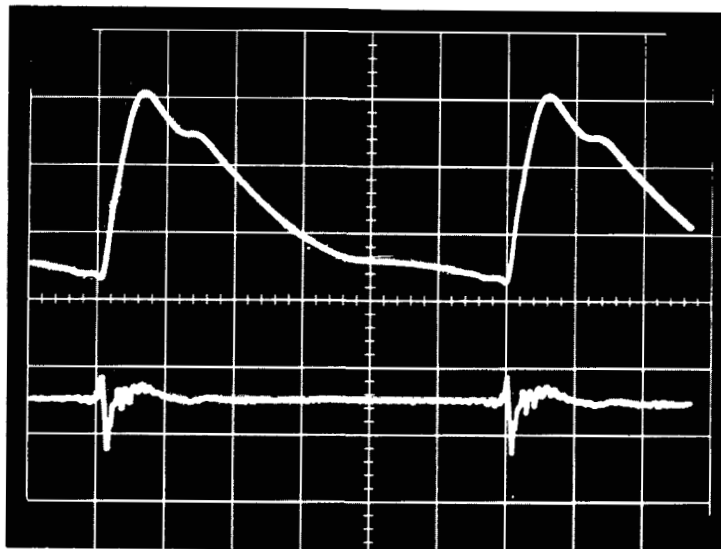
Figure 11.- Relative location of stethoscope and pressure transducer.

As noted above, the sounds from the stethoscope were "essentially" but not "indistinguishably" the same as those from the speaker. Since a stethoscope has resonant frequencies ("organ pipe" frequencies) that result

from the tubing connecting the head of the stethoscope to the earpiece, it was decided to investigate the possibility that the "difference" of sounds could be due to the stethoscope resonances. This study was made by varying the length of tubing connecting the stethoscope head- and earpiece. The variation of sounds as a function of tube length was observed by two methods - listening through the earpiece and using a microphone inside the earpiece so that the sound pressure waveforms could be observed on an oscilloscope (fig. 12). The oscilloscope traces showed the expected trend of increasing frequency in pressure fluctuations as the stethoscope tube length was decreased. The sounds appeared to become correspondingly "sharper" as the length of

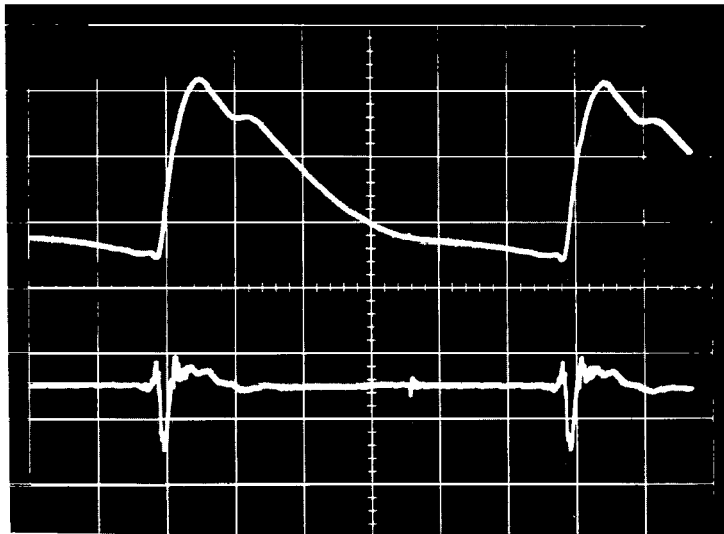


(a) 11 feet of tubing.

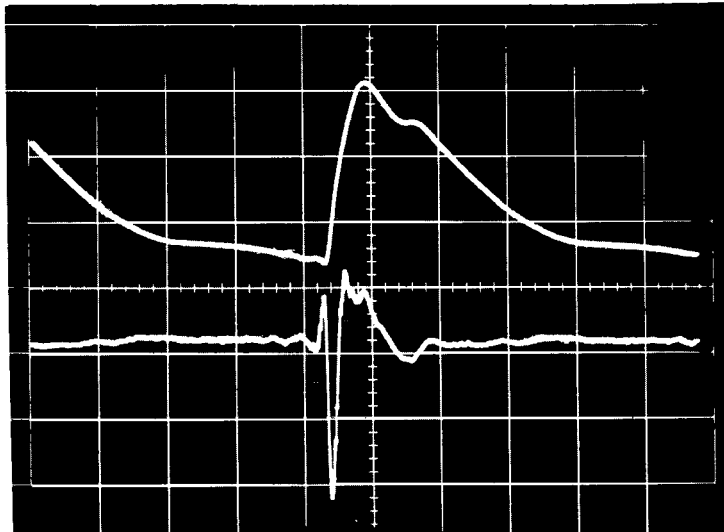


(b) 5.5 feet of tubing.

Figure 12.- Typical pulse wave and simulated Korotkoff sounds for various lengths of tubing connecting stethoscope head- and earpiece. Upper trace is pressure recorded from pressure transducer inside tube under stethoscope (see fig. 11). Lower trace is signal from microphone mounted inside stethoscope tubing. Time base 0.1 sec per division.



(c) 2 feet (standard) of tubing.



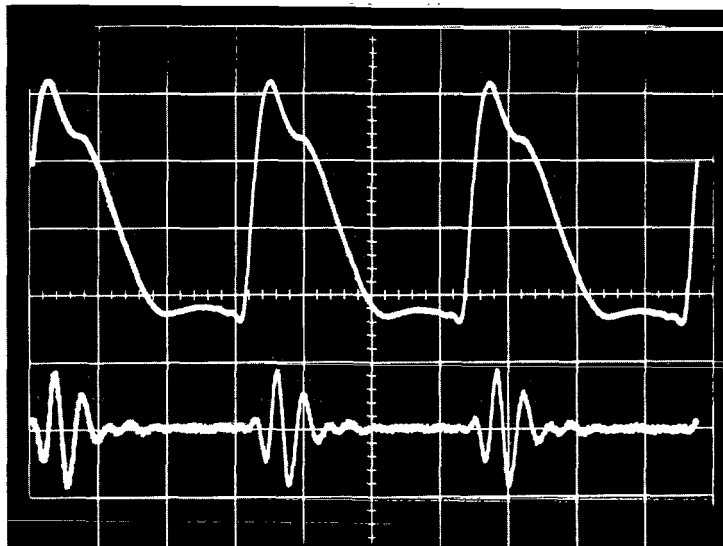
(d) 0.4 feet of tubing.

Figure 12.- Concluded.

stethoscope tubing was decreased and more closely approximated the speaker system sounds (from the pressure pulse wave). The author concluded that the differences in the sounds from the stethoscope and the audio system (with pulse wave as input) were accounted for by the characteristics (resonances) of the stethoscope.

The range of frequencies⁸ of the Korotkoff sounds reported in the literature is quite varied and includes the threshold of hearing (approximately 20 cps) to about 500 cps (refs. 8, 22, 30, and 31). During the development of an automatic blood pressure monitoring system for the Mercury manned space flights (ref. 29) it was discovered that the microphone signal containing the Korotkoff sounds could be filtered with negligible loss of accuracy in blood pressure determination. It was determined that if the frequency band pass were set so that only components between 32 and 40 cps were passed, the indirect blood pressure determination was essentially the same as if the unfiltered signal were used. This result implies either that the predominant oscillatory components of the sounds were in the frequency band of the filter (32 to 40 cps) or that the microphone filter was not "filtering" the sounds but was "responding" to a sharp input. In light of the previous discussion the latter explanation seemed more plausible to the author and a test was conducted to verify this suspicion.

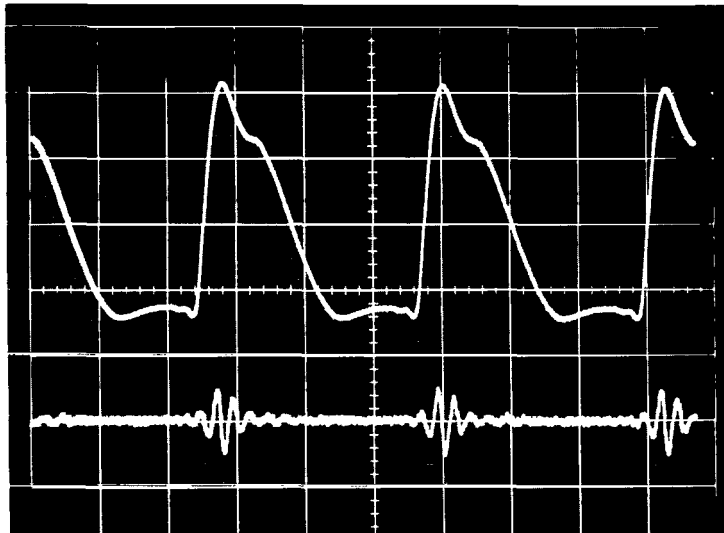
The pressure transducer signal was filtered by a band-pass filter and then introduced into the audio speaker system. The pressure waveform before and after filtering was monitored on an oscilloscope (fig. 13). It was found that the filter changed the input waveform from a sharp rise and slow decay to a momentary decayed oscillation. The sounds were not eliminated but their quality changed so that they did not seem as "sharp" as the unfiltered sounds. The frequency of the filtered signal was that of the filter (about 36 cps when the bandwidth was 32 to 40 cps). In order to confirm that the oscillatory decay was due to the response of the filter (and not the frequency components



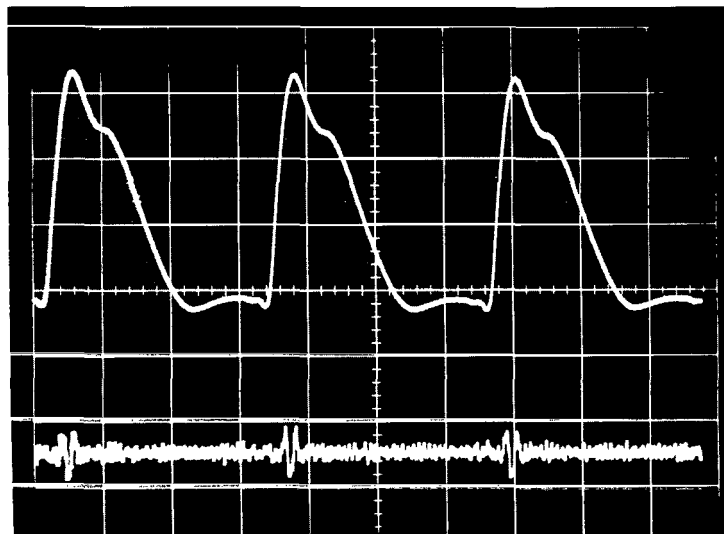
(a) 12-16 cps band-pass filter.

Figure 13.- Typical pulse wave and pulse wave after narrow band filtering. Upper trace is pressure recorded from pressure transducer inside tube. Lower trace is same signal after narrow band filtering. Time base 0.2 sec per division.

⁸The frequencies referred to are those of the oscillatory components of the Korotkoff sounds when displayed on an oscilloscope or other recording device (ref. 28, for example) and should not be confused with the Fourier components of a pulse wave type record as reported by Ware (ref. 29).



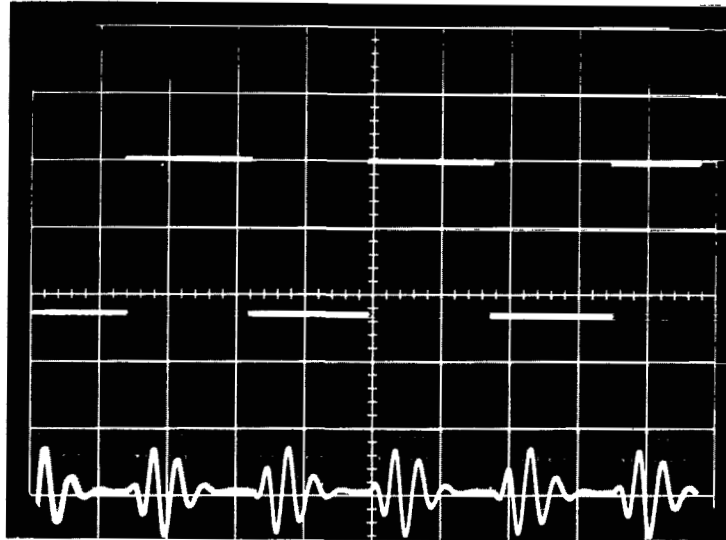
(b) 22-26 cps band-pass filter.



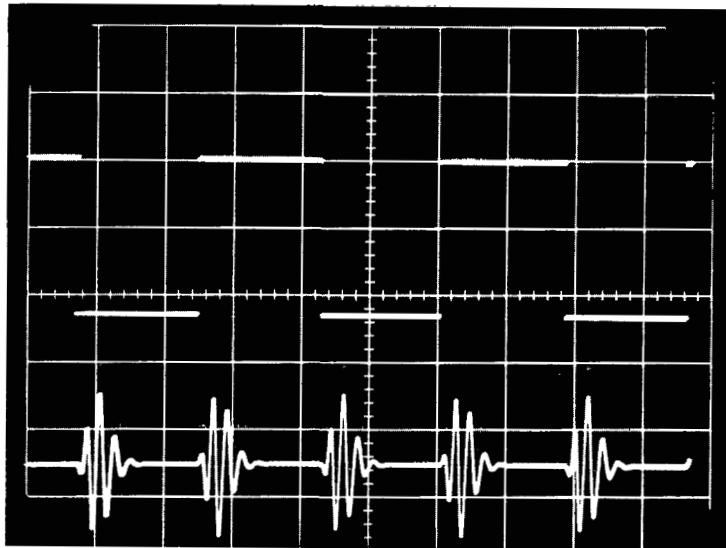
(c) 32-36 cps band-pass filter.

Figure 13.- Concluded.

of the Korotkoff sounds) a square wave function generator was used in place of the pulse pressure. The filtered response once again was a series of short bursts of decayed oscillations with frequencies of the filter "center" frequency (fig. 14).

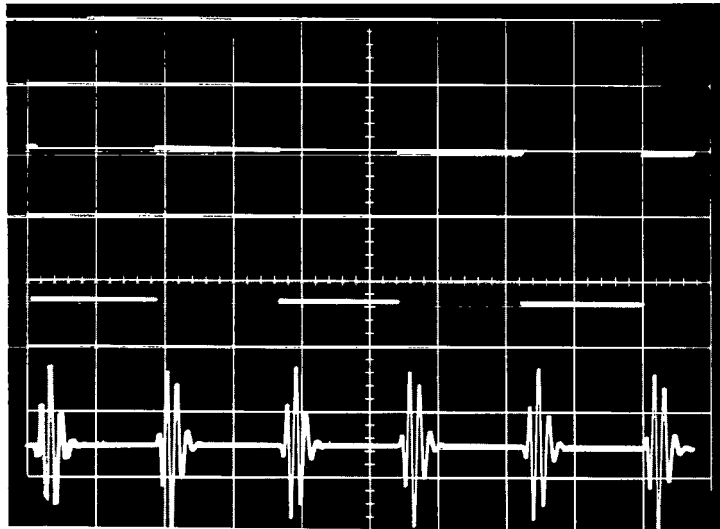


(a) 12-16 cps band-pass filter.



(b) 22-26 cps band-pass filter.

Figure 14.- Square wave and narrow band-pass filtered square wave. Time base 0.2 sec per division. Upper trace is square wave from function generator. Lower trace is same signal after narrow band-pass filtering.



(c) 32-36 cps band-pass filter.

Figure 14.- Concluded.

Much work remains to be done to confirm that all the sounds observed in sphygmomanometry are due to the phenomenon of the sharp wave. The author believes that the preliminary experiments and analysis presented in this report offer sufficient evidence for proposing a hypothesis for the sounds. This is, of course, only a hypothesis that must be evaluated by further investigation both analytical and experimental.

Hypothesis for Sounds

On the basis of the analysis for the mechanism of the Korotkoff sounds presented previously and experiments by the author, it is concluded that:

The Korotkoff sounds are the response of the measuring system to a steep (possibly sharp or shock) wave input; that is, the frequency components of the sounds are characteristic of the measuring system and are not characteristic of the dynamics of the blood-filled artery. This does not mean that the sounds produced are independent of the properties of the blood-filled artery, but the dependence is only on the sharpness of the pulse (which depends on the properties listed in the mechanism hypothesis) which excites the measuring system.

The term "measuring system" means the ear alone or the aided ear (i.e., with a stethoscope, or some microphone speaker system). Sounds, by definition, require the ear as part of the sound measuring system.

This hypothesis implies that the "frequencies" of the blood artery system cannot be obtained directly from the frequency spectral analysis of the sounds. In fact, those frequencies obtained from such analyses reflect the inherent

frequencies of the measuring system. This could account for the wide variation of "frequencies" for the Korotkoff sounds reported in the literature.

Experimental Corroboration

Although Erlanger and Bramwell had meager analytical support of their hypothesis, they did extensive experimental work. A summary of their work on mechanical (as opposed to biological) systems was given by Bramwell (ref. 32) as follows:

"Sounds produced by waves in closed tubes. How is it that the deflation of the armlet causes the changes in the arterial sounds described by Korotkoff? To answer that question I must first refer to some experimental observations made on an artificial schema of the circulation. This consisted of the inner tube of a bicycle tyre containing fluid but no air. The distal end of the tube was closed by a wooden bung and the proximal end connected with a pump capable of generating waves in the fluid. The tyre represented a portion of the arterial tree and the pump the heart. Erlanger (1924) found that a wave passing along a rubber tube filled with water produced a sound which varied both in quality and in intensity at different points along the tube. The change in quality of the sounds, as heard with a stethoscope, enables one to map out three regions in the tube.

- (1) In the proximal region, only a dull thud is heard; this becomes louder as one passes farther down the tube.
- (2) In the intermediate region, the sound is very loud; here it is no longer dull but sharp like the crack of a whip.
- (3) In the distal region, the sound loses its cracking character and becomes progressively fainter as we pass towards the distal end of the tube.

"Erlanger pointed out that, in the intermediate region of the tube where the loud crack is heard, the wall of the tube is subject to a severe shock. This he demonstrated by sprinkling fine sand on the upper surface of the tube. With the passage of a wave the sand is violently thrown off from that region in the tube where the shock is most severe and where the loud cracking is heard on auscultation, but lies undisturbed in the segments on either side.

"Working in Erlanger's laboratory in 1923, I made some further observations on this phenomenon (Bramwell 1937). I found that the position of the disturbed area - i.e., the area in which the sand is thrown off the tube - varied with the pressure of the fluid in the tube. As one raises the pressure, the disturbed area recedes from the proximal towards the distal end of the tube, at high pressures the wave passes down the whole length of the tube without producing sufficient disturbance to throw off the sand. In other words, the

higher the initial pressure, the farther has the wave to travel before the conditions arise which lead to the throwing off of sand and the production of sharp cracking sounds."

Bramwell presented sufficient detail of his experiments (ref. 20) for a comparison with the theory of section II. Using mercury as the fluid, Bramwell introduced the pressure wave shown in figure 15. He determined directly the wave velocity pressure curve from the distortion of the wave. The result is shown in figure 16.

The calculation of the critical length of tube (eq. (56)) is particularly simple for the given pulse wave (almost linear in its rise) and the minimum critical length is also easily obtained (at minimum $c^2/(dc/dp)$). If it is now assumed that the minimum critical length corresponds to the "disturbed area" determined experimentally by Bramwell, a comparison of theory and experiment is in order. Because of apparatus restrictions, Bramwell could determine only bounds on the "disturbed area" and these are shown as vertical lines in figure 17. The calculated minimum critical length is shown as a solid line.

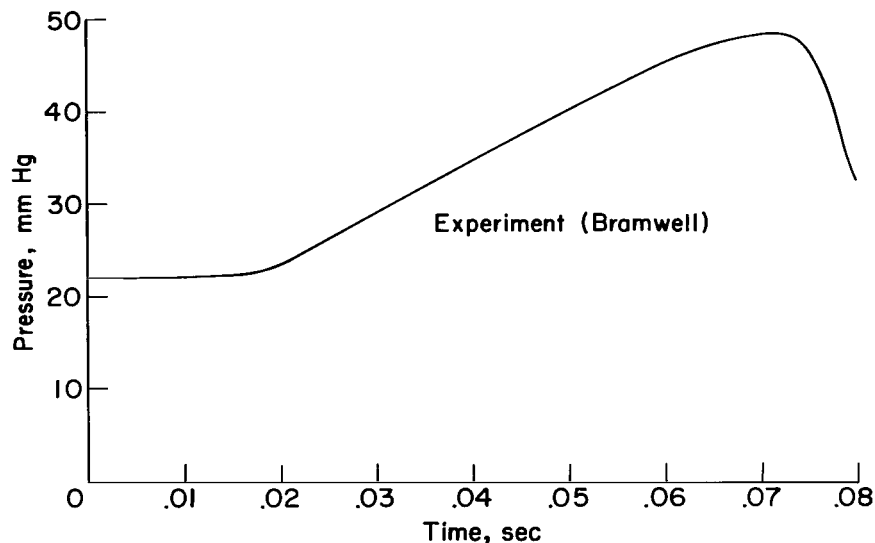


Figure 15.- Experimental input pulse wave (Bramwell data). The pressure is the differential pressure across the tube wall.

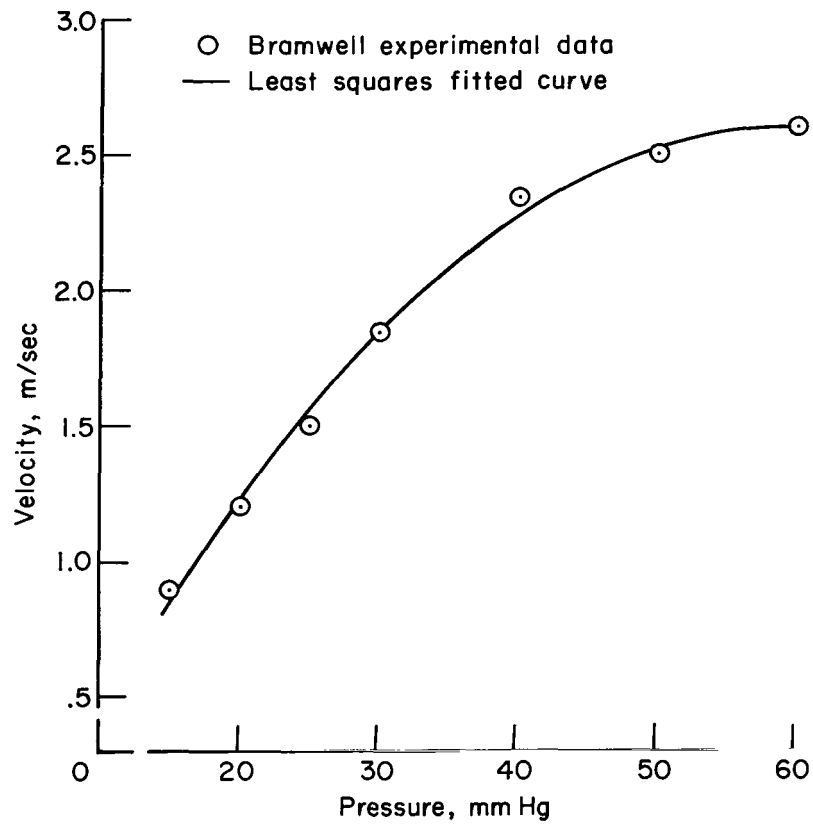


Figure 16.- Experimental pressure-wave velocity curve (Bramwell data). The pressure is the differential pressure across the tube wall.

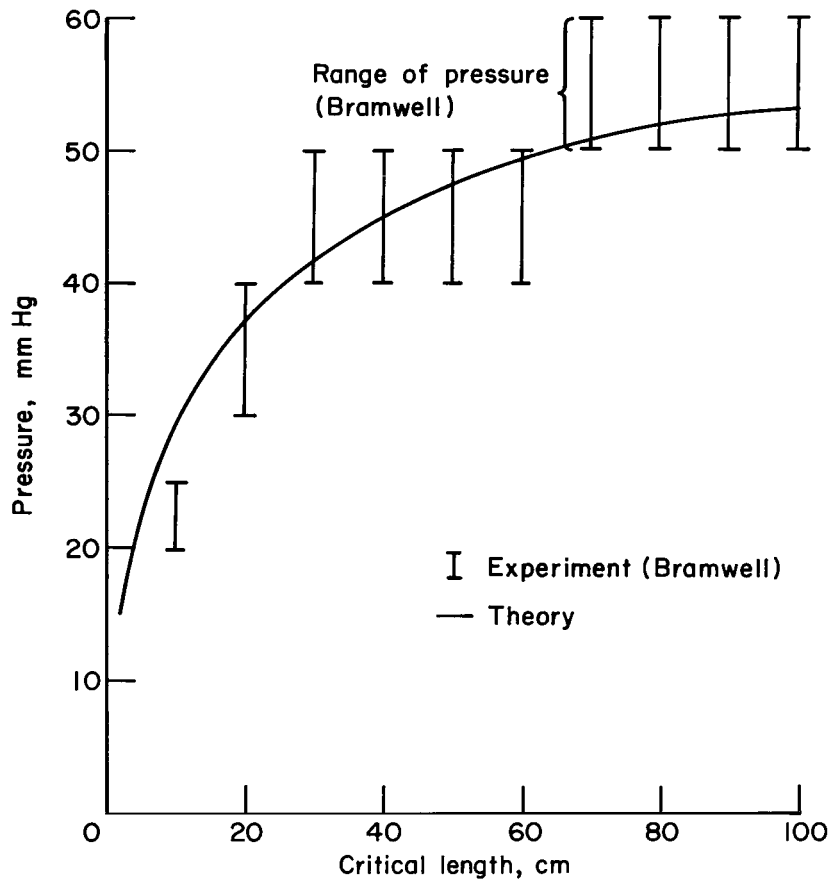


Figure 17.- Comparison of experimental (Bramwell) and theoretical (this report) determination of critical tube length. The pressure is the differential pressure across the tube wall.

V. RESUMÉ

The results of the analytical and experimental work presented in this report can best be summarized by a series of conclusive statements.

1. A solution for the finite amplitude wave equation with general tube pressure-area relation can be obtained for simple (nonreflecting) waves.
2. The finite amplitude nonlinear wave equation reduces to the linear wave equation when the pressure-area relation for a longitudinally constrained rubber tube with circular cross section is introduced.
3. The set of criteria for wave distortion (steepening or nonsteepening) can be stated as a function of the pressure-area relation of the tube.

4. The distortion of waves and the corresponding critical length of tube can be calculated if the initial shape of the wave (entering the tube) and the pressure-area (or wave velocity-pressure) relation of the tube are known.

5. The development of shock waves in fluid-filled elastic tubes is theoretically feasible (from the simplified theory). The shock-wave velocity and energy loss are readily obtainable if the amplitude of the wave and the pressure-area relation for the tube are known.

6. The analysis in this report supports the wave distortion hypothesis and experimental work of Erlanger and Bramwell.

Statements 7, 8, and 9 are based on new hypotheses presented in this report. These hypotheses have not been proven but are supported by the analysis and experiments discussed in this report.

7. The development of steep waves produces sharp waves or possibly shock waves and not "breakers" as hypothesized by Bramwell.

8. In the indirect method of blood pressure measurement (sphygmomanometry) the cuff pressures at which Korotkoff sounds are heard correspond to artery conditions suitable for the pulse wave to steepen sufficiently to excite a response in the measuring system.

9. The frequency components of the Korotkoff sounds are characteristic of the measuring system (stethoscope or other) and not characteristic of the blood-filled artery. More precisely, the pressure variation inside the artery is very nearly a step (with possibly a "blip" or two) while the pressure variations to which the ear is exposed (through a stethoscope or otherwise) contain oscillatory components of the measuring system response (to the steep wave).

Ames Research Center

National Aeronautics and Space Administration

Moffett Field, Calif., 94035, May 27, 1968

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