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# FINITE COMPLEX REFLECTION GROUPS

BY ARJEH M. COHEN

## Introduction

In 1954 G. C. Shephard and J. A. Todd published a list of all finite irreducible complex reflection groups (up to conjugacy). In their classification they separately studied the imprimitive groups and the primitive groups. In the latter case they extensively used the classification of finite collineation groups containing homologies as worked out by G. Bagnera (1905), H. F. Blichfeldt (1905), and H. H. Mitchell (1914). Furthermore, Shephard and Todd determined the degrees of the reflection groups, using the invariant theory of the corresponding collineation groups in the primitive case. In 1967 H. S. M. Coxeter (*cf.* [6]) presented a number of graphs connected with complex reflection groups in an attempt to systematize the results of Shephard and Todd.

This paper is another attempt to obtain a systematization of the same results. The complex reflection groups are classified by means of new methods (without use of the old literature). Furthermore, we give some new results concerning these groups.

Chapter 1 contains a number of familiar facts about reflection groups and is of a preparatory nature.

Chapter 2 deals with the imprimitive case and contains a study of systems of imprimitivity.

In Chapter 3 we look for all complex reflection groups among the finite subgroups of $G\,l_2(\mathbf{C})$.

As to Chapter 4, inspired by Coxeter's "complex" graphs (*see* above) and by root systems associated with real reflection groups, we define root graphs and root systems connected with finite reflection groups. Furthermore, we show how these root graphs may be useful by constructing for a given reflection group G a root graph with corresponding reflection group H in such a way that H is a subgroup of G with properties resembling those of G. Furthermore, a number of root graphs are brought together. Using these root graphs, we build root systems and study the associated reflection groups. These groups are all primitive and complex.

Thanks to T. A. Springer's work on regular elements of reflection groups (*cf.* [18]), we are able to determine the degrees of these groups in a manner analogous to the one in Bourbaki.

In Chapter 5, a theorem of Blichfeldt concerning finite primitive groups is discussed. From this theorem we deduce several necessary conditions for a primitive subgroup of $Gl_n(\mathbf{C})$ ($n \geqq 3$) to be a reflection group. The classification is completed by manipulating with root graphs.

This article is the greater part of a thesis written at the State University of Utrecht.

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## 1. Generalities about reflection groups

Let V be a complex vector space of dimension $n$.

(1.1) DEFINITIONS. — A *reflection* in V is a linear transformation of V of finite order with exactly $n-1$ eigenvalues equal to 1. A *reflection group* in V is a finite group generated by reflections in V. From (1.6) on we will assume a reflection (group) to be unitary with respect to a unitary inner product. A *reflection subgroup* of a group G of linear transformations of V is a subgroup of G which is a reflection group in V. A reflection group G in V is called a *real group* or *Coxeter group* if there is a G-invariant $\mathbf{R}$-subspace $V_0$ of V such that the canonical map $\mathbf{C} \otimes_{\mathbf{R}} V_0 \to V$ is bijective. If this is not the case, G will be called *complex* (note that, according to this definition, a real reflection group is not complex). A reflection group G is called *r-dimensional* if the dimension of the subspace $V^G$ of points fixed by G is $n-r$. We will say that G is *irreducible in dimension r* (also *irreducible r-dimensional* or merely *irreducible* if no confusion is possible) if G is $r$-dimensional and the restriction of G to a G-invariant complement of $V^G$ in V is irreducible. We will use the same convention for other properties than irreducibility (e. g. primitivity, *see* § 2).

(1.2) Let G be a finite group of linear transformations of V and let $S = S(V)$ be the algebra of the polynomial functions on V with G-action defined by $(g.f)(v) = f(g^{-1}v)$ for any $v \in V$, $f \in S$, $g \in G$. The subalgebra of G-invariant polynomials will be denoted by $S^G$. The theorem below is a well-known characterization of reflection groups (*cf.* [18]).

THEOREM. — *The following three statements are equivalent:*

(i) G *is a reflection group in* V*;*

(ii) *there are n algebraically independent homogeneous polynomials* $f_1, f_2, \ldots, f_n \in S^G$ *with* $|G| = \deg(f_1).\deg(f_2)\ldots\deg(f_n)$*;*

(iii) *there are n algebraically independent homogeneous polynomials* $f_1, f_2, \ldots, f_n \in S^G$ *which generate* $S^G$ *as an algebra over* $\mathbf{C}$ (*together with* 1).

*Furthermore, let* $f_1, f_2, \ldots, f_n$ *be a family of algebraically independent homogeneous polynomials in* $S^G$ *such that* $d_i \leqq d_{i+1}$ $(i \in \underline{n-1})$ *where* $d_j$ *is the degree of* $f_j$ $(j \in \underline{n})$*; then*

*$f_1, f_2, \ldots, f_n$ satisfy* (ii) *if and only if they satisfy* (iii). *In this situation, the sequence $d_1, d_2, \ldots, d_n$ is independent of the particular choice of such a family $f_1, f_2, \ldots, f_n$.*

DEFINITION. — $d_1, d_2, \ldots, d_n$ are called the *degrees* of G.

(1.3) Let G be a reflection group with degrees $d_1, d_2, \ldots, d_n$. Suppose $f_1, f_2, \ldots, f_n \in S^G$ satisfy (1.2) (ii) and $d_i = \deg(f_i)$. Let I be the ideal in S generated by $f_1, f_2, \ldots, f_n$. Then I is a G-invariant graded ideal, so S/I is a graded G-module. According to [4], G acts on S/I by the regular representation. Let $\psi$ be an irreducible character of G; denote by $a_i(\psi)$ the multiplicity of $\psi$ in the $i$-th homogeneous component $(S/I)_i$ of S/I ($i \geq 0$). Adopting an idea of [18], we define

$$p_\psi(T) = \sum_{i=0}^{\infty} a_i(\psi) T^i.$$

Note that $p_\psi(T)$ is a polynomial specifying for which $i$ the representation corresponding to $\psi$ occurs in $(S/I)_i$.

Now the identity

$$(1) \qquad |G|^{-1} \sum_{g \in G} \psi(g).\det(1-gT)^{-1} = p_\psi(T) \prod_{i=1}^{n} (1-T^{d_i})^{-1}$$

is obtained by writing out left- and right-hand side as a formal power series in T.

The following result is due to R. Steinberg (*cf.* [3], p. 127): if G is irreducible, then, for each $j \in \underline{n}$, the action of G on the $j$-th exterior power of V is also irreducible. Write $\chi_a^j$ for the corresponding character ($j = 1, 2, \ldots, n$). The coefficient of $Y^j$ in Solomon's formula (*see* [3], p. 136):

$$(2) \qquad \frac{1}{|G|} \sum_{g \in G} \frac{\det(1+Yg)}{\det(1-gT)} = \prod_{i=1}^{n} \frac{(1+YT^{d_i-1})}{(1-T^{d_i})}$$

equals $p_{\chi_a^j}(T). \prod_{i=1}^{n} (1 - T^{d_i})^{-1}$.

(1.4) We mention a few corollaries of these statements.

*Notations as before.* — *Put* $N = \sum_{i=1}^{n} (d_i - 1)$, *and let* $\chi$ *be the character of the given representation of* G *in* V. *Suppose* G *is irreducible. Then:*

(i) $p_\chi(T) = \sum_{i=1}^{n} T^{d_i-1}$;

(ii) $|Z(G)| = gcd(d_1, d_2, \ldots, d_n)$;

(iii) $p_{\bar{\psi}\delta}(T) = T^N p_\psi(T^{-1})$, *where* $\delta(g) = \det(g)$ $(g \in G)$;

(iv) *there is a homogeneous polynomial* J *of degree* N *such that* $g(J) = \det(g).J$ *for any* $g \in G$;

(v) N *is the number of reflections in* G, *in fact* $\sum_{i=0}^{n} h_i T^i = \prod_{i=1}^{n} (d_i - 1 + T)$, *where* $h_i$ *is the number of elements in* G *with exactly i eigenvalues equal to* 1;

(vi) $d_1 \geqq 2$; *furthermore, the following three statements are equivalent:* (*a*) G *is complex*, (*b*) $\chi$ *has complex values*, (*c*) $d_1 > 2$.

Since (i), (ii), ..., (v) appear elsewhere in the literature (*cf.* [3], [18]), it is left as an exercise to the reader to deduce them from (1.3).

For a character $\psi$ of an irreducible representation $\rho$ of G we define

$$\nu(\psi) = |G|^{-1} \sum_{g \in G} \psi(g^2).$$

It is a known fact that

$$\nu(\psi) = \begin{cases} 1 \text{ if } \rho \text{ is a real representation,} \\ -1 \text{ if } \rho \text{ is not real, but conjugate to } \bar{\rho}, \\ 0 \text{ otherwise, i. e. if } \psi \text{ has complex values.} \end{cases}$$

For further details we refer to [11].

We will now prove the essential parto f (vi). Suppose $\chi$ takes only real values; then $(\chi \mid \bar{\chi}) = 1$, so $d_1 = 2$. Considering the coefficient of $T^2$ in formula (1) of (1.3), we get $(\chi_\sigma^2 \mid 1) = \# \{ i \mid d_i = 2 \}$, where $\chi^2$ is the second symmetric character (in the notation of [15]). But $(\chi_\sigma^2 \mid 1) = 1/2 ((\chi \mid \bar{\chi}) + \nu(\chi)) = 1/2 (1 + \nu(\chi))$, so $1 + \nu(\chi) = 2 (\chi_\sigma^2 \mid 1) \in 2\mathbf{N}$, and therefore $\nu(\chi) = 1$. Hence G is a real reflection group. This shows that (*a*) implies (*b*). The rest of the proof of (vi) is easy.

(1.5) G is an *n*-dimensional reflection group in V. Let P be a subset of V. Put $G_P = \{ g \in G \mid gp = p \text{ for all } p \in P \}$. Then $G_P$ is a reflection sub-group of G (*see* [20]). If *m* is the dimension of the vector space spanned by P, then $G_P$ is at most $(n-m)$-dimensional. If G is reducible, then G is a direct product of reflection subgroups which are irreducible in dimension smaller than *n*. Therefore we can restrict ourselves to the determination of irreducible reflection groups. It is clear that it is only the conjugacy class of G we are interested in.

(1.6) It is well known that there exists a unitary inner product $( \mid )$ on V invariant under G; hence we may assume that G is a subgroup of U (V), the group of all unitary transformations with respect to a unitary inner product. From now on we will make this assumption. One can prove that two finite subgroups of U (V) are conjugate in U (V) if and only if they are conjugate in G *l* (V). Furthermore, G is real if and only if there is an orthonormal basis of V such that the matrix of any element in G with respect to this basis has real coefficients only.

In the sequel **U** will stand for the set of unitary complex numbers.

DEFINITIONS. — A (*unitary*) *root of a reflection* in V is an eigenvector (of length 1) corresponding to the unique nontrivial eigenvalue of the reflection.

A (*unitary*) *root of* G is a (unitary) root of a reflection in G. Let $s$ be a reflection in V of order $d > 1$; there is a nonzero vector $a \in V$ and a primitive $d$-th root of unity $\zeta$ such that, putting

(1) $$s_{a,\zeta} x = x - (1-\zeta)(a \mid a)^{-1}(x \mid a)\, a \qquad (x \in V),$$

we have $s = s_{a,\zeta}$;

(2) we will also write $s_{a,d}$ instead of $s_{a,\zeta}$ if $\zeta = \exp(2\pi i d^{-1})$, and $s_a$ instead of $s_{a,2}$.

If $t$ is any unitary transformation of V, we have the equality;

(3) $$t s_{a,\zeta}\, t^{-1} = s_{ta,\zeta}.$$

Define $o_G : V \to \mathbf{N}$ by $o_G(v) = |G_W|$ where $W = v^{\perp}$ ($v \in V$).

Then $o_G(v) > 1$ if and only if $v$ is a root of G. In this case $o_G(v)$ is the order of the cyclic group generated by the reflections in G with root $v$; if $a$ is a root of G, the number $o_G(a)$ will be called the *order* of $a$ (*with respect to* G).

In the rest of this chapter, G is a reflection group in the unitary space V [as always $G \subseteq U(V)$] with degrees $d_1, d_2, \ldots, d_n$. The following observations concerning linear characters of G are due to T. A. Springer.

(1.7) If $a$ is a nonzero element of V, we denote by $l_a$ the linear (homogeneous) polynomial defined by $l_a(x) = (x \mid a)$ ($x \in V$).

LEMMA. — *Let $a$, $b$ be roots of* G, *let $\zeta$ be a root of unity, and let $c \in \mathbf{C}^*$ be such that $s_{a,\zeta}.l_b = c l_b$. Then either $c = 1$, or $c = \zeta^{-1}$ and $a \in \mathbf{C}\, b$.*

*Proof.* — Since $l_{\bar{c}b} = s_{a,\zeta}.l_b = l_{s_{a,\zeta}b}$, we have that $b$ is an eigenvector of $s_{a,\zeta}$ with eigenvalue $c$. Therefore $c = 1, \zeta^{-1}$. If $c = \zeta^{-1}$, the dimension of the eigenspace of $s_{a,\zeta}$ corresponding to $\zeta$ is 1, so $b$ is a multiple of $a$.

(1.8) For each reflection $s$ of G we fix a unitary root $a_s$ in such a way that if $s$ and $s'$ are reflections of G with $\mathbf{U}\, a_s = \mathbf{U}\, a_{s'}$, we have $a_s = a_{s'}$.

Put $U = \{ a_s \mid s \text{ is a reflection of } G \}$ and $P = \{ \mathbf{U}\, a_s \mid s \text{ is a reflection of } G \}$. Note that G acts on P and that there is a natural map $\tau : P \to U$ such that, for $L \in P$, we have $\tau(L) = a \Leftrightarrow a \in L \cap U$.

If O is an orbit of G in P, define $f_O \in S$ by $f_O = \prod_{L \in O} l_{\tau(L)}$; moreover, define

$$\chi_O : G \to \mathbf{U} \text{ by } \chi_O(g) = \prod_{\mathbf{U} a_{s_i} \in O} (\det s_i)^{-1}$$

if $s_1, s_2, \ldots, s_r$ are reflections of G with $g = s_1 s_2 \ldots s_r$. It follows from (i) of the proposition below that $\chi_O$ is well defined.

PROPOSITION. — *Fix an orbit* $O_1$ *of* G *in* P:

(i) *if $s$ is a reflection in* G *with nontrivial eigenvalue $\zeta$, then*

$$s f_{O_1} = \begin{cases} f_{O_1} & \text{if } \mathbf{U} a_s \notin O_1, \\ \zeta^{-1} f_{O_1} & \text{if } \mathbf{U} a_s \in O_1; \end{cases}$$

(ii) $\chi_{O_1}$ *is a linear character of* G *with* $p_{\chi_{O_1}}(T) = T^{|O_1|}$;

(iii) *any linear character of* G *is the product of some* $\chi_O$ (O *orbit in* P).

*Proof*. — (i) let $L_1, L_2, \ldots, L_r$ be an orbit of $s$ in P, so $s L_i = L_{i+1}$ ($i \in \underline{r-1}$) and $s L_r = L_1$. Put $h = \prod_{i \in \underline{r}} l_{\tau(L_i)}$, and $L_{r+1} = L_1$.

Since there are $c_i \in C^*$ such that $s\,\tau(L_i) = c_i\,\tau(L_{i+1})$, we have

$$s.h = (\prod_{i \in \underline{r}} \bar{c}_i)\, h \tag{1}$$

and

$$s^r . l_{\tau(L_1)} = (\prod_{i \in \underline{r}} \bar{c}_i)\, l_{\tau(L_1)}. \tag{2}$$

If $\prod_{i \in \underline{r}} \bar{c}_i \neq 1$, then (2) and (1.7) imply that $s^r$ (and therefore $s$, too) has unitary root $a_s = \tau(L_1)$, whence $h = l_{a_s}$, and (using (1)) $s.h = \zeta^{-1} h$.

As to (iii), let $\varphi$ be a nontrivial linear character of G, and let $f \in S$ be a nonzero homogeneous polynomial of minimal degree such that $g.f = \varphi(g) f$ ($g \in G$). It is clear from (1.3) that such a polynomial exists. Suppose $s \in G$ is a reflection with $\varphi(s) \neq 1$. For any $v \in V$ with $(v \mid a_s) = 0$, we have $f(v) = f(s^{-1} v) = (s.f)(v) = \varphi(s) f(v)$, so $f(v) = 0$. Therefore $f$ must be divisible by $l_{a_s}$, and also by $l_{\tau(L)}$ for any L in the G-orbit O of U $a_s$. Hence $f$ is divisible by $f_O$. Write $f_1$ for the quotient of $f$ by $f_O$, and $\varphi_1$ for the quotient of $\varphi$ by $\chi_O$. If $f_1$ is a non-constant, then $f_1$ is a nonzero homogeneous polynomial of minimal degree such that $g.f_1 = \varphi_1(g) f_1$ ($g \in G$), and of degree strictly lower than the degree of $f$. Induction finishes the proof.

Finally, (ii) is a consequence of the preceding.

(1.9) $o_G(\tau(L))$ being independent of the choice of $L \in O$ for a given G-orbit O of P, we will also write $o_G(O)$ for this number.

COROLLARY. — $G/[G, G]$ *is a direct product of cyclic groups of orders* $o_G(O)$, O *running through the* G-*orbits in* P.

The proof follows from $G/[G, G] \cong \mathrm{Hom}(G, U)$ and the fact that if $O_1, O_2, \ldots, O_m$ are different orbits in P, and $\lambda_1, \lambda_2, \ldots, \lambda_m$ are integers such that $\chi_{O_1}^{\lambda_1} \chi_{O_2}^{\lambda_2} \ldots \chi_{O_m}^{\lambda_m} = 1$, then $\chi_{O_1}^{\lambda_1} = \chi_{O_2}^{\lambda_2} \ldots = \chi_{O_m}^{\lambda_m} = 1$.

(1.10) DEFINITIONS. — A *vector* $v \in V$ is called *regular* with respect to G, if $(v \mid a) \neq 0$, for any root $a$ of G [in other words, if no reflection of G fixes $v$, or equivalently, if $G_{\{v\}} = 1$, *cf.* (1.5)]. A *transformation* $g \in G$ is called *regular* if $g$ has a regular eigenvector. The *regular degrees* of G form the set of numbers, minimal with respect to inclusion, such that the order of any regular element of G is a divisor of an element in this set and, vice-versa, any divisor of an element in this set is the order of a regular element of G. Of the many interesting properties of regular elements we will only mention a few. For their proofs and other properties we refer to [18].

THEOREM. — *Let* $\zeta$ *be a primitive d-th root of unity. Let* $g \in G$ *be regular with regular eigenvector* $v \in V$ *and related eigenvalue* $\zeta$. *Denote by* W *the eigenspace* $\{ x \in V \mid gx = \zeta x \}$ *of* $g$ *in* V. *Then:*

(i) $d$ *is the order of* $g$; *moreover,* $g$ *has eigenvalues* $\zeta^{1-d_1}, \zeta^{1-d_2}, \ldots, \zeta^{1-d_n}$;

(ii) $\dim W = \# \{ i \mid d \text{ is divisor of } d_i \}$;

(iii) *the restriction to* W *of the centralizer of* $g$ *in* G *defines an isomorphism onto a reflection group in* W, *whose degrees are the* $d_i$ *divisible by* $d$ *and whose order is* $\prod_{d \mid d_i} d_i$;

(iv) *the conjugacy class of* $g$ *consists of all elements of* G *having* dim W *eigenvalues* $\zeta$.

## 2. The imprimitive case

V is the unitary space $\mathbf{C}^n$ equipped with standard unitary inner product $(\ |\ )$, and S is as in (1.2). The standard basis of V is denoted by $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$. In accordance with (1.6), a reflection (group) is always assumed to be unitary.

(2.1) DEFINITIONS. — A group G of unitary automorphisms of V is called *imprimitive* if V is a direct sum $V = V_1 \oplus V_2 \oplus \ldots \oplus V_t$ of nontrivial proper linear subspaces $V_i$ $(1 \leqq i \leqq t)$ of V such that $\{ V_i \mid i \in \underline{t} \}$ is invariant under G. In this situation the family $(V_i)_{1 \leqq i \leqq t}$ is called a *system of imprimitivity* for G. If such a direct splitting of V does not exist, G is called *primitive* (*cf.* [7], [9]).

A polynomial $p \in S$ is called *semi-invariant* with respect to G if it affords a linear character $\varepsilon$ of G, i. e. if $g.p = \varepsilon(g) p$ for any $g \in G$.

(2.2) PROPOSITION. — *Let* G *be an irreducible imprimitive reflection group in* V $(n \geqq 2)$, *and let* $(V_i)_{1 \leqq i \leqq t}$ *be a system of imptimitivity for* G. *Then:*

(i) $\dim V_i = 1$ *for each* $i \in \underline{t}$, *and* $t = n$; *there are distinct linear homogeneous polynomials* $l_1, l_2, \ldots, l_n$ (*not even equal up to a constant factor*) *such that* $l_1, l_2, \ldots, l_n$ *is a semi-invariant homogeneous polynomial of degree n in* S;

(ii) *for any reflection* $s \in G$ *we have either* $s V_i = V_i$ *for all* $i \in \underline{n}$, *or there are* $i \neq j$ $(1 \leqq i, j \leqq n)$ *such that any root of* $s$ *is contained in* $V_i + V_j$, $s V_i = V_j$, $s V_k = V_k$ *for all* $k \neq i, j$, *and* $s$ *is of order* 2;

(iii) *let* $\psi : G \to S_n$ *be the homomorphism that assigns to* $g \in G$ *the permutation* $\sigma \in S_n$ *defined by* $g V_i = V_{\sigma(i)}$ *for any* $i \in \underline{n}$. *Then* $\psi$ *is surjective and admits a section* $\tau : S_n \to G$, *which is a homomorphism;*

(iv) $V_i \perp V_j$ *for all* $i, j$ $(i \neq j,\ 1 \leqq i, j \leqq n)$;

(v) *if* $v$ *is a unitary root of* G *of order* 2, *and* $w$ *is a unitary root of* G *of order* $> 2$, *then* $|(v \mid w)| \in \{ 0, 2^{-1/2} \}$.

*Proof:*

(i) let $i$ be such that $\dim V_i > 1$; because G is irreducible, there is a $j \neq i$ and a reflection $\in G$ such that $s V_i = V_j$. It follows that $\dim (V_j \cap V_i) > 0$, contradicting $V_i \cap V_j = \{ 0 \}$. To establish the last part of (i), fix unitary $a_i \in V_i$, and take for $l_i$ the linear homogeneous polynomial with $l_i (a_i) = 1$ and $l_i (a_j) = 1$ for $j \neq i$;

(ii) let $s \in G$ be a reflection with unitary root $a$ and nontrivial eigenvalue $\zeta$ such that $(a \mid V_1) \neq 0$ and $a \in V_1$. Then (up to a permutation of indices) $s V_1 = V_2$. Take $0 \neq x_i \in V_i$ $(i = 1, 2)$ such that $s x_1 = x_2$. There is a $j \in \underline{n}$ with $s^2 x_1 \in V_j$; thus

$$s^2 x_1 \in (\mathbf{C}\, a + \mathbf{C}\, x_1) \cap V_j = (V_1 + V_2) \cap V_j.$$

This implies that $j = 1$ or 2. Since $a \notin V_1 \cup V_2$, we have $s^2 x_1 = x_1$ and $\zeta^2 = 1$; so $\zeta = -1$ and $s$ is of order 2. Furthermore, the root $a$ is a scalar multiple of $x_1 - x_2$, in particular $a \in V_1 + V_2$. If $(a \mid V_i) \neq 0$ for some $i > 2$, there is a $j > 2$ such that $s V_i = V_j$; thus $a \in (V_i + V_j) \cap (V_1 + V_2) = \{ 0 \}$, which is impossible. So $(a \mid V_i) = 0$ for any $i > 2$. In particular $s V_k = V_k$ for $k > 2$, and (ii) is proved;

(iii) the irreducibility of G implies that for each $j > 1$ there exists a reflection $s_j$ with $s_j V_1 = V_j$ (necessarily of order 2). The image of $s_j \in G$ under $\psi$ is the element $(1\, j) \in S_n$ [according to (ii)]. It is known that the set $\{ (1\, j) \mid j = 2, 3, \ldots, n \}$ generates $S_n$. Finally, not that the restriction of $\psi$ to the reflection subgroup $\langle s_2, s_3, \ldots, s_n \rangle$ of G is an isomorphism;

(iv) note that $(x, y) \mapsto \sum_{i=1}^{n} l_i (x) \overline{l_i (y)}$ [where $l_1, l_2, \ldots, l_n$ are as in (i)] defines a unitary inner product on V, fixed by G. Since such a unitary inner product must be a striclty positive scalar multiple of $( \mid )$, the required result is readily deduced;

(v) is clear from (ii) and (iv).

(2.3) If we do not mention an explicit basis, we shall always identify a linear transformation of $\mathbf{C}^n$ with its matrix with respect to the standard basis.

Let $\Pi_n$ be the group of all $n \times n$-permutation matrices; let $A(m, p, n)$, where $p \mid m$ $(m, p \in \mathbf{N})$, be the group of all $n \times n$-matrices $(a_{ij})_{1 \leqq i,\, j \leqq n}$ such that $a_{ij} = \theta_i \delta_{ij}$, where $\theta^m = 1$ for each $i \in \underline{n}$, and $(\det (a_{ij}))^{m/p} = 1$. Then $\Pi_n$ normalizes $A(m, p, n)$. Define $G(m, p, n) = A(m, p, n) \Pi_n$; this is a semi-direct product. It is not hard to see that $G(m, p, n)$ is an imprimitive reflection group in $\mathbf{C}^n$, with system of imprimitivity $(\mathbf{C}\, \varepsilon_i)_{i \in \underline{n}}$.

(2.4) THEOREM. — *Let $n \geqq 2$, and let G be an irreducible imprimitive reflection group in V. Then G is conjugate to $G(m, p, n)$ for some $m, p \in \mathbf{N}$ with $p \mid m$. Furthermore, $G(m, p, n)$ is irreducible if and only if $m > 1$ and $(m, p, n) \neq (2, 2, 2)$. [By conjugacy we mean conjugacy within* U (V).]

*Proof.* — Let G be as stated. There is an orthonormal basis $e_1, e_2, \ldots, e_n$ with the properties that the $V_i = \mathbf{C}_i\, e_i$ $(1 \leqq i \leqq n)$ form a system of imprimitivity for G, and that for each $j > 1$ there is a reflection $s_j \in G$ such that $s e_1 = e_j$ [*cf.* (2.2)]. Without changing the conjugacy class of G we may put $e_i = \varepsilon_i$ (i. e. $e_1, e_2, \ldots, e_n$ is the standard basis). It follows from (2.2) that $\Pi_n$ is a subgroup of G. Let $q$ be the order of the cyclic group generated by the reflections that leave $V_1^\perp$ pointwise fixed (*so* $q = o_G(e_1)$ with notations of (1.6)). Then $A(q, 1, n)$ is a subgroup of G.

According to (2.2), the only reflections outside $A(q, 1, n).\Pi_n$ are the $s' \in G$ with $s' e_i = \theta e_j$ where $\theta \in U \setminus \{1\}$ and $i \neq j$, and $s' e_k = e_k$ for $k \neq i, j$. Up to conjugacy by an element of $\Pi_n$, we may take $i = 1$ and $j = 2$. Let $s = s_2 \in G$ be the reflection with $se_1 = e_2$. Then $(ss') e_1 = \theta e_1$ and $(ss') e_2 = \theta^{-1} e_2$, so $\theta$ is a root of unity. Let $m$ be the maximum of all orders of elements $st \in G$ where $t$ is a reflection such that $t V_1 = V_2$, It is not difficult to see that $A(m, m, n)$ is a subgroup of G, commuting with $A(q, 1, n)$. and that $q \mid m$. Putting $p = q^{-1}.m$, we have $A(m, p, n) = A(q, 1, n).A(m, m, n)$; so $G(m, p, n) = A(m, p, n)\Pi_n$ is a subgroup of G. Since all reflections of G are contained in this subgroup, the subgroup must be equal to G itself, in other words $G = G(m, p, n)$.

In order to prove the second statement of the theorem, suppose that $G = G(m, p, n)$ leaves invariant a nontrivial proper linear subspace W of V. Since W is also a $\Pi_n$-invariant subspace of V, we know from [15; p. 29, 30], for instance, that $W = \mathbf{C}(\varepsilon_1+\varepsilon_2+\ldots+\varepsilon_n)$ up to an interchange of W and $W^\perp$. As $A(m, p, n)$ stabilizes $\mathbf{C}(\varepsilon_1+\varepsilon_2+\ldots+\varepsilon_n)$, all diagonal coefficients of an element in $A(m, p, n)$ must be equal. It is not hard to deduce from this that $(m, p, n) \in \{(1, 1, n), (2, 2, 2)\}$.

On the other hand, it is obvious that $G(1, 1, n)$ and $G(2, 2, 2)$ are reducible in V.

(2.5) *Remarks:*

(i) $G(m, m, 2)$ is conjugate to $W(I_2(m))$, the Coxeter group corresponding to type $I_2(m)$ (notation of [3]). This group is reducible only if $m \leqq 2$. For the other Coxeter groups the notation will be similar;

$G(1, 1, n) = \Pi_n$, operating on the hyperplane $(\varepsilon_1+\varepsilon_2+\ldots+\varepsilon_n)^\perp$ of $\mathbf{C}^n$, represents $W(A_{n-1})$ $(n \geqq 2)$.

$G(2, 1, n)$ represents $W(B_n) = W(C_n)$ $(n \geqq 2)$.

$G(2, 2, n)$ represents $W(D_n)$ $(n \geqq 3)$.

$W(A_n)$ is primitive if $n > 2$, and $W(A_2)$ is conjugate to $W(I_2(3))$.

The above groups form the set if all real reflection groups appearing in (2.4);

(ii) let $X_1, X_2, \ldots, X_n \in S$ be as in (1.11) (ii). The first $n-1$ elementary symmetric polynomials in $(X_i^m)_{i \in \underline{n}}$ (i. e. $X_1^m+X_2^m+\ldots+X_n^m$, $\sum_{i<j} X_i^m X_j^m, \ldots, \sum_{i=1}^{n} \prod_{j \neq i} X_j^m$) and $(X_1 X_2 \ldots X_n)^q$, where $q = p^{-1}.m$, form a set of $G(m, p, n)$-invariant homogeneous algebraically independent polynomials; the product of their degrees equals

$$m.2m\ldots(n-1)m.qn = q.m^{n-1}.n! = p^{-1}.m^n.n! = |G(m, p, n)|.$$

By (1.2) the degrees of $G(m, p, n)$ are $m, 2m, \ldots, (n-1)m, qn$. One of the consequences is that $|Z(G(m, p, n))| = q.gcd(p, n)$. Finally, $X_1 X_2 \ldots X_n$ is the semi-invariant associated with the canonical system of imprimitivity;

(iii) $G(m, m, n)$ and $G(p^{-1} m, 1, n)$ are reflection subgroups of $G(m, p, n)$;

(iv) $G(4, 4, 2)$ is conjugate to $G(2, 1, 2)$. These two groups form the only pair of conjugates in the set of all irreducible $G(m, p, n)$, as can be seen with the help of the invariant polynomials of (ii);

(v) if $p = 1$ or $m$, it is possible to choose $n$ generating reflections for $G(m, p, n)$: take the reflections of order 2 with roots $\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \ldots, \varepsilon_{n-1} - \varepsilon_n$, and

$$\begin{cases} \text{the reflection of order 2 with root } (\varepsilon_1 - \exp(2\pi i m^{-1})\varepsilon_2) & \text{if } p = m, \\ \text{a reflection of order } m \text{ with root } \varepsilon_1 & \text{if } p = 1. \end{cases}$$

If $p \neq 1$, $m$ take $n$ generating reflections for $G(m, m, n)$ and an additional reflection of order $p^{-1} m$ with root $\varepsilon_1$ to obtain $n+1$ generating reflections for $G(m, p, n)$;

(vi) put $G = G(m, p, n)$ and, as always, $q = p^{-1} m$. Let P be as in (1.8). Suppose $n = 2$; P consists of $gcd(2, m)(gcd(2, q))^{-1}$ G-orbits of length $m.gcd(2, q).(gcd(2, m))^{-1}$ and, if $p \neq m$, of one more G-orbit, in fact $\{U\varepsilon_1, U\varepsilon_2\}$ of length 2. If $n > 2$, then G admits one orbit in P of length $1/2\, mn(n-1)$; if, moreover, $p = m$, this is the single orbit in P; if $p \neq m$, P contains one more orbit, of length $n$;

(vii) if $q$ is even or $m$ is odd, then the $f_0$ of $G(m, p, 2)$ are $X_1^m - X_2^m$ and, unless $p = m$, $X_1 X_2$. If $q$ is odd and $m = 2k$, then the $f_0$ of $G(m, p, 2)$ are $X_1^k - X_2^k$, $X_1^k + X_2^k$ and, unless $p = m$, $X_1 X_2$. Finally, if $n > 2$, then the $f_0$ of $G(m, p, n)$ are $\prod_{i<j} (X_i^m - X_j^m)$ and, unless $p = m$, $X_1 X_2 \ldots X_n$.

(2.6) LEMMA. — *Let* G *be an irreducible reflection group in* V. *If* G *has a reflection subgroup which is primitive in dimension* $r > 1$ *and not conjugate to* $W(A_r)$, *then* G *itself is primitive.*

*Proof.* — Let H be a reflection subgroup of G as described in the assumptions, and denote by W the orthogonal complement of $V^H$. We may assume that $r < n$.

Suppose that G is imprimitive with system of imprimitivity $L_1, L_2, \ldots, L_n$. Since $\dim W = r$, we have that H is primitive, and therefore irreducible, in W. If $L_i \subseteq W$ for some $i \in \underline{n}$, then $HL_i$ spans W, so the $L_j$ with $L_j \subseteq W$ form a system of imprimitivity for H in W, unless $W = L_i$; but $W = L_i$ would imply that $r = 1$ [because of (2.2) (i)], which is assumed to be false. Thus

$$L_j \nsubseteq W \quad \text{for all } j \in \underline{n}. \tag{1}$$

Let $s \in H$ be a reflection with root $a \in V$. Note that $a \in W$. If $s L_i = L_i$ for every $i \in \underline{n}$, then $a \in L_j$ for some $j \in \underline{n}$, and $L_j \subseteq W$, which is impossible because of (1); so $s L_1 = L_2$ (up to a permutation of indices), and $s$ is of order 2 [*see* (2.2) (ii)]. If $a' \in V$ is a root of another reflection $s'$ of H such that $s' L_1 = L_2$, then [by (2.2) (ii)] $\mathbf{C}\, a + \mathbf{C}\, a' = L_1 + L_2$; since $a, a' \in W$, this yields $L_1, L_2 \subseteq W$, contradicting (1).

We conclude that there are no reflections $s \in H$ such that $s L_i = L_i$ for any $i \in \underline{n}$, and that for $i, j \in \underline{n}$ $(i \neq j)$ there is at most one reflection $s \in H$ with $s L_i = L_j$. By now, it is obvious from (2.2) (iii) that there exists $t \in \underline{n}$ such that H is conjugate to G (1, 1, $t$). Because H is $r$-dimensional, we have that $t = r+1$, and that H is conjugate to W ($A_r$).

(2.7) LEMMA. — *Suppose* $G = G(m, p, n)$ *is irreducible* ($p \mid m$ *and* $n \geqq 2$). *Then* G *has a unique system of imprimitivity that* $(m, p, n) \notin \{ (2, 1, 2), (4, 4, 2), (3, 3, 3), (2, 2, 4) \}$.

*Proof.* — The $L_i = C\, \varepsilon_i$ ($i \in \underline{n}$) constitute a system of imprimitivity for G ($m, p, n$). Let P be as in (1.8). First of all, we will pay attention to the case that an orbit of P gives rise to another system of imprimitivity. By (2.5) (vi), we then have either

(1) $$n = 2 = m\, gcd(2, q).(gcd(2, m))^{-1},$$

or

(2) $$n > 2 \qquad \text{and} \qquad mn(n-1) = 2n.$$

If (1) occurs, we have $(m, p, n) \in \{ (2, 1, 2), (4, 4, 2) \}$; (2) leads to a contradiction with $m > 1$ [*cf.* (2.4)]. The conclusion is that none of the groups G in question has a system of imprimitivity afforded by roots different from the canonical system.

Let us assume that $V_1, V_2, \ldots, V_n$ is a system of imprimitivity different from $L_1, L_2, \ldots, L_n$ and not corresponding to an orbit of P. Let $l_1, l_2, \ldots, l_n$ be defined with respect to $V_1, V_2, \ldots, V_n$ as in the proof of (2.2) (i), and put $f = l_1, l_2, \ldots, l_n$. Suppose that $f$ is a semi-invariant but not an invariant. It follows, by an argument similar to the one on the proof of (1.8) (iii), that $f$ is the product of an invariant and some $f_O$ (O orbit in P). Since deg ($f$) = $n$, the irreducibility of G implies that there is an orbit O in P of length $n$ such that $f = f_O$; this is contradictory to our assumption. Therefore $f$ is an invariant homogeneous polynomial. Because $f \notin C.X_1 X_2 \ldots X_n$, there must be an $\alpha \in C$ such that $f - \alpha X_1 X_2 \ldots X_n$ is a nonzero homogeneous G-invariant polynomial in $X_1^m, X_2^m, \ldots, X_n^m$ [*cf.* (2.5) (ii)]. Hence $m$ divides $n$.

Put

$$l_1 = \alpha_1 X_1 + \alpha_2 X_2 + \ldots + \alpha_n X_n, \qquad r_j = \# \{ i \in \underline{n} \mid \alpha_i = \pi_j \} \quad \text{for } j \in \underline{n},$$

and $r_0 = \# \{ i \in \underline{n} \mid \alpha_i \neq 0 \}$. Let $j \in \{ 0, 1, 2, \ldots, n \}$. Since the stabilizer in $\Pi_n$ of $C\, l_1$ is of order $\leqq r_j\,!\,(n-r_j)\,!$, and since the $\Pi_n$-orbit of $C\, l_1$ has at most $n$ elements, we have

(3) $$n \geqq n!.(r_j!)^{-1}.((n-r_j)!)^{-1} = \binom{n}{r_j}.$$

This implies that $r_j = 1, n-1, n$ for any $j \in \{ 0, 1, \ldots, n \}$.

Note that $r_0 \neq 1$. Suppose $r_0 = n-1$. Using a combinatorial argument, we get that the stabilizer of $C\, l_1$ in G ($m, p, n$) is of order $\leqq mq\,(n-1)\,!$, whence

$$n \geqq m^{n-1}.q.n!.(m.q.(n-1)!)^{-1} = m^{n-2}.n;$$

so $n \leqq 2$, and $l_1 \in C\, X_1$, contrary to the assumption that $V_1, V_2, \ldots, V_n$ is different from $L_1, L_2, \ldots, L_n$. We conclude that $r_0 = n$. The order of the $C\, l_1$-stabilizer in

$G(m, p, n)$ is smaller than or equal to $m.n\,!$, so $n \geqq m^{n-1}.q.n\,!.(m.n\,!)^{-1} = m^{n-2}.q$. Together with $m \mid n$, this implies that $(m, p, n) \in \{(2, 1, 2), (2, 2, 4), (3, 3, 3)\}$.

(2.8) *Remark.* — To $G(2, 1, 2)$ correspond, apart from the canonical one, two systems of imprimitivity, namely $\mathbf{C}(\varepsilon_1+\varepsilon_2)$, $\mathbf{C}(\varepsilon_1-\varepsilon_2)$ with semi-invariant $X_1^2-X_2^2$, and $\mathbf{C}(\varepsilon_1+i\,\varepsilon_2)$, $\mathbf{C}(\varepsilon_1-i\,\varepsilon_2)$ with invariant $X_1^2+X_2^2$ [compare (1.11) (iii)]. To $G(3, 3, 3)$ corresponds $\mathbf{C}(\varepsilon_1+\varepsilon_2+\varepsilon_3)$, $\mathbf{C}(\varepsilon_1+\omega\varepsilon_2+\omega^2\,\varepsilon_3)$, $\mathbf{C}(\varepsilon_1+\omega^2\,\varepsilon_2+\omega\varepsilon_3)$ with invariant $X_1^3+X_2^3+X_3^3-3\,X_1\,X_2\,X_3$.

To $G(2, 2, 4)$ correspond $\mathbf{C}(\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4)$, $\mathbf{C}(\varepsilon_1+\varepsilon_2-\varepsilon_3-\varepsilon_4)$, $\mathbf{C}(\varepsilon_1-\varepsilon_2+\varepsilon_3-\varepsilon_4)$, $\mathbf{C}(\varepsilon_1-\varepsilon_2-\varepsilon_3+\varepsilon_4)$ with invariant

$$X_1^4+X_2^4+X_3^4+X_4^4-2(X_1^2X_2^2+X_1^2X_3^2+X_1^2X_4^2+X_2^2X_3^2+X_2^2X_4^2+X_3^2X_4^2)+8\,X_1\,X_2\,X_3\,X_4,$$

and the system that can be obtained from the preceding one by substitution of $-\varepsilon_1$ for $\varepsilon_1$ with invariant that can be obtained from the preceding one by substitution of $-X_1$ for $X_1$.

Moreover one can prove that these are *all* non-canonical systems of imprimitivity in the respective cases.

(2.9) Proposition. — *Let* $1 < m < n$, *let* G *be a primitive reflection group in* V, *and let* H *be an imprimitive irreducible m-dimensional* (*i. e. imprimitive irreducible in dimension m as defined in* (1.1)) *reflection subgroup of* G. *Suppose, moreover, that* H *has a unique system of imprimitivity* $L_1, L_2, \ldots, L_m$ *in* $(V^H)^\perp$. *Then* G *contains a reflection* $s_0$ *such that* $\langle H, s_0 \rangle$ *is a primitive* $(m+1)$*-dimensional reflection subgroup of* G.

*Proof.* — Put $W = (V^H)^\perp$. Now $\dim W = m$; the proof goes by induction with respect to $n-m$.

Suppose $m = n-1$. Put $L_n = V^H$. Note that $L_1, L_2, \ldots, L_n$ form a system of imprimitivity for H in V. The required result is a direct consequence of the observation that this is the only system of imprimitivity for H in V consisting of 1-dimensional linear subspaces. In order to prove this observation, let $V_1, V_2, \ldots, V_n$ be another such system. Reasoning as in the proof of (2.6), we obtain that either H is conjugate to $G(1, 1, n)$ or there is a $j \in \underline{n}$ with $V_j \subset W$. As $G(1, 1, n)$ is primitive in dimension $n-1$, we must have that $V_i \subset W$ for all but one $i \in \underline{n}$; the uniqueness of $L_1, L_2, \ldots, L_n$ readily follows.

Suppose $m < n-1$. Assume that there is no $s_0$ as required. Let $s$ be a reflection in G with unitary root $a$ such that $a \notin W \cup W^\perp$. Note that $\langle H, s \rangle$ is $(m+1)$-dimensional. Now $\langle H, s \rangle$ is irreducible and imprimitive in $W+s\,W$. Furthermore, $\langle H, s \rangle$ has a unique system of imprimitivity in $W+s\,W$, namely $L_1, L_2, \ldots, L_m, L_{m+1} = W^\perp \cap s\,W$. Choose unitary vectors $a_i \in L_i$ ($i \in \underline{m+1}$), and permute the indices to obtain $sa_i = a_i$ ($i \in \underline{m-1}$) and $sa_m = a_{m+1}$. Clearly, $s$ is of order 2. Application of the induction hypothesis to $\langle H, s \rangle$ provides a reflection $s' \in G$ with unitary root, say, $b \in V$ such that $\langle H, s, s' \rangle$ is primitive in $W' = W+s\,W+s'\,W+s'\,s\,W$.

Let $a_{m+2}$ be a unitary vector in $W' \cap (W+s\,W)^\perp$. There is an $i \in \underline{m+1}$ with $s'\,a_i \notin W+s\,W$; because the $L_j$ ($j \in \underline{m+1}$) form a single $\langle H, s \rangle$-orbit, there are $g \in \langle H, s \rangle$ and $\alpha \in \mathbf{U}$ with $ga_i = \alpha\,a_{m+1}$. Replacing $s'$ by $gs'\,g^{-1}$, we see that the assumption

$s' a_m \notin W+s W$ does not harm the generality. Because $\langle H, s' \rangle$ is imprimitive in dimension $m+1$, we may assume that there are $\lambda, \mu \in \mathbf{C}$ with $b = 2^{-1/2}(a_m - \lambda a_{m+1} - \mu a_{m+2})$. The imprimitivity of $\langle H, ss's \rangle$ in dimension $m+1$ implies that

$$1 - |\lambda|^2 = |(ss'sa_m \mid a_m)| \in \{0, 1\}.$$

Therefore $|\lambda| = 0, 1$, and $b = 2^{-1/2}(a_m - \lambda a_{m+1})$ or $2^{-1/2}(a_m - \mu a_{m+2})$, contradicting the primitivity of $\langle H, s, s' \rangle$ in dimension $m+2$.

(2.10) *Remark.* — The knowledge of the orders of regular elements is useful for the determination of conjugacy classes and character tables of the reflection groups. Since much of this is contained in [1], we will not pursue this matter here beyond the presentation of the regular degrees.

(2.11) PROPOSITION. — *The regular degrees of* $G(m, p, n)$ *with* $p \mid m$, $m > 1$, $n > 2$ *are*

$$\begin{cases} (n-1)m,\ n & \text{if } p = m \text{ and } n \nmid m, \\ (n-1)m & \text{if } p = m \text{ and } n \mid m, \\ mn/p & \text{if } p \neq m. \end{cases}$$

(2.12) LEMMA. — *Let* $G \subseteq Gl_n(\mathbf{R})$ *be a finite irreducible group, and let* $t \in Gl_n(\mathbf{C})$ *be such that* $tGt^{-1} = G$. *Then there are* $\eta \in \mathbf{C}$ *and* $u \in Gl_n(\mathbf{R})$ *such that* $t = \eta . u$.

*Proof.* — Let $\zeta \in \mathbf{C}$ be an eigenvalue of $t^{-1}\bar{t}$ corresponding to the eigenvector $w \in \mathbf{C}^n$. Let $\eta \in \mathbf{C}$ be such that $\eta^{-2} = \zeta$. Put $u = \eta^{-1} t$. Then $\bar{u}w = uw$, so

$$W = \{ x \in \mathbf{C}^n \mid \bar{u}x = ux \}$$

is a nonzero G-invariant subspace of $\mathbf{C}^n$. Therefore $W = \mathbf{C}^n$, whence $\bar{u} = u$.

We will look for all finite subgroups H in $U_n(\mathbf{C})$ that H normalizes a reflection group G in case $n \geqq 3$ (the case $n = 2$ can easily be handled without use of the specific properties of reflection groups). In view of the previous lemma, the results for a real reflection group G are to be found in [3] (p. 232, ex. 16).

(2.13) Put $\mu_\infty = \bigcup_{n=1}^{\infty} \mu_n$ for the set of all roots of unity in $\mathbf{C}$.

PROPOSITION. — *Suppose* $n \geqq 2$ *and let* $G(m, p, n)$ *be irreducible. Let* $H \subseteq U_n(\mathbf{C})$ *be a finite group such that* $G(m, p, n) \trianglelefteq H$. *If*

$$(m, p, n) \notin \{(2, 1, 2), (3, 3, 3), (2, 2, 4)\} \quad \textit{then} \quad H \subseteq \mu_\infty . G(m, 1, n).$$

*If*

$$(m, p, n) = (3, 3, 3) \quad \textit{then} \quad H \subseteq \mu_\infty . W(M_3);$$

*if*

$$(m, p, n) = (2, 2, 4) \quad \textit{then} \quad H \subseteq \mu_\infty . W(F_4).$$

*Proof.* – Let $t$ be a unitary transformation of $\mathbf{C}^n$ normalizing $G(m, p, n)$. Note that $t$ transforms one system of imprimitivity of $G(m, p, n)$ into another.

Suppose $t$ leaves invariant the canonical system of imprimitivity $(\mathbf{C}\,\varepsilon_i)_{1 \leq i \leq n}$ of $G(m, p, n)$. One easily sees that $t$ has a diagonal matrix after replacement of $t$ by a suitable element of $t\,G(1, 1, n)$. Let $\eta \in \mathbf{U}$ be such that $\eta^{-1}\,t\,\varepsilon_1 = \varepsilon_1$, and let $j > 1$. As $\eta^{-1}\,t(\varepsilon_1 - \varepsilon_j) = \varepsilon_1 - \eta^{-1}\,t\,\varepsilon_j$ is a root of $G(m, p, n)$, it follows that $\eta^{-1}\,t\,\varepsilon_j \in \mathbf{Q}(e^{2\pi i/m})\,\varepsilon_j$. Hence all coefficients of $\eta^{-1}\,t$ are in $\mathbf{Q}(e^{2\pi i/m})$, and $\eta^{-1}\,t \in G(m, 1, n)$.

This settles the proposition in the case where $G(m, p, n)$ has only one system of imprimitivity [*cf.* (2.7)], including the case $(m, p, n) = (4, 4, 2)$.

Thanks to (2.8), one immediately finds all possible cosets $t.\mu_\infty\,G(m, 1, n)$ in the remaining cases [recall that $G(2, 1, 2)$ is conjugate to $G(4, 4, 2)$].

## 3. The two-dimensional case

$V = \mathbf{C}^2$ with standard unitary inner product.

In this chapter we shall identify $\mu_m$ and $\mu_m.I_2$ for $m \in \mathbf{N}$.

(3.1) We present a description of all finite subgroups in $G\,l_2(\mathbf{C})$ (*cf.* [10], [13]).

Let H, K be finite subgroups of $S\,l_2(\mathbf{C})$ such that $K \lhd H$ and $H/K$ is a cyclic group of order $w$ and assume an isomorphism $\varphi : \mu_{wd}/\mu_d \to H/K$ is given.

Some definitions: $Z\,l_2(\mathbf{C}) = Z(G\,l_2(\mathbf{C}))$ is the center of $G\,l_2(\mathbf{C})$,

$$\psi : Z\,l_2(\mathbf{C}) \times S\,l_2(\mathbf{C}) \to G\,l_2(\mathbf{C})$$

is the usual product map,

$$\mu_{wd} \times_\varphi H = \{(m, s) \in \mu_{wd} \times H \mid \varphi(m\,\mu_d) = s\,K\}$$

and

$$(\mu_{wd} \mid \mu_d;\, H \mid K)_\varphi = \psi(\mu_{wd} \times_\varphi H).$$

The latter group is a finite subgroup of $G\,l_2(\mathbf{C})$. Every finite subgroup G of $G\,l_2(\mathbf{C})$ can be gotten in this way: put

$$\mu_{wd} = G.S\,l_2(\mathbf{C}) \cap Z\,l_2(\mathbf{C}), \qquad \mu_d = G \cap Z\,l_2(\mathbf{C}),$$
$$H = S\,l_2(\mathbf{C}) \cap G.Z\,l_2(\mathbf{C}), \qquad K = S\,l_2(\mathbf{C}) \cap G,$$

and let $\varphi : \mu_{wd}/\mu_d \to H/K$ be the composition of the natural isomorphisms:

$$\mu_{wd}/\mu_d \to (Z\,l_2(\mathbf{C}) \cap G.S\,l_2(\mathbf{C}))\,G/G = (G.Z\,l_2(\mathbf{C}) \cap S\,l_2(\mathbf{C}))\,G/G \to H/K.$$

Conjugation of G does not alter $\mu_{wd}$ and $\mu_d$, and changes H and K into conjugates by the same element.

For each conjugacy class of finite subgroups of $S\,l_2(\mathbf{C})$ we fix a representing element (*cf. loc. cit.*):

the cyclic group of order $m$:

$$\mathbf{C}_m = \left\langle \begin{pmatrix} e^{2\pi i m^{-1}} & 0 \\ 0 & e^{-2\pi i m^{-1}} \end{pmatrix} \right\rangle,$$

the binary dihedral group of order $4m$:

$$\mathbf{D}_m = \left\langle \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \mathbf{C}_{2m} \right\rangle,$$

the binary tetrahedral group of order 24:

$$\mathbf{T} = \left\langle \frac{1}{\sqrt{2}} \begin{pmatrix} \varepsilon & \varepsilon^3 \\ \varepsilon & \varepsilon^7 \end{pmatrix}, \mathbf{D}_2 \right\rangle,$$

the binary octahedral group of order 48:

$$\mathbf{O} = \left\langle \begin{pmatrix} \varepsilon^3 & 0 \\ 0 & \varepsilon^5 \end{pmatrix}, \mathbf{T} \right\rangle,$$

the binary icosahedral group of order 120:

$$\mathbf{I} = \left\langle \frac{1}{\sqrt{5}} \begin{pmatrix} \eta^4-\eta & \eta^2-\eta^3 \\ \eta^2-\eta^3 & \eta-\eta^4 \end{pmatrix}, \frac{1}{\sqrt{5}} \begin{pmatrix} \eta^2-\eta^4 & \eta^4-1 \\ 1-\eta & \eta^3-\eta \end{pmatrix} \right\rangle,$$

Where $\varepsilon = \exp(\pi i 4^{-1})$, $\eta = \exp(2\pi i 5^{-1})$.

The choice of the representing elements is such that each group is in $SU_2(\mathbf{C})$ and such that $\mathbf{C}_{2m} \lhd \mathbf{D}_m \lhd \mathbf{D}_{2m}$ and $\mathbf{D}_2 \lhd \mathbf{T} \lhd \mathbf{O}$. Apart from $\mathbf{C}_m \lhd \mathbf{C}_{mr}$, these are the only normal inclusions with cyclic quotient. Thanks to this observation, it is readily checked that if H is not cyclic, the conjugacy class of $G = (\mu_{wd} \mid \mu_d; H \mid K)_\varphi$ is independent of the choice of $\varphi$. In that case we shall drop the index $\varphi$ and write $(\mu_{wd} \mid \mu_d; H \mid K)$ for G.

Note that G is irreducible if and only if H is non-cyclic. By now it is not hard to prove the following well-known

THEOREM. — *Any irreducible finite subgroup of* $G\,l_2(\mathbf{C})$ *is conjugate to one of the following subgroups of* $U_2(\mathbf{C})$:

$$(\mu_{4q} \mid \mu_{2q}; \mathbf{D}_m \mid \mathbf{C}_{2m}), \qquad \mu_{2m}\cdot\mathbf{T},$$
$$(\mu_{4q} \mid \mu_{2q}; \mathbf{D}_{2m} \mid \mathbf{D}_m) \qquad (\mu_{6m} \mid \mu_{2m}; \mathbf{T} \mid \mathbf{D}_2),$$
$$(\mu_{2q} \mid \mu_{2q}; \mathbf{D}_m \mid \overline{\mathbf{D}}_m) = \mu_{2q}\cdot\mathbf{D}_m, \qquad \mu_{2m}\cdot\mathbf{O},$$
$$(\mu_{4q} \mid \mu_q; \mathbf{D}_m \mid \mathbf{C}_m) \textit{ for } (m, 2) = 1 \qquad (\mu_{4m} \mid \mu_{2m}; \mathbf{O} \mid \mathbf{T}), \quad \mu_{2m}\mathbf{I}.,$$

*where* $m, q \in \mathbf{N}$.

(3.2) Let H, K be subgroups of $U_2(\mathbf{C})$ occurring in the list of (3.1) such that $K \trianglelefteq H$; suppose that H is non-cyclic. The following statements concerning $G = (\mu_{wd} \mid \mu_d; H \mid K)$ are easily verified:

(i) G *is imprimitive if and only if* $H = \mathbf{D}_m$ *for some* $m \in \mathbf{N}$;

(ii) $Z(G) = G \cap Z\,l_2(\mathbf{C}) = \mu_d$ *and* $G/Z(G) \cong H/Z(H) = H/\mu_2$;

(iii) *let* $m = pq > 1$; *then*

$$G(m, p, 2) \text{ is conjugate to } \begin{cases} (\mu_{4q} \mid \mu_{2q}; \mathbf{D}_{m/2} \mid \mathbf{C}_m) & \text{if } p \text{ even, } q \text{ odd,} \\ (\mu_{2q} \mid \mu_q; \mathbf{D}_m \mid \mathbf{D}_{m/2}) & \text{if } p \text{ odd, } q \text{ even,} \\ \mu_{2q}.\mathbf{D}_m & \text{if } p, q \text{ even,} \\ (\mu_{4q} \mid \mu_q; \mathbf{D}_m \mid \mathbf{C}_m) & \text{if } m \text{ odd.} \end{cases}$$

(3.3) Let $G = (\mu_{wd} \mid \mu_d; H \mid K)$ be as in (3.2). Suppose moreover that G is an irreducible reflection group with degrees $d_1$, $d_2$. Denote by $\Pi$ the projective group $G/Z(G)$ operating on the projective complex line, and by $n_1$, $n_2$, $n_3$ the orders of three non-conjugate non-trivial isotropy subgroups of $\Pi$ (*cf.* [13]). We state without proof:

PROPOSITION:

(i) $2.d.(d^{-1}d_1)(d^{-1}d_2) = |H|$ *and* $d = |Z(G)| = gcd(d_1, d_2)$;

(ii) $\mu_l.G$ *is a reflection group if* $l = 1/2\,\text{lcm}(2, wd)$;

(iii) $\mu_{wd}.H$ *is a reflection group if and only if* $wd \mid d_1 d_2$; *in this situation, the degrees of* $\mu_{wd}.H$ *are* $d.\text{lcm}(w, d^{-1}d_i)$ $(i = 1, 2)$;

(iv) *the order of any reflection in* G *is a divisor of some* $n_i$ $(i = 1, 2, 3)$;

(v) $wd \mid 2\,lcm(n_1, n_2, n_3)$.

(3.4) A case-by-case argument, involving either invariants or generating reflections, yields the following theorem (*cf.* [17]).

THEOREM. — *Up to conjugacy the primitive* 2-*dimensional reflection groups are*

$$\left.\begin{array}{c} \mu_{6m}.\mathbf{T} \\ (\mu_6 \mid \mu_{2m}; \mathbf{T} \mid \mathbf{D}_2) \end{array}\right\} m = 1, 2,$$

$$\left.\begin{array}{c} \mu_{4m}.\mathbf{O} \\ (\mu_{4m} \mid \mu_{2m}; \mathbf{O} \mid \mathbf{T}) \end{array}\right\} m = 1, 2, 3, 6,$$

$$\mu_{2m}.\mathbf{I} \qquad m = 2, 3, 5, 6, 10, 15, 30.$$

(3.5) The primitive 2-dimensional reflection groups are listed in (3.6) together with some properties. In order to obtain the column of regular degrees, make the following observations [*cf.* (1.10)]:

(i) *if* G *is an irreducible reflection group, and* $g \in G$ *is regular, then the same holds for any element in g.* $Z(G)$; *moreover, the order of g is a divisor of one of the degrees of* G;

(ii) *if* G *is a* 2-*dimensional reflection group and s is a reflection in* G, *then s is regular if and only if any reflection* $(\neq 1)$ *in* G *commuting with s has the same set of roots as s.*

(3.6) In the following table we have listed the primitive 2-dimensional reflection groups together with some properties. In the first column is written the number Shephard and Todd gave to the corresponding reflection group in [17] (p. 301). Later on the group $(\mu_6 \mid \mu_2; \mathbf{T} \mid \mathbf{D}_2)$ will also be denoted by W $(L_2)$ [compare (4.4) (iv)].

TABLE

| Shephard-Todd number | Group | Degrees | Order | Order of center | Regular degrees | Number of reflections of order 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|---|---|---|
| 4. . . . . . . . . . . . . | $(\mu_6 \mid \mu_2; \mathbf{T} \mid \mathbf{D}_2)$ | 4,6 | 24 | 2 | 4,6 | – | 8 | – | – |
| 5. . . . . . . . . . . . . | $\mu_6 . \mathbf{T}$ | 6,12 | 72 | 6 | 12 | – | 16 | – | – |
| 6. . . . . . . . . . . . . | $(\mu_{12} \mid \mu_4; \mathbf{T} \mid \mathbf{D}_2)$ | 4,12 | 48 | 4 | 12 | 6 | 8 | – | – |
| 7. . . . . . . . . . . . . | $\mu_{12} . \mathbf{T}$ | 12,12 | 144 | 12 | 12 | 6 | 16 | – | – |
| 8. . . . . . . . . . . . . | $(\mu_8 \mid \mu_4; \mathbf{O} \mid \mathbf{T})$ | 8,12 | 96 | 4 | 8,12 | 6 | – | 12 | – |
| 9. . . . . . . . . . . . . | $\mu_8 . \mathbf{O}$ | 8,24 | 192 | 8 | 24 | 18 | – | 12 | – |
| 10. . . . . . . . . . . . . | $(\mu_{24} \mid \mu_{12}; \mathbf{O} \mid \mathbf{T})$ | 12,24 | 288 | 12 | 24 | 6 | 16 | 12 | – |
| 11. . . . . . . . . . . . . | $\mu_{24} . \mathbf{O}$ | 24,24 | 576 | 24 | 24 | 18 | 16 | 12 | – |
| 12. . . . . . . . . . . . . | $(\mu_4 \mid \mu_2; \mathbf{O} \mid \mathbf{T})$ | 6,8 | 48 | 2 | 6,8 | 12 | – | – | – |
| 13. . . . . . . . . . . . . | $\mu_4 . \mathbf{O}$ | 8,12 | 96 | 4 | 12 | 18 | – | – | – |
| 14. . . . . . . . . . . . . | $(\mu_{12} \mid \mu_6; \mathbf{O} \mid \mathbf{T})$ | 6,24 | 144 | 6 | 24 | 12 | 16 | – | – |
| 15. . . . . . . . . . . . . | $\mu_{12} . \mathbf{O}$ | 12,24 | 288 | 12 | 12 | 18 | 16 | – | – |
| 16. . . . . . . . . . . . . | $\mu_{10} . \mathbf{I}$ | 20,30 | 600 | 10 | 20,30 | – | – | – | 48 |
| 17. . . . . . . . . . . . . | $\mu_{20} . \mathbf{I}$ | 20,60 | 1200 | 20 | 60 | 30 | – | – | 48 |
| 18. . . . . . . . . . . . . | $\mu_{30} . \mathbf{I}$ | 30,60 | 1800 | 30 | 60 | – | 40 | – | 48 |
| 19. . . . . . . . . . . . . | $\mu_{60} . \mathbf{I}$ | 60,60 | 3600 | 60 | 60 | 30 | 40 | – | 48 |
| 20. . . . . . . . . . . . . | $\mu_6 . \mathbf{I}$ | 12,30 | 360 | 6 | 12,30 | – | 40 | – | – |
| 21. . . . . . . . . . . . . | $\mu_{12} . \mathbf{I}$ | 12,60 | 720 | 12 | 60 | 30 | 40 | – | – |
| 22. . . . . . . . . . . . . | $\mu_4 . \mathbf{I}$ | 12,20 | 240 | 4 | 12,20 | 30 | – | – | – |

## 4. Root graphs and root systems

If $n \geqq m$, we think of $\mathbf{C}^m$ as the subspace of $\mathbf{C}^\infty$ spanned by the first $m$ standard basis vectors $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m$. Moreover, $\mathbf{C}^\infty$ is endowed with the standard unitary inner product $( \mid )$. If G is a reflection group in a complex vector space V of dimension $n$, we can assume, after choosing coordinates in V, that $G \subseteq G\,l_n(\mathbf{C})$, and, thanks to (1.6), even that $G \subseteq U_n(\mathbf{C})$. Since roots of G are vectors in $\mathbf{C}^n \subset \mathbf{C}^{n+1}$, there is a natural way to view G as a subgroup of $U_{n+1}(\mathbf{C})$. If G is $r$-dimensional, then $r$ is the smallest number such that a conjugate of G is contained in $U_r(\mathbf{C})$.

(4.1) DEFINITIONS. — A *vector graph* is a pair (B, $w$) where B is a nonempty finite subset of $\mathbf{C}^\infty$ such that for all $a, b \in B$ we have $|(a \mid b)| = 1 \Leftrightarrow a = b$, and $w$ is a map from B to $\mathbf{N} \setminus \{1\}$. In this situation B is called the set of *points*, *vectors*, or *vertices* of the vector graph and $w(a)$, for $a \in B$, the *order* of $a$ [with respect to (B, $w$)].

Two vector graphs (B, $w$) and (B′, $w'$) are called *isomorphic* if there is a bijection $\sigma : B \to B'$ such that for all $a,\ b \in B$:

$$w(a) = w'(\sigma a) \qquad \text{and} \qquad (a \mid b) = (\sigma a \mid \sigma b);$$

or equivalently, if there is a unitary transformation $t$ of $\mathbf{C}^\infty$ such that $t\,B = B'$ and $w(a) = w'(ta)$ for any $a \in B$.

We shall identify a vector graph (B, $w$) with a "directed valued graph" in the following way. The points of the graph are the elements of B. For any set $\{a, b\} \subseteq B$ with $(a \mid b) \neq 0, 1$ we fix a direction, i. e. we prescribe which point is starting point and which one is end point. Now the directed edges of the graph are the $(a, b) \in B \times B$ with $(a \mid b) \neq 0, 1$ such that $a$ is starting point according to the direction of $\{a, b\}$. Finally, to any point $a \in B$ we assign the value $w(a)$, and to any directed edge $(a, b) \in B \times B$ we assign the value $(a \mid b)$.

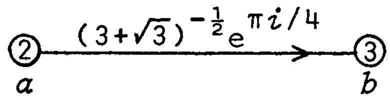
Note that the set of directed edges of a vector graph is not uniquely determined.

To provide an example, let $a = \varepsilon_1$ and

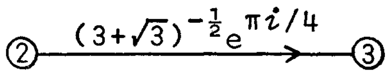
$$b = (3+\sqrt{3})^{-1/2}(e^{-\pi i/4}\varepsilon_1 + (e^{\pi i/4} + \sqrt{2}\,e^{2\pi i/3})\,\varepsilon_2).$$

Put $B = \{a, b\}$, $w(a) = 2$, and $w(b) = 3$.

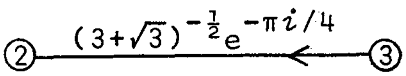
We represent the vector graph (B, $w$) by




or, if we are only interested in the isomorphism class of the vector graph, by




The vector graph may also be represented by




with edge $(b, a)$.

We shall often use the following conventions when drawing a vector graph (B, $w$). Let $a$, $b \in$ B:

(1) if $w(a) = 2$, the number 2 in the vertex $a$ is omitted;

(2) if $(a \mid b) \in \mathbf{R}^*$, the arrow indicating the direction of the edge connecting $a$ and $b$ is left out;

(3) if $(a \mid b) = -1/2$ and $w(a) = w(b) = 2$, the value $-1/2$ associated with the edge connecting $a$ and $b$ is omitted.

Let $\Gamma = (\mathrm{B}, w)$ be a vector graph. Put $\mathrm{E} = \{ \{ a, b \} \mid a, b \in \mathrm{B}, \mid (a \mid b) \mid \neq 0, 1 \}$. Now (B, E) is a graph. All usual definitions concerning graphs (like *cycle* and *connectedness*) applied to $\Gamma$ are with respect to (B, E). A cycle of a vector graph consisting of 3 points will often be called a *triangle*.

If $v \in \mathbf{C}^\infty$, we denote by $\bar{v}$ the complex conjugate of $v$. Let $\bar{\mathrm{B}} = \{ \bar{b} \mid b \in \mathrm{B} \}$, and let $\bar{w}: \bar{\mathrm{B}} \to \mathbf{N} \setminus \{ 1 \}$ be defined by $\bar{w}(\bar{b}) = w(b)$ $(b \in \mathrm{B})$. We shall say that (B, $w$) is the *complex conjugate of* $\Gamma$.

(4.2) DEFINITIONS. – Let $\Gamma = (\mathrm{B}, w)$ be a vector graph. We denote by $\dim(\Gamma)$ (or *dimension of* $\Gamma$) the dimension of the vector space spanned by B, and by W($\Gamma$) the group generated by all reflections $s_{a, w(a)}$ with $a \in$ B. Thus, if $\Gamma$ is isomorphic to $\Gamma'$, then W($\Gamma$) is conjugate to W($\Gamma'$).

$\Gamma$ is called a *root graph* if:

(1) $\dim(\Gamma) = \mid \mathrm{B} \mid$ (in other words, the elements of B are linearly independent);

(2) W($\Gamma$) is a finite group (and therefore a reflection group).

Note that if $a$ is a point of a root graph $\Gamma$, then $a$ is a unitary root of W($\Gamma$).

Let $\Gamma = (\mathrm{B}, w)$ be a root graph. We say that $\Gamma$ is *irreducible* if W($\Gamma$) is irreducible in dimension $\dim(\Gamma)$, or equivalently, if $\Gamma$ is connected. $\Gamma$ is called *real* (*complex*, *primitive*) if W($\Gamma$) is real (complex, primitive).

Let $\Gamma' = (\mathrm{B}', w')$ be another root graph. If $\mathrm{B} \subset \mathrm{B}'$ and $w' |_{\mathrm{B}} = w$, we say that $\Gamma'$ is an *extension* of $\Gamma$, or that $\Gamma$ is a *sub-root-graph* of $\Gamma'$. If W($\Gamma$) is conjugate to W($\Gamma'$), we say that $\Gamma$ is *equivalent* to $\Gamma'$. Furthermore, $\Gamma$ is said to be *congruent* to $\Gamma'$ if there is $t \in \mathrm{G}l(\mathbf{C}^\infty)$ such that $w'(ta) = w(a)$ for any $a \in$ B and the elements of B are eigen vectors of $t$.

If the roots $v$, $v'$ span a root graph $\Gamma$ and are of order 2, then there exists $m \in \mathbf{N}$ such that W($\Gamma$) is conjugate to G($m, m, 2$); so $\mid (v \mid v') \mid = \cos(\pi k/m)$ for some $k$ prime to $m$.

Let $\Gamma = (\mathrm{B}, w)$ be a root graph. Put

$$\mathrm{M} = \{ m \in \mathbf{N} \mid \text{there exist } a, b \in \mathrm{B} \text{ with } w(a) = w(b) = 2 \text{ and } |(a \mid b)| = \cos(\pi m^{-1}) \}.$$

We define $d(\Gamma)$ by

$$d(\Gamma) = \begin{cases} \max(\mathrm{M}) & \text{if } \mathrm{M} \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

Let G be a reflection group. Put

$$N = \{ \text{order}\,(ss') \mid s, s' \text{ are reflections of G of order } 2 \}.$$

Now $d(G)$ is defined by

$$d(G) = \begin{cases} \max(N) & \text{if } N \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

(4.3) *Remarks:*

(i) The complex conjugate of a root graph is, again, a root graph;

(ii) the definition of a real (complex, primitive, irreducible) root graph depends only on the isomorphism class of the root graph;

(iii) there exist root graphs $\Gamma$ with $d(\Gamma) < d(W(\Gamma))$. This will be clear from (4.4) (v);

(iv) if $\Gamma$ is a root graph, then $W(\Gamma)$ is a reflection group generated by $\dim(\Gamma)$ reflections. On the other hand, every reflection group G in $\mathbf{C}^n$ that is generated by $n$ reflections can be obtained as follows. Fix a unitary root for each of the $n$ generating reflections in G. Let B be the set of these unitary roots and let $w : B \to N \setminus \{1\}$ be given by $w(a) = o_G(a)$ $(a \in B)$ [notation of (1.6)]. Then $\Gamma = (B, w)$ is a root graph with $W(\Gamma) = G$;

(v) if two root graphs are congruent, they are equivalent;

(vi) we will frequently use the following observation. If $\Gamma = (B, w)$ is a root graph and $a, b \in B$, then $\Gamma$ is equivalent to the root graph

$$\Gamma' = (B', w'), \qquad \text{where} \quad B' = \{ s_{a, w(a)} b \} \cup B \setminus \{ b \},$$

and $w' : B \to N \setminus \{1\}$ is given by

$$w'(x) = \begin{cases} w(x) & \text{if } x \in B \setminus \{b\}, \\ w(b) & \text{if } x = s_{a, w(a)} b; \end{cases}$$

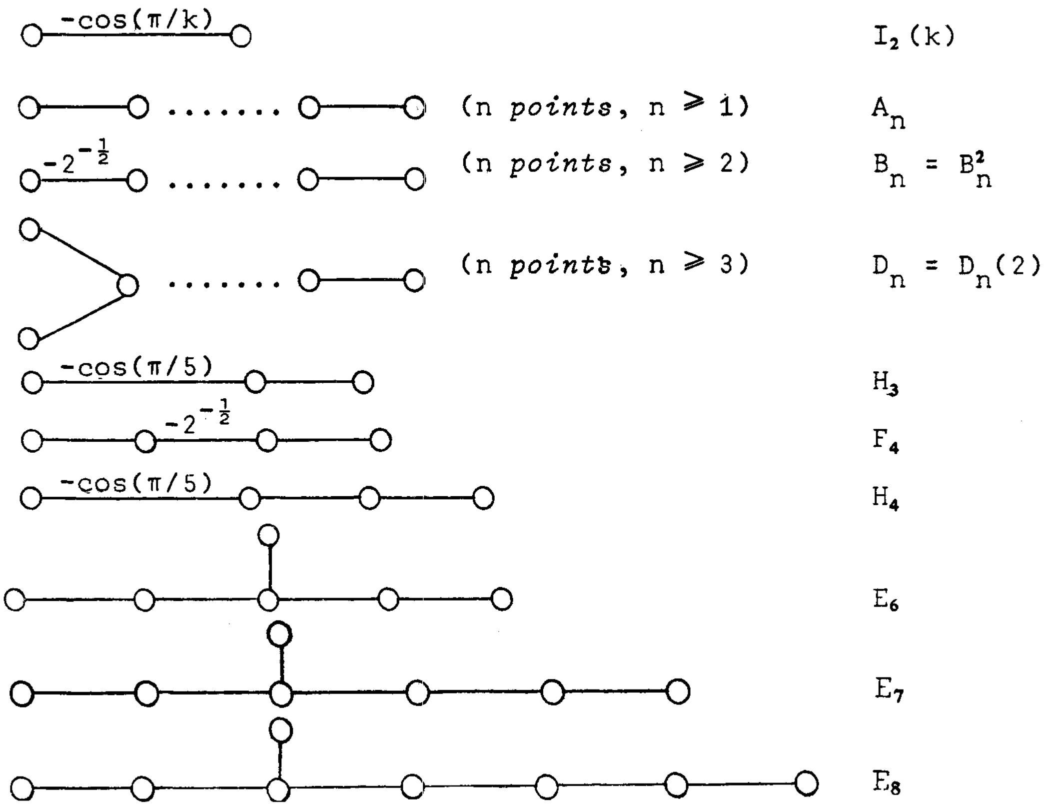
(vii) if $(B, w)$ is a vector graph with $B = \{ e_1, e_2, \ldots, e_n \}$, then $\det((e_i \mid e_j))$ is a real number $\geqq 0$.

Equality holds if and only if $e_1, e_2, \ldots, e_n$ are linearly dependent. If this is the case, $(B, w)$ cannot be a root graph.

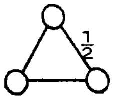
(4.4) *Examples:*

(i) A slight adaptation of the usual Coxeter graphs (*cf.* [3]) provides root graphs in the sense of (4.2), namely: Replace any number $m$ assigned to an edge by $-\cos(\pi m^{-1})$. By a *Coxeter graph* we mean here a root graph obtained in this way.

The classification of Coxeter graphs (*loc. cit.*) gives us the following result. *The only irreducible root graphs* $\Gamma = (B, w)$ *with* $w(B) = \{2\}$ *and without cycles are the Coxeter graphs:*

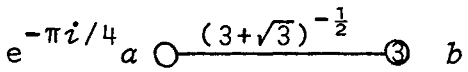



It is well known that if $\Gamma$ is one of these root graphs, then $d(\Gamma) = d(W(\Gamma))$; furthermore, $d(I_2(k)) = k$, $d(A_n) = d(D_n) = d(E_n) = 3$, $d(B_n) = d(F_n) = 4$, $d(H_n) = 5$. As a consequence of our definitions, we have that $\Gamma$ is a real root graph if and only if $\Gamma$ is equivalent to a Coxeter graph;

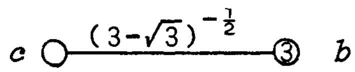
(ii) let $\Gamma = (B, w)$ be a root graph. If $w(B) = \{2\}$ and $\Gamma$ has no cycles, then $\Gamma$ is a real root graph, as we saw in (i). But  is a real root graph too [equivalent to , as can be deduced with the process of (4.3) (vi)];

(iii) the example in (4.1) is a root graph: the reflections $s_a = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and $s_{b,3} = 1/2\,\omega^2(i-1)\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}$ satisfy $(s_{b,3}\, s_a)^3 = -i\, I_2$ and generate a finite group; in fact, the corresponding reflection group is conjugate to $(\mu_{12} \mid \mu_4; \mathbf{T} \mid \mathbf{D}_2)$ [*cf.* (3.6)].

The root graph is congruent to



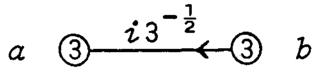

and equivalent to




for some $c \in \mathbf{U}\, s_a\, s_{b,3}\, a$.

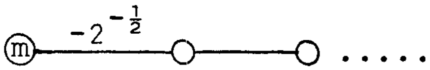
The latter statement follows from $\langle s_{b,3}, s_a\, s_{b,3}\, s_a\, s_{b,3}^2\, s_a \rangle = \langle s_a, s_{b,3} \rangle$;

(iv) put $a = \varepsilon_3$ and $b = i\, 3^{-1/2} (\varepsilon_1 + \varepsilon_2 + \varepsilon_3)$, and let $L_2$ be the vector graph



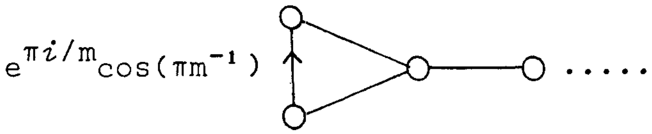

Then $L_2$ is a 2-dimensional complex primitive root graph with reflection group $W(L_2)$ conjugate to $(\mu_6 \mid \mu_2 ; \mathbf{T} \mid \mathbf{D}_2)$; furthermore $d(L_2) = d(W(L_2)) = 0$ [*cf.* (3.6)];

(v) let $m, n > 1$. By $B_n^m$ or $\Gamma(m, 1, n)$ we denote the vector graph




(n points)

and by $D_n(m)$ or $\Gamma(m, m, n)$ the vector graph




(n points, $n \geqslant 3$)

so $B_n^2 = B_n$ and $D_n(2) = D_n$, in accordance with (i).

It can be deduced with the help of (2.5) (v) that $\Gamma(m, p, n)$ is a root graph, with $W(\Gamma(m, p, n))$ conjugate to $G(m, p, n)$ $(p = 1, m)$. Furthermore

$$d(\Gamma(m, m, n)) = \max(3, m),$$

and

$$d(\Gamma(m, 1, n)) = \begin{cases} 4 & \text{if } m = 2, \\ 3 & \text{otherwise,} \end{cases}$$

while

$$d(G(m, p, n)) = \begin{cases} \max(3, m) & \text{if } p^{-1} m \text{ is odd,} \\ \max(4, m) & \text{if } p^{-1} m \text{ is even.} \end{cases}$$

Finally, $\Gamma(m, m, n)$ is equivalent to its complex conjugate.

(4.5) All usual definitions concerning root graphs (like *complex*) applied to a cycle C of a root graph (B, $w$) are meant to hold for the sub-root-graph of (B, $w$) spanned by C [i. e. for (C, $w|_C$)].

LEMMA. — *Let* $\Gamma = (B, w)$ *be a root graph:*

(i) *let* $C = \{e_1, e_2, \ldots, e_m\}$ *be a cycle of* $\Gamma$*; put* $e_{m+1} = e_1$. *If* $\prod_{i=1}^{m} (e_i \mid e_{i+1}) \in \mathbf{C}\setminus\mathbf{R}$, *then* C *spans a complex root graph;*

(ii) *if* $w(B) = \{2\}$ *and* $\Gamma$ *is complex, then* $\Gamma$ *contains a cycle* $\{e_1, e_2, \ldots, e_m\}$ *such that* $\prod_{i=1}^{m} (e_i \mid e_{i+1}) \in \mathbf{C}\setminus\mathbf{R}$ *(where* $e_{m+1} = e_1$*); hence* $\dim(\Gamma) \geqq 3$*;*

(iii) *let* $\Gamma$ *be irreducible and let* G *be an irreducible n-dimensional reflection group with* $W(\Gamma) \subseteq G$*; then* $\Gamma$ *can be extended to a root graph* $\Gamma'$ *such that* $W(\Gamma')$ *is an irreducible n-dimensional reflection subgroup of* G*;*

(iv) *suppose* $\Gamma$ *is irreducible and complex,* $w(B) = 2$, $d(\Gamma) = d(W(\Gamma))$, *and* $n = \dim(\Gamma) \geqq 3$*; then there is a 3-dimensional irreducible complex root graph* $\Gamma_0$ *with* $W(\Gamma_0) \subseteq W(\Gamma)$ *and* $d(\Gamma_0) = d(W(\Gamma))$.

*Proof:*

(i) One easily sees that a cycle C as in (i) is not embeddable in a real vector space. This proves (i);

(ii) if $w(B) = \{2\}$ and $\Gamma$ does not contain any cycle as described in (ii), then $(a \mid b) \in \mathbf{R}$ for any $a, b \in B$, up to congruency of $\Gamma$. It is immediate that we have $W(\Gamma) \subseteq Gl(\mathbf{R}^{\infty})$, up to conjugacy of $W(\Gamma)$;

(iii) let W be the subspace of $\mathbf{C}^n$ spanned by the roots of $\Gamma$. If W is a proper subspace of $\mathbf{C}^n$ such that $x \in W \cup W^{\perp}$ for any root $x$ of G, then G is reducible; so there is a unitary root $v \notin W \cup W^{\perp}$. Let $\Gamma_1$ be the root graph spanned by $\Gamma$ and $v$. Now $\dim(\Gamma_1) = \dim(\Gamma)+1$. Continue with $\Gamma_1$ instead of $\Gamma$, and so on, until the newly obtained root graph is $n$-dimensional.

(iv) if $n = 3$, we have nothing to prove. We use induction on $n$. Suppose $n > 3$. Because of induction it suffices to construct a complex irreducible root graph $\Gamma_1$ of dimension $< n$ with $W(\Gamma_1) \leqq W(\Gamma)$ and $d(\Gamma_1) = d(\Gamma)$. Let $a, b \in B$ be such that $|(a \mid b)| = \cos(\pi\, d(\Gamma)^{-1})$. Let C be a complex cycle of $\Gamma$ with a minimal number of points. Let $\{e_1, e_2, \ldots, e_m\}$ form the set of points of C, numbered such that $(e_i \mid e_{i+1}) \neq 0$ for any $i \in \underline{m}$ $(e_{m+1} = e_1)$. Note that C does not have any subcycles (so $(e_i \mid e_j) = 0$ if both $|i-j| > 1$ and $\{i, j\} \neq \{1, m\}$) and that C is as described in (ii). Without loss of generality we may (and shall) assume that $e_1$ is end point of a minimal path connecting $\{a, b\}$ and C, and that $a$ is the starting point of this minimal path; thus $\{a, b\} \cap C \neq \emptyset$ implies that $e_1 = a$. If $m \geqq 4$, then $\{s_{e_3} e_2\} \cup B\setminus\{e_2, e_3\}$ spans a root graph $\Gamma_1$ as wanted. Therefore we may assume that $m = 3$. If $\{a, b\} \subseteq C$, we are through. We are left with the case that $\{a, b\} \cap C$ consists of at most one point. If $\{a, b\} \cap C = \emptyset$, let $c$ be the (unique) point with $(c \mid e_1) \neq 0$ in the minimal path given above, otherwise put $c = b$ (then $(a \mid c) = (a \mid b) \neq 0$). If both $(c \mid e_2) \neq 0$ and $(c \mid e_3) \neq 0$, then there is an $i \in \underline{3}$ such that the triangle spanned by $\{ c \} \cup C \setminus \{ e_i \}$ is complex; now $\{ c \} \cup B \setminus \{ e_i \}$ spans a root graph $\Gamma_1$ as wanted.

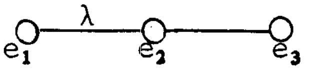
Thus we may assume $(e_3 \mid c) = 0$. Replacing $e_2$ by $s_{e_1} e_2$, if necessary, we obtain $(e_2 \mid c) \neq 0$. If the triangle $\{ e_1, e_2, c \}$ is complex, we can take for $\Gamma_1$ the root graph spanned by $B \setminus \{ e_3 \}$. If the triangle is real, then the root graph spanned by $\{ s_{e_3} e_1 \} \cup B \setminus \{ e_1, e_3 \}$ yields the reduction to dimension $< n$.

(4.6) THEOREM. — *Suppose* G *is an n-dimensional complex irreducible reflection group, all reflections having order* 2 *and* $n \geq 3$. *Then there is a* 3-*dimensional complex irreducible root graph* $\Gamma$ *with* $d(\Gamma) = d(G)$ *and* $W(\Gamma) \subseteq G$.

*Proof.* — We proceed by induction on $n$. The order of a point in any vector graph that will be considered here is 2.

Let $n = 3$. Put $\lambda = \cos(\pi\, d(G)^{-1})$. If follows from the definition of $d(G)$ that there are unitary roots $e_1$, $e_2$ of G with $(e_1 \mid e_2) = \lambda$. We add a unitary root $e_3$ in order to obtain an irreducible root graph $\Gamma_0$ spanned by $\{ e_1, e_2, e_3 \}$ [*cf.* (4.5) (iii)]. If $\Gamma_0$ is complex we can take $\Gamma = \Gamma_0$.

Consider the case that $\Gamma_0$ is real. After replacing G by a suitable conjugate, we may assume that $\Gamma_0$ is a Coxeter graph; in fact, $\Gamma_0$ is




(cf. (4.4)(i))

If for every unitary root $v$ of G there is an $\alpha \in \mathbf{U}$ such that $\alpha (v \mid e_i) \in \mathbf{R}$ ($i \in \underline{3}$), then every reflection of G has a real matrix with respect to $e_1$, $e_2$, $e_3$, contradictory to G being complex; so there is a unitary root $v$ of G with the following property:

(1) for any $\alpha \in \mathbf{U}$ there is an $i \in \underline{3}$ with $\alpha (v \mid e_i) \in \mathbf{C} \setminus \mathbf{R}$.

Take such a unitary root $v$ with the additional property that $(v \mid e_1) \in \mathbf{R}^*$. If $v$, $e_1$, $e_2$ are linearly dependent, then either $v$, $e_1$, $s_{e_3} e_2$ or $s_{e_3} v$, $e_1$, $e_2$ are linearly independent. After replacing $e_2$ by $s_{e_3} e_2$ or $v$ by $s_{e_3} v$ in the respective cases if necessary, we still have that $(v \mid e_1) \in \mathbf{R}^*$, and that (1) holds, but also that $v$, $e_1$, $e_2$ are linearly independent. Put

$$B_1 = \{ e_1, e_2, v \}, \qquad B_2 = \{ e_1, s_{e_2} e_3, v \}, \qquad B_3 = \{ s_{e_2} e_1, e_3, v \},$$
$$B_4 = \{ e_2, e_3, v \} \quad \text{and} \quad B_5 = \{ e_2, e_3, s_{e_2} v \}.$$

Let $\Gamma_i$ be the vector graph spanned by $B_i$. Suppose none of these five vector graphs $\Gamma_i$ is a complex irreducible root graph with $d(\Gamma_i) = d(G)$.

Now $\Gamma_1$ is an irreducible root graph, so $\Gamma_1$ is real; but $(e_1 \mid e_2)$, $(v \mid e_1) \in \mathbf{R}^*$, so $(v \mid e_2) \in \mathbf{R}$ by (4.5) (i). This implies that $(v \mid s_{e_2} e_1) \in \mathbf{R}$ and $(v \mid e_3) \in \mathbf{C} \setminus \mathbf{R}$ because of (1).

If $e_1$, $s_{e_2} e_3$, $v$ were linearly independent, then $\Gamma_2$ would be a complex root graph. Thus there are $\lambda_1 \neq 0$, $\lambda_2 \in \mathbf{C} \setminus \mathbf{R}$ such that

$$v = \lambda_1 e_1 + \lambda_2 e_2 + \lambda_2 e_3. \tag{2}$$

Suppose $(v \mid s_{e_2} e_1) \neq 0$. Reasoning with $\Gamma_3$ as with $\Gamma_2$, we get that $v$ is also linearly dependent on $s_{e_2} e_1$ and $e_3$. From this and (2) we derive that $v \in \mathbf{C}(s_{e_2} e_1 - 2\lambda e_3)$; this however contradicts (1).

We now have $(v \mid s_{e_2} e_1) = 0$; so $\lambda_1 (1 - 2\lambda^2) = 0$. Since $\lambda_1 \neq 0$, this is equivalent to $\lambda = 2^{-1/2}$. Thus $d(\Gamma^*) = d(G) = 4$.

Note that $(v \mid e_2) \in \mathbf{R}^*$. Now $v$, $e_2$, $e_3$ are linearly independent, so $\Gamma_4$ is a complex root graph. If $|(v \mid e_i)| = 2^{-1/2}$ for some $i \in \{2, 3\}$, we are through. Therefore we may assume, after replacement of $v$ by $-v$ if necessary, that $(v \mid e_2) = |(v \mid e_3)| = 1/2$. This means $\lambda_1 \sqrt{2} + \lambda_2 = 1$ and $|\lambda_2| = 1$. Reasoning with $\Gamma_5$ as with $\Gamma_4$, we also get $|(s_{e_2} v \mid e_3)| = 1/2$, so $|1 + \lambda_2| = 1$, and $\lambda_2 \in \{\omega, \omega^2\}$. But now it is clear that there is a unitary transformation $t$ of $\mathbf{C}^\infty$ such that $te_1 = -\varepsilon_3$, $te_2 = 2^{-1/2}(\varepsilon_2 - \varepsilon_3)$, $te_3 = 2^{-1/2}(\varepsilon_1 - \varepsilon_2)$, and $v = 2^{-1/2}(\lambda_2 \varepsilon_1 - \varepsilon_3)$. The conclusion is that the reflection group $\langle s_v, s_{e_1}, s_{e_2}, s_{e_3} \rangle$ is conjugate to G (6, 3, 3) [this follows from the proof of (2.4)]. By (4.4) (v), we have $4 = d(G) \geqq d(G(6, 3, 3)) = 6$. This contradiction establishes the 3-dimensional case.

Suppose $n > 3$. Let H be a real irreducible reflection subgroup of G with $d(H) = d(G)$ and maximal with respect to these properties. Then H is of dimension $\geqq 3$. Up to conjugacy, there is a Coxeter graph $\Gamma_0 = (B_0, w_0)$ with $B_0 = \{e_1, e_2, \ldots, e_r, \ldots, e_t\}$ such that $(e_1 \mid e_2) = -\cos(\pi\, d(G)^{-1})$, $(e_{i-1} \mid e_i) = -1/2$ $(3 \leqq i \leqq r)$, $e_r$ is end point of $\Gamma_0$, and $W(\Gamma_0) = H$.

Since $G \setminus H \neq \emptyset$, there is a unitary root $v$ of G such that $s_v \in G \setminus H$. If either $v$ or $s_{e_r} v$ is linearly dependent on $e_1, \ldots, e_{r-1}, e_{r+1}, \ldots, e_t$, the induction hypothesis, applied to the subgroup of G generated by $\{s_{e_i} \mid i \neq r\}$ together with $s_v$ or $s_{e_r} s_v s_{e_r}$, provides a root graph as wanted. (Note that such a subgroup is complex, for otherwise the roots of this group together with $e_r$ could be embedded in $\mathbf{R}^\infty$, in contradiction with the maximality of H.)

Thanks to this argument and (4.5) (iv), we are left with the case:

(3) Both $v, e_1, \ldots, e_{r-1}, e_{r+1}, \ldots, e_t$ and $s_{e_r} v, e_1, \ldots, e_{r-1}, e_{r+1}, \ldots, e_t$

span irreducible real root graphs.

After replacing $v$ by a suitable scalar multiple, we have that $(v \mid e_i) \in \mathbf{R}$ for $i \neq r$, and $(v \mid e_r) \notin \mathbf{R}$. Furthermore, $(v \mid e_j) \neq 0$ for some $j \neq r$, so we can change $v$ by an element of $\langle s_{e_i} \mid i \neq r, r-1 \rangle$ to make $(v \mid e_{r-1}) \neq 0$. This implies that $(s_{e_r} v \mid e_{r-1}) \in \mathbf{C} \setminus \mathbf{R}$ and $(s_{e_r} v \mid e_i) \in \mathbf{R}$ for $i \neq r-1, r$. Because of (4.5) (i) and (3), we must have that $(s_{e_r} v \mid e_i) = 0$, so $(v \mid e_i) = 0$ if $i \neq r-1, r$.

Put $u = e_2 + e_3 + \ldots + e_{r-1} = s_{e_2} \ldots s_{e_{r-2}} e_{r-1}$. Observe that $u$ is a unitary root of H with $(e_1 \mid u) = -\cos(\pi\, d(G)^{-1})$. If $v$ is a linear combination of $e_1$, $u$ and $e_r$, then the subgroup K of G generated by the reflections with roots $v$, $e_1$, $u$, $e_r$ is complex irreducible in dimension 3 and satisfies $d(K) = d(G)$, so we are done.

Suppose $v$ is linearly independent of $e_1$, $u$ and $e_r$. Since

$$(s_{e_r} s_u v \mid e_1) = -2(e_2 \mid e_1)(v \mid e_{r-1}) \in \mathbf{R}^* \quad \text{and} \quad (s_{e_r} s_u v \mid u) = (v \mid e_r) \in \mathbf{C} \setminus \mathbf{R},$$

the root graph spanned by $e_1$, $u$ and $s_{e_r} s_u v$ fulfils the demands for $\Gamma$.

(4.7) COROLLARY. — G *as in* (4.6). *Suppose moreover that* G *is primitive and* $n \geqq 8-d(G) \geqq 4$. *Then there is a primitive complex* $(8-d(G))$*-dimensional root graph* $\Gamma$ *with* $d(\Gamma) = d(G)$ *and* $W(\Gamma) \subseteq G$. *In fact,* $\Gamma$ *can be obtained as an extension of any* 3*-dimensional irreducible complex root graph* $\Gamma_0$ *with* $d(\Gamma_0) = d(G)$ *and* $W(\Gamma_0) \subseteq G$.

*Proof.* — By the theorem there exists $\Gamma_0$ as described. By Lemma (4.5) we can extend $\Gamma_0$ to a complex irreducible $(7-d(G))$-dimensional root graph $\Gamma_1$ with $W(\Gamma_1) \subseteq G$. Since $G(2, 2, 4)$ is real and $d(G(3, 3, 3)) = 3$, we have by (4.4) (v) and Lemma (2.7) that $W(\Gamma_1)$ is either primitive or has a unique system of imprimitivity. By (2.6) and (2.9), there is a unitary root $v$ of G which extends $\Gamma_1$ to a root graph as wanted.

(4.8) *Remark.* — If we replace the inequalities in the hypotheses of the preceding corollary by $n \geqq 4$ and $d(G) = 5$, we get, using the same arguments as before, that there exists a primitive complex 4-dimensional root graph $\Gamma$ with $d(\Gamma) = 5$ and $W(\Gamma) \subseteq G$; however, a similar root graph of dimension 3 would be more useful [compare (6.6)].

(4.9) DEFINITIONS. — Let $\Sigma = (R, f)$ be a pair consisting of:

(1) a finite set R of nonzero elements of $\mathbf{C}^\infty$

(2) a map $f : R \to \mathbf{N} \setminus \{ 1 \}$ such that for all $a, b \in R$

$$s_{a, f(a)} R = R \quad \text{and} \quad f(s_{a, f(a)} b) = f(b).$$

In this situation $\Sigma$ is called a *pre-root-system*. If R is a subset of a linear subspace V of $\mathbf{C}^\infty$, we say that $\Sigma$ is a *pre-root-system in* V.

To $\Sigma = (R, f)$ is associated the reflection group $W(\Sigma)$ defined by

$$W(\Sigma) = \langle s_{a, f(a)} \mid a \in R \rangle.$$

[Since $W(\Sigma)$ fixes $R^\perp$ pointwise, the restriction of $W(\Sigma)$ to the vector space spanned by R is faithful, so $W(\Sigma)$ can be viewed as a group of permutations of R; hence $W(\Sigma)$ is finite.]

A pre-root-system $\Sigma = (R, f)$ is called a *root system* if for all $a \in R$

$$(3) \qquad \alpha a \in R \iff \alpha a \in W(\Sigma) a.$$

We shall say that $\Sigma$ is *isomorphic* to the root system $(R', f')$ if there is a unitary transformation $t$ of $\mathbf{C}^\infty$ such that $t(R) = R'$ and $f'(ta) = f(a)$ for all $a \in R$.

(4.10) *Remarks:*

(i) If $\Gamma = (B, w)$ is a root graph, then the pair $\Sigma = (R, f)$ where $R = W(\Gamma).B$ and the map $f: R \to \mathbf{N}\setminus\{1\}$ is induced by the order function $o_{W(\Gamma)}$ defines a pre-root-system with $W(\Sigma) = W(\Gamma)$;

(ii) let G be a finite reflection group in a linear subspace V of $\mathbf{C}^\infty$. For any reflection $s \in G$, we choose a unitary root $a_s \in V$. Let $R_0$ be the set of elements $a_s \in V$ obtained in this way, and define $f_0 : R_0 \to \mathbf{N}\setminus\{1\}$ by $f_0(a) = o_G(a)$ $(a \in R_0)$. Now $R = G.R_0$ furnished with the extension $f: R \to \mathbf{N}\setminus\{1\}$ of $f_0$ given by $f(ga) = f_0(a)$ for $a \in R_0$, $g \in G$ (note that $f$ is well defined) yields a pre-root-system $(R, f)$ in V;

(iii) if $\Sigma$, $\Sigma'$ are two isomorphic root systems, then $W(\Sigma)$ is conjugate to $W(\Sigma')$.

(4.11) LEMMA. — *Suppose $\Sigma = (R, f)$ is a pre-root-system. We have:*

(i) $\{ s^j_{a, f(a)} \mid a \in R, 0 < j < f(a) \}$ *is the set of all reflections in* $W(\Sigma)$;

(ii) *there is a root system* $\Phi = (S, g)$ *with* $W(\Phi) = W(\Sigma)$, $S \subset R$ *and* $g = f|_S$.

*Suppose moreover that $\Sigma$ is a root system;*

(iii) *if* $\Delta \subset R$ *is such that* $W(\Sigma) = \langle s_{a, f(a)} \mid a \in \Delta \rangle$, *then every reflection of* $W(\Sigma)$ *is conjugate to* $s^j_{a,m}$ *for some* $j$, $m \in \mathbf{N}$ *and* $a \in \Delta$; *furthermore,* R *consists of* $W(\Sigma)$*-orbits of elements in* $\Delta$;

(iv) *let* A *be a subring of* $\mathbf{C}$ *with* $\exp(2\pi i/m) \in A$ *for each* $m \in f(R)$, *and let* $\Delta$ *be as in* (iii) *but with the additional property that* $(a \mid a)$, $(b \mid a)(a \mid a)^{-1} \in A$ *for all* $a, b \in \Delta$; *then* $(b \mid a)(a \mid a)^{-1} \in A$ *for all* $a, b \in R$ *and* $W(\Sigma)$ *is defined over the quotient field of* A;

(v) *if $g$ is a regular element of* $W(\Sigma)$, *then the order of $g$ is a divisor of* $|R|$.

*Proof:*

(i) put $T = \{ v \in \mathbf{C}^\infty \mid \mathbf{C}v \cap R \neq \emptyset \}$. Suppose $u \in \mathbf{C}^\infty\setminus T$ is a root of $W(\Sigma)$ of order $m > 1$. Now $W(\Sigma)$ leaves T invariant; so by (1.8) there is a linear character $\chi : W(\Sigma) \to \mathbf{C}$ such that, for any reflection $r \in W(\Sigma)$, we have

$$\chi(r) = \begin{cases} \det(r) & \text{if } r \text{ has a root in T,} \\ 1 & \text{otherwise.} \end{cases}$$

On the other hand, $W(\Sigma)$ is generated by the $s_{a, f(a)}$ with $a \in R$, so $\chi = \det$, and $1 = \chi(s_{u,m}) = \det(s_{u,m})$; this is absurd. Therefore $\mathbf{C}^\infty\setminus T$ does not contain any roots of $W(\Sigma)$, whence T is the set of roots of $W(\Sigma)$.

Let $s$ be a reflection in $W(\Sigma)$ with eigenvalue $\zeta \neq 1$. We have just seen that $s$ has a root $a \in R$. Put $Q = W(\Sigma).(\mathbf{C}^* a)$. Using (1.8) once again, we obtain a linear character $\psi : W(\Sigma) \to \mathbf{C}^*$ such that for any reflection $r \in W(\Sigma)$, we have

$$\psi(r) = \begin{cases} \det r & \text{if } r \text{ has a root in Q,} \\ 1 & \text{otherwise.} \end{cases}$$

There exist $a_1, a_2, \ldots, a_l \in \mathrm{R}$ with $s = s_{a_1, f(a_1)} s_{a_2, f(a_2)} \cdots s_{a_l, f(a_l)}$; now, the order of $\zeta = \det s = \sum_{a_i \in \mathrm{Q}} \det s_{a_i, f(a_i)}$ is a divisor of $f(a)$, and we are done;

(ii) put $\mathrm{U} = \{ \mathbf{C}^* u \mid u$ is a root of $\mathrm{W}(\Sigma) \}$. Let $u_1, u_2, \ldots, u_l \in \mathrm{R}$ be such that $\{ \mathbf{C}^* u_1, \mathbf{C}^* u_2, \ldots, \mathbf{C}^* u_l \}$ is a set of representatives of the $\mathrm{W}(\Sigma)$-orbits in U. Define $\mathrm{S} = \bigcup_{i=1}^{l} \mathrm{W}(\Sigma) u_i$ and $g = f|_{\mathrm{S}}$. Putting $\Phi = (\mathrm{S}, g)$, we have that $\mathrm{W}(\Phi) = \mathrm{W}(\Sigma)$ and that $\Phi$ is a root system;

(iii) $\mathrm{R}_1 = \mathrm{W}(\Sigma).\Delta$ together with $f_1 : \mathrm{R}_1 \to \mathbf{N} \setminus \{1\}$ defined in the obvious way gives a pre-root-system $\Sigma_1$ with $\mathrm{W}(\Sigma_1) = \mathrm{W}(\Sigma)$. If $a \in \mathrm{R}$, an application of (i) to $\Sigma_1$ yields $a \in \mathrm{W}(\Sigma).\Delta$. The rest of (iii) is clear from formula (3) of (1.6);

(iv) since

$$(s_{a, f(a)}^k b \mid c) = (b \mid c) - (1 - \exp(2\pi i k f(a)^{-1})(b \mid a)(a \mid c)(a \mid a)^{-1} \in \mathrm{A}$$
$$\text{for all} \quad a, b, c \in \Delta \quad \text{and} \quad k \in \mathbf{N},$$

the proof comes down to a straightforward induction argument;

(v) a regular $g \in \mathrm{W}(\Sigma)$ permutes the elements of R and, unless $g = 1$, fixes no element of R [*cf.* (1.10) (i)]; so all $g$-orbits have the same number of elements, i. c. the order of $g$.

(4.12) *Examples:*

(i) put

$$\mathrm{R}(m, m, n) = \mu_2 . \mu_m . \{ e^{2\pi i l/m} \varepsilon_i - \varepsilon_j \mid i, j, l \in \mathbf{N}, i \neq j, 1 \leqq i, j \leqq n \},$$

and let $f_{m,m,n} : \mathrm{R}(m, m, n) \to \mathbf{N} \setminus \{1\}$ be the constant map 2; then

$$\Sigma(m, m, n) = (\mathrm{R}(m, m, n), f_{m,m,n})$$

is a root system with

$\mathrm{W}(\Sigma(m, m, n)) = \mathrm{G}(m, m, n)$ and $|\mathrm{R}(m, m, n)| = m^2 . n(n-1) . \gcd(m, 2)^{-1}$.

Let $q = p^{-1} m \in \mathbf{N} \setminus \{1\}$. Put $\mathrm{R}(m, p, n) = \mathrm{R}(m, m, n) \cup \mu_q \{ \varepsilon_k \mid 1 \leqq k \leqq n \}$, and let $f_{m,p,n} : \mathrm{R}(m, p, n) \to \mathbf{N} \setminus \{1\}$ be the extension of $f_{m,m,n}$ determined by $f(\varepsilon_k) = q$ for all $k$; then $\Sigma(m, p, n) = (\mathrm{R}(m, p, n), f_{m,p,n})$ is a root system with

$$\mathrm{W}(\Sigma(m, p, n)) = \mathrm{G}(m, p, n);$$

(ii) put $\mathrm{R} = \mu_6 \{ \varepsilon_1, 1/3(2\omega + 1)(\omega^j \varepsilon_1 + \varepsilon_2 + \varepsilon_3) \mid j = 0, 1, 2 \}$, and let $f : \mathrm{R} \to \mathbf{N} \setminus \{1\}$ be the constant map 3; then $\Sigma = (\mathrm{R}, f)$ is a root system such that $\mathrm{W}(\Sigma)$ is conjugate to $(\mu_6 \mid \mu_2; \mathbf{T} \mid \mathbf{D}_2)$ [compare (4.4) (iv) and (4.10) (i)].

(4.13) DEFINITION. — Let $\Sigma = (R, f)$ be a pre-root-system. A *neat extension* of $\Sigma$ is a root system $(S, g)$ with the properties $S \supsetneq R$, $g|_R = f$, $g(S) = f(R)$, and

$$\{ |(a \mid b)| . |a|^{-1} . |b|^{-1} \mid a, b \in S \cap g^{-1}(2) \} \subset \{ |\cos(\pi k/m)| \mid k \in \mathbf{Z}, 0 < m \leqq d(W(\Sigma)) \}.$$

(4.14) LEMMA:

(i) *If* G *is a reflection group, and* $\Sigma$ *is a root system such that* $W(\Sigma)$ *is conjugate to a proper reflection subgroup of* G *with* $o(W(\Sigma)) = o(G)$ *and* $d(W(\Sigma)) = d(G)$, *then* $G = W(\Sigma')$ *for some root system* $\Sigma'$ *which is isomorphic to a neat extension of* $\Sigma$*;*

(ii) *no root system in* $\mathbf{C}^3$ *is a neat extension of* $\Sigma(3, 3, 3)$.

*Proof:*

(i) is obvious;

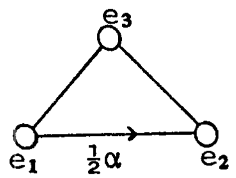
(ii) suppose $x \in \mathbf{C}^3$ is an element outside $\Sigma(3, 3, 3)$ contained in a neat extension. Changing the length of $x$ if necessary, we may assume $|x| = \sqrt{2}$. Then

$$|(x \mid \varepsilon_i - \omega \varepsilon_j)| = 0, 1$$

for any pair $i \neq j$ $(1 \leqq i, j \leqq 3)$. An easy computation shows that this is impossible.

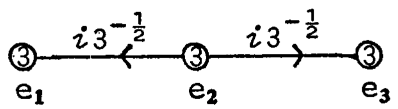
(4.15) We now present a number of vector graphs in order to construct primitive reflection groups.

By $J_3(4)$ we denote the vector graph



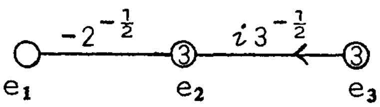

where $\alpha = 1/2(1 + i\sqrt{7})$ (this is a root of $X^2 - X + 2 = 0$), $e_1 = \varepsilon_2$, $e_2 = 1/2\,\bar{\alpha}(\varepsilon_2 + \varepsilon_3)$, $e_3 = -1/2(\varepsilon_1 + \varepsilon_2 - \alpha\varepsilon_3)$.

By $L_3$ we denote the vector graph



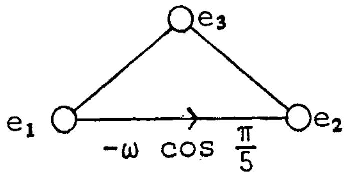

where $e_1 = \varepsilon_3$, $e_2 = i\,3^{-1/2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)$, $e_3 = \varepsilon_2$.

By $M_3$ we denote the vector graph




with $e_1 = 2^{-1/2}(\varepsilon_2 - \varepsilon_3)$, $e_2 = \varepsilon_3$, $e_3 = i\,3^{-1/2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)$.

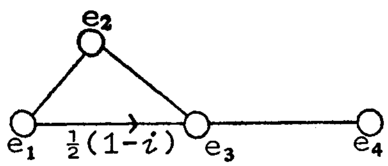
By $J_3(5)$ we denote the vector graph




where

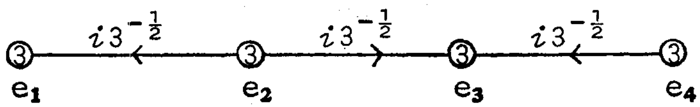
$$e_1 = \varepsilon_1, \qquad e_2 = -(\omega^2 \cos(\pi/5)\,\varepsilon_1 - \cos(3\pi/5)\,\varepsilon_2 + 1/2\,\omega\varepsilon_3),$$
$$e_3 = -(1/2\,\varepsilon_1 + \cos(3\pi/5)\,\varepsilon_2 + \cos(\pi/5)\,\varepsilon_3).$$

By $N_4$ we denote the vector graph



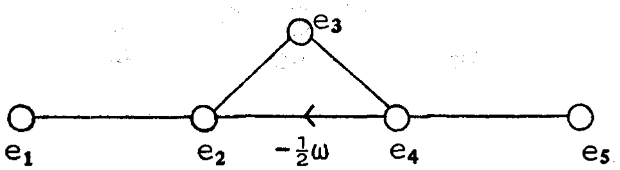

where $e_1 = 1/2\,(1+i)(\varepsilon_2 + \varepsilon_4)$, $e_2 = 1/2\,(1+i)(\varepsilon_3 - \varepsilon_2)$, $e_3 = 1/2\,(-\varepsilon_1 + i\,\varepsilon_2 - \varepsilon_3 + i\,\varepsilon_4)$, $e_4 = \varepsilon_1$.

By $L_4$ we denote the vector graph




where $e_1 = \varepsilon_3$, $e_2 = i\,3^{-1/2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)$, $e_3 = \varepsilon_2$, $e_4 = i\,3^{-1/2}(-\varepsilon_1 + \varepsilon_2 + \varepsilon_4)$.

By $K_5$ we denote the vector graph

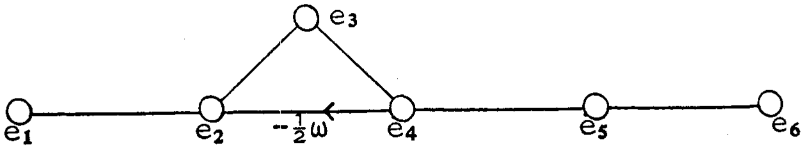


where

$$e_1 = \omega\, 2^{-1/2}(\varepsilon_5+\varepsilon_6), \qquad e_2 = -\omega\, 2^{-3/2}(-\varepsilon_1+(1+2\,\omega)\,\varepsilon_2+\varepsilon_3+\varepsilon_4+\varepsilon_5+\varepsilon_6),$$
$$e_3 = 2^{-1/2}(\varepsilon_1-\varepsilon_2), \qquad e_4 = 2^{-1/2}(\varepsilon_2-\varepsilon_3), \qquad e_5 = 2^{-1/2}(\varepsilon_3-\varepsilon_4).$$

Note that $e_i \in (\varepsilon_5-\varepsilon_6)^{\perp}$ for $i \in \underline{5}$.

By $K_6$ we denote the graph




where $e_1, e_2, e_3, e_4, e_5$ are as in (5.9) and $e_6 = -2^{-3/2}\omega^2(\varepsilon_1+\varepsilon_2+\varepsilon_3+{}_{(1+2\omega)}\varepsilon_4+\varepsilon_5-\varepsilon_6)$.

THEOREM. — *Let* $\Gamma = (B, w)$ *be any of the above vector graphs, of dimension, say, n. Put* $G = W(\Gamma)$, $R = G(B)$, *and* $\Sigma = (R, f)$ *where* $f: R \to \mathbf{N}\setminus\{1\}$ *is determined by* $w: B \to \mathbf{N}\setminus\{1\}$. *Then:*

(i) $\Gamma$ *is a root graph,* G *is a primitive reflection group in* $\mathbf{C}^n$, *and* $\Sigma$ *a root system in* $\mathbf{C}^n$ *with* $G = W(\Sigma)$;

(ii) *if* $\Gamma \neq N_4$, *then* $\Sigma$ *does not admit a neat extension in* $\mathbf{C}^n$;

(iii) *if* $\Gamma = N_4$, *then* $\Sigma$ *admits exactly one neat extension in* $\mathbf{C}^n$, *namely* $\Delta = (S, g)$ *where* $S = G(B) \cup G(\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4)$ *and* $g: S \to \mathbf{N}\setminus\{1\}$ *is the constant map 2;*

(iv) $\Gamma$ *is equivalent to its complex conjugate.*

The reflection group associated with the root system $\Delta$ in (iii) will be denoted by $EW(N_4)$. Thus $EW(N_4) = W(\Delta)$. Note that $EW(N_4)$ is primitive since it contains $W(N_4)$ as a 4-dimensional primitive subgroup.

We sketch a case-by-case proof:

(i) since the proofs in the different cases are rather similar, we shall only deal with the case $\Gamma = K_5$. As

$$s_{e_2}s_{e_1}s_{e_4}s_{e_2}s_{e_1}s_{e_5}s_{e_4}s_{e_3}\sqrt{2}\,e_2 \in \mu_6(\varepsilon_3+\varepsilon_4)$$

there is no problem in verifying that

$$(1)\quad R = G e_1 = \mu_6\Big\{\varepsilon_j \pm \varepsilon_k,\ \varepsilon_5+\varepsilon_6,\ \frac{1}{2}((-1-2\,\omega)(-1)^{k_1}\varepsilon_j+(-1)^{k_2}\varepsilon_k$$
$$+(-1)^{k_3}\varepsilon_l+(-1)^{k_4}\varepsilon_m-\varepsilon_5-\varepsilon_6)\ \Big|\ \prod_{p=1}^{4}(-1)^{k_p}=1,\ \{j,k,l,m\} \neq \underline{4}\Big\}$$

and that

$$|R| = 270.$$

It follows that $\Sigma$ is a root system, that G is a reflection group and that $\Gamma$ is a root graph. It remains to prove that G is primitive. To this end, note that $\Sigma$ is a neat extension of the pre-root-system spanned by the subset

$$\mu_6\left\{\varepsilon_j \pm \varepsilon_k, \varepsilon_5+\varepsilon_6, \frac{1}{2}((-1-2\omega)(-1)^{k_1}\varepsilon_1+(-1)^{k_2}\varepsilon_2+(-1)^{k_3}\varepsilon_3 +(-1)^{k_4}\varepsilon_4-\varepsilon_5-\varepsilon_6) \;\middle|\; \prod_{p=1}^{4}(-1)^{k_p}=1; j, k\in\{2, 3, 4\}, j\neq k\right\} \text{ of R.}$$

As the reflection group associated to this pre-root-system is conjugate to W $(A_5)$, the group G contains a primitive subgroup, and is therefore primitive itself;

(ii) again, we shall restrict ourselves to a single case, in fact to the case $\Gamma = J_3(4)$.

The assertion can be stated as follows.

(2) There is no nonzero $v \in \mathbf{C}^3 \setminus \mathbf{C}.\mathrm{R}$ with $|(v \mid x)|.|v|^{-1}.|x|^{-1} \in \{0, 1/2, 2^{-1/2}\}$ for every $x \in \mathrm{R}$.

To prove (2) we choose $v = \sum_{i=1}^{3} v_i \varepsilon_i \neq 0$ such that $|v| = 2$ and $|(v \mid x)|^2 \in \{0, 4, 8\}$ for any $x \in \mathrm{R}$. It is enough to show that the existence of $v$ leads to a contradiction. Setting $x = \varepsilon_i$, we get $|v_i|^2 \in \{0, 1, 2\}$ for each $i \in \underline{3}$. Furthermore $\sum_{i=1}^{3} |v_i|^2 = 4$, so up to a permutation of the coördinates (i. e. modulo action of G) we have either

(3) $$|v_1|^2 = |v_2|^2 = 2 \quad \text{and} \quad v_3 = 0,$$

(4) $$|v_1|^2 = 2 \quad \text{and} \quad |v_2| = |v_3| = 1.$$

Setting $x = \alpha(\varepsilon_i \pm \varepsilon_j)$, $\alpha\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3$ in the above condition on $v$, we get

(5) $$|v_1 \pm v_2|^2 \in \{0, 2, 4\},$$

(6) $$|v_2 \pm v_3|^2 \in \{0, 2, 4\},$$

(7) $$|v_1 \pm v_3|^2 \in \{0, 2, 4\},$$

(8) $$|\bar{\alpha} v_1 \pm v_2 \pm v_3|^2 \in \{0, 4, 8\}.$$

Suppose (3) holds; then (5) implies $v_1 = \pm i v_2$. Application of (8) leads to

$$3 \pm \sqrt{7} = |\bar{\alpha} i \pm 1|^2 \in \{0, 2, 4\},$$

which is absurd. Suppose (4) holds; then (6) implies $v_3 = \pm i v_2, \pm v_2$. From (5) we get that $v_1 = \pm \alpha v_2, \pm \bar{\alpha} v_2$ and from (7) that $v_1 = \pm \alpha v_3, \pm \bar{\alpha} v_3$.

So $v_3 = \pm v_2$, and $v = v_2(\alpha\varepsilon_1+\varepsilon_2+\varepsilon_3)$ or $v = v_2(\bar{\alpha}\varepsilon_1+\varepsilon_2+\varepsilon_3)$ (modulo action of G). The first case does not occur, for otherwise $v \in \mathrm{R}$, up to a scalar multiple. Thus we are left with the case

$$v = v_2(\bar{\alpha}\varepsilon_1+\varepsilon_2+\varepsilon_3) \quad \text{where} \quad v_2 \in \mathrm{U}.$$

Now (8) yields $\sqrt{2} = |\bar{\alpha}^2+2| \in \{0, 4, 8\}$, which is impossible. So (2) holds indeed, proving (iii);

(iii) first of all, note that $\Delta$ is a root system (this results from (1.6) (3) and $s_{\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4} \mathrm{S} = \mathrm{S}$).

Suppose $x = (x_1, x_2, x_3, x_4) \in \mathbf{C}^4 \setminus \mathrm{U.R}$ is an element of a neat extension of $\Sigma$ with $|x| = 2$. Then $|x_i| \in \{0, 1, \sqrt{2}\}$ for each $i \in \underline{4}$. Since $\sum_{i=1}^{4} |x_i|^2 = 4$, we have (up to the choice of $x$ within its G-orbit) one of the following three possibilities

(1) $$x_1 = x_2 = 0 \quad \text{and} \quad |x_3| = |x_4| = \sqrt{2},$$

(2) $$|x_j| = 1 \quad (j \in \underline{4}),$$

(3) $$x_1 = 0, \quad |x_2| = |x_3| = 1, \quad |x_4| = \sqrt{2}.$$

Calculation of inner products with $\varepsilon_i \pm \varepsilon_j$ yields

(4) $$|x_i \pm x_j| \in \{0, \sqrt{2}, 2\} \quad \text{for all} \quad i \neq j.$$

If $x_i = 0$ and $|x_j| = 1$ for $i \neq j$, then (4) leads to a contradiction; hence case (3) cannot occur. Suppose we have (1). From (4) we deduce $Re\, x_3 \bar{x}_4 = 0$; so $x \in \mathbf{C}(\varepsilon_3 \pm i \varepsilon_4)$ and a nonzero scalar multiple of $x$ is contained in R′.

In case (2) we have $Re\, x_i \bar{x}_j \in \{\pm 1, 0\}$. Therefore we may assume, by adapting $x$ if necessary, that

$$x \in \mathbf{C}(\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4) \cup \mathbf{C}(i \varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4).$$

If $x \in \mathbf{C}(i \varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4)$, then

$$|(x \mid \varepsilon_1+\varepsilon_2-i\varepsilon_3-i\varepsilon_4)| = |3i+1| \notin 4.\left\{0, \frac{1}{2}, 2^{-1/2}\right\},$$

so $x \in \mathbf{C}(\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4)$ and, again, a multiple of $x$ is contained in R′. Since $\mathrm{R} \cup \mathrm{R}' = \mathrm{S}$ consists of only one EW$(\mathrm{N}_4)$-orbit [for $s_{\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4}(2\varepsilon_1) = \varepsilon_1-\varepsilon_2-\varepsilon_3-\varepsilon_4$], there is no other neat extension of $\Sigma$ which is a root system but $\Delta$;

(iv) in case $\Gamma = \mathrm{K}_6$, for instance, $\Gamma$ is equivalent to the root graph spanned by $e_1, e_2, e_3, -s_{e_3} e_4, -e_5, -e_6$.

(4.16) Corollary. — The degrees, the regular degrees, and the numbers of reflections of given order of the reflection groups discussed in (4.15), as well as corresponding isomorphisms with classical groups, are as indicated in the following table.

$q$ is a non-degenerate quadratic form of index $[1/2(n-1)]$, and $h$ is a hermitian form on $\mathbf{F}_4^4$.

As to the proof, we will only treat the case $\Gamma = \mathrm{K}_5$ (in the notation of (4.15)). Put $\mathrm{R}_0 = \sqrt{2}\,\mathrm{R}$, and put $\mathrm{V}_0 = (\varepsilon_5-\varepsilon_6)^\perp \cap \mathbf{C}^6$. First of all we shall determine the degrees $d_1, d_2, \ldots, d_5$ of G in $\mathrm{V}_0$.

TABLE

| Shephard-Todd number | Group | Degrees | Regular degrees | Number of reflections of order 2 | 3 | Associated isomorphisms |
|---|---|---|---|---|---|---|
| 24. . . . . . . . . . . | $W(J_3(4))$ | 4, 6, 14 | 6,14 | 21 | – | $W(J_3(4)) \cong O_3(\mathbf{F}_7, q)$ |
| 25. . . . . . . . . . . | $W(L_3)$ | 6, 9, 12 | 9,12 | – | 24 | |
| 26. . . . . . . . . . . | $W(M_3)$ | 6, 12, 18 | 18 | 9 | 24 | |
| 27. . . . . . . . . . . | $W(J_3(5))$ | 6, 12, 30 | 30 | 45 | – | $W(J_3(5)) \cong O_3(\mathbf{F}_9, q)$ |
| 29. . . . . . . . . . . | $W(N_4)$ | 4, 8, 12, 20 | 20 | 40 | – | |
| 31. . . . . . . . . . . | $EW(N_4)$ | 8, 12, 20, 24 | 20,24 | 60 | – | |
| 32. . . . . . . . . . . | $W(L_4)$ | 12, 18, 24, 30 | 24,30 | – | 80 | $W(L_4)/\mu_2 \cong U_4(\mathbf{F}_4 \mid \mathbf{F}_2, h)$ |
| 33. . . . . . . . . . . | $W(K_5)$ | 4, 6, 10, 12, 18 | 10,18 | 45 | – | $W(K_5) \cong O_5(\mathbf{F}_3, q)$ |
| 34. . . . . . . . . . . | $W(K_6)$ | 6, 12, 18, 24, 30, 42 | 42 | 126 | – | $W(K_6)/\mu_3 \cong O_6(\mathbf{F}_3, q)$ |

Put

$$c = r_5 r_4 r_3 r_2 r_1 = \frac{1}{4}\begin{bmatrix} i\sqrt{3} & 1 & -i\sqrt{3} & -i\sqrt{3} & i\sqrt{3} & i\sqrt{3} \\ 1 & i\sqrt{3} & 3 & -1 & 1 & 1 \\ 1 & i\sqrt{3} & -1 & 3 & 1 & 1 \\ 3 & -i\sqrt{3} & 1 & 1 & -1 & -1 \\ 1 & i\sqrt{3} & -1 & -1 & 1 & -3 \\ 1 & i\sqrt{3} & -1 & -1 & -3 & 1 \end{bmatrix}.$$

Then $c^3$ has eigenvalues $-\omega, -\omega, -\omega, -1, -1, 1$; so $c$ has order 18. Let $\zeta_{18}$ be both a primitive 18-th root of unity and an eigenvalue of $c$. Then $\zeta_{18}^3 = -\omega$. Now $\zeta_{18}^7$ and $\zeta_{18}^{13}$ are conjugates of $\zeta_{18}$ over $\mathbf{Q}(\omega)$, and eigenvalues of $c$. Furthermore, $c$ has an eigenvalue 1 (since $c$ fixes $\varepsilon_5 - \varepsilon_6$) and two eigenvalues of the form $\zeta_{18}^{3k}$, $\zeta_{18}^{3j}$ with $j, k \in \{1, 3, 5\}$. Now

$$\begin{aligned} v &= (-1-4\omega)\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + (1-2\omega)\varepsilon_4 - \varepsilon_5 - \varepsilon_6 \\ &\quad + \zeta_{18}((-1-2\omega)\varepsilon_1 + \varepsilon_2 + 3\varepsilon_3 + 3\varepsilon_4 - \varepsilon_5 - \varepsilon_6) + \zeta_{18}^2(2\varepsilon_1 + 2\varepsilon_3 + 4\varepsilon_4) \end{aligned}$$

is a regular eigenvector in $V_0$ of $c$ corresponding to the eigenvalue $\zeta_{18}$. This implies that $(d_i)_{i \in \underline{5}} = 18+k_1 18,\ 12+k_2 18,\ 6+k_3 18,\ 19-3k+k_4 18,\ 19-3j+k_5 18$ where $k_l \geqq o\ (l \in \underline{5})$. Since there are exactly 45 reflections in G, the equality $\sum_{i=1}^{5} d_i = 50$ holds; hence $j+k = 8+9\left(\sum_{l=1}^{5} k_l\right)$. But $j+k \leqq 10$, so $k_l = 0$ for each $l \in \underline{5}$, and $j+k = 8$. This implies that $\{j, k\} = \{3, 5\}$; thus the degrees are as stated.

We have just seen that 18 is a regular degree. Since $|R| = 270$, we have by (4.11) (v) that 4 does not divide a regular degree. We shall now establish that 5 is the order of a regular element. By Sylow's theorem, $W(K_5)$ contains an element $g$ of order 5. Since $W(K_5) \subseteq Gl(\mathbf{Q}(\omega))$, the eigenvalues of $g$ are all distinct 5-th roots of unity.

Suppose now that $g$ is not regular. Then there is an eigenvector of $g$ (corresponding to the eigenvalue 1) which is a root of $W(K_5)$. Up to conjugacy of $g$, we may assume this eigenvector to be $\varepsilon_5+\varepsilon_6$. It results from (1.5) that $g$ is an element of the group generated by the reflections with roots in $R_0 \cap \{\varepsilon_5+\varepsilon_6\}^{\perp}$, which is a group conjugate to $W(D_4)$.

As 5 does not divide $|W(D_4)|$, this leads to a contradiction. We conclude that there is a regular element of order 5 and (since $|Z(W(D_4))| = 2$) one of order 10, too.

We leave the determination of the number of reflections of order 2 and 3 to the reader and finish by settling the associated isomorphisms.

Let $\sigma : \mathbf{Z}[1/2, \omega] \to \mathbf{F}_3$ be the homomorphisms "reduction mod 3" determined by $\sigma(\omega) = \sigma(-1/2) = 1$, and let $\varphi : (\mathbf{Z}[1/2, \omega])^6 \to \mathbf{F}_3^6$ be induced by $\sigma$. Put $W_0 = \varphi(V_0)$, and $\varepsilon'_5 = \varepsilon_5+\varepsilon_6$. Define $b_0 : W_0 \times W_0 \to \mathbf{F}_3$ by

$$b_0\left(\sum_{i=1}^{4} x_i\varepsilon_i+x_5\varepsilon'_5,\ \sum_{i=1}^{4} y_i\varepsilon_i+y_5\varepsilon'_5\right) = \sum_{i=1}^{4} x_i y_i - x_5 y_5 \quad (x_i,\ y_i \in \mathbf{F}_3).$$

Then

$$b_0(\varphi x, \varphi y) = \sigma((x \mid y)) \qquad \text{for} \quad x, y \in V_0 \cap \left(\mathbf{Z}\left[\frac{1}{2}, \omega\right]\right)^3.$$

This leads to a non-degenerate quadratic form $q_0$ of index 2. One easily establishes that $\varphi(R) = \{x \in W_0 \mid q_0(x) = -1\}$. Denoting by $\psi$ the homomorphism from $Gl_6(\mathbf{Z}[1/2, \omega])$ to $Gl_6(\mathbf{F}_3)$ defined in the obvious way with the help of $\varphi$, we get $\psi(G) \trianglelefteq O_5(\mathbf{F}_3, q_0)$ and $\psi(G^+) \trianglelefteq \Omega_5(\mathbf{F}_3, q_0)$. The latter group if the known simple one of order 25,920 (*cf.* [8]). Since $\psi(G^+) \neq 1$, we have $\psi(G^+) = \Omega_5(\mathbf{F}_3, q_0)$ and $\psi(G) = O_5(\mathbf{F}_3, q_0)$. Comparison of the orders of both G and $O_5(\mathbf{F}_3, q_0)$ shows that the restriction of $\psi$ to G is the desired isomorphism.

## 5. The primitive case

In this chapter we shall show that for any complex primitive $n$-dimensional reflection group G ($n \geqq 3$) there is a root system $\Sigma$ as discussed in (4.15) such that $W(\Sigma)$ is conjugate to G, thus completing the classification of all complex reflection groups. Prior to this classification we shall give a useful theorem of Blichfeldt (*see* [9]; the idea of the proof goes back to Frobenius, [12]).

The main goal of this chapter is the proof of (5.12). Put $V = \mathbf{C}^n$.

(5.1) THEOREM (Blichfeldt). — *Let* G *be a finite group of unitary automorphisms of* V. *Let* $g \in G$ *and let* $\{\zeta_1, \zeta_2, \ldots, \zeta_m\}$ *be the set of distinct eigenvalues of* $g$. *Denote* $V_j$ *the eigenspace of* $g$ *corresponding to* $\zeta_j$. *Suppose* $|\arg \zeta_j \zeta_1^{-1}| \leqq \pi/3$ *for all* $j \in \underline{m}$:

(i) *if* $k \in G$ *has eigenvalues* $\eta_1, \eta_2, \ldots, \eta_n$ *such that there is an* $\alpha \in U$ *with* $|\arg \alpha\eta_i| < 1/2\,\pi$ *for all* $i \in \underline{n}$, *then* $k V_1 = V_1$;

(ii) *if* G *is primitive, then* $g \in Z(G)$;

(iii) *if* $g$ *has only* 2 *different eigenvalues and* $k$ *is as in* (i), *then* $gk = kg$.

(5.2) COROLLARY. — *Let* $\dim V = n \geqq 3$, *and let* G *be a finite primitive group of unitary transformations of* V:

(i) *if* H *is a primitive* 2*-dimensional reflection subgroup of* G *then* H *is conjugate to* $W(L_2)$ (*i. e. to* $(\mu_6 \mid \mu_2; \mathbf{T} \mid \mathbf{D}_2)$);

(ii) G *does not contain any reflection of order* $\geqq 4$; *if* G *contains a reflection of order* 3, *then* G *contains a reflection subgroup conjugate to* $W(L_2)$;

(iii) *suppose* H *is an irreducible r-dimensional reflection subgroup of* G, *and* $1 < r < n$; *then* $|Z(H)| < 6$; *if* $r = n-1$, *then* $|Z(H)| < 4$; *if* $r = n-1$ *and* $|Z(H)| = 3$, *then* $\langle G, \omega I_n \rangle$ *contains a reflection subgroup conjugate to* $W(L_2)$;

(iv) *if* $m, p, r \in \mathbf{N}$ *are such that* $p \mid m$, $m > 1$, $r \geqq 2$, *and* $G(m, p, r)$ *is a reflection subroup of* G, *then* $m \leqq 5$ *and* $m \leqq 3p$; *in particular,* $d(G) \leqq 5$;

(v) *if* H *is a primitive r-dimensional reflection subgroup of* G *with* $r < n$, *then* $|Z(H)| < 4$;

(vi) *any irreducible* 2*-dimensional root graph* $\Gamma$ *with* $W(\Gamma) \subseteq G$ *is equivalent to* $I_2(m)$ $(3 \leqq m \leqq 5)$, $B_2^3$, *or* $L_2$.

*Proof.* — By (ii) of the previous theorem, there is no element $g \in G$ having the set $\{1, \zeta\}$ of distinct eigenvalues if $\zeta$ is a primitive root of unity of order $\geqq 6$. This implies the first part of (iii), the absence of reflections of order $\geqq 6$, and the absence of 2-dimensional primitive reflection subgroups not conjugate to $(\mu_6 \mid \mu_2; \mathbf{T} \mid \mathbf{D}_2)$, $(\mu_{12} \mid \mu_4; \mathbf{T} \mid \mathbf{D}_2)$, $(\mu_8 \mid \mu_4; \mathbf{O}, \mathbf{T})$, $(\mu_4 \mid \mu_2; \mathbf{O} \mid \mathbf{T})$, $\mu_4.\mathbf{O}$, $\mu_4.\mathbf{I}$ [*cf.* (3.6)].

Since $(\mu_8 \mid \mu_4; \mathbf{O} \mid \mathbf{T})$, $(\mu_4 \mid \mu_2; \mathbf{O} \mid \mathbf{T})$, and $\mu_4.\mathbf{O}$ contain the element

$$2^{-1/2}.\exp\frac{\pi i}{4}\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \in \mathbf{T}$$

[*cf.* (3.1)], and since $\mu_4.\mathbf{I}$ contains the element $-1/\sqrt{5}\begin{pmatrix} \eta^2-\eta^4 & \eta^4-1 \\ 1-\eta & \eta^3-\eta \end{pmatrix}$ where $\eta = \exp(2\pi i/5)$ [*cf.* (3.1)], each of these four groups has an element with eigenvalues $-\omega$, $-\omega^2$. Another application of Theorem (6.2) shows that a group H as described in (i) must be conjugate to $(\mu_6 \mid \mu_2; \mathbf{T} \mid \mathbf{D}_2)$ or $(\mu_{12} \mid \mu_4; \mathbf{T} \mid \mathbf{D}_2)$. Suppose now that G contains reflections of order $\geqq 3$. Denote the orbit of a root of such a reflection by B. If we have $(x \mid y) = 0$ for each linearly independent pair $x, y \in B$, then the orbit B gives rise to a system of imprimitivity of G. Thus there are $x, y \in B$ with $(x \mid y) \neq 0$ and $\mathbf{C}x \neq \mathbf{C}y$. Considering the subgroup of G generated by all reflections with root $x$ or $y$, we obtain from (2.2) that there exists a primitive 2-dimensional reflection subgroup in G. This subgroup must be conjugate to $(\mu_6 \mid \mu_2; \mathbf{T} \mid \mathbf{D}_2)$ or $(\mu_{12} \mid \mu_4; \mathbf{T} \mid \mathbf{D}_2)$, as we have previously seen. As a consequence, there are no roots of G of order $\geqq 4$ (for neither $(\mu_6 \mid \mu_2; \mathbf{T} \mid \mathbf{D}_2)$ nor $(\mu_{12} \mid \mu_4; \mathbf{T} \mid \mathbf{D}_2)$ does contain reflections of order $\geqq 4$). This proves the first part of (ii).

If G contains an $(n-1)$-dimensional reflection subgroup with center of order $m$, then $\langle G, e^{2\pi i/m} I_n \rangle$ is a primitive group containing reflections of order $m$; hence $m < 4$.

Let H be as in (v). We may assume that $r > 1$. Denote by K the normal subgroup of G generated by all reflections of G. The primitivity of G implies that K is irreducible in dimensions $n$. Note that $\mid Z(H) \mid < 4$ if H is conjugate to $W(A_r)$. If H is not conjugate to $W(A_r)$, apply (2.6) to an irreducible $(r+1)$-dimensional reflection subgroup of K containing H, and use the above argument to obtain $\mid Z(H) \mid < 4$, and (v).

As $\mid Z((\mu_{12} \mid \mu_4; \mathbf{T} \mid \mathbf{D}_2)) \mid = 4$, there is no reflection subgroup of G or $\langle G, \omega I_n \rangle$ conjugate to this one. If H is as in (iii) and $r = n-1$, then there is a reflection of order 3 in $\langle G, \omega I_n \rangle$; it follows by the argument before that this group contains a subgroup conjugate to $(\mu_6 \mid \mu_4; \mathbf{T} \mid \mathbf{D}_2) = W(L_2)$. This finishes the proofs of (i), (ii) and (iii).

As to the proof of (iv), remark that $G(m, p, r)$ has reflections of order $p^{-1}.m$. So $m \leqq 3p$. Use the element

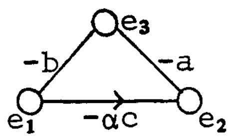
$$\begin{pmatrix} e^{2\pi i/m} & & \\ & e^{-2\pi i/m} & \\ & & I_{n-2} \end{pmatrix} \in G(m, m, r)$$

[considered as subgroup of $U_n(\mathbf{C})$] to derive from (6.2) (ii) that $m \leqq 5$.

Finally, (vi) is obtained by putting (i), (iv) and (4.4) (iv), (v) together (note that G (4, 2, 2) is not generated by 2 reflections).

We shall use the convention to write $r_i$ instead of $s_{e_i,\, w(e_i)}$ throughout the rest of this chapter.

(5.3) Let G be a complex 3-dimensional finite group generated by reflections of order 2. Put $m = d(G)$. By (4.6) there is a root graph $\Gamma$:




with $a, b, c \in \mathbf{R}_{>0}$, $\alpha \in \mathbf{U} \setminus \mathbf{R}$, $c = \cos(\pi/m)$, $m = d(\Gamma)$, and $W(\Gamma) \subseteq G$.

PROPOSITION:

(i) *if* $m \leqq 5$, *then* $\Gamma$ *is equivalent to* $D_3(m)$ $(m \geqq 3)$ *or* $J_3(m)$ $(m \geqq 4)$;

(ii) *if* G *is primitive, then* G *is conjugate to* $W(J_3(m))$ $(m = 4, 5)$;

(iii) *no irreducible n-dimensional reflection group contains a reflection subgroup conjugate to* $W(J_3(5))$ *if* $n \geqq 4$.

The proof will be given in a number of steps.

Write

$$p = |(r_1 e_2 \mid e_3)|, \qquad q = |(r_2 e_1 \mid e_3)|, \qquad r = |(r_3 e_1 \mid e_2)|.$$

We shall frequently use that

(1) $$p,\ q,\ r,\ a,\ b,\ c \in \{ |\cos(\pi k/l)| \ \big|\ k \in \mathbf{Z},\ 0 < l \leqq m \} \setminus \{1\}.$$

Note that $p$, $q$, $r$ can be expressed in the data $a$, $b$, $c$, $\alpha$; for example,

(2) $$\begin{cases} p^2 = |(e_2 \mid e_3 + 2b e_1)|^2 = a^2 + 4b^2c^2 + 4\,Re\,\alpha\, abc \\ \text{and} \\ \qquad q^2 = b^2 + 4a^2c^2 + 4\,Re\,\alpha\, abc. \end{cases}$$

We shall refer to any formula of this kind by (2).

Subtraction gives

(3) $$p^2 - q^2 = (a^2 - b^2)(1 - 4c^2).$$

Again, (3) stands for all similar expressions.

A translation of $\det((e_i \mid e_j)) > 0$ [*cf.* (4.3) (vii)] in the present case is

(4) $$1 - a^2 - b^2 - c^2 - 2\,Re\,\alpha\, abc > 0.$$

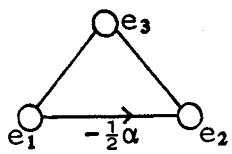
Elimination of $Re\,\alpha\, abc$ with the help of (2) yields

(5) $$a^2 + p^2 < 1 + (1 - 2b^2)(1 - 2c^2).$$

As $Re\,\alpha > -1$, it is evident from (2) that

(6) $$pqr \neq 0.$$

If $m = 3$, then $\Gamma$ is the root graph




with $\alpha \in \{\omega, \omega^2\}$; so $\Gamma$ is equivalent to $D_3$ (3) [this follows from

$$a = b = c = p = q = r = \frac{1}{2} \text{ and } Re\,\alpha = -\frac{1}{2},$$

*see* (1), (2), and (6)].

Now W ($D_3$ (3)) = G (3, 3, 3) is imprimitive. If G is primitive, then (4.14) leads to a contradiction with the existence of G. Therefore the primitivity of G implies that $m \geqq 4$.

Suppose $m = 4$; so $c = \cos(\pi/4) = 2^{-1/2}$. Denote by $\Gamma_0$ the root graph spanned by $e_1$ and $e_2$. We assert:

(iv) *there is a root* $v \in W(\Gamma_0) e_3$ *of* W ($\Gamma$) *such that* $\Gamma$ *is equivalent to the root graph spanned by* $e_1, e_2, v$ *and such that* $|(v \mid e_1)| = |(v \mid e_2)| = 1/2$.

*Proof of* (iv). — Assume that $a = b = 2$ does not hold (otherwise we are done).

If $a = 1/2$, $b = 2^{-1/2}$, we obtain from (3) that $q = r$ and $p^2 - q^2 = 1/4$.

Together with (1) and (6), it follows that $p = 2^{-1/2}$ and $q = r = 1/2$. Thus we can take $v = r_2 e_3$.

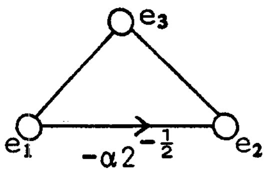
If $a = 2^{-1/2}$, $b = 1/2$ symmetry in the root graph leads to a choice of $v$ analogous to the one in the previous case.

If $a = b = 2^{-1/2}$, then the root graph spanned by $e_1$, $e_2$, $r_1 e_3$ is equivalent to $\Gamma$ [*cf.* (4.3) (vi)] and congruent to the root graph dealt with in the preceding case [as can be seen by computation of $p$ from (3), (5) and (6)]. This finishes the proof of (iv).

Calculating the inner products for the root graph spanned by $v$, $e_1$, $e_2$ [with the help of (2)] in the case $a = 2$, $b = 2^{-1/2}$ of the previous proof, we obtain the following corollary:

(v) *if* $2^{-1/2} \in \{p, q, a, b\}$, *then* $\Gamma$ *is equivalent to* $J_3$ (4).

We conclude from (iv) that $\Gamma$ is equivalent to




If $p = 1/2$ (or $2^{-1/2}$), then $Re\,\alpha = -2^{-1/2}$ (or $-2^{-3/2}$), and $\Gamma$ is equivalent to $D_3$ (4) (or $J_3$ (4)). Hence G contains a reflection group conjugate to either G (4, 4, 3) or W ($J_3$ (4)).

Suppose that G is primitive. If G contains a reflection group conjugate to G (4, 4, 3), then $\langle G, i I_3 \rangle$ is a primitive 3-dimensional finite group containing a conjugate of

$$\begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix},$$

i. e. a reflection of order 4. This contradicts (5.2) (ii). As a result, G has a reflection subgroup conjugate to W ($J_3$ (4)). From (4.14) (i) and (4.15) (ii), we obtain that G itself is conjugate to W ($J_3$ (4)). By now we have established (i) and (ii) in the case $m \leqq 4$.

Suppose $m = 5$; so $c = \cos(\pi/5) = (1+\sqrt{5})/4$. Note that

$$\{ |\cos(\pi k/5)| \mid k \in \mathbf{Z} \} = \{ 0, \cos(\pi/5), \cos(2\pi/5), 1 \} = \left\{ 0, \frac{1}{4}(\pm 1 + \sqrt{5}), 1 \right\}.$$

We assert

(vi) $a, b, p, q \notin \{ \cos(\pi/5), \cos(2\pi/5) \}$.

*Proof of* (vi). — By an argument similar to the one at the end of the proof of (iv), it suffices to prove that $a, b \notin \{ \cos(\pi/5), \cos(2\pi/5) \}$.

Suppose that either the group generated by $r_1$ and $r_3$ or the group generated by $r_2$ and $r_3$ is conjugate to $W(I_2(5)) = G(5, 5, 2)$. We may assume that $a = \cos(\pi/5)$ (after interchanging $e_1$ and $e_2$, replacing $e_3$ by $r_3 r_2 e_3$ and changing some signs if necessary). We have the following possibilities for $b$.

(*a*) $b = \cos(2\pi/5)$. Using (3), we get $q^2 - p^2 = (5+\sqrt{5})/8$; but this is impossible because of (1).

(*b*) $b = \cos(\pi/5)$. Bow (5) gives $p^2 < 1 - 1/4\sqrt{5} < 1/2$; so $p = 1/2, \cos(2\pi/5)$. The latter case ($p = \cos(2\pi/5)$) implies that the root graph spanned by $e_1, r_1 e_2, e_3$ is (up to congruence) as described in the previous case, which is absurd. So $p = q = r = 1/2$ [*cf.* (3)]. Application of (3) to the triangle associated with $e_1, e_2, r_2 e_3$ leads to

$$|(r_1 e_2 \mid r_2 e_3)|^2 - |(e_1 \mid e_3)|^2 = -\cos^2(\pi/5) \quad \text{and} \quad (r_1 e_2 \mid r_2 e_3) = 0.$$

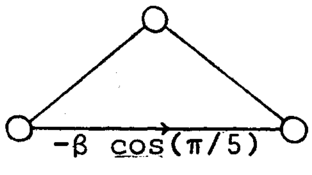
We find that $\Gamma$ is equivalent to a Coxeter graph, namely the one congruent to the root graph spanned by $r_2 e_3, e_1, r_1 e_2$. But this contradicts our assumption of $\Gamma$ being complex.

(*c*) $b = \cos(\pi/4)$. Because $e_1, e_2, r_3 e_1$ span a Coxeter graph (up to congruence) with $|(e_1 \mid e_2)| = \cos(\pi/5)$, it follows that $r = |(e_2 \mid r_3 e_1)| = 1/2$. From (3) we get $p = r = 1/2$ and $q^2 = p^2 + (b^2 - a^2)(1 - 4c^2) = 0$; but this is in contradiction with (6).

(*d*) $b = \cos(\pi/3)$. Thanks to (3), we have $p^2 - q^2 = -\cos^2(\pi/5)$, which is impossible because of (1) and (6). This contradiction establishes (vi).

The conclusion is that $a, b, p, q \in \{ 1/2, 2^{-1/2} \}$. We shall now finish the proof of the proposition in case $m = 5$.

Suppose $b = \cos(\pi/4)$. The root graph associated with $e_1, r_3 e_1, e_2$ is congruent to a Coxeter graph; so $r = |(r_3 e_1 \mid e_2)| = 1/2$. Note that (3) yields $q^2 = (1+\sqrt{5})/8$ if $a = \cos(\pi/4)$; this contradicts (1). Therefore we have $a = 1/2$, and, using (3) once more, $q = 1/2$. Now $\Gamma$ is equivalent to the root graph spanned by $e_1, e_2, r_2 e_3$. This root graph on its turn is congruent to




for some $\beta \in \mathbf{U}$. Thus the proof of (i) is complete in the case $b = \cos(\pi/4)$.

If we have $a = \cos(\pi/4)$, we can reason similarly, thanks to symmetry. Therefore we may assume that $a = b = 1/2$ (up to the equivalence of $\Gamma$).

In view of (1), (2), (6), and (vi), we have but two possibilities:

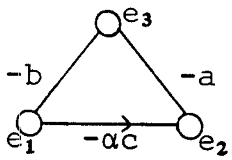
$$\begin{cases} p = \dfrac{1}{2} & \Rightarrow \quad Re\,\alpha = -\cos(\pi/5) \quad \text{and } \Gamma \text{ is equivalent to } D_3(5), \\ p = 2^{-1/2} & \Rightarrow \quad Re\,\alpha = -\dfrac{1}{2} \quad \text{and } \Gamma \text{ is equivalent to } J_3(5). \end{cases}$$

If G is primitive, we can argue as in the case where G contains a conjugate of G (4, 4, 3) in order to obtain that G does not obtain a conjugate of G (5, 5, 3). So far, we have established (i) and (ii) in case $m \leqq 5$.

Since $m \leqq 5$ whenever G is primitive [*see* (5.2) (iv)], we are done with (i) and (ii).

As to (iii) of the proposition, note that $|Z(W(J_3(5)))| = 6$ [*cf.* (4.16) (ii) and (1.4) (ii)]. Now (iii) is a direct consequence of (2.6), and (5.2) (iii).

(5.4) LEMMA. — *Let* $\Gamma$ *be the real root graph*




*where* $a, b, c \in \mathbf{R}_{>0}$, $\alpha \in \{-1, 1\}$, $c = \cos(\pi m^{-1})$, *and* $m = d(\Gamma) = d(W(\Gamma))$*:*

(i) *if* $m \leqq 4$, *then at least two of the* 3 *values* $(r_1 e_2 \mid e_3)$, $(r_2 e_3 \mid e_1)$, $(r_3 e_1 \mid e_2)$ *are* 0.

*Suppose* $m = 5$. *Denote by* $\Gamma_0$ *the sub-root-graph of* $\Gamma$ *spanned by* $e_1, e_2$*;*

(ii) *at least one of the* 2 *values* $(r_1 e_2 \mid e_3)$, $(r_2 e_1 \mid e_3)$ *equals* 0*;*

(iii) *there is* $v \in W(\Gamma_0) e_3$ *with* $(v \mid e_1) = 0$.

*Proof.* — We will adopt the notation of (5.3) concerning $p, q, r$. Note that (1), (2), (3), (4), (5) of (5.3) hold in this case too.

(i) from (1) and (4) of (5.3), we obtain $Re\,\alpha < 1$. Hence $\alpha = Re\,\alpha = -1$. By (5.3) (2), we have $p = |a - 2bc|$. Similar equalities hold for $q$ and $r$. From this it is immediate that $a = b = c = 1/2$ implies $p = q = r = 0$. From (5.3) (1) and computation of $p$, it is clear that we are left with the case $a = 1/2$, $b = c = 2^{-1/2}$ (up to a permutation of $a, b, c$). But then $p = 1/2$ and $q = r = 0$; so we are through with (i);

(ii) suppose $pq \neq 0$. Since $\Gamma$ is real and irreducible, $\Gamma$ is equivalent to the Coxeter graph $H_3$, and $|(x \mid y)| \neq 2^{-1/2}$ for any pair $x, y$ of unitary roots of $W(\Gamma)$. We will check (ii) for all possible values of $a$ and $b$ up to symmetry.

(*a*) $a = \cos(2\pi/5)$, $b = 1/2$. Then $q = |1/2 \pm 2\cos(2\pi/5)\cos(\pi/5)| = 0, 1$ [by (2) of (6.4)]; but this is absurd.

(*b*) $a = b = 1/2$. Now
$p = | 1/2 \pm \cos(\pi/5) | = (3+\sqrt{5})/4, (-1+\sqrt{5})/4.$
So $p = (-1+\sqrt{5})/4 = \cos(2\pi/5)$. Replacement of $e_3$ by $r_1 e_3$ leads to the previous case.

(*c*) $a = \cos(\pi/5)$, $b = 1/2$ leads to the absurdity $p = | \cos(\pi/5) \pm \cos(\pi/5) |$.

(*d*) $a = \cos(\pi/5)$, $b = \cos(2\pi/5)$ yields

$$q = |\cos(2\pi/5) \pm 2\cos^2(\pi/5)| = 1, \frac{1}{2}(1+\sqrt{5}),$$

which is, again, a contradiction.

(*e*) $a = b = \cos(2\pi/5)$. Since $p = | \cos(2\pi/5) \pm 1/2 |$, we must have that $p = \cos(\pi/5)$. After replacing $e_3$ by $r_1 e_3$, we are back in the preceding case.

(*f*) $a = b = \cos(\pi/5)$. Since $p = 1+1/2\sqrt{5}$, $1/2$, we have $p = 1/2$. Therefore replacing $e_3$ by $r_1 e_3$ reduces the check to case (*c*);

(iii) from (ii) we obtain that $pq = 0$. If $q = 0$, we can take $v = r_2 e_3$. If $p = 0$, then by $r_1 r_2 r_1 r_2 e_1 \in \mathrm{U}\, e_2$ we have

$$(r_2 r_1 r_2 e_3 \mid e_1) = (r_1 e_3 \mid r_1 r_2 r_1 r_2 e_1) = 0,$$

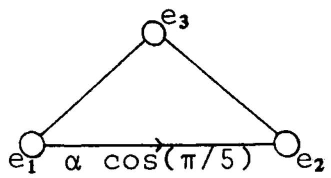
and $v = r_2 r_1 r_2 e_3$ is a proper choice.

(5.5) PROPOSITION. — *Let* G *be a primitive n-dimensional complex reflection group* ($n \geqq 3$) *generated by reflections of order* 2, *and let* $m = d(\mathrm{G}) \geqq 5$. *Then* $n = 3$, *and* G *is conjugate to* W (J$_3$ (5)).

*Proof.* — Because of (5.2) (iv), we only need consider the case $m = 5$. In view of (4.14) (i), (4.15) (ii) and (5.3) (ii), (iii), it suffices to show that there is a complex primitive 3-dimensional reflection subgroup $\mathrm{G}_0$ of G with $d(\mathrm{G}_0) = 5$.

According to (4.6), there is an irreducible 3-dimensional complex root graph $\Gamma$ with $d(\Gamma) = 5$ and $\mathrm{W}(\Gamma) \subseteq \mathrm{G}$. If $\Gamma$ is primitive, we can take $\mathrm{G}_0 = \mathrm{W}(\Gamma)$.

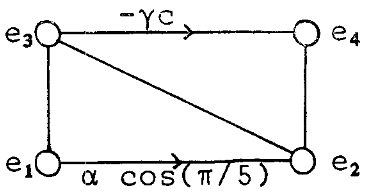
The case that $\Gamma$ is imprimitive remains to be considered. Up to conjugacy of G, we may assume [*cf.* (2.4) and (4.4) (v)] that $\Gamma$ is the root graph




where $\alpha \in \mathrm{U}$ with $Re\, \alpha = \cos(\pi/5)$.

By (2.9) there is a unitary root $e'_4$ of G such that $e_1$, $e_2$, $e'_4$ span a primitive root graph $\Gamma'$ with $w(\Gamma') \subseteq \mathrm{G}$ and $m(\Gamma') = 5$. If $\Gamma'$ is complex, then $\mathrm{G}_0 = \mathrm{W}(\Gamma')$ is as required.

Suppose that $\Gamma'$ is real. By (5.4) (iii) there is a unitary root $e_4$ of $W(\Gamma')$ such that $e_1, e_2, e_4$ span a root graph congruent to a Coxeter graph and equivalent to $\Gamma'$. We may assume that $(e_1 \mid e_4) = 0$ and $(e_2 \mid e_4) = -1/2$. Putting $(e_3 \mid e_4) = -\gamma c$ ($c \in \mathbf{R}_{>0}$, $\gamma \in \mathbf{U}$), we have that $e_1, e_2, e_3, e_4$ span the vector graph $\Delta$:




(Note that we have drawn a connection between $e_3$ and $e_4$, although the possibility $\gamma c = 0$ is not yet excluded.)

Now $W(\Delta)$ is a reflection subgroup of G. If $e_1, e_2, e_3, e_4$ are linearly dependent, then $G_0 = W(\Delta)$ is as required. We are left with the case where

(1) $\Delta$ is a root graph.

Because $\det((e_i \mid e_j)) > 0$ [*see* (4.3) (vii)], we have the inequality

$$(2) \qquad 5/16 - c^2 + \left(c^2 - \frac{1}{2}\right)\cos^2(\pi/5) - \frac{1}{2} c \, Re\,\gamma + \frac{1}{2} c . \cos(\pi/5) . Re\,\alpha\bar{\gamma} > 0.$$

Consequently

(3) $\gamma c \notin \{0, -1/2, 1/2\,\alpha^2\}$,

as will be used later on.

Another way of expressing that $|(r_1 e_3 \mid r_2 e_4)| = \cos(\pi k/l)|$ for some $k \leqq l \leqq 5$ is

$$\left|\alpha \cos(\pi/5) - \frac{1}{2} - \gamma c\right| \in \left\{0, \frac{1}{2}, 2^{-1/2}, \cos(\pi/5), \cos(2\pi/5), 1\right\}.$$

As a result we have that

(4) $\gamma c \notin \{1/2\,\omega, 1/2\,\omega^2\}$.

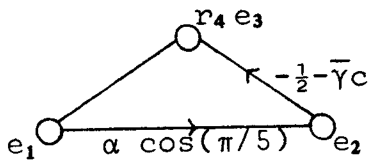
From (1) and (3) it is clear that $r_3 e_4, e_1, e_2$ span a root graph $\Gamma''$ (with $m(\Gamma'') = 5$) which is a cycle. If $\Gamma''$ is complex and primitive, we have found our $G_0$ (namely $W(\Gamma'')$).

On the other hand, if $\Gamma''$ is complex and imprimitive, we have by (5.3) (i) that $\Gamma''$ is equivalent to $D_3(5)$. Since $|(x \mid y)| \neq \cos(\pi/4)$ for any pair $x, y$ of unitary roots of $W(D_3(5)) = G(5, 5, 3)$, it follows from (5.3) (vi) that $|(r_3 e_4 \mid e_2)| = |(r_3 e_4 \mid e_1)| = 1/2$. This implies that $c = |1/2 + \gamma c| = 1/2$; so $Re\,\gamma = -1/2$. According to (4), this is impossible.

Thus we may (and shall) assume that $\Gamma''$ is real; by (4.5) (i) this comes down to

(5) $\alpha(\bar{\gamma} + 2c) \in \mathbf{R}$.

A similar reasoning with the triangle




instead of $\Gamma''$ shows that we may also assume:

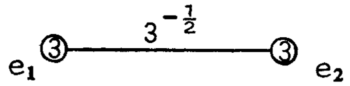
(6) $\alpha(1+2\bar{\gamma} c) \in \mathbf{R}$.

From (5) and (6), we obtain (since $c > 0$) that

(7) $\alpha(\bar{\gamma}+1) \in \mathbf{R}$.

If $c \neq 1/2$, then $\alpha \in \mathbf{R}$, which is impossible. Hence $c = 1/2$. Putting $\gamma = -1+\alpha r$ ($r \in \mathbf{R}$), we get from (3) that $\gamma \neq -1, \alpha^2$ and $r \neq 0,\ 2\, Re\, \alpha$. This contradiction with $1 = |\gamma|^2 = |-1+\alpha r|^2$ completes the proof.

(5.6) Let G be a complex primitive 3-dimensional reflection group generated by reflections of order 3. It follows from (6.3) (vi) and (4.12) (ii) that $|(v \mid w)| \in \{0, 3^{-1/2}\}$ if $v$, $w$ are two distinct unitary roots of G of order 3. In view of (5.2) (ii) and (4.4) (iv), there is a root graph $\Gamma_0$:




with $W(\Gamma_0) \subseteq G$. Let $e_3$ be a unitary root of G not contained in $\mathbf{C}e_1+\mathbf{C}e_2$ such that $(e_2 \mid e_3) \neq 0$ (since G is irreducible there is a unitary root $w \notin \mathbf{C}e_1+\mathbf{C}e_2$ such that $(w \mid e_1) \neq 0$ or $(w \mid e_2) \neq 0$; if $(w \mid e_2) = 0$, take $e_3 = r_1 e_2$; this shows that such a unitary root $e_3$ exists). Let $\Gamma$ be the root graph spanned by $e_1, e_2, e_3$ (in other words, let $\Gamma = (B, w)$ be the root graph with $B = \{e_1, e_2, e_3\}$ and $w(B) = \{3\}$).

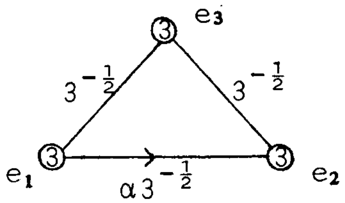
PROPOSITION:

(i) $\Gamma$ *is equivalent to* $L_3$;

(ii) G *is conjugate to* $W(L_3)$.

*Proof:*

(i) If $\Gamma$ is a root graph without cycles, then $\Gamma$ is clearly congruent to $L_3$. Suppose $\Gamma$ is a triangle; then $\Gamma$ is the root graph




where $\alpha \in \mathbf{U}$ (up to congruence).

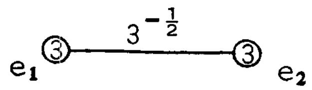
If $(r_3^j e_1 \mid e_2) = 0$ for some $j \in \underline{2}$, then $\Gamma$ is equivalent to the root graph spanned by $r_3^j e_1, e_2, e_3$; but the latter root graph has no cycles. Therefore we can restrict ourselves to the case $(r_3^j e_1 \mid e_2) \neq 0$ $(j = 1, 2)$. But now

$$|(r_3^j e_1 \mid e_2)| = 3^{-1/2}, \quad \text{or} \quad |\alpha - 3^{-1/2}(1-\omega^j)| = 1 \quad (j = 1, 2),$$

which is impossible. This brings the proof of (i) to an end;

(ii) follows from (i) and (4.15) (ii).

(5.7) Suppose G is a complex primitive 3-dimensional reflection group containing reflections of order 2 as well as of order 3. Again, there is a root graph $\Gamma_0$:




with $W(\Gamma_0) \subset G$. In view of (2.2) (v) and (5.2) (vi) we have that $|(v \mid w)| \in \{0, 2^{-1/2}\}$ if $v$ is a unitary root of order 2 and $w$ is a unitary root of order 3.—Take a unitary root $e_3$ of order 2 such that $(e_2 \mid e_3) \neq 0$ (if all roots of order 2 were perpendicular to $e_2$, they would span a proper G-invariant linear subspace of V; hence such a unitary root $e_3$ does exist). Denote by $\Gamma$ the root graph spanned by $e_1, e_2, e_3$ (the orders of the roots being as indicated above).

PPROPOSITION:

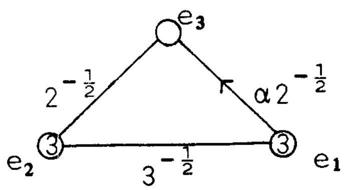
(i) $\Gamma$ *is equivalent to* $M_3$*;*

(ii) G *is equivalent to* $W(M_3)$*;*

(iii) *if* $n > 3$ *and* H *is an irreducible n-dimensional reflection group, then* H *does not contain a reflection group conjugate to* $W(M_3)$*; if* H *is primitive,* H *does not have reflections of order* 2 *and* 3 *at the same time.*

*Proof:*

(i) if $(e_1 \mid e_3) = 0$, then $\Gamma$ is necessarily congruent to $M_3$. Suppose $(e_1 \mid e_3) \neq 0$. Now $\Gamma$ is the root graph



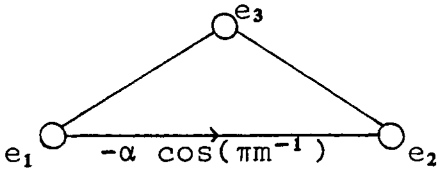

for some $\alpha \in \mathbf{U}$ (up to congruence). As in (5.6), we need only consider the case that $(r_2^j e_1 \mid e_3) \neq 0$, or $|(r_2^j e_1 \mid e_3)| = 2^{-1/2}$ $(j \in \underline{2})$.

This leads to the same absurdity as in (5.6); hence (i).

(ii) is a direct consequence of (4.15) (ii);

(iii) follows from $|Z(W(M_3))| = 6$ [*see* (4.16)], (5.2) (iii), (2.6), and the above.

(5.8) Let G be a complex primitive finite subgroup of $U_4(\mathbf{C})$ generated by reflections of order 2. Put $m = d(G)$. By (4.6) and (5.3) (i) there is a root graph $\Gamma_0$:




with $\alpha \in U \setminus \mathbf{R}$, $m = d(\Gamma_0)$, and $W(\Gamma_0) \subseteq G$.

By (4.5) (iii) and (4.7) there is a unitary root $e_4$ of G such that $e_1, e_2, e_3, e_4$ span an irreducible root graph $\Gamma$ which is primitive if $m = 4$.

Furthermore, (5.5) implies that $m \leqq 4$.

PROPOSITION :

(i) $m = 4$, *and* $\Gamma$ *is equivalent to* $N_4$*;*

(ii) G *is conjugate to* $W(N_4)$ *or* $EW(N_4)$*;*

(iii) *no irreducible n-dimensional reflection group* $(n \geqq 5)$ *contains a group conjugate to* $W(N_4)$ (or $EW(N_4)$)*;*

(iv) *if* H *is a primitive n-dimensional reflection group* $(n \geqq 5)$, *then* $d(H) = 3$.

The rest of this section is made up of the proof of the proposition. Because of (5.3) (i), (ii), (2) we need only consider three distinct cases:

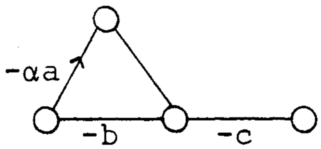
(*a*) $m = 3$, $\Gamma_0 = D_3(3)$, and $Re\, \alpha = -1/2$.

(*b*) $m = 4$, $\Gamma_0 = D_3(4)$, $Re\, \alpha = -2^{-1/2}$, and G does not contain a primitive 3-dimensional complex reflection group H with $d(H) = 4$.

(*c*) $m = 4$, $\Gamma_0 = J_3(4)$, and $Re\, \alpha = -2^{-3/2}$;

(v) *if* $\# \{ i \in \underline{3} \mid (e_i \mid e_4) \neq 0 \} = 1$, *then* $\Gamma$ *is congruent to* $D_4(m)$ *or* $W(N_4)$.

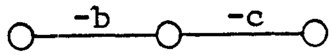
*Proof of* (v). – If $\Gamma$ is of the form




then $3/4 - a^2 - b^2 - c^2 + a^2 c^2 - ab\, Re\, \alpha > 0$ [by (4.3) (vii)]. If (*a*) or (*b*) holds, it follows that $c = 1/2$, and we are through. Suppose (*c*) holds, so $Re\, \alpha = -2^{-3/2}$. Up to symmetry we have either

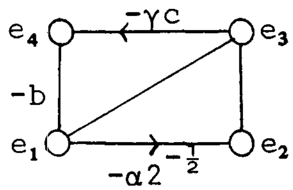
$$b = \frac{1}{2}, \quad a = 2^{-1/2} \quad \text{and} \quad c \in \left\{ \frac{1}{2}, 2^{-1/2} \right\} \quad \text{or} \quad b = 2^{-1/2} \quad \text{and} \quad a = c = \frac{1}{2}.$$

Note that $c \neq 2^{-1/2}$ in the latter case, since




is a real sub-root-graph of $\Gamma$. The above inequality shows that none of these cases occurs, whence (v).

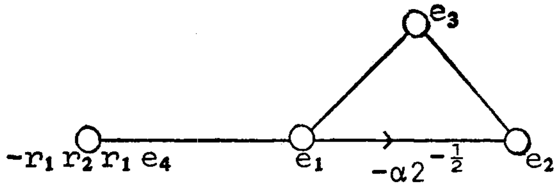
Suppose $\Gamma$ is the root graph



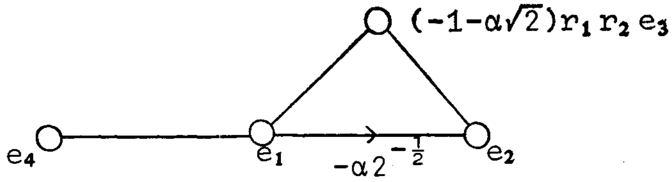

where $b, c \in \{1/2, 2^{-1/2}\}$, $\alpha, \gamma \in \mathrm{U}$, and $Re\,\alpha \in \{-2^{-1/2}, -2^{-3/2}\}$.

We assert:

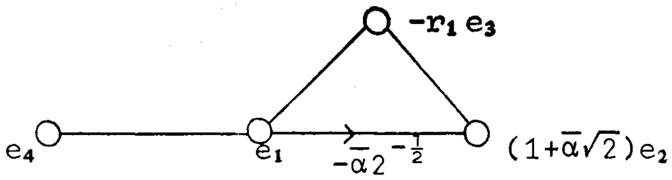
(vi) $b = 1/2$ *and* $Re\,\alpha = -2^{-1/2}$. *If* $c = 2^{-1/2}$, *then* $\gamma = \alpha$ *and* $\Gamma$ *is equivalent to*




*If* $c = 1/2$, *then either* $\gamma = 1 + \alpha\sqrt{2}$ *and* $\Gamma$ *is equivalent to*




*or* $\gamma = -1$ *and* $\Gamma$ *is equivalent to*




$\Gamma$ *is equivalent to* $N_4$ *in all cases.*

*Proof of* (vi). — Since $e_1, e_2, e_4$ span a root graph congruent to a Coxeter graph, we have $b = 1/2$. From (4.3) (vii), we get

$$(1)\qquad -\frac{3}{16}-\frac{1}{2}c^2-2^{-3/2}\,Re\,\alpha-\frac{1}{2}c\,Re\,\gamma-c\,2^{-3/2}\,Re\,\alpha\gamma>0.$$

As $m = 4$, we are in case (*b*) or (*c*). First of all we will show that case (*c*) does not occur. Suppose, in order to do this, that $\Gamma_0 = J_3(4)$ and $Re\,\alpha = -2^{-3/2}$. The root graph associated with $e_4, r_1 e_3, e_2$ is congruent to a Coxeter graph, while

$$|(r_1 e_3 \mid e_2)| = 2^{-1/2}.$$

This implies $|(e_4 \mid r_1 e_3)| = |1/2+\bar{\gamma}\,c| \in \{0, 1/2\}$. Since $c \in \{1/2, 2^{-1/2}\}$, there are three possibilities:

(*c* 1) $c = 1/2$ and $Re\,\gamma = -1$. But then equality holds in (1).

(*c* 2) $c = 1/2$ and $Re\,\gamma = -1/2$. As $(r_2 e_3 \mid e_1) = \alpha\,2^{-1/2}$ and $e_1, r_2 e_3, e_4$ span a root graph, we must have by (5.3) (i), (2) and (5.4) (i) that $Re\,\alpha\bar{\gamma} \in \{\pm 1, -2^{-1/2}, -2^{-3/2}\}$. But a straightforward computation shows that $Re\,\alpha\bar{\gamma} = 2^{-5/2}(1 \pm \sqrt{21})$.

(*c* 3) $c = 2^{-1/2}$ and $Re\,\gamma = -2^{-1/2}$. Now

$$Re\,\alpha\gamma = (1 \pm \sqrt{7})/4, \qquad \text{and} \qquad |(r_2 e_1 \mid r_4 e_3)|^2 = (3 \pm \sqrt{7})/4.$$

On the other hand, $|(r_2 e_1 \mid r_4 e_3)|^2 \in \{0, 1/4, 1/2\}$.

All three possibilities lead to an absurdity. We conclude that $\Gamma_0 \neq J_3(4)$, and that we are in case (*b*).

Now $\Gamma_0 = D_3(4)$. Let $c = 2^{-1/2}$. Application of (5.3) (2), (v) to the triangle corresponding to $e_1, e_3, e_4$ gives $\gamma = \alpha$ or $\bar{\alpha}$. In the latter case, inequality (1) yields the absurdity $-3/16 > 0$; so $\gamma = \alpha$. The assertions of (vi) in this situation are now easily checked. Suppose $c = 1/2$. Since $|(r_3 e_1 \mid e_4)| = 1/2\,|1+\gamma| \in \{0, 1/2, 2^{-1/2}\}$, there are three possible values for $Re\,\gamma$:

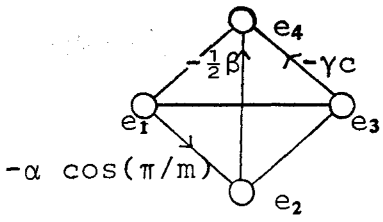
(*b* 1) $Re\,\gamma = -1$. This case is further discussed in the assertions.

(*b* 2) $Re\,\gamma = -1/2$. This implies the absurdity $|(r_1 e_2 \mid r_3 e_4)|^2 = (2 \pm \sqrt{3})/4$.

(*b* 3) $Re\,\gamma = 0$. If $\gamma = 1+\bar{\alpha}\sqrt{2}$, then $|(r_1 e_2 \mid r_3 e_4)|^2 = 5/4$. Hence $\gamma = 1+\alpha\sqrt{2}$. This case, too, is further described in the assertions.

By now, the proof of (vi) is complete;

(vii) *if* $\Gamma$ *is the root graph*




*where* $\alpha, \beta, \gamma \in \mathbf{U}$ *and* $c \in \mathbf{R}_{>0}$, *then* $\Gamma$ *has a real triangle.*

*Proof of* (vii). – Assume that all triangles of $\Gamma$ are complex. We will show that this leads to a contradiction.

Suppose $c = 1/2$. Applying (5.3) (2), (6) to the complex root graph spanned by $e_1, e_3, e_4$, we get $|(r_1 e_3 | e_4)| = 1/2 |1+\gamma| \in \{1/2, 2^{-1/2}\}$. Hence

(2) $$Re\,\gamma \in \left\{0, -\frac{1}{2}\right\}.$$

A similar argument with the triangle corresponding to $e_2, e_3, e_4$ yields

(3) $$Re\,\beta\bar{\gamma} \in \left\{0, -\frac{1}{2}\right\}.$$

Suppose $\beta = 1$, then

$$\det((e_i \mid e_j)) = \frac{3}{4}(1-\cos^2(\pi/m)) - (1+\cos(\pi/m)\,Re\,\alpha)\left(1+\frac{1}{2}Re\,\gamma\right).$$

Hence, if (*a*) holds, then $Re\,\gamma = -1/2$, and $\det((e_i | e_j)) = 0$; if (*b*) or (*c*) holds, then $Re\,\gamma \geqq -1/2$ and $Re\,\alpha = -2^{-1/2}$, so $\det((e_i | e_j)) \leqq 0$, in contradiction with (4.3) (vii). Therefore $\beta \neq 1$.

We now derive a contradiction for each of the three distinct $\Gamma_0$.

(*a*) $$m = 3 \quad \text{and} \quad Re\,\gamma = Re\,\alpha = Re\,\alpha\beta = Re\,\beta\bar{\gamma} = -\frac{1}{2}.$$

Put $\alpha = \omega$ (note that this does not harm the generality). Now $\alpha\beta = \omega$ or $\omega^2$. So $\beta = \omega$ and $\gamma\omega^2 = \omega$ or $\omega^2$. Since $\gamma = \omega$ or $\omega^2$, too, we have $\gamma = \omega^2$ and $\alpha = \beta = \bar{\gamma} = \omega$. This leads to the absurdity $\det((e_i | e_j)) = 0$.

(*b*) $$m = 4 \quad \text{and} \quad Re\,\alpha = -2^{-1/2}.$$

From (5.3) (2), (6), (v) we know that $|(r_1 e_4 | e_2)| = 1/2$. This implies $\alpha\beta = \bar{\alpha}$, and $\beta = \bar{\alpha}^2 = \pm i$. Moreover, $\beta\bar{\gamma} \in \{\omega, \omega^2, \pm i\}$ by (3). So $\gamma \in \{\pm i\omega, \pm i\omega^2, \pm 1\}$, in contradiction with (2).

(*c*) $$m = 4 \quad \text{and} \quad Re\,\alpha = -2^{-3/2}.$$

Put $\alpha = -2^{-3/2}(1+i\sqrt{7})$. If $|(r_1 e_4 | e_2)| = 1/2$, then $Re\,\alpha\beta = -2^{-1/2}$ by (5.3) (2). and $\beta = (1+\sqrt{7})/4 + i(1-\sqrt{7})/4$ or $(1-\sqrt{7})/4 - i(1+\sqrt{7})/4$, which is not in agreement with (2) and (3). The remaining possibility is $|(r_1 e_4 | e_2)| = 2^{-1/2}$. So $Re\,\alpha\beta = -2^{-3/2}$, Now $\beta = \bar{\alpha}^2 = (-3-i\sqrt{7})/4$, and again we end up in contradiction with (2) and (3).

The conclusion is that $c \neq 1/2$. This means that $c = 2^{-1/2}$. Arguing as in the case $c = 1/2$, we get

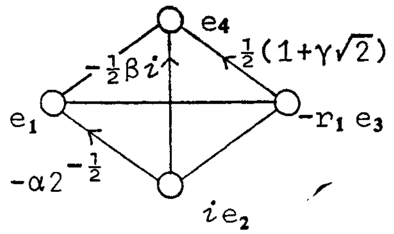
(4) $$Re\,\alpha,\quad Re\,\gamma,\quad Re\,\beta\bar{\gamma},\quad Re\,\alpha\beta \in \{-2^{-1/2}, -2^{-3/2}\}.$$

If $\beta = 1$, we have $\det((e_i | e_j)) = -5/8 - (1+2^{-1/2}\,Re\,\alpha)(1+2^{-1/2}\,Re\,\gamma) \leqq 0$. Hence $\beta \neq 1$.

As $m = 4$, case (*a*) does not occur. It remains to check the other two cases.

(*b*) $$Re\,\alpha = Re\,\gamma = Re\,\beta\bar{\gamma} = Re\,\alpha\beta = -2^{-1/2}$$

(for otherwise it follows from (4) that there is a sub-root-graph of $\Gamma$ equivalent to W ($J_3$ (4))). Put $\alpha = -2^{-1/2}(1+i)$. Now the root graph



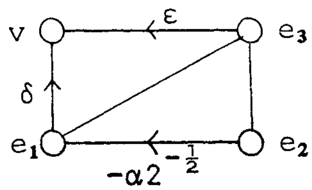

is of the type described in the case $c = 1/2$ provided all triangles are complex (note that $|1+\gamma\sqrt{2}| = 1$). Therefore one of the triangles must be real. No triangle but $ie_2$, $e_4$, $-r_1 e_3$ qualifies for this property [*cf.* (4.5) (i)]. It follows that

$$i\,\beta\,(1+\bar{\gamma}\sqrt{2}) = \pm 1, \qquad \text{whence} \quad \beta = \pm 1.$$

But $Re\,\alpha\beta = -2^{-1/2}$ then implies $\beta = 1$, which was excluded beforehand.

(*c*) $$Re\,\alpha = 2^{-3/2}.$$

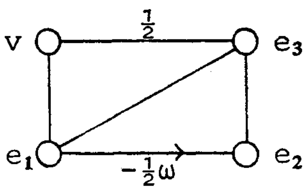
Put $v = r_3 r_2 r_4 e_3$. The roots $v, e_1, e_2, e_3$ (being linearly independent) span the root graph




for certain $\delta, \varepsilon \in \mathbf{C}$, contradicting (v) or (vi).

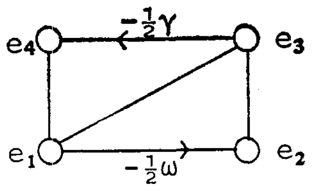
This brings the proof of (vii) to an end;

(viii) *suppose that* (*a*) *holds. Then there is a unitary root* $v \in W(\Gamma_0)\,U\,e_4$ *such that either* $\Gamma_0$ *and* $v$ *span the root graph*



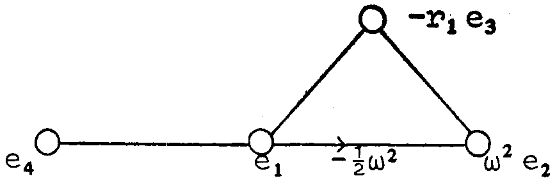

*or* $(v \mid e_i) \neq 0$ *for exactly one* $i \leqq 3$. *In any case,* $\Gamma$ *is equivalent to* $D_4$ (3).

*Proof of* (viii). – Because of (5.4) (i) and (5.8), we can restrict ourselves to the case where $\# \{ i \in \underline{3} \mid (e_i \mid e_4) \neq 0 \} = 2$. If $(e_1 \mid e_4) = 0$ (or $(e_3 \mid e_4) = 0$), replace $e_4$ by $r_1 r_2 e_4$ (or $r_3 r_2 e_4$) in order to obtain $(e_1 \mid e_4) \neq 0$, $(e_2 \mid e_4) = 0$, and $(e_3 \mid e_4) \neq 0$. Replacing $e_4$ by a suitable scalar multiple, we obtain that $\Gamma$ is



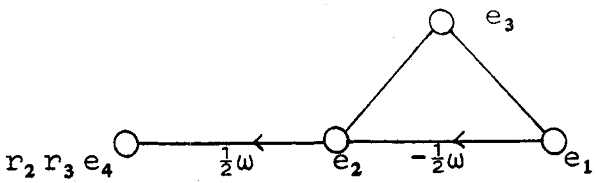

where $\gamma \in \mathbf{U}$. Application of (5.3) (1), (2), (6) to the triangle spanned by $e_1, e_3, e_4$ yields that $\gamma \in \{ -1, \omega, \omega^2 \}$. We verify (viii) for these three distinct values $\gamma$.

($c$ 4) $\gamma = -1$. Now $\Gamma$ is equivalent to




($c$ 5) $\gamma = \omega^2$. We have that $\mid (r_4 e_3 \mid r_2 e_1) \mid = \mid 1/2+\omega^2 \mid = 1/2\sqrt{3}$, which is absurd.

($c$ 6) $\gamma = \omega$. Now $\Gamma$ is equivalent to




This finishes the proof of (viii).

We will now prove that ($a$) does not occur. Suppose it does, then it is direct from (viii) that G contains a reflection group conjugate to $W(D_4(3)) = G(3, 3, 4)$. Hence $\langle G, \omega I_4 \rangle$ is a primitive finite group containing reflections of order 2 and 3, contradicting (5.7) (iii).

The conclusion is that $m = 4$. In order to prove the second part of (i), note that $\Gamma$ cannot be equivalent to $D_4(4)$, as $\Gamma$ is primitive. We may (and shall) assume that $(e_1 \mid e_4) \in \mathbf{R}_{>0}$. If $(e_2 \mid e_4) = 0$, then (i) follows from (v) and (vi). Let $(e_2 \mid e_4) \neq 0$. If $(e_3 \mid e_4) = 0$, replace $e_4$ by $r_3 r_2 e_4$ and apply (v) and (vi) once more. Thus we are left with the case that $(e_3 \mid e_4) \neq 0$ too. By (5.3) (iv) and (5.4) (i), we may assume that $(e_1 \mid e_4) = -1/2$, $\mid (e_2 \mid e_4) \mid = 1/2$, and that all triangles are complex (for otherwise we can reduce the situation to a previous one). But (vii) asserts that there is no such root graph.

By now, (ii) is a direct consequence of (i) and (4.15) (iii). Furthermore, (iii) follows from (5.2) (v), $|Z(W(N_4))| = 4$ [*cf.* (4.16)], and (2.6). Note that (iv) holds if H is real. Let H be complex. By (5.5), we must have $d(H) \leqq 4$. If $d(H) = 4$, then (4.7) together with (ii) implies that H contains a group conjugate to $W(N_4)$. But this contradicts (iii). Hence $d(H) = 3$, proving (iv).

(5.9) Let G be a primitive 4-dimensional reflection group containing reflections of order > 2. In view of (5.2) (ii) and (5.7) (iii), there is no reflection of order $\neq 3$ in G.

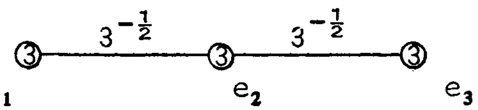
PROPOSITION :

(i) *there is a root graph* $\Gamma$ *equivalent to* $L_4$ *such that* $W(\Gamma) \subseteq G$;

(ii) G *is conjugate to* $W(L_4)$;

(iii) *no n-dimensional primitive reflection group* ($n \geqq 5$) *contains reflections of order* > 2.

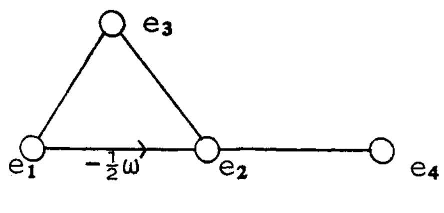
*Proof.* — By (5.2) (ii) and (5.6), there is a root graph $\Gamma_0$:




with $W(\Gamma_0) \subseteq G$. Let $e_4$ be a unitary root of G such that $e_1, e_2, e_3, e_4$ is a basis of $\mathbf{C}^4$ and such that $(e_2 \mid e_4) \in \mathbf{R}_{>0}$. Then $(e_2 \mid e_4) = 3^{-1/2}$. Let $\Gamma$ be the root graph spanned by this basis.

If $(e_1 \mid e_4) \neq 0$, then $(e_1 \mid r_2^j e_4) = 0$ for some $j \in \underline{2}$ [*see* (5.6)]. Therefore we may assume $(e_1 \mid e_4) = 0$ and (still) $(e_2 \mid e_4) = 3^{-1/2}$. If $(e_3 \mid e_4) = 0$, then $\det((e_i \mid e_j)) = 0$; so $|(e_3 \mid e_4)| = 3^{-1/2}$ by (4.3) (vii). Replacing $e_4$ by a suitable root in $r_3 \mathbf{U} e_4 \cup r_3^2 \mathbf{U} e_4$, we see that $\Gamma$ is equivalent to $L_4$. Thus (i) is proved. Now (ii) is a consequence of (i) and (4.15) (ii), while (iii) follows from (5.2) (ii), (iii), $|Z(W(L_4))| = 6$, and (ii).

(5.10) Suppose G is a 5-dimensional complex primitive reflection group. Then $d(G) = 3$ by (5.8) (iv), and all reflection orders equal 2 by (5.9) (iii). It is clear that G contains a complex irreducible 4-dimensional reflection group. From (5.8) (i), we obtain that this group is imprimitive. Hence $G(3, 3, 4) \subset G$, up to conjugacy. By (4.4) (v) there is a root graph $\Gamma_0$:




with $W(\Gamma_0) \subseteq G$. Let $e_5$ be a unitary root of G such that $e_1, e_2, e_3, e_4, e_5$ span a primitive root graph [*cf.* (2.7) and (2.9)]. We shall denote this root graph by $\Gamma$.

PROPOSITION:

(i) $\Gamma$ *is equivalent to* $K_5$;

(ii) G *is conjugate to* W ($K_5$).

*Proof.* — By (5.8) (viii) and some symmetry arguments, we may assume that $e_5$ is such that one of the following cases prevails.

(*a*) $$(e_3 \mid e_5) = -\frac{1}{2} \quad \text{and} \quad (e_1 \mid e_5) = (e_2 \mid e_5) = 0.$$

(*b*) $$(e_2 \mid e_5) = -\frac{1}{2} \quad \text{and} \quad (e_1 \mid e_5) = (e_3 \mid e_5) = 0.$$

(*c*) $$(e_1 \mid e_5) = \frac{1}{2}, \quad (e_2 \mid e_5) = 0, \quad \text{and} \quad (e_3 \mid e_5) = -\frac{1}{2}.$$

We will deal with these cases in the given order.

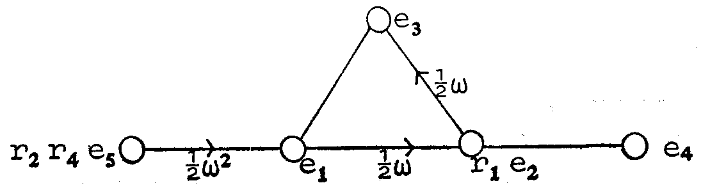
(*a*) If $(e_4 \mid e_5) = 0$, then $\Gamma$ is congruent to $K_5$.

Suppose $(e_4 \mid e_5) = -1/2\,\gamma$ for some $\gamma \in U$. Then $(r_3\, r_2\, r_4\, e_5 \mid e_1) = -1/2\,(1-\bar{\gamma}\omega)$. Hence $\gamma \in \{-1, \omega, -\omega^2\}$. But

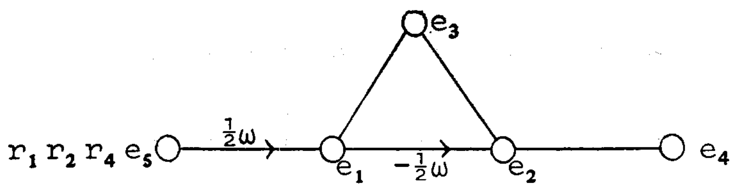
$$|(e_3 \mid r_2\, r_4\, e_5)| = 1/2\,|1+\gamma| \in \{0, 1/2\}$$

implies $\gamma \neq -\omega^2$.

If $\gamma = -1$, then $\Gamma$ is equivalent to

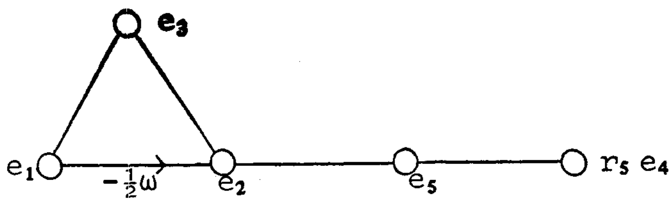



If $\gamma = \omega$, then $\Gamma$ is equivalent to




In both cases, $\Gamma$ is equivalent to $K_5$.

(*b*) Now (4.3) (vii) implies that $(e_4 \mid e_5) \neq 0$ and that the triangle spanned by $e_2, e_4, e_5$ is real, in other words $(e_4 \mid e_5) = 1/2$. Hence $\Gamma$ is equivalent to

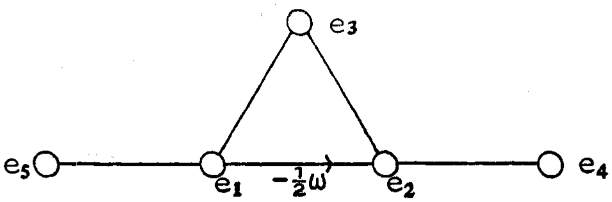



and therefore also to $D_5$ (3), which is impossible since the latter root graph is imprimitive.

(*c*) comes down to one of the previous cases, as will become clear if one replaces $e_1$ by $r_3 e_1$.

So far for the proof of (i). From this and (4.15) (ii) one readily deduces (ii).

(5.11) Let G be a 6-dimensional complex primitive reflection group. Then, again, $d(G) = 3$ and all reflections in G have order 2. Using (4.7) and (5.10), we obtain a root graph $\Gamma_0$:




with $W(\Gamma_0) \subseteq G$. Let $e_6$ be a unitary root such that $e_1, e_2, \ldots, e_6$ span an irreducible root graph. This root graph will be called $\Gamma$. Note that $\Gamma$ is primitive [*cf.* (2.6)].

PROPOSITION:

(i) $\Gamma$ *is equivalent to* $K_6$;

(ii) G *is conjugate to* $W(K_6)$;

(iii) *no n-dimensional complex reflection group* ($n \geqq 7$) *is primitive.*

*Proof.* — Symmetry and (5.8) (viii) allow us to assume (without loss of generality) that we have one of the following situations:

(*a*) $$(e_3 \mid e_6) = -\frac{1}{2} \quad \text{and} \quad (e_1 \mid e_6) = (e_2 \mid e_6) = 0.$$

(*b*) $$(e_2 \mid e_6) = -\frac{1}{2} \quad \text{and} \quad (e_1 \mid e_6) = (e_3 \mid e_6) = 0.$$

(*c*) $$(e_1 \mid e_6) = \frac{1}{2}, \quad (e_2 \mid e_6) = 0 \quad \text{and} \quad (e_3 \mid e_6) = -\frac{1}{2}.$$

We will check these cases separately.

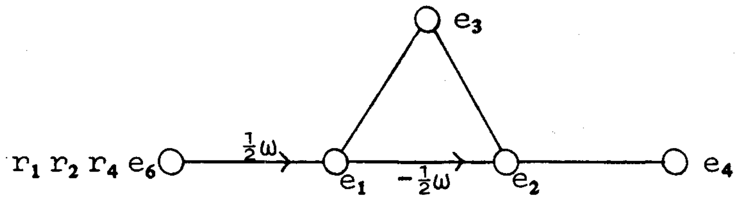
(*a*) If $(e_4 \mid e_6) = (e_5 \mid e_6) = 0$, then $\det((e_i \mid e_j)) < 0$. Assume that there is $\gamma \in \mathbf{U}$ with $(e_4 \mid e_6) = -1/2\,\gamma$ (thanks to symmetry in the root graph, this does not harm the generality). Comparison of (5.10) (*a*) with the root graph spanned by $e_1, e_2, e_3, e_4, e_6$, gives that either:

(*a* 1) $$\gamma = \omega,$$

or

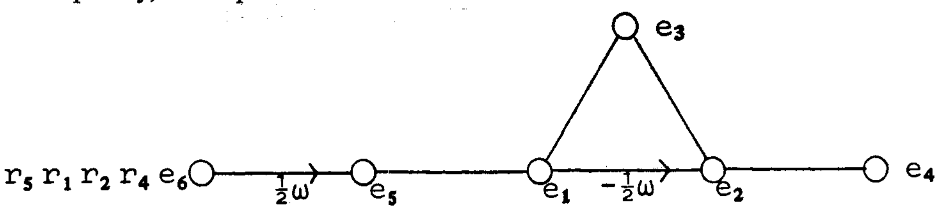
(*a* 2) $$\gamma = -1.$$

If (*a* 1) holds, we have that $\Gamma$ is equivalent to the root graph spanned by



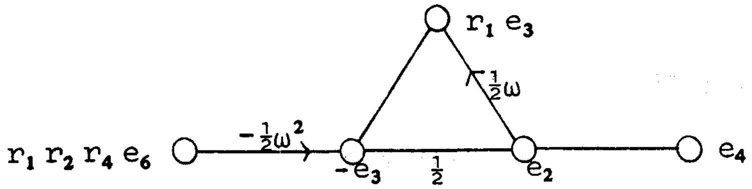

and $e_5$. As we have seen before in an analogous situation, this implies that $r_1 r_2 r_4 e_6$, $e_1$, and $e_5$ span a real triangle, i. e. $(e_5 \mid r_1 r_2 r_4 e_6) = -1/2\,\omega^2$.

Consequently, $\Gamma$ is equivalent to




hence to $K_6$.

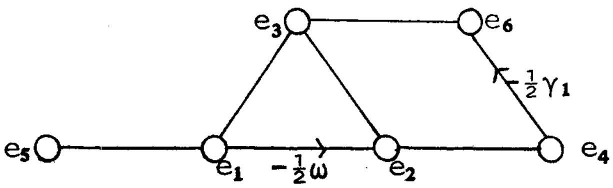
Suppose (*a* 2) holds, i. e. $\gamma = -1$. Then $\Gamma$ is equivalent to the root graph spanned by $\Gamma_1$:




and $e_5$, while $(e_4 \mid e_5) = 0$. But $\Gamma_1$ is a root graph congruent to $\Gamma_0$. Thus we have reduced this case to the situation:

(*a* 3) $$(e_5 \mid e_6) = 0 \quad \left(\text{and } (e_4 \mid e_6) = -\frac{1}{2}\gamma_1 \text{ with } \gamma_1 \in \mathbf{U}\right).$$

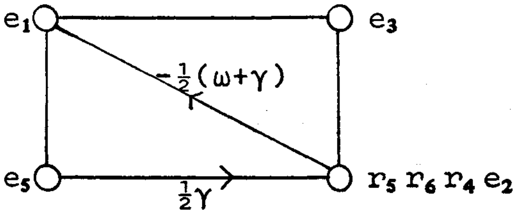
So $\Gamma$ is




If $\gamma_1 = \omega$, we are back in case (*a* 1). If $\gamma_1 = -1$, then $\det((e_i \mid e_j)) < 0$, which is impossible.

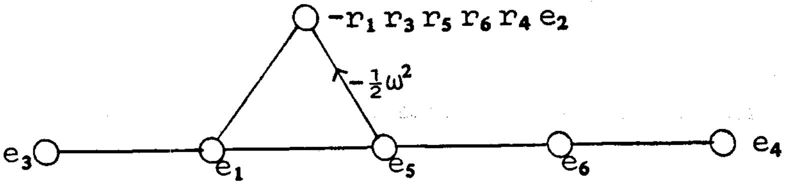
(*b*) Leaving out $e_5$ for a while, we see that the triangle spanned by $e_2, e_4, e_6$ is a real one; so $(e_4 \mid e_6) = 1/2$. Replacing $e_6$ by $r_4 e_6$ shows that we can alter the situation in order to get $(e_1 \mid e_6) = (e_2 \mid e_6) = (e_3 \mid e_6) = 0$ and $(e_4 \mid e_6) = -1/2$. If $(e_5 \mid e_6) = 0$, it suffices for the proof to refer to (4.15) (iv).

Let $(e_5 \mid e_6) = -1/2\,\gamma$ with $\gamma \in \mathbf{U}$. Since

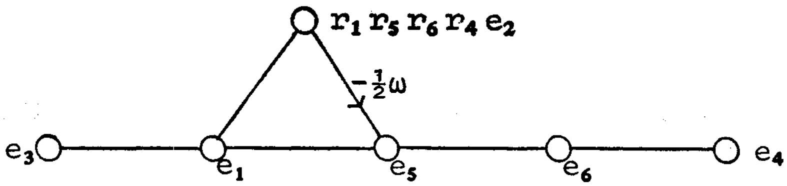



is a root graph, we obtain $\gamma = -\omega$ or $\omega^2$.

If $\gamma = -\omega$, then $\Gamma$ is equivalent to




If $\gamma = \omega^2$, then $\Gamma$ is equivalent to




In both cases $\Gamma$ is equivalent to $K_6$.

(c) Replacement of $e_1$ by $r_3 e_1$ brings us back in one of the previous cases.

Now (ii) follows from (i) and (4.15) (ii), whereas (iii) is a consequence of $|Z(W(K_6))| = 6$ [*see* (4.16)], (5.2) (iii), (4.7), (5.8) (iii), and (5.9) (iii).

(5.12) The conclusion from (5.3), (5.4), ..., (5.11) is the following

THEOREM. — *If $n \geqq 3$, then any complex primitive reflection group in $\mathbf{C}^n$ is conjugate to one of the reflection groups discussed in* (4.15).

We refer to (4.16) for a list of these groups. The rest of this chapter gives some properties concerning these groups.

(5.13) PROPOSITION. — *Every primitive complex n-dimensional reflection group* G *generated by reflections of order* 2 ($n \geqq 3$) *contains an irreducible n-dimensional real reflection group* H *with* $d(H) = d(G)$.

*Proof.* — It suffices to prove the corresponding statement for the root systems of type $J_3(4)$, $J_3(5)$, $N_4$, $K_5$, and $K_6$. It is immediate that the root systems of type $J_3(4)$, $J_3(5)$, $N_4$, $K_6$ contain systems of type $B_3$, $H_3$, $B_4$, $D_6$ respectively.

Finally, $K_5$ is dealt with in the proof of (4.15) (i).

(5.14) Put $\mu_\infty = \bigcup_{n=1}^{\infty} \mu_n$.

PROPOSITION. — *Let* G *be a complex primitive n-dimensional reflection group* ($n \geqq 3$), *and let* $H \subseteq U_n(\mathbf{C})$ *be a finite group such that* $G \trianglelefteq H$. *Then* $H \subseteq \mu_\infty . G$ *except for the following two cases:*

$$\begin{cases} G = W(L_3) \quad \textit{and} \quad H \subseteq \mu_\infty . W(M_3), \\ G = W(L_4) \quad \textit{and} \quad H \subseteq \mu_\infty . \left\langle W(L_4), \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 1 & 0 \\ & & 0 & -1 \end{pmatrix} \right\rangle . \end{cases}$$

The proof consists of a case-by-case argument along the following lines. Fix a unitary transformation $t$ of $\mathbf{C}^n$ that normalizes G. Since $t$ permutes the reflections of G [*cf.* (3) of (1.6)] and since we are only interested in $t . \mu_\infty . G$, we may assume $tv = v$ for some root $v$ provided G does not have more than one orbit of the same length in $\{ \mathbf{C} x \mid x \text{ is root of } G \}$ (this, though occurring in $F_4$, never happens in the complex primitive case). Furthermore, $t$ leaves invariant the set $\{ \mathbf{C} x \mid x \text{ is root of G in } v^\perp \}$. This implies that $t$ normalizes $G_v$, a lower dimensional reflection group, and we have reduced our problem.

(5.15) *Remark.* — Blichfeldt's list of finite primitive collineation groups in 4 variables given in [2] is not complete, as the non-trivial extension $H/(\mu_\infty \cap H)$ of $W(L_4)/\mu_6$ [notation as in (5.14)] is omitted. The mistake is due to an incorrect conclusion on the last page of his article.

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