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Finite and Covariant Formulation of Local Field Theory Based on Improved Operator Products

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On the basis of Wilson's operator product expansions, local improved operator products are defined as well-defined parts of usual operator products. The local products thus defined are independent of the order of the operators in the original products. All expressions, identities, relations and equations to be considered in the quantum field theory are uniquely, by virtue of the order independence, given by these local products. They are exactly of the same form as those of the formal theory except the case of the trace identity anomaly. The Adler-Bell-Jackiw anomaly is an example of the trace anomaly. When the improved operator products are rewritten in terms of the usual products, our theory reduces to the renormalization theory.

§ 1. Introduction and summary

In general, the product of the Heisenberg operators

$$A_1(x_1)\,A_2(x_2)\cdots A_n(x_n) \tag{1·1}$$

in the covariant field theory is singular at $x_1=x_2=\cdots=x_n$ owing to the singularities of the canonical commutation relations and then it is impossible to consider the local product

$$A_1(x)\,A_2(x)\cdots A_n(x) \tag{1·2}$$

in a naive sense. It is not easy to regularize the commutation relations. So we propose, without regularizing them, the adoption of the improved and regularized operator products. For the singularities of operator products Wilson[1] has presented a powerful hypothesis which states that any operator product allows an expansion in the form

$$A_1(x_1)\,A_2(x_2)\cdots A_n(x_n)=E_0(x_1,x_2,\cdots,x_n)$$
$$+\sum E_i(x_1,x_2,\cdots,x_n)\,O_i(x_1)+R(x_1,x_2,\cdots,x_n). \tag{1·3}$$

Here the E_i's are c-number functions which may be singular at $x_1=x_2=\cdots=x_n$, the O_i's are local operators (anti-) commuting with themselves and other local operator of the model at y if $x-y$ is space-like and R is an operator which becomes weakly zero as $x_1=x_2=\cdots=x_n$.

On the basis of this hypothesis, we define, in the next section, the non-local improved product (non-local IP)

$$I_\kappa[A_1(x_1)A_2(x_2)\cdots A_n(x_n)] \tag{1·4}$$

as a well-defined part of the expansion (1·3), where κ is a constant, with the dimension of mass, necessary in the process of the subtraction of a logarithmically divergent part. Then the improved product (IP) is κ-dependent. The local improved product (local IP)

$$I_\kappa[A_1(x)A_2(x)\cdots A_n(x)] \tag{1·5}$$

exists, contrary to the formal product (1·2), and is the one which takes the place of (1·2). The IP thus defined has the following important properties. Firstly, the local IP is independent of the order of the operators in the original product, for example,

$$\mathrm{I}_\kappa[A_1(x)A_2(x)\cdots]=\mathrm{I}_\kappa[A_2(x)A_1(x)\cdots], \tag{1·6}$$

but if $x-y$ is time-like

$$\mathrm{I}_\kappa[A_1(x)A_2(y)\cdots]\neq\mathrm{I}_\kappa[A_2(y)A_1(x)\cdots]. \tag{1·7}$$

This is because the IP has no singular part to bring about the order dependence. This property is indispensable for the correct definition of the product, since the Feynman rules with regularization procedure[2] always give correct results and the manifestly covariant Feynman rules possess the independence of this kind. Secondly, for the operators with the Lorentz index, A_μ and B_μ, in some cases, the trace identity is modified as

$$\delta_{\mu\nu}\mathrm{I}_\kappa[A_\mu B_\nu\cdots]=\mathrm{I}_\kappa[A_\nu B_\nu\cdots]+\text{anomalous term}\,. \tag{1·8}$$

From the definition of the IP it is obvious that the anomalous term is always finite and covariant. This is also indispensable because the regularization scheme brings[3] the anomaly of this kind.

Our basic proposition is that all the quantities to be considered in the quantum field theory, such as each term of the equations of motion and the generators of the transformation groups, are given by the local IP without any modification. The product like $\mathrm{I}_\kappa[\cdots]\mathrm{I}_\kappa[\cdots]$ is unnecessary. According to this, the quantum equation for a formal one, $\partial_\mu J_\mu(x)=O(x)$ is $\mathrm{I}_\kappa[\partial_\mu J_\mu(x)]=\mathrm{I}_\kappa[O(x)]$. Then, in some cases, the divergence equation becomes

$$\partial_\mu\mathrm{I}_\kappa[J_\mu(x)]=\mathrm{I}_\kappa[O(x)]+\text{anomalous term} \tag{1·9}$$

on account of the trace anomaly seen in (1·8). This is the operator form of Adler-Bell-Jackiw's result.[4] We emphasize that, by virtue of (1·6), there is no arbitrariness in the process of the quantization. We can rigorously prove that the generators thus obtained satisfy the necessary algebraic relations.

We study the IP formulation of a free field theory in four dimensions in § 2, of the π-N system in two dimensions in § 3 and of the vector-meson nucleon system in two dimensions in § 4. In § 5 the connection between our theory and the renormalization theory is discussed.

Finally we want to remark on the constant κ. Our formalism explicitly depends on κ. Therefore, all the results depend on κ. However, even in our theory, the finite renormalization is necessary. The matrix element, renormalizable in the sense of the usual renormalization theory, will be κ-independent, so long as we write it by the renormalized mass and coupling. The unrenormalizable matrix element, however, remains κ-dependent although it becomes finite.

§ 2. Finite formulation of free field theory based on IP

For the case of a free complex scalar field with mass μ in four dimensions, we investigate the following subjects:

A) Definition of IP,

B) How to formulate finite theory,

C) Properties of IP,

D) How to express generators of the Poincaré group and the gauge group,

E) Transformation properties of IP,

F) Algebra of generators.

Since there is no problem in the free field theory, we can safely use the conventional ones for the equations of motion, the equal-time commutators, the definition of vacuum and so on. Most of the discussion in this section holds also for interacting cases.

A) *Definition of* IP

In the free field theory, the operator products $\phi(x)\phi(y)$ and $\phi^*(x)\phi^*(y)$ have no singularity at $x=y$. Only the products $\phi^*(x)\phi(y)$ and $\phi(x)\phi^*(y)$ have c-number singularities. Its form is obtained from the vacuum expectation value

$$\langle\phi^*(x)\phi(y)\rangle_0=\langle\phi(x)\phi^*(y)\rangle_0=\frac{\mu}{4\pi^2}\frac{K_1(\mu\sqrt{\lambda^2})}{\sqrt{\lambda^2}}$$
$$=\frac{1}{4\pi^2}\left[\frac{1}{\lambda^2}-\frac{\mu^2}{4}+\cdots+\left(\frac{\mu^2}{2}+\frac{\mu^4}{16}\lambda^2+\cdots\right)\log\frac{\gamma\mu\sqrt{\lambda^2}}{2}\right], \qquad (2\cdot1)$$

where

$$\lambda_\mu=(\boldsymbol{x}-\boldsymbol{y},\, x_0-y_0-i\epsilon). \qquad (2\cdot2)$$

At this stage, it will be worthwhile to note that

$$\frac{\mu}{4\pi^2}\left[\partial_0\left(\frac{K_1(\mu\sqrt{\lambda^2})}{\sqrt{\lambda^2}}-\frac{K_1(\mu\sqrt{(\lambda^2)^*})}{\sqrt{(\lambda^2)^*}}\right)\right]_{x_0=y_0}=\frac{1}{4\pi^2}\left[\partial_0\left(\frac{1}{\lambda^2}-\frac{1}{(\lambda^2)^*}\right)\right]_{x_0=y_0}$$
$$=-\frac{i}{\pi^2}\frac{\epsilon}{[(\boldsymbol{x}-\boldsymbol{y})^2+\epsilon^2]^2}=-i\delta(\boldsymbol{x}-\boldsymbol{y}). \qquad (2\cdot3)$$

Therefore $(2\cdot1)$ reproduces the canonical commutation relation.

Now, taking the case of $\partial_\mu\phi^*(x)\partial_\nu\phi(y)$, let us illustrate how to define the IP. Using $(2\cdot1)$, we extract the ill-defined part of $\partial_\mu\phi^*(x)\partial_\nu\phi(y)$ as

$$a_{\mu\nu}(\lambda)=\frac{1}{4\pi^2}\delta_{\mu\nu}\left[\frac{2}{(\lambda^2)^2}-\frac{\mu^2}{2\lambda^2}-\frac{\mu^4}{8}\log\kappa\sqrt{\lambda^2}\right]$$
$$-\frac{1}{4\pi^2}\frac{\lambda_\mu\lambda_\nu}{\lambda^2}\left[\frac{8}{(\lambda^2)^2}-\frac{\mu^2}{\lambda^2}+\frac{\mu^4}{8}\right]. \qquad (2\cdot4)$$

In $(2\cdot4)$ κ is any constant with the dimension of mass. The last term in the second square bracket of $(2\cdot4)$ is not well-defined, since it depends on the direction of the limit. The well-defined part of $\partial_\mu\phi^*(x)\partial_\nu\phi(y)$, which is called the non-local IP and denoted by $\mathrm{I}_\kappa[\partial_\mu\phi^*(x)\partial_\nu\phi(y)]$, is

$$\mathrm{I}_\kappa[\partial_\mu\phi^*(x)\partial_\nu\phi(y)]=\partial_\mu\phi^*(x)\partial_\nu\phi(y)-a_{\mu\nu}(\lambda). \qquad (2\cdot5)$$

As is seen in $(2\cdot4)$, in general, the unique separation of the ill-defined part is done in the form

$$\sum_m A_m\frac{1}{(\lambda^2)^m}+\sum_m B_m(\log\kappa\sqrt{\lambda^2})^m+\sum_m C_m\frac{\lambda_\mu\lambda_\nu}{(\lambda^2)^m}, \qquad (2\cdot6)$$

where A_m, B_m and C_m are x, y-independent constants or the non-local IP already defined. Only after subtracting these terms from $\partial_\mu\phi^*(x)\partial_\nu\phi(y)$, we can define the local product

$$\mathrm{I}_\kappa[\partial_\mu\phi^*(x)\partial_\nu\phi(x)]. \qquad (2\cdot7)$$

The IP thus defined depends on the choice of κ and hence we denote it by the suffix.

Next let us show that the ill-defined part $(2\cdot5)$ is derivable from only the informations of the canonical commutation relations and the equations of motion. First we put

$$\mathrm{I}_\kappa[\phi^*(x)\phi(y)]=\phi^*(x)\phi(y)-\frac{1}{4\pi^2\lambda^2}-L(x,y). \qquad (2\cdot8)$$

From the definition, the singularities of $\phi^*(x)\phi(y)$ are subtracted by the second and the third terms. The second term $-1/4\pi^2\lambda^2$ is the one responsible for the canonical commutation relation between $\partial_\mu\phi^*$ and ϕ as is seen in the discussion around $(2\cdot3)$. The third term $L(x,y)$ is the rest of the ill-defined part of $\phi^*(x)\times\phi(y)$ and therefore

$$[L(x,y)]_{qs}=0, \qquad (2\cdot9)$$

where $[\ \ ]_{qs}$ denotes the extraction of the singular part, at least quadratically. Operating $\Box_x-\mu^2$ in $(2\cdot8)$, we have

$$(\Box_x-\mu^2)\mathrm{I}_\kappa[\phi^*(x)\phi(y)]=\frac{\mu^2}{4\pi^2\lambda^2}-(\Box_x-\mu^2)L(x,y). \qquad (2\cdot10)$$

From the definition, in general, $\mathrm{I}_\kappa[\cdots]$ is less singular than $\log \kappa\sqrt{\lambda^2}$ and $\lambda_\mu\lambda_\nu/\lambda^2$, then

$$[(\Box_x-\mu^2)\mathrm{I}_\kappa(\cdots)]_{qs}=0\,. \tag{2·11}$$

Using (2·9), (2·10) and (2·11), we obtain

$$[\Box_x L(x,y)]_{qs}=\frac{\mu^2}{4\pi^2\lambda^2}\,. \tag{2·12}$$

This determines $L(x,y)$ uniquely, because of (2·9), as

$$L(x,y)=\frac{\mu^2}{8\pi^2}\log\kappa\sqrt{\lambda^2}\,. \tag{2·13}$$

Secondly, using

$$\left[\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial y_\nu}\mathrm{I}_\kappa\{\phi^*(x)\phi(y)\}\right]_{qs}=0 \tag{2·14}$$

and (2·8) with (2·13), we can fix the singularities of $\partial_\mu\phi^*(x)\partial_\nu\phi(y)$ up to the quadratic order. Thus

$$\mathrm{I}_\kappa[\partial_\mu\phi^*(x)\partial_\nu\phi(y)]=\partial_\mu\phi^*(x)\partial_\nu\phi(y)-\frac{1}{2\pi^2}\left[\frac{\delta_{\mu\nu}}{(\lambda^2)^2}-4\frac{\lambda_\mu\lambda_\nu}{(\lambda^2)^3}\right]$$
$$+\frac{\mu^2}{8\pi^2}\left[\frac{\delta_{\mu\nu}}{\lambda^2}-2\frac{\lambda_\mu\lambda_\nu}{(\lambda^2)^2}\right]-L_{\mu\nu}(x,y), \tag{2·15}$$

where $L_{\mu\nu}$ is the rest of the ill-defined part of $\partial_\mu\phi^*(x)\partial_\nu\phi(y)$ and therefore

$$[L_{\mu\nu}(x,y)]_{qs}=0\,. \tag{2·16}$$

Operating $\Box_x-\mu^2$ again in (2·15), we have

$$(\Box_x-\mu^2)\mathrm{I}_\kappa[\partial_\mu\phi^*(x)\partial_\nu\phi(y)]$$
$$=-\frac{\mu^4}{8\pi^2}\left[\frac{\delta_{\mu\nu}}{\lambda^2}-2\frac{\lambda_\mu\lambda_\nu}{(\lambda^2)^2}\right]-(\Box_x-\mu^2)L_{\mu\nu}(x,y)\,. \tag{2·17}$$

From (2·11), (2·16) and (2·17) we find

$$[\Box_x L_{\mu\nu}(x,y)]_{qs}=-\frac{\mu^4}{8\pi^2}\left[\frac{\delta_{\mu\nu}}{\lambda^2}-2\frac{\lambda_\mu\lambda_\nu}{(\lambda^2)^2}\right]. \tag{2·18}$$

This determines $L_{\mu\nu}(x,y)$ uniquely. The ill-defined part of $\partial_\mu\phi^*(x)\partial_\nu\phi(y)$ thus obtained is exactly $a_{\mu\nu}(\lambda)$ given by (2·4).

The IP has some similarities to the normal ordered product. The former is, however, defined by subtracting only ill-defined terms from the usual operator product, whereas in the latter for two operators in a free theory the whole vacuum expectation value is subtracted. We emphasize that the ill-defined terms are directly derivable from the informations of the canonical commutation relations and the equations of motion, but the whole vacuum expectation value is not.

Finally from (2·5) we have

$$\begin{aligned}
&\mathrm{I}_\kappa[\partial_\mu\phi^*(x)\partial_\nu\phi(y)]-\mathrm{I}_\kappa[\partial_\nu\phi(y)\partial_\mu\phi^*(x)] \\
&\quad=[\partial_\mu\phi^*(x),\partial_\nu\phi(y)]-a_{\mu\nu}(\lambda)+a^*_{\nu\mu}(\lambda) \\
&\quad=-\frac{\mu}{4\pi^2}\partial_\mu\partial_\nu\left[\frac{K_1(\mu\sqrt{\lambda^2})}{\sqrt{\lambda^2}}-\frac{K_1(\mu\sqrt{(\lambda^2)^*})}{\sqrt{(\lambda^2)^*}}\right]-a_{\mu\nu}(\lambda)+a^*_{\nu\mu}(\lambda). \qquad (2\cdot19)
\end{aligned}$$

Then it is obvious that

$$\mathrm{I}_\kappa[\partial_\mu\phi^*(x)\partial_\nu\phi(x)]=\mathrm{I}_\kappa[\partial_\nu\phi(x)\partial_\mu\phi^*(x)] \qquad (2\cdot20)$$

which is the special case of (1·6), but if $x-y$ is time-like

$$\mathrm{I}_\kappa[\partial_\mu\phi^*(x)\partial_\nu\phi(y)]\neq\mathrm{I}_\kappa[\partial_\nu\phi(y)\partial_\mu\phi^*(x)] \qquad (2\cdot21)$$

which is the special case of (1·7). This statement is valid also for the interacting cases. Thus we see that the local IP is independent of the order of the operators in the original product, although the non-local IP depends. For simplicity in the following we omit $i\epsilon$ which comes from the order of operators.

B) *Proposition for finite and covariant formulation of local field theory*

Our basic proposition is that all quantities in the finite theory are obtained from those in the formal theory by replacing them by corresponding local IP's. It should be noted that this proposition is meaningless without (1·6). Identities, relations and equations in the formal theory are valid also in the finite theory without any modification except the case of the trace anomaly seen in (1·8). When we consider the expression involved in the trace anomaly, the trace should be always taken in the inside of the IP.

C) *Some properties of* IP

It is clear from the discussion in A) that

$$\mathrm{I}_\kappa[\partial_\mu\phi^*(x)\partial_\mu\phi(y)]=\partial_\mu\phi^*(x)\partial_\mu\phi(y)-a_{\mu\mu}(\lambda)-\frac{\mu^4}{32\pi^2}. \qquad (2\cdot22)$$

The origin of the last term is that $-(\mu^4/32\pi^2)\lambda_\mu\lambda_\nu/\lambda^2$ in (2·4) becomes $-\mu^4/32\pi^2$ for $\partial_\mu\phi^*(x)\partial_\mu\phi(y)$ and becomes well-defined. So it should not be subtracted from $\partial_\mu\phi^*(x)\partial_\mu\phi(y)$. Comparing (2·22) with (2·5), we have

$$\delta_{\mu\nu}\mathrm{I}_\kappa[\partial_\mu\phi^*(x)\partial_\nu\phi(y)]=\mathrm{I}_\kappa[\partial_\mu\phi^*(x)\partial_\mu\phi(y)]+\frac{\mu^4}{32\pi^2}. \qquad (2\cdot23)$$

This is a simple example[3] of the trace anomaly (1·8). We do not have this kind of anomalous equation for the normal product of free field operators.

From (2·1), we easily verify that

$$\left(\frac{\partial}{\partial x_\mu}+\frac{\partial}{\partial y_\mu}\right)\mathrm{I}_\kappa[\phi^*(x)\phi(y)]=\mathrm{I}_\kappa[\partial_\mu\phi^*(x)\phi(y)]+\mathrm{I}_\kappa[\phi^*(x)\partial_\mu\phi(y)]. \qquad (2\cdot24)$$

Obviously this identity holds for the IP of any operators.

Using the definition of the IP, we can easily calculate the equal-time commutator between the IP's, for example,

$$[\mathrm{I}_\kappa\{\phi^*(x)\phi(y)\},\quad \mathrm{I}_\kappa\{\partial_0\phi^*(x')\phi(y')\}]_{x_0=y_0=x_0'=y_0'}$$
$$=i\phi^*(x)\phi(y')\delta(\boldsymbol{x}'-\boldsymbol{y}). \tag{2·25}$$

This indicates that the commutator between the IP's is not always written by IP alone. Therefore the commutator between the local IP's is meaningless. This is the reason why we consider the non-local IP as well as the local IP in spite of our proposition that all the physical quantities are given by the local IP. This is not an obstacle to formulate the theory.

D) *Generators of transformation groups*

According to the proposition in B), the energy-momentum tensor and the conserved current are

$$T_{\mu\nu}(x)=\mathrm{I}_\kappa[\partial_\mu\phi^*(x)\partial_\nu\phi(x)]+\mathrm{I}_\kappa[\partial_\nu\phi^*(x)\partial_\mu\phi(x)]$$
$$-\delta_{\mu\nu}\{\mathrm{I}_\kappa[\partial_\rho\phi^*(x)\partial_\rho\phi(x)]+\mu^2\mathrm{I}_\kappa[\phi^*(x)\phi(x)]\}$$
$$-c\delta_{\mu\nu}, \tag{2·26}$$

$$J_\mu(x)=i\mathrm{I}_\kappa[\partial_\mu\phi^*(x)\phi(x)]-i\mathrm{I}_\kappa[\phi^*(x)\partial_\mu\phi(x)]. \tag{2·27}$$

When the vacuum is defined, we should determine the last term of (2·26) so that the vacuum expectation value of $T_{\mu\nu}(x)$ may be zero.

Using (2·26) and (2·27), we can construct the generators of the Poincaré group and the gauge group. Then we obtain the usual commutation relations between the generators and each of $\phi(x)$, $\partial_0\phi(x)$, $\phi^*(x)$, $\partial_0\phi^*(x)$.

E) *Transformation properties of* IP

As an illustration we study the transformation property of $\mathrm{I}_\kappa[\partial_i\phi^*(x)\partial_j\phi(x)\partial_0\phi(x)]$ under the time translation. The definition of $\mathrm{I}_\kappa[\partial_i\phi^*\partial_j\phi\partial_0\phi]$ is

$$\mathrm{I}_\kappa[\partial_i\phi^*(x)\partial_j\phi(y)\partial_0\phi(z)]=\partial_i\phi^*(x)\partial_j\phi(y)\partial_0\phi(z)$$
$$-a_{ij}(x-y)\partial_0\phi(z)+a_{i0}(x-z)\partial_j\phi(y) \tag{2·28}$$

with the singular function $a_{\mu\nu}$ (2·4). Since the Hamiltonian H has already been defined in D), we can calculate the commutator between H and $\mathrm{I}_\kappa[\partial_i\phi^*\partial_j\phi\partial_0\phi]$ as

$$i[H,\mathrm{I}_\kappa\{\partial_i\phi^*(x)\partial_j\phi(y)\partial_0\phi(z)\}]$$
$$=\partial_0\partial_i\phi^*(x)\partial_j\phi(y)\partial_0\phi(z)+\partial_i\phi^*(x)\partial_0\partial_j\phi(y)\partial_0\phi(z)$$
$$+\partial_i\phi^*(x)\partial_j\phi(y)(\nabla^2-\mu^2)\phi(z)-a_{ij}(x-y)(\nabla^2-\mu^2)\phi(z)$$
$$+a_{i0}(x-z)\partial_0\partial_j\phi(y). \tag{2·29}$$

Now we try to write, by the IP, each term of (2·29):

$$\text{the first term} = \mathrm{I}_\kappa[\partial_0\partial_i\phi^*(x)\partial_j\phi(y)\partial_0\phi(z)] + \partial_0 a_{ij}(x-y)\partial_0\phi(z) + \partial_i a_{00}(x-z)\partial_j\phi(y), \qquad (2\cdot30)^{*)}$$

$$\text{the second term} = \mathrm{I}_\kappa[\partial_i\phi^*(x)\partial_0\partial_j\phi(y)\partial_0\phi(z)] - \partial_0 a_{ij}(x-y)\partial_0\phi(z) - a_{i0}(x-z)\partial_0\partial_j\phi(y), \qquad (2\cdot31)$$

$$\text{the third term} = \mathrm{I}_\kappa[\partial_i\phi^*(x)\partial_j\phi(y)(\nabla^2-\mu^2)\phi(z)] + a_{ij}(x-y)(\nabla^2-\mu^2)\phi(z) - \partial_i a_{kk}(x-z)\partial_j\phi(y) - \frac{\mu^2}{4\pi^2}\partial_i\left[\frac{1}{(x-z)^2} + \frac{\mu^2}{2}\log\kappa\sqrt{(x-z)^2}\right]\partial_j\phi(y), \qquad (2\cdot32)$$

where $a_{kk} = a_{11} + a_{22} + a_{33}$. Substituting $(2\cdot30)\sim(2\cdot32)$ into $(2\cdot29)$ and using the explicit form $(2\cdot4)$ of $a_{\mu\nu}$, we obtain

$$i[H, \mathrm{I}_\kappa\{\partial_i\phi^*(x)\partial_j\phi(y)\partial_0\phi(z)\}] = \mathrm{I}_\kappa[\partial_0\partial_i\phi^*(x)\partial_j\phi(y)\partial_0\phi(z)] + \mathrm{I}_\kappa[\partial_i\phi^*(x)\partial_0\partial_j\phi(y)\partial_0\phi(z)] + \mathrm{I}_\kappa[\partial_i\phi^*(x)\partial_j\phi(y)(\nabla^2-\mu^2)\phi(z)]. \qquad (2\cdot33)$$

This contains no ill-defined term. Then taking the local limit of $(2\cdot33)$, we have

$$\partial_0\mathrm{I}_\kappa[\partial_i\phi^*(x)\partial_j\phi(x)\partial_0\phi(x)] = \mathrm{I}_\kappa[\partial_0\partial_i\phi^*(x)\partial_j\phi(x)\partial_0\phi(x)] + \mathrm{I}_\kappa[\partial_i\phi^*(x)\partial_0\partial_j\phi(x)\partial_0\phi(x)] + \mathrm{I}_\kappa[\partial_i\phi^*(x)\partial_j\phi(x)(\nabla^2-\mu^2)\phi(x)]. \qquad (2\cdot34)$$

This is the rigorous quantum equation corresponding to the formal equation

$$\partial_0[\partial_i\phi^*(x)\partial_j\phi(x)\partial_0\phi(x)] = \partial_0\partial_i\phi^*(x)\partial_j\phi(x)\partial_0\phi(x) + \partial_i\phi^*(x)\partial_0\partial_j\phi(x)\partial_0\phi(x) + \partial_i\phi^*(x)\partial_j\phi(x)(\nabla^2-\mu^2)\phi(x). \qquad (2\cdot35)$$

The derivation of the equation like $(2\cdot34)$ for the other kind of transformation is much easier than that of $(2\cdot34)$.

Equation $(2\cdot34)$ tells us the following two facts. Firstly, the rigorous equation corresponding to the formal equation is obtained by simply replacing the formal operator products with the local IP's. In this case we have Eq. $(2\cdot33)$ for the non-local IP, too. However, the equation we want to have is not the non-local one but the local one on account of the order dependence of the non-local IP. Secondly, although

$$[\mathrm{I}_\kappa(\cdots), \mathrm{I}_\kappa(\cdots)] \qquad (2\cdot36)$$

cannot be written by the IP as shown in $(2\cdot25)$,

$$[\text{generator}, \mathrm{I}_\kappa(\cdots)] \qquad (2\cdot37)$$

*) Throughout this paper ∂_μ is the differentiation with respect to the argument of the nearest operator or function unless $\partial_\mu[\cdots]$.

can be written by the IP alone. From the local limit of the commutator (2·37) written by the IP, we see that the local IP has the transformation property expected from its constituents.

F) *Algebra of generators*

Since the local IP has the expected transformation property and the generator is constructed from the IP's, the algebra of generators is exactly the one which is expected from the formal theory.

§ 3. Finite theory for π-N system in two dimensions

According to the proposition in B) in § 2, the equations of motion and the symmetrical energy-momentum tensor for the π-N system in two dimensions are

$$(\gamma\cdot\partial+m)\psi(x)=-ig\mathrm{I}_\kappa[\gamma_5\psi(x)\phi(x)], \tag{3·1}$$ *)

$$(\square-\mu^2)\phi(x)=ig\mathrm{I}_\kappa[\bar\psi(x)\gamma_5\psi(x)] \tag{3·2}$$

and

$$\begin{aligned}\Theta^{\pi N}_{\mu\nu}(x)=&\tfrac{1}{4}\mathrm{I}_\kappa[\bar\psi(x)\gamma_\mu(\vec\partial_\nu-\overleftarrow\partial_\nu)\psi(x)]+\tfrac{1}{4}\mathrm{I}_\kappa[\bar\psi(x)\gamma_\nu(\vec\partial_\mu-\overleftarrow\partial_\mu)\psi(x)]\\&-\tfrac{1}{2}\delta_{\mu\nu}\mathrm{I}_\kappa[\bar\psi(x)\gamma\cdot(\vec\partial-\overleftarrow\partial)\psi(x)]-m\delta_{\mu\nu}\mathrm{I}_\kappa[\bar\psi(x)\psi(x)]\\&+\mathrm{I}_\kappa[\partial_\mu\phi(x)\partial_\nu\phi(x)]-\tfrac{1}{2}\delta_{\mu\nu}\mathrm{I}_\kappa[\partial_\rho\phi(x)\partial_\rho\phi(x)]-\tfrac{1}{2}\mu^2\delta_{\mu\nu}\mathrm{I}_\kappa[\phi(x)\phi(x)]\\&-ig\delta_{\mu\nu}\mathrm{I}_\kappa[\bar\psi(x)\gamma_5\psi(x)\phi(x)]-c'\delta_{\mu\nu}\,.\end{aligned} \tag{3·3}$$

The last term in (3·3) is the one corresponding to $c\delta_{\mu\nu}$ in (2·26). In order to understand (3·1)～(3·3) we need the definition of each IP. For these, after the straightforward but rather lengthy calculations explained in the Appendix, we have

$$\mathrm{I}_\kappa[\psi(x)\phi(y)]=\psi(x)\phi(y), \tag{3·4}$$

$$\mathrm{I}_\kappa[\phi(x)\phi(y)]=\phi(x)\phi(y)-\text{IDCT's}\,, \tag{3·5}$$

$$\mathrm{I}_\kappa[\psi(x)\bar\psi(y)]=\psi(x)\bar\psi(y)+\frac{1}{2\pi}\Big\{-\gamma\cdot\partial+m-\frac{1}{2}ig\gamma_5[\phi(x)+\phi(y)]\Big\}\times\log\kappa\sqrt{\lambda^2}\,, \tag{3·6}$$

$$\begin{aligned}\mathrm{I}_\kappa[\partial_\mu\psi(x)\bar\psi(y)]=&\partial_\mu\psi(x)\bar\psi(y)\\&-\frac{ig}{4\pi}\gamma_5\Big[\frac{2\lambda_\mu}{\lambda^2}\phi(x)-\frac{\lambda_\mu\gamma\cdot\lambda}{\lambda^2}\gamma\cdot\partial\phi(x)+\gamma\cdot\partial\phi(x)\gamma_\mu\log\kappa\sqrt{\lambda^2}\Big]\\&-\frac{g^2}{4\pi}\Big[\gamma_\mu\log\kappa\sqrt{\lambda^2}+\frac{\lambda_\mu\gamma\cdot\lambda}{\lambda^2}\Big]\mathrm{I}_\kappa[\phi(x)\phi(y)]-\text{IDCT's},\end{aligned} \tag{3·7}$$

$$\mathrm{I}_\kappa[\partial_\mu\phi(x)\partial_\nu\phi(y)]=\partial_\mu\phi(x)\partial_\nu\phi(y)-\text{IDCT's}\,, \tag{3·8}$$

*) For the Dirac matrices we take $\gamma_1=\sigma_1$, $\gamma_4=\sigma_2$ and $\gamma_5=\sigma_3$.

$$i\mathrm{I}_\kappa[\psi(x)\bar{\psi}(y)\phi(z)] = i\psi(x)\bar{\psi}(y)\phi(z) - \frac{i}{2\pi}\left[\frac{\gamma\cdot\lambda}{\lambda^2} - m\log\kappa\sqrt{\lambda^2}\right]\phi(z)$$

$$+\frac{g}{4\pi}\gamma_5\log\kappa\sqrt{\lambda^2}\{\mathrm{I}_\kappa[\phi(x)\phi(z)] + \mathrm{I}_\kappa[\phi(y)\phi(z)]\} - \text{IDCT's}, \qquad (3\cdot 9)$$

where $\lambda_\mu = (x_1 - y_1, x_0 - y_0 - i\epsilon)$ and IDCT's are ill-defined c-number terms.

When we use $(3\cdot 4)\sim(3\cdot 9)$ for the tensor like $\Theta_{\mu\nu}$, we must be careful about the trace identity anomaly $(1\cdot 8)$. Especially, in the two-dimensional space-time this problem becomes perplexing because of the identity for any tensor $T_{\mu\nu}$,

$$T_{11} = T_{\mu\mu} - T_{44}\,. \qquad (3\cdot 10)$$

We can easily avoid this by examining in the n-dimensional space-time free from $(3\cdot 10)$.

Now we try to derive the equations of motion $(3\cdot 1)$ and $(3\cdot 2)$ from the Hamiltonian

$$H = -\int dx_1 \Theta_{44}^{\pi N}(x) \qquad (3\cdot 11)$$

by calculating

$$[H, \psi], \qquad [H, \phi] \quad \text{and} \quad [H, \partial_4\phi]. \qquad (3\cdot 12)$$

The energy-momentum tensor $(3\cdot 3)$ is sufficient so far as the matrix elements between some states are concerned. However, the special care is required in calculating the commutators $(3\cdot 12)$. As discussed in E) in § 2, $(3\cdot 12)$ must be written by IP alone, that is, the singularities which appear in the commutators between each term of $\Theta_{44}^{\pi N}$ and ψ (or ϕ, $\partial_4\phi$) must be cancelled out each other. Therefore the local limit of each term of $\Theta_{44}^{\pi N}$ should be related to each other so that the cancellation may take place correctly. For example, the logarithmic factors as coefficients of $\mathrm{I}_\kappa[\phi\phi]$ in $(3\cdot 7)$ and $(3\cdot 9)$ should take the common value. Then the Hamiltonian $(3\cdot 11)$ is defined as the local limit of an integral over one variable x, keeping the relative coordinates fixed so as to satisfy the condition for the cancellation. We confine the fixed relative coordinates to space components alone from necessity of deriving the equations of motion from $(3\cdot 12)$. Using the Hamiltonian thus defined, we have $(3\cdot 1)$ and $(3\cdot 2)$ through

$$[H, \psi(x)] = -\gamma_4(\gamma_1\partial_1 + m)\psi(x) - ig\gamma_4\gamma_5\mathrm{I}_\kappa[\psi(x)\phi(x)], \qquad (3\cdot 13)$$

$$[H, \phi(x)] = \partial_4\phi(x), \qquad (3\cdot 14)$$

$$[H, \partial_4\phi(x)] = -\partial_1{}^2\phi(x) + \mu^2\phi(x) + ig\mathrm{I}_\kappa[\bar{\psi}(x)\gamma_5\psi(x)]. \qquad (3\cdot 15)$$

Now we examine the transformation property of IP. As an illustration we study the transformation property of $\mathrm{I}_\kappa[\psi(x)\bar{\psi}(y)]$ under the time translation. Using $(3\cdot 6)$ and $(3\cdot 1)$, we find

$$\frac{\partial}{\partial x_4}\mathrm{I}_\kappa[\psi(x)\bar{\psi}(y)] + \mathrm{I}_\kappa[\psi(x)\bar{\psi}(y)]\frac{\overleftarrow{\partial}}{\partial y_4} = -\gamma_4\gamma_1\partial_1\psi(x)\bar{\psi}(y) - \psi(x)\bar{\psi}(y)\gamma_1\overleftarrow{\partial}_1\gamma_4$$

$$
\begin{aligned}
&-m\gamma_4\psi(x)\bar{\psi}(y)+m\psi(x)\bar{\psi}(y)\gamma_4 \\
&-ig\gamma_4\gamma_5\psi(x)\phi(x)\bar{\psi}(y)+ig\psi(x)\phi(y)\bar{\psi}(y)\gamma_5\gamma_4 \\
&-\frac{ig}{4\pi}\gamma_5[\partial_4\phi(x)+\partial_4\phi(y)]\log\kappa\sqrt{\lambda^2}\,.
\end{aligned} \tag{3·16}
$$

Using (3·6), (3·7) and (3·9), we rewrite each term of (3·16) into the corresponding IP with care to the trace anomaly about $\gamma\cdot\partial$. The result is

$$
\begin{aligned}
&\mathrm{I}_\kappa[\gamma_4\gamma\cdot\partial\psi(x)\bar{\psi}(y)]+\mathrm{I}_\kappa[\psi(x)\bar{\psi}(y)\gamma\cdot\overleftarrow{\partial}\gamma_4] \\
&\quad=-m\mathrm{I}_\kappa[\gamma_4\psi(x)\bar{\psi}(y)]+m\mathrm{I}_\kappa[\psi(x)\bar{\psi}(y)\gamma_4] \\
&\qquad-ig\mathrm{I}_\kappa[\gamma_4\gamma_5\psi(x)\phi(x)\bar{\psi}(y)]+ig\mathrm{I}_\kappa[\psi(x)\phi(y)\bar{\psi}(y)\gamma_5\gamma_4] \\
&\qquad+\frac{g^2}{4\pi}\gamma_4\{\mathrm{I}_\kappa[\phi(x)\phi(x)]-\mathrm{I}_\kappa[\phi(y)\phi(y)]\}\log\kappa\sqrt{\lambda^2}\,.
\end{aligned} \tag{3·17}
$$

Although in (3·7) and (3·9) we have not written the explicit forms of the IDCT's, (3·17) is correct including the IDCT. The local limit is

$$
\begin{aligned}
&\mathrm{I}_\kappa[\gamma_4\gamma\cdot\partial\psi(x)\bar{\psi}(x)]+\mathrm{I}_\kappa[\psi(x)\bar{\psi}(x)\gamma\cdot\overleftarrow{\partial}\gamma_4] \\
&\quad=-m\mathrm{I}_\kappa[\gamma_4\psi(x)\bar{\psi}(x)]+m\mathrm{I}_\kappa[\psi(x)\bar{\psi}(x)\gamma_4] \\
&\qquad-ig\mathrm{I}_\kappa[\gamma_4\gamma_5\psi(x)\bar{\psi}(x)\phi(x)]+ig\mathrm{I}_\kappa[\psi(x)\bar{\psi}(x)\gamma_5\gamma_4\phi(x)]\,.
\end{aligned} \tag{3·18}
$$

Note that $\gamma\cdot\partial$ is placed inside of $\mathrm{I}_\kappa[\cdots]$. This is exactly the equation we have proposed in B) in § 2.

§ 4. Vector-meson nucleon system in two dimensions and the Adler-Bell-Jackiw anomaly

Let us consider the system whose Lagrangian is

$$
\begin{aligned}
\mathcal{L}=&-\bar{\psi}(\gamma\cdot\partial+m)\psi-\tfrac{1}{4}F_{\mu\nu}F_{\mu\nu}-\tfrac{1}{2}\mu^2A_\mu A_\mu \\
&-\chi\partial_\mu A_\mu+\tfrac{1}{2}\alpha^2\chi^2-ie\bar{\psi}\gamma_\mu\psi A_\mu\,,
\end{aligned} \tag{4·1}
$$

where

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu\,. \tag{4·2}$$

The canonical commutation relations are

$$[A_\mu(x),\,A_\nu(y)]_{x_0=y_0}=0\,, \tag{4·3}$$

$$[A_\mu(x),\,\pi_\nu(y)]_{x_0=y_0}=i\delta_{\mu\nu}\delta(x_1-y_1), \tag{4·4}$$

$$[\pi_\mu(x),\,\pi_\nu(y)]_{x_0=y_0}=0\,, \tag{4·5}$$

where

$$\pi_1(x)=iF_{41}(x), \tag{4·6}$$

$$\pi_0(x)=i\pi_4(x)=-\chi(x). \tag{4·7}$$

The proposed equations of motion are

$$(\gamma\cdot\partial+m)\psi(x)=-ie\mathrm{I}_\kappa[\gamma_\mu\psi(x)A_\mu(x)], \tag{4·8}$$

$$(\Box-\mu^2)A_\mu(x)-\partial_\mu\partial_\nu A_\nu(x)+\partial_\mu\chi(x)=ie\mathrm{I}_\kappa[\bar\psi(x)\gamma_\mu\psi(x)], \tag{4·9}$$

$$\partial_\mu A_\mu(x)-\alpha^2\chi(x)=0\,. \tag{4·10}$$

Then

$$(\Box-\alpha^2\mu^2)\chi(x)=0\,. \tag{4·11}$$

The symmetrical energy-momentum tensor, whose (4, 4) component does not contain $\partial_4\psi$, $\partial_4\bar\psi$, $\partial_4^2A_\mu$ and $\partial_4\chi$, is

$$\begin{aligned}\Theta_{\mu\nu}^{VN}(x)=&\tfrac14\mathrm{I}_\kappa[\bar\psi(x)\gamma_\mu(\vec\partial_\nu-\overleftarrow\partial_\nu)\psi(x)]\\&-\tfrac14\delta_{\mu\nu}\mathrm{I}_\kappa[\bar\psi(x)\gamma\cdot(\vec\partial-\overleftarrow\partial)\psi(x)]-\tfrac12 m\delta_{\mu\nu}\mathrm{I}_\kappa[\bar\psi(x)\psi(x)]\\&+\tfrac12\mathrm{I}_\kappa[F_{\mu\rho}(x)F_{\nu\rho}(x)]+\tfrac14\mathrm{I}_\kappa[A_\mu(x)(\vec\Box+\overleftarrow\Box)A_\nu(x)]\\&-\tfrac12\mathrm{I}_\kappa[A_\mu(x)\partial_\nu\partial_\rho A_\rho(x)]-\tfrac12\mathrm{I}_\kappa[\partial_\mu\chi(x)A_\nu(x)]\\&-\tfrac18\delta_{\mu\nu}\mathrm{I}_\kappa[F_{\rho\sigma}(x)F_{\rho\sigma}(x)]-\tfrac14\mu^2\delta_{\mu\nu}\mathrm{I}_\kappa[A_\rho(x)A_\rho(x)]\\&+\tfrac12\delta_{\mu\nu}\mathrm{I}_\kappa[\partial_\rho\chi(x)A_\rho(x)]+\tfrac14\alpha^2\delta_{\mu\nu}\mathrm{I}_\kappa[\chi(x)\chi(x)]\\&-\tfrac12 ie\delta_{\mu\nu}\mathrm{I}_\kappa[\bar\psi(x)\gamma_\rho\psi(x)A_\rho(x)]+(\mu\leftrightarrow\nu).\end{aligned} \tag{4·12}$$

After the same procedure, used the canonical commutation relations (4·3) ~(4·5) and the equations of motion (4·8)~(4·10), as that in the π-N case in the previous section, we have

$$\mathrm{I}_\kappa[A_\mu(x)A_\nu(y)]=A_\mu(x)A_\nu(y)-\text{IDCT's}, \tag{4·13}$$

$$\mathrm{I}_\kappa[\partial_\rho A_\mu(x)\partial_\sigma A_\nu(y)]=\partial_\rho A_\mu(x)\partial_\sigma A_\nu(y)-\text{IDCT's}, \tag{4·14}$$

$$\mathrm{I}_\kappa[A_\mu(x)\partial_\rho\partial_\sigma A_\nu(y)]=A_\mu(x)\partial_\rho\partial_\sigma A_\nu(y)-\text{IDCT's}, \tag{4·15}$$

$$\mathrm{I}_\kappa[\partial_\mu\chi(x)A_\nu(y)]=\partial_\mu\chi(x)A_\nu(y)-\text{IDCT's}, \tag{4·16}$$

$$\mathrm{I}_\kappa[\chi(x)\chi(x)]=\chi(x)\chi(y)-\text{IDCT's}, \tag{4·17}$$

$$\mathrm{I}_\kappa[\psi(x)A_\mu(y)]=\psi(x)A_\mu(y)\,, \tag{4·18}$$

$$\mathrm{I}_\kappa[\psi(x)\bar\psi(y)]=\psi(x)\bar\psi(y)+\frac{ie}{4\pi}\frac{\lambda_\rho\gamma\cdot\lambda}{\lambda^2}[A_\rho(x)+A_\rho(y)]-\text{IDCT's}, \tag{4·19}$$

$$\begin{aligned}\mathrm{I}_\kappa[\partial_\mu\psi(x)\bar\psi(y)]=&\partial_\mu\psi(x)\bar\psi(y)\\&+\frac{ie}{8\pi}\Big[4\Big(\frac{\gamma_\mu\lambda_\rho+\delta_{\rho\mu}\gamma\cdot\lambda}{\lambda^2}-2\frac{\lambda_\mu\lambda_\rho\gamma\cdot\lambda}{(\lambda^2)^2}\Big)[A_\rho(x)-\frac12\lambda\cdot\partial A_\rho(x)]\\&+[\gamma\cdot\partial A_\rho(x)\gamma_\rho\gamma_\mu-\gamma_\mu\gamma_\rho\gamma\cdot\partial A_\rho(x)]\log\kappa\sqrt{\lambda^2}\\&+2\frac{\lambda_\rho\partial_\mu A_\rho(x)\gamma\cdot\lambda}{\lambda^2}+\frac{\lambda_\mu\gamma\cdot\partial A_\rho(x)\gamma_\rho\gamma\cdot\lambda}{\lambda^2}-\frac{\lambda_\mu\gamma\cdot\lambda\gamma_\rho\gamma\cdot\partial A_\rho(x)}{\lambda^2}\end{aligned}$$

$$-4m\left(\delta_{\rho\mu}\log\kappa\sqrt{\lambda^2}+\frac{\lambda_\mu\lambda_\rho}{\lambda^2}\right)A_\rho(x)\Big]$$

$$+\frac{e^2}{4\pi}\left[\frac{\gamma_\mu\lambda_\rho\lambda_\sigma}{\lambda^2}+2\frac{\delta_{\rho\mu}\lambda_\sigma\gamma\cdot\lambda}{\lambda^2}-2\frac{\lambda_\mu\lambda_\rho\lambda_\sigma\gamma\cdot\lambda}{(\lambda^2)^2}\right]\mathrm{I}_\kappa[A_\rho(x)A_\sigma(y)]$$

$$-\mathrm{IDCT's}\,, \tag{4·20}$$

$$\mathrm{I}_\kappa[\psi(x)\bar{\psi}(y)A_\mu(z)]=\psi(x)\bar{\psi}(y)A_\mu(z)-\frac{1}{2\pi}\left(\frac{\gamma\cdot\lambda}{\lambda^2}-m\log\kappa\sqrt{\lambda^2}\right)A_\mu(z)$$

$$+\frac{ie}{4\pi}\frac{\lambda_\rho\gamma\cdot\lambda}{\lambda^2}\{\mathrm{I}_\kappa[A_\rho(x)A_\mu(z)]+\mathrm{I}_\kappa[A_\rho(y)A_\mu(z)]\}-\mathrm{IDCT's}. \tag{4·21}$$

By using these the consistency between the equations of motion and the energy-momentum tensor is justified in the same way as the π-N case.

Through the same process of the calculation as done in the derivation of (3·18), we get

$$\mathrm{I}_\kappa[\partial_\mu\{\bar{\psi}(x)\gamma_\mu\psi(x)\}]=0 \tag{4·22}$$

and

$$\mathrm{I}_\kappa[\partial_\mu\{\bar{\psi}(x)\gamma_5\gamma_\mu\psi(x)\}]=-2m\mathrm{I}_\kappa[\bar{\psi}(x)\gamma_5\psi(x)]. \tag{4·23}$$

By virtue of (2·24) we can take out ∂_μ from the IP so far as the differentiation is concerned. Only problem is the trace anomaly. The careful evaluation gives us

$$\partial_\mu\mathrm{I}_\kappa[\bar{\psi}(x)\gamma_\mu\psi(x)]=0\,, \tag{4·24}$$

$$\partial_\mu\mathrm{I}_\kappa[\bar{\psi}(x)\gamma_5\gamma_\mu\psi(x)]=-2m\mathrm{I}_\kappa[\bar{\psi}(x)\gamma_5\psi(x)]$$

$$-\frac{ie}{2\pi}\epsilon_{\mu\nu}F_{\mu\nu}(x), \tag{4·25}$$

where $\epsilon_{14}=-\epsilon_{41}=i$ and

$$\mathrm{I}_\kappa[\bar{\psi}(x)\gamma_\mu\psi(y)]=\bar{\psi}(x)\gamma_\mu\psi(y)-\frac{1}{\pi}\frac{\lambda_\mu}{\lambda^2}$$

$$-\frac{ie}{2\pi}\frac{\lambda_\mu\lambda_\rho}{\lambda^2}[A_\rho(x)+A_\rho(y)], \tag{4·26}$$

$$\mathrm{I}_\kappa[\bar{\psi}(x)\gamma_5\gamma_\mu\psi(y)]=\bar{\psi}(x)\gamma_5\gamma_\mu\psi(y)-\frac{1}{\pi}\epsilon_{\mu\nu}\frac{\lambda_\nu}{\lambda^2}$$

$$-\frac{ie}{2\pi}\epsilon_{\mu\nu}\frac{\lambda_\nu\lambda_\rho}{\lambda^2}[A_\rho(x)+A_\rho(y)], \tag{4·27}$$

which are the special cases of (4·19). Equation (4·25) is the two-dimensional analog of Adler-Bell-Jackiw's anomalous equation.[4)]

Finally we remark that (4·24) and (4·25) are derivable directly from (4·26), (4·27) and the equations of motion (4·8)～(4·10).

§ 5. Connection with renormalization theory

In our finite theory all operator products have been expressed by the local IP's. If we wish, we can write these products by the usual products in a formal but not a rigorous way. For example, the Hamiltonian densities in the models of § 3 and § 4 are written, apart from the c-number terms, as

$$
\begin{aligned}
-\Theta_{44}^{\pi N}(x) &= \frac{1}{2}\bar{\psi}(x)\gamma_1(\vec{\partial}_1-\overleftarrow{\partial}_1)\psi(x)+m\bar{\psi}(x)\psi(x) \\
&+\frac{1}{2}\pi(x)\pi(x)+\frac{1}{2}\partial_1\phi(x)\partial_1\phi(x)+\frac{1}{2}\mu^2\phi(x)\phi(x) \\
&+ig\bar{\psi}(x)\gamma_5\psi(x)\phi(x)-\frac{g^2}{2\pi}(\log\kappa\delta)\phi(x)\phi(x), \qquad (5\cdot1)
\end{aligned}
$$

$$
\begin{aligned}
-\Theta_{44}^{VN} &= \frac{1}{2}\bar{\psi}(x)\gamma_1(\vec{\partial}_1-\overleftarrow{\partial}_1)\psi(x)+m\bar{\psi}(x)\psi(x) \\
&+\frac{1}{2}\pi_1(x)\pi_1(x)-i\partial_1\pi_1(x)A_4(x)+i\partial_1\pi_4(x)A_1(x)+\frac{1}{2}\alpha^2\pi_4(x)\pi_4(x) \\
&+\frac{1}{2}\mu^2A_\mu(x)A_\mu(x)+ie\bar{\psi}(x)\gamma_\mu\psi(x)A_\mu(x)+\frac{e^2}{2\pi}A_1(x)A_1(x). \qquad (5\cdot2)
\end{aligned}
$$

In (5·1), $\pi=\partial_0\phi$ and $\log\kappa\delta$ are the abbreviation of $[\log\kappa\sqrt{(x-y)^2}]_{x\to y}$. Equations (5·1) and (5·2) are the Hamiltonian in the usual renormalization theory. The last term of (5·1) is only one divergent quantity in that model and is the counter term to the meson mass. Since the vector-meson nucleon system in two dimensions has no divergent Feynman graph, there is no counter term in (5·2). The last term of (5·2) is necessary to assure[5] the Lorentz covariance (the integrability condition in the interaction representation or the Schwinger condition in the Heisenberg representation). Thus our theory is closely connected with the renormalization theory. However, (5·1) and (5·2) are the Hamiltonian written only in a formal way.

Similarly, rewriting (3·1), (3·2) (4·8)~(4·10) by the usual product, we get the formal equations of motion in the renormalization theory.

Appendix

——Derivation of (3·4)~(3·9)——

In this Appendix we discuss how to derive (3·4)~(3·9). For this purpose we explain the method of the derivation of (3·6) and (3·7).

Now we put

$$
\mathrm{I}_\kappa[\psi(x)\bar{\psi}(y)]=\psi(x)\bar{\psi}(y)-\frac{1}{2\pi}\frac{\gamma\cdot\lambda}{\lambda^2}-S(x,y). \qquad (\mathrm{A}\cdot1)
$$

The singularities of $\psi(x)\bar{\psi}(y)$ are subtracted by the second and the third terms. The second term $-\gamma\cdot\lambda/2\pi\lambda^2$, satisfying

$$\left[-\frac{1}{2\pi}\frac{\gamma\cdot\lambda}{\lambda^2}+\frac{1}{2\pi}\frac{\gamma\cdot\lambda^*}{(\lambda^2)^*}\right]_{x_0=y_0}=-\gamma_4\delta(x_1-y_1), \tag{A·2}$$

is the one responsible for the canonical commutation relation between ψ and $\bar{\psi}$. The third term is the rest of the ill-defined part of $\psi(x)\bar{\psi}(y)$ and therefore

$$[S(x,y)]_{ls}=0, \tag{A·3}$$

where $[\ \ ]_{ls}$ denotes the extraction of the singular part, at least linearly. Operating $(\gamma\cdot\partial+m)$ in (A·1), we have

$$\left[\gamma\cdot\frac{\partial}{\partial x}S(x,y)\right]_{ls}=-[m+ig\gamma_5\phi(x)]\frac{\gamma\cdot\lambda}{2\pi\lambda^2}, \tag{A·4}$$

since for any IP

$$\{\partial_\mu \mathrm{I}_\kappa[\cdots]\}_{ls}=0 \tag{A·5}$$

and

$$[\phi(x)\psi(x)\bar{\psi}(y)]_{ls}=\phi(x)\frac{\gamma\cdot\lambda}{2\pi\lambda^2}. \tag{A·6}$$

From (A·3) and (A·4), we obtain (3·6). In (3·6) we use the form $-(1/4\pi)\times ig\gamma_5[\phi(x)+\phi(y)]\log\kappa\sqrt{\lambda^2}$ instead of $-(1/2\pi)ig\gamma_5\phi(x)\log\kappa\sqrt{\lambda^2}$. This difference is not essential so long as the ill-defined part is concerned. We use the former one to maintain the symmetry between $\psi(x)$ and $\bar{\psi}(y)$.

Next using (A·5) and (3·6), we can write

$$\begin{aligned}\mathrm{I}_\kappa[\partial_\mu\psi(x)\bar{\psi}(y)]&=\partial_\mu\psi(x)\bar{\psi}(y)\\&\quad+[-\gamma\cdot\partial+m-ig\gamma_5\phi(x)]\frac{\lambda_\mu}{2\pi\lambda^2}-L_\mu(x,y)\end{aligned} \tag{A·7}$$

with

$$[L_\mu(x,y)]_{ls}=0. \tag{A·8}$$

Operating $(\gamma\cdot\partial/\partial y-m)$ from the right in (A·7) and using (A·5), we get

$$\begin{aligned}\left[L_\mu(x,y)\gamma\cdot\frac{\overleftarrow{\partial}}{\partial y}\right]_{ls}&=ig[\partial_\mu\psi(x)\bar{\psi}(y)\gamma_5\phi(y)]_{ls}\\&\quad+[-\gamma\cdot\partial+m-ig\gamma_5\phi(x)]\frac{\lambda_\mu}{2\pi\lambda^2}\left(\gamma\cdot\frac{\overleftarrow{\partial}}{\partial y}-m\right).\end{aligned} \tag{A·9}$$

The linear singularities are closely connected with the equal-time commutators as is seen in (A·2). Then, we can easily obtain

$$[\partial_\mu\psi(x)\bar{\psi}(y)\gamma_5\phi(y)]_{ls}=-[-\gamma\cdot\partial+m-ig\gamma_5\phi(x)]\frac{\lambda_\mu}{2\pi\lambda^2}\gamma_5\phi(y)$$

$$= -(-\gamma\cdot\partial + m)\gamma_5 \frac{\lambda_\mu}{2\pi\lambda^2}\phi(y)$$
$$+ig\frac{\lambda_\mu}{2\pi\lambda^2}\left\{\mathrm{I}_\kappa[\phi(x)\phi(y)] - \frac{1}{2\pi}\log\kappa\sqrt{\bar{\lambda}^2}\right\}. \tag{A·10}$$

Equations (A·9) and (A·10) are sufficient to derive (3·7).

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