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20. Abstract (continued)

and can be read independently of each other. In each part, after providing the main ingredients of the direct model and a statement of the conservation laws, a rapid outline is given of the derivation of the basic equations and nonlinear constitutive equations for elastic materials. Each part also includes a discussion of constrained theories and an account of recent developments on the subject.

# Finite Deformation of Elastic Rods and Shells

by

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Abstract

The objective of this paper is to present an account of recent developments in the direct formulation of theories of rods and shells based on 1 and 2-dimensional continuum models originating in the works of Duhem and E. and F. Cosserat. Following some preliminaries and description of (3-dimensional) shell-like and rod-like bodies, the rest of the paper is arranged in two parts, namely Part A (for shells) and Part B (for rods) and can be read independently of each other. In each part, after providing the main ingredients of the direct model and a statement of the conservation laws, a rapid outline is given of the derivation of the basic equations and nonlinear constitutive equations for elastic materials. Each part also includes a discussion of constrained theories and an account of recent developments on the subject.

## 1. Introduction

Rods and shells are a class of 3-dimensional bodies whose boundary surfaces have special characteristic features. In general, two entirely different approaches may be adopted for the construction of 1-dimensional and 2-dimensional mechanical theories of rods and shells and similarly these two approaches may be used in the construction of theories of fluid jets and fluid sheets. One approach starts with the 3-dimensional equations of the classical continuum mechanics and by applying approximation procedures strives to obtain 1-dimensional (in the case of rods and jets) and 2-dimensional (in the case of shells and sheets) field equations and constitutive equations for the medium under consideration. In the other approach, the medium response is *modelled* as a 1-dimensional and a 2-dimensional *directed* continuum, called a Cosserat curve and a Cosserat surface, respectively; and one then proceeds to the development of the field equations and the appropriate constitutive equations*. If *full* information is desired regarding the motion and deformation of the continuum under study in the context of the classical 3-dimensional theory, then there would be no need to develop a particular 1-dimensional and a 2-dimensional theory. In fact, the aim of 1-dimensional and 2-dimensional theories of the type mentioned above is to provide only partial information in some sense: for example, in the case of shells information concerning quantities which can be regarded as representing the medium response confined to a surface or its neighborhood as a consequence of the (3-dimensional) motion of the body, or the determination of certain weighted averages of quantities resulting from the (3-dimensional) motion of the body.

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*Other 2-dimensional and 1-dimensional models may also be used to construct direct theories of shells and rods but we postpone further remarks on this until later in this section.

The nature of the difficulties in the development of both the theory of shells and the theory of rods from the full 3-dimensional equations is well known and has been elaborated upon in various contexts by Green, Laws and Naghdi (1968), Green and Naghdi (1970), Naghdi (1972, Secs. 1,4; 1974; 1979a) and Ericksen (1979). In view of these it is reasonable to attempt to formulate 1-dimensional and 2-dimensional theories of the types described above by replacing the continuum characterizing the (3-dimensional) medium in question with an alternative model which would reflect the main features of the response of the 3-dimensional medium and which would then permit the formulation of appropriate 1-dimensional and 2-dimensional theories by a <u>direct</u> approach and without the appeal to special assumptions or approximations generally employed in the derivation from the 3-dimensional equations. It should be emphasized that a Cosserat surface and a Cosserat curve are not, respectively, just a 2-dimensional surface and a 1-dimensional curve; but are, in fact, endowed with some structure in the form of additional primitive kinematical vector fields.

The concept of 'directed' or 'oriented' media originated in the work of Duhem (1893) and a first systematic development of theories of oriented media in one, two and three dimensions was carried out by E. and F. Cosserat (1909). In their work, the Cosserats represented the orientation of each point of their continuum by a set of mutually perpendicular rigid vectors. The purely kinematical aspects of oriented bodies characterized by ordinary displacement and the independent deformation of N deformable vectors in N-dimensional space has been discussed by Ericksen and Truesdell (1958), who also introduced the terminology of <u>directors</u>.

A complete general theory of a Cosserat surface with a single deformable director given by Green, Naghdi and Wainwright (1965) was developed within the framework of thermomechanics. This derivation (Green <u>et</u> <u>al</u>. 1965) is carried out mainly from an appropriate energy equation, together with invariance requirements under superposed rigid body motions. A related development utilizing three directors at each point of the surface, in the context of a purely mechanical theory and with the use of a virtual work principle, is given by Cohen and DeSilva (1966b). A further development of the basic theory of a Cosserat surface along with certain general considerations regarding the construction of nonlinear constitutive equations for elastic shells is given by Naghdi (1972, Sec. 8), which also contains additional historical remarks relevant to oriented continua and to the theory of thin elastic shells. Hierarchical theory of Cosserat surfaces, namely that comprising a material surface with K ($\geq 1$) directors, is contained in a paper by Green and Naghdi (1976a) which deals with fluid sheets and its application to water waves.

A parallel development in the theory of a Cosserat curve with two deformable directors begins with a paper of Green and Laws (1966) whose derivation is carried out mainly from an appropriate energy equation, together with invariance requirements under superposed rigid body motions. A related development of a directed curve with three deformable directors at each point of the curve, in the context of a purely mechanical theory and with the use of a virtual work principle, is given by Cohen (1966). A further development of the basic theory of a Cosserat curve along with certain general developments regarding the construction of nonlinear constitutive equations for elastic rods is given by Green, Naghdi and Wenner (1974b). Hierarchical theory of Cosserat curves, namely that comprising a material curve with L($\geq 2$) directors, is contained in a paper by Naghdi (1979b) which is concerned with applications to Newtonian and non-Newtonian flows in pipes.

Of course, the introduction of an alternative model and formulation of

1-dimensional and 2-dimensional theories by the direct approach do not mean that one ignores the nature of the field equations in the 3-dimensional theory. In fact, some of the developments of the field equations by direct procedure are materially aided or influenced by available information which can be obtained from the 3-dimensional theory. For example, the integrated equations of motion from the 3-dimensional equations provide guidelines for a statement of 1- and 2-dimensional conservation laws in conjunction with the 1- and 2-dimensional models, and also provide some insight into the nature of inertia terms and the kinetic energy in the direct formulation of the 1-dimensional and 2-dimensional theories.

Inasmuch as most of the difficulties associated with the derivation of the 1-dimensional and 2-dimensional theories from the 3-dimensional equations occur in the construction of the constitutive equations, it is in fact here that the direct approach offers a great deal of appeal. These constructions, as well as the entire development by the direct approach, are exact in the sense that they rest on (1-dimensional and 2-dimensional) postulates valid for nonlinear behavior of materials but clearly they cannot be expected to represent all the features that could only be predicted by the relevant full 3-dimensional equations. Theories constructed via a direct approach necessarily satisfy the requirements of invariance under superposed rigid body motions that arise from physical considerations and, of course, they are also consistent and fully invariant in the mathematical sense. Moreover, the development by the direct approach is conceptually simple and does not have the difficulties associated with approximations usually made in the development of the theory of thin shells or the theories of slender rods from their corresponding 3-dimensional equations.

Although the direct approach to shells and rods employed in this paper is based on the 2-dimensional and 1-dimensional <u>directed</u> continuum models, respectively, other direct 2-dimensional and 1-dimensional models may also be used to construct theories of shells and rods. For example in the case of shells, instead of developing a theory based on a Cosserat surface, we may consider only a material surface and construct a direct theory in which the basic kinematical ingredients are the position vector of the surface together with its first and second gradients. A theory of this kind has been discussed by Balaban, Green and Naghdi (1967) and a somewhat less general theory by Cohen and DeSilva (1966a,1968). Although these developments have some overlapping features with corresponding results in the theory of Cosserat surfaces, they are more restrictive. Additional related remarks are made in Sec. 6 of this paper.

Following some general background information and definitions of shell-like and rod-like bodies in Sec. 2, the remainder of the paper is arranged in two parts which can be read independently of each other: one part (Part A) is concerned with the theory of shells and the other (Part B) is devoted to the theory of rods. In Part A (Secs. 3-8), first a concise development of the basic theory of a Cosserat surface with a single director followed by its generalization is presented. For a Cosserat surface with a single director, constitutive equations are discussed in the context of finite deformation of elastic shells and a procedure is indicated for identification of the assigned fields and the inertia coefficients which occur in the basic theory. Next, a fairly detailed account of constrained theories of shells is presented which includes the construction of an interesting nonlinear constrained theory not discussed previously in the literature. This is followed by an account of recent developments pertaining to elastic shells and a representation of the basic equations of a Cosserat surface in direct (coordinate-free) notation. A table of contents for Part A is listed in the introductory paragraph of Sec. 3.

Similarly, in Part B (Secs. 9-13), first a concise development of the Cosserat curve with two directors and its generalization is presented. Next, with reference to a Cosserat curve with two directors, constitutive equations are discussed for finite deformation of elastic rods and a procedure is indicated for identification of the assigned fields and the inertia coefficients which occur in the basic theory. This is followed by some additional remarks pertaining to elastic rods, together with a brief discussion of the constrained theories of rods, and a representation of the basic equations for a Cosserat curve in direct (coordinate-free) notation. A table of contents for Part B is listed in the introductory paragraph of Sec. 9.

## 2. General Background

In this section, we provide appropriate definitions for shell-like and rod-like bodies. To this end, consider a finite three-dimensional body $\mathcal{B}$ in a Euclidean 3-space, and let convected (or Lagrangian) coordinates $\theta^i$ $(i = 1,2,3)$, be assigned to each particle (or material point) of $\mathcal{B}$. Further, let[†] $\underset{\sim}{r}^*$ be the position vector, from a fixed origin, of a typical particle of $\mathcal{B}$ in the present configuration at time t. Then, a motion of the (three-dimensional) body is defined by a vector-valued function $\hat{\underset{\sim}{r}}^*$ which assigns position $\underset{\sim}{r}^*$ to each particle of $\mathcal{B}$ at each instant of time, i.e.[§],

$$\underset{\sim}{r}^* = \hat{\underset{\sim}{r}}^*(\theta^1,\theta^2,\theta^3,t) \quad . \tag{2.1}$$

We assume that the vector function $\hat{\underset{\sim}{r}}^*$ -- a 1-parameter family of configurations with t as the real parameter -- is sufficiently smooth in the sense that it is differentiable with respect to $\theta^i$ and t as many times as required. In some developments, it is convenient to set $\theta^3 = \xi$ and adopt the notation

$$\theta^i = (\theta^\alpha,\xi) \quad , \quad \theta^3 = \xi \quad . \tag{2.2}$$

We recall the formulas

$$\begin{aligned} &\underset{\sim}{g}_i = \frac{\partial \hat{\underset{\sim}{r}}^*}{\partial \theta^i} \quad , \quad g_{ij} = \underset{\sim}{g}_i \cdot \underset{\sim}{g}_j \quad , \quad g = \det(g_{ij}) \quad , \\ &\underset{\sim}{g}^i \cdot \underset{\sim}{g}_j = \delta^i_j \quad , \quad \underset{\sim}{g}^i = g^{ij}\underset{\sim}{g}_j \quad , \quad \underset{\sim}{g}^i \cdot \underset{\sim}{g}^j = g^{ij} \quad , \end{aligned} \tag{2.3}$$

$$dv = g^{\frac{1}{2}} d\theta^1 d\theta^2 d\theta^3 \tag{2.4}$$

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[†] The use of an asterisk attached to various symbols is for later convenience. The corresponding symbols without the asterisks are reserved for different definitions or designations to be introduced later.

[§] Recall that when the particles of a continuum are referred to a convected coordinate system, the numerical values of the coordinates associated with each particle remain the same for all time.

and further assume that†

$$g^{\frac{1}{2}} = [\underset{\sim}{g}_1 \underset{\sim}{g}_2 \underset{\sim}{g}_3] > 0 \quad . \tag{2.5}$$

In (2.4), $\underset{\sim}{g}_i$ and $\underset{\sim}{g}^i$ are the covariant and the contravariant base vectors at time t, respectively, $g_{ij}$ is the metric tensor, $g^{ij}$ is its conjugate, $\delta^i_j$ is the Kronecker symbol in 3-space and dv the volume element in the present configuration.

The velocity vector $\underset{\sim}{v}^*$ of a particle of the three-dimensional body in the present configuration is defined by

$$\underset{\sim}{v}^* = \dot{\underset{\sim}{r}}^* \quad , \tag{2.6}$$

where a superposed dot denotes material time differentiation with respect to t holding $\theta^i$ fixed. The stress vector $\underset{\sim}{t}$ across a surface in the present configuration with outward unit normal $\underset{\sim}{\nu}^*$ is given by

$$\underset{\sim}{t} = \nu^*_i \frac{\underset{\sim}{T}^i}{g^{\frac{1}{2}}} = \nu^*_i \tau^{ik} \underset{\sim}{g}_k \quad , \quad \underset{\sim}{T} = \underset{\sim}{g}_i \otimes g^{-\frac{1}{2}} \underset{\sim}{T}^i = \tau^{ik} \underset{\sim}{g}_i \otimes \underset{\sim}{g}_k \quad , \tag{2.7}$$

where

$$\underset{\sim}{T}^i = g^{\frac{1}{2}} \tau^{ij} \underset{\sim}{g}_j = g^{\frac{1}{2}} \tau^i_j \underset{\sim}{g}^j \quad , \quad \underset{\sim}{\nu}^* = \nu^*_i \underset{\sim}{g}^i = \nu^{*i} \underset{\sim}{g}_i \quad , \quad \tau^{ij} = \underset{\sim}{g}^i \cdot \underset{\sim}{T} \underset{\sim}{g}^j \quad , \tag{2.8}$$

where $\underset{\sim}{T}$ is the symmetric Cauchy stress tensor, $\tau^{ik}$ its contravariant components and ⊗ denotes the tensor product of two vectors. In terms of quantities defined in (2.5)-(2.8), the local field equations which follow from the integral forms of the three-dimensional conservation laws for mass, linear momentum and moment of momentum, respectively, are

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† The choice of positive sign in (2.5) is for definiteness. Alternatively, for physically possible motions we only need to assume that $g^{\frac{1}{2}} \neq 0$ with the understanding that in any given motion $[\underset{\sim}{g}_1 \underset{\sim}{g}_2 \underset{\sim}{g}_3]$ is either $> 0$ or $< 0$. The condition (2.5) also requires that $\theta^i$ be a right-handed coordinate system.

$$\overline{\rho^* g^{*\frac{1}{2}}}^{\cdot} = 0 \quad ,$$

$$\mathbf{T}^i{}_{,i} + \rho^* \mathbf{f}^* g^{*\frac{1}{2}} = \rho^* g^{*\frac{1}{2}} \dot{\mathbf{v}}^* \quad , \quad \mathbf{g}_i \wedge \mathbf{T}^i = 0 \quad , \tag{2.9}$$

where $\rho^*$ is the 3-dimensional mass density, $\mathbf{f}^*$ is the body force field per unit mass and a comma denotes partial differentiation with respect to $\theta^i$. For later reference, we note that for an incompressible medium, the condition of incompressibility may be expressed as

$$\dot{\overline{g^{\frac{1}{2}}}} = 0 \quad \text{or} \quad \operatorname{div} \mathbf{v}^* = 0 \quad . \tag{2.10}$$

A material surface in $\mathcal{B}$ can be defined by the equation $\xi = \xi(\theta^\alpha)$; the equation resulting from (2.1) with $\xi = \xi(\theta^\alpha)$ represents the parametric form of this material surface in the current configuration and defines a 1-parameter family of surfaces in space, each of which we assume to be smooth and non-intersecting. We refer to the surface $\xi = 0$ in the current configuration by $s$. Any point of the surface $s$ is specified by the position vector $\mathbf{r}$, relative to the same fixed origin to which $\mathbf{r}^*$ is referred, where

$$\mathbf{r} = \hat{\mathbf{r}}(\theta^\alpha, t) = \hat{\mathbf{r}}^*(\theta^\alpha, 0, t) \quad . \tag{2.11}$$

Let $\mathbf{a}_\alpha$ denote the base vectors along the $\theta^\alpha$-curves on the surface $s$. By (2.11) and $(2.3)_1$,

$$\mathbf{a}_\alpha = \mathbf{a}_\alpha(\theta^\gamma, t) = \frac{\partial \hat{\mathbf{r}}}{\partial \theta^\alpha} = \mathbf{g}_\alpha(\theta^\gamma, 0, t) \quad , \tag{2.12}$$

and the unit normal $\mathbf{a}_3 = \mathbf{a}_3(\theta^\gamma, t)$ to $s$ may be defined by**

$$\mathbf{a}_\alpha \cdot \mathbf{a}_3 = 0 \quad , \quad \mathbf{a}_3 \cdot \mathbf{a}_3 = 1 \quad , \quad \mathbf{a}_3 = \mathbf{a}^3 \qquad [\mathbf{a}_1 \mathbf{a}_2 \mathbf{a}_3] > 0 \quad . \tag{2.13}$$

We also recall the formulas

$$\begin{gathered} a_{\alpha\beta} = \mathbf{a}_\alpha \cdot \mathbf{a}_\beta \quad , \quad a = \det(a_{\alpha\beta}) \quad , \\ \mathbf{a}^\alpha = a^{\alpha\beta} \mathbf{a}_\beta \quad , \quad \mathbf{a}^\alpha \cdot \mathbf{a}^\beta = a^{\alpha\beta} \quad , \quad a^{\alpha\gamma} a_{\gamma\beta} = \delta^\alpha_\beta \end{gathered} \tag{2.14}$$

and

$$\begin{gathered} b_{\alpha\beta} = b_{\beta\alpha} = -\mathbf{a}_\alpha \cdot \mathbf{a}_{3,\beta} = \mathbf{a}_3 \cdot \mathbf{a}_{\alpha,\beta} \quad , \\ \mathbf{a}_{\alpha|\beta} = b_{\alpha\beta} \mathbf{a}_3 \quad , \quad \mathbf{a}_{3,\alpha} = -b^\gamma_\alpha \mathbf{a}_\gamma \quad , \quad b_{\alpha\beta|\gamma} = b_{\alpha\gamma|\beta} \quad , \end{gathered} \tag{2.15}$$

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**The use of the same symbols for base vectors of a surface in (2.12)-(2.13) and for the triad of a space curve in (2.17)-(2.18) should not give rise to confusion. The main developments for shells and rods are dealt with separately in the rest of the paper; this permits the use of the same symbol for different quantities in the case of shells and rods without confusion.

where $\underset{\sim}{a}^{\alpha}$ denote the reciprocal base vectors of the surface s, $a_{\alpha\beta}$ and $b_{\alpha\beta}$ are the components of its first and second fundamental forms, a comma denotes partial differentiation with respect to the surface coordinates $\theta^{\gamma}$, a vertical bar stands for covariant differentiation with respect to $a_{\alpha\beta}$ and $\delta^{\alpha}_{\beta}$ is the Kronecker symbol in 2-space.

A material line (not necessarily a straight line) in $\mathcal{B}$ can be defined by the equations $\theta^{\alpha} = \theta^{\alpha}(\xi)$; the equation resulting from (2.1) with $\theta^{\alpha} = \theta^{\alpha}(\xi)$ represents the parametric form of this material line in the current configuration and defines a 1-parameter family of curves in space, each of which we assume to be smooth and nonintersecting. We refer to the space curve $\theta^{\alpha} = 0$ in the current configuration by c. Any point of this curve is specified by the position vector $\underset{\sim}{r}$, relative to the same fixed origin to which $\underset{\sim}{r}^{*}$ is referred, where

$$\underset{\sim}{r} = \hat{\underset{\sim}{r}}(\xi,t) = \hat{\underset{\sim}{r}}^{*}(0,0,\xi,t) \quad . \tag{2.16}$$

Let $\underset{\sim}{a}_3$ denote the tangent vector along the $\xi$-curve.[§] By (2.16) and $(2.3)_1$,

$$\underset{\sim}{a}_3 = \underset{\sim}{a}_3(\xi,t) = \frac{\partial\hat{\underset{\sim}{r}}}{\partial\xi} = \underset{\sim}{g}_3(0,0,\xi,t) \tag{2.17}$$

and the unit principal normal $\underset{\sim}{a}_1$ and the unit binormal vector $\underset{\sim}{a}_2$ to c may be introduced as

$$\underset{\sim}{a}_1 = \underset{\sim}{a}_1(\xi,t) = \frac{\partial\underset{\sim}{a}_3/\partial\xi}{|\partial\underset{\sim}{a}_3/\partial\xi|} \quad , \quad \underset{\sim}{a}_2 = \underset{\sim}{a}_2(\xi,t) = \frac{\underset{\sim}{a}_3}{|\underset{\sim}{a}_3|} \times \underset{\sim}{a}_1 \quad ,$$
$$|\underset{\sim}{a}_3| = (a_{33})^{\frac{1}{2}} \quad , \quad a_{33} = \underset{\sim}{a}_3 \cdot \underset{\sim}{a}_3 \quad , \quad [\underset{\sim}{a}_1\underset{\sim}{a}_2\underset{\sim}{a}_3] > 0 \quad , \tag{2.18}$$

where the notation $|\underset{\sim}{a}_3|$ stands for the magnitude of $\underset{\sim}{a}_3$. The system of base vectors $\underset{\sim}{a}_i$ are oriented along the Serret-Frenet triad and satisfy the differential equations

$$\frac{\partial\underset{\sim}{a}_1}{\partial\xi} = \tau(a_{33})^{\frac{1}{2}}\underset{\sim}{a}_2 - \kappa\underset{\sim}{a}_3 \quad , \quad \frac{\partial\underset{\sim}{a}_2}{\partial\xi} = -\tau(a_{33})^{\frac{1}{2}}\underset{\sim}{a}_1 \quad , \quad \frac{\partial\underset{\sim}{a}_3}{\partial\xi} = a_{33}\kappa\underset{\sim}{a}_1 + \frac{1}{2a_{33}}\frac{\partial a_{33}}{\partial\xi}\underset{\sim}{a}_3 \ , \tag{2.19}$$

where $\kappa$ and $\tau$ denote, respectively, the curvature and the torsion of c. In the special case that c is a plane curve, we may choose $\underset{\sim}{a}_1$ as the unit normal to the curve and then $\underset{\sim}{a}_2$ will be perpendicular to the plane of $\underset{\sim}{a}_1$ and $\underset{\sim}{a}_3$. If c is a straight curve, then there is no unique Serret-Frenet triad and $\underset{\sim}{a}_i$ may be chosen as any orthogonal triad with $\underset{\sim}{a}_1,\underset{\sim}{a}_2$ as unit vectors. Equations (2.19) are not

[§]The designation of the tangent vector to a curve by $\underset{\sim}{a}_3$ should not be confused with the use of the same symbol for a different purpose in (2.13). In this connection, see the preceding footnote.

identical to the formulas of Frenet because the parameter $\xi$ is not necessarily the arc length of c. It may be noted here that the convected coordinate $\xi$ may be chosen to coincide with the arc length in any one configuration of the material curve, e.g., in the present configuration. However, in a general motion (involving different configurations) the arc length between any pair of particles changes while the convected coordinates of each particle must remain the same. Therefore, arc length would not qualify as a convected coordinate.

In the next four paragraphs (identified as subsections 2A and 2B) we provide appropriate definitions for shell-like and rod-like bodies in fairly precise terms.

2A. Definition of a shell-like body. A representation for the motion of a thin shell.

Consider a two-dimensional surface s defined by the parametric equation $\xi = 0$, over a finite coordinate patch $\alpha' \leq \theta^1 \leq \alpha''$, $\beta' \leq \theta^2 \leq \beta''$. Let $\underset{\sim}{r}$ and $\underset{\sim}{a}_3$ denote, respectively, the position vector and the unit normal to s. At each point of s, imagine material filaments projecting normally above and below the surface s. The surface formed by the material filaments constructed at the points of the closed boundary curve of s is called the <u>lateral surface</u>. Such a 3-dimensional body (depicted in Fig. 1) is called a shell if the dimension of the body along the normals, called the height and denoted by h, is <u>small</u>. A shell is said to be <u>thin</u> if its thickness is much smaller than a certain characteristic length L(s) of the surface s, for example, the local minimum radius of curvature of the surface, or the smallest dimension of s in the case of a plane. If h is constant, the shell is said to be of uniform thickness, otherwise of variable thickness. Since a material surface in the three-dimensional body can be defined by the equation $\xi = \xi(\theta^\alpha)$, it follows that the equation resulting from (2.1) with $\xi = \xi(\theta^\alpha)$ represents the parametric form of the material surface in the present configuration. In particular, the equation $\xi = 0$ defines a surface in space at time t, which we assume to be smooth and nonintersecting. Every point of this surface has a position vector r specified by (2.11). Let the boundary of the three-dimensional continuum be specified by the material surfaces

$$\xi = \xi_1(\theta^1,\theta^2) \quad , \quad \xi = \xi_2(\theta^1,\theta^2) \quad , \quad \xi_1 < \xi_2 \quad , \tag{2.20}$$

with the surface $\xi = 0$ lying either on one of the two surfaces $(2.20)_{1,2}$ or between them (see, for example, Fig. 1), and a material surface

$$f(\theta^1,\theta^2) = 0 \quad , \tag{2.21}$$

which is chosen such that $\xi = \text{const.}$ form closed smooth curves on the surface (2.21). As pointed out previously by Naghdi (1975a), in the development of a general theory, it is preferable to leave unspecified the choice of the relation of the surface s $(\xi = 0)$ to the major surfaces $s^+$ and $s^-$. In special cases of the general theory or in specific applications, however, it is necessary to fix the relation of s to the surfaces $(2.20)_{1,2}$.

We now suppose that $\hat{\underset{\sim}{r}}^*$ in (2.1) can be represented by the Taylor expansion in the bounded region $\xi_1 \leq \xi \leq \xi_2$ with coefficients which are continuous functions of $\theta^\alpha$,t and have continuous space and time derivatives of order 2. Thus, for shell-like bodies, we write

$$\hat{\underset{\sim}{r}}^* = \underset{\sim}{r} + \sum_{N=1}^{K} \xi^N \underset{\sim}{d}_N \quad , \quad \underset{\sim}{d}_N = \underset{\sim}{d}_N(\theta^\alpha,t) \tag{2.22}$$

and by $(2.3)_1$ and (2.6) we also have

$$\underset{\sim}{g}_\alpha = \underset{\sim}{a}_\alpha + \sum_{N=1}^{K} \xi^N \frac{\partial \underset{\sim}{d}_N}{\partial \theta^\alpha} \quad , \quad \underset{\sim}{g}_3 = \sum_{N=1}^{K} N\xi^{N-1} \underset{\sim}{d}_N \quad , \tag{2.23}$$

$$\underset{\sim}{v}^* = \underset{\sim}{v} + \sum_{N=1}^{K} \xi^N \underset{\sim}{w}_N \quad , \quad \underset{\sim}{v} = \dot{\underset{\sim}{r}} \quad , \quad \underset{\sim}{w}_N = \dot{\underset{\sim}{d}}_N \quad , \tag{2.21}$$

where $\underset{\sim}{r}$ is defined by (2.1) and a superposed dot in (2.22) denotes material time differentiation with respect to t holding $\theta^\alpha$ fixed. A special case of (2.22) which is of particular interest in subsequent developments is when $N = 1$, namely

$$\hat{\underset{\sim}{r}}^* = \underset{\sim}{r} + \xi \underset{\sim}{d} \quad , \tag{2.25}$$

where we have set $\underset{\sim}{d}_1 = \underset{\sim}{d}$.

2B. Definition of a rod-like body. A representation for the motion of a slender rod.

Consider a space curve c defined by the parametric equations $\theta^\alpha = 0$, over a finite interval $\xi_1 \leqslant \xi \leqslant \xi_2$. Let $\underset{\sim}{r}$ be the position vector of any point of c and let $\underset{\sim}{a}_1, \underset{\sim}{a}_2$ and $\underset{\sim}{a}_3$ denote its unit principal normal, unit binormal and the tangent vector, respectively. At each point of c, imagine material filaments lying in the normal plane, i.e., the plane perpendicular to $\underset{\sim}{a}_3$, and forming the normal cross-section $\mathcal{A}_n$. The surface swept out by the closed boundary curve $\partial\mathcal{A}_n$ of $\mathcal{A}_n$ is called the lateral surface. Such a 3-dimensional body (depicted in Fig. 2) is called rod-like if the dimensions in the plane of the normal cross-section are <u>small</u> compared to some characteristic dimension L(c) of c (see Fig. 2), e.g., its local radius of curvature $1/\kappa$, or the length of c in the case of a straight curve. A rod-like body is said to be <u>slender</u> if the largest dimension of $\mathcal{A}_n$ is much smaller than L(c). If $\mathcal{A}_n$ is independent of $\xi$, the body is said to be of uniform cross-section, otherwise of variable cross-section. Since a material curve in the three-dimensional body $\mathcal{B}$ can be defined by the equations $\theta^\alpha = \theta^\alpha(\xi)$, it follows that the equation resulting from (2.1) with $\theta^\alpha = \theta^\alpha(\xi)$ represents the parametric form of the material curve in the present configuration and defines a curve c in space at time t, which we assume to be sufficiently smooth and nonintersecting. Every point of this curve has a position vector specified by (2.14). Let the (3-dimensional) rod-like body in some neighborhood of c be bounded by material surfaces $\xi = \xi_1$, $\xi = \xi_2$ (indicated in Fig. 2) and a material surface of the form

$$F(\theta^1, \theta^2, \xi) = 0 \quad , \tag{2.26}$$

which is chosen such that $\xi$ = constant are curved sections of the body bounded by closed curves on this surface with c lying on or within (2.26). In the development of a general theory, it is preferable to leave unspecified the choice of the relation of the curve c to one on the boundary surface (2.26). In special cases or in specific applications, however, it is necessary to fix the relation of c to the surface (2.26).

We now suppose that $\hat{\underset{\sim}{r}}^*$ in (2.1) can be represented by the Taylor expansion in the bounded region lying inside the surface (2.26) and between $\xi = \xi_1$, $\xi = \xi_2$, with coefficients which are continuous functions of $\xi$,t and have continuous space and time derivatives of order 2. Thus, for rod-like bodies, we write

$$\hat{\underset{\sim}{r}}^* = \underset{\sim}{r} + \sum_{N=1} \theta^{\alpha_1}\theta^{\alpha_2} \ldots \theta^{\alpha_N} \underset{\sim}{d}_{\alpha_1 \ldots \alpha_N} \quad , \quad \underset{\sim}{d}_{\alpha_1 \ldots \alpha_N} = \underset{\sim}{d}_{\alpha_1 \ldots \alpha_N}(\xi, t) \tag{2.27}$$

and by $(2.3)_1$ and (2.6) we also have

$$\underset{\sim}{g}_\beta = \underset{\sim}{d}_\beta + \sum_{N=1}^{K} N\theta^{\alpha_2} \ldots \theta^{\alpha_N} \underset{\sim}{d}_{\beta\alpha_2 \ldots \alpha_N} \quad , \quad \underset{\sim}{g}_3 = \underset{\sim}{a}_3 + \sum_{N=1}^{K} \theta^{\alpha_1} \ldots \theta^{\alpha_N} (\partial \underset{\sim}{d}_{\alpha_1 \ldots \alpha_N} / \partial \xi) \quad , \tag{2.28}$$

$$\underset{\sim}{v}^* = \underset{\sim}{v} + \sum_{N=1}^{K} \theta^{\alpha_1} \ldots \theta^{\alpha_N} \underset{\sim}{w}_{\alpha_1 \ldots \alpha_N} \quad , \quad \underset{\sim}{w}_{\alpha_1 \ldots \alpha_N} = \dot{\underset{\sim}{d}}_{\alpha_1 \ldots \alpha_N} \quad , \tag{2.29}$$

where $\underset{\sim}{r}$ in (2.27) is defined by (2.16), $\underset{\sim}{d}_{\alpha_1 \ldots \alpha_N}$ is symmetric with respect to indices $\alpha_1 \ldots \alpha_N$ and a superposed dot in (2.29) denotes material time differentiation with respect to t holding $\xi$ fixed. A special case of (2.27) which is of particular interest in subsequent developments is when $N = 1$, namely

$$\hat{\underset{\sim}{r}}^* = \underset{\sim}{r} + \theta^\alpha \underset{\sim}{d}_\alpha \quad , \tag{2.30}$$

where we have put $\alpha_1 = \alpha$.

## Part A

## Elastic shells: A direct formulation

In Part A (Secs. 3-8), we summarize the main kinematics and the basic principles of the theory of Cosserat (or directed) surfaces and then discuss the constitutive equations for elastic shells, as well as several related aspects of the basic theory and recent developments on the subject. Although we are concerned here mainly with the purely mechanical theory involving appropriate forms of the conservation laws for mass, linear momentum, director momentum and moment of momentum, we also include a statement of the conservation of energy. The latter provides motivation in the development of certain constitutive equations, such as those for an elastic material, and in the discussion of aspects of some special solutions involving jump in energy. The contents of Part A are as follows:

3. The basic theory of Cosserat surfaces
   - 3.1 Kinematics of a Cosserat surface $\mathcal{C}$.
   - 3.2 Basic principles of a Cosserat surface $\mathcal{C}$.
   - 3.3 Hierarchical theories of Cosserat surfaces.
4. Elastic shells.
5. Identification of the assigned fields and the inertia coefficients.
6. Constrained theories of shells
   - 6.1 Incompressible Cosserat surface $\mathcal{C}$.
   - 6.2 A constrained theory with director along the normal to the surface of $\mathcal{C}$.
7. Additional remarks on shells.
8. Basic equations for a Cosserat surface in direct notation.

## 3. The basic theory of Cosserat surfaces

Having introduced the notion of a (three-dimensional) shell-like body in section 2, we now formally define a direct model for such a body. Thus, deformable media which are modelled by a material surface $\mathcal{S}$ embedded in a Euclidean 3-space, together with K (K = 1,2,...,N) deformable vector fields -- called directors -- attached to every point of the material surface are called Cosserat surfaces or directed surfaces and may be conveniently referred to as $\mathcal{C}_K$. The directors which are not necessarily along the unit normals to the surface have, in particular, the property that they remain unaltered in length under superposed rigid body motions.

In the absence of the directors, we merely have a 2-dimensional material surface $\mathcal{S}$ which can serve as a model for the construction by direct approach of the membrane theory of shells. With K = 1, the directed medium is a body $\mathcal{C}_1 = \mathcal{C}$ comprising a material surface and a single deformable director attached to every point of the material surface of $\mathcal{C}$. The latter is the simplest model for the construction of a general bending theory of thin shells; and, for simplicity, we restrict attention to this particular model in most of the development* of section 3.

### 3.1 Kinematics of a Cosserat surface $\mathcal{C}$.

Let the particles of the material surface $\mathcal{S}$ of $\mathcal{C}$ be identified by means of a system of convected coordinates $\theta^\alpha$ $(\alpha = 1,2)$ and let the 2-dimensional region occupied by the material surface $\mathcal{S}$ in the present configuration of at time t be referred to as $\mathfrak{s}$. Let $\underset{\sim}{r}$ and $\underset{\sim}{d}$ denote the position vector of a typical point of $\mathfrak{s}$ and the director at the same point, respectively. Also, let $\underset{\sim}{a}_\alpha, \underset{\sim}{a}_3$ designate, respectively, the base vectors along the $\theta^\alpha$-curves on $\mathfrak{s}$

*A brief account of the more general theory for Cosserat surfaces $\mathcal{C}_K$ is indicated at the end of this section.

and the outward unit normal to $\mathcal{s}$. Then, a motion of the Cosserat surface is defined by vector-valued functions which assign position $\underset{\sim}{r}$ and director $\underset{\sim}{d}$ to each particle of $\mathcal{C}$ at each instant of time, i.e.**,

$$\underset{\sim}{r} = \hat{\underset{\sim}{r}}(\theta^\alpha, t) \quad , \quad \underset{\sim}{d} = \hat{\underset{\sim}{d}}(\theta^\alpha, t) \quad , \quad [\underset{\sim}{a}_1 \underset{\sim}{a}_2 \underset{\sim}{d}] > 0 \quad , \tag{3.1}$$

where

$$\underset{\sim}{a}_\alpha = \underset{\sim}{a}_\alpha(\theta^\alpha, t) = \frac{\partial \hat{\underset{\sim}{r}}}{\partial \theta^\alpha} \tag{3.2}$$

and the condition $(3.1)_3$ ensures that the director $\underset{\sim}{d}$ is nowhere tangent to $\mathcal{s}$. The velocity and the director velocity vectors are defined by

$$\underset{\sim}{v} = \dot{\underset{\sim}{r}} \quad , \quad \underset{\sim}{w} = \dot{\underset{\sim}{d}} \tag{3.3}$$

and since the coordinate curves on $\mathcal{s}$ are convected from (3.2), we have

$$\dot{\underset{\sim}{a}}_\alpha = \underset{\sim}{v}_{,\alpha} \quad , \tag{3.4}$$

where a superposed dot denotes differentiation with respect to t holding $\theta^\alpha$ fixed.

It is convenient to introduce here a slightly different notation than that adopted in Naghdi (1972) and a number of earlier papers on the subject. Thus, we put

$$\underset{\sim}{d}_\alpha = \underset{\sim}{a}_\alpha \quad , \quad \underset{\sim}{d}_3 = \underset{\sim}{d} \tag{3.5}$$

and observe that, in view of $(3.1)_3$ and (3.4), $\underset{\sim}{d}_1, \underset{\sim}{d}_2, \underset{\sim}{d}_3$ are linearly independent vectors. Hence, we may introduce a set of reciprocal vectors

---

** For convenience, we adopt the notation for $\underset{\sim}{r}$ in (2.11) and (2.25) also for the surface $(3.1)_1$. This permits an easy identification of the two surfaces, if desired. The choice of positive sign in $(3.1)_3$ is for definiteness. Alternatively, it will suffice to assume that $[\underset{\sim}{a}_1 \underset{\sim}{a}_2 \underset{\sim}{d}] \neq 0$ with the understanding that in any given motion the scalar triple product $[\underset{\sim}{a}_1 \underset{\sim}{a}_2 \underset{\sim}{d}]$ is either $> 0$ or $< 0$.

$\mathbf{d}^i$ such that

$$\mathbf{d}_i \cdot \mathbf{d}^j = \delta_i^j \quad , \tag{3.6}$$

where $\delta_i^j$ is the Kronecker symbol in 3-space. Whenever desirable, the notations $\mathbf{d}_i = (\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3)$ and $(\mathbf{a}_\alpha, \mathbf{d})$ will be used interchangeably throughout Part A depending on the particular context. Consider now a reference configuration, not necessarily the initial configuration, of the Cosserat surface $\mathcal{C}$. In the reference configuration, let the material surface of $\mathcal{C}$ be referred to by $\mathcal{S}_R$ with $\mathbf{R}$ as its position vector; let $\mathbf{D}$ be the director at $\mathbf{R}$; and let $\mathbf{A}_\alpha, \mathbf{A}_3$ denote, respectively, the base vectors along the $\theta^\alpha$-curves on $\mathcal{S}_R$ and the unit normal to $\mathcal{S}_R$. Then, in the reference configuration we have

$$\mathbf{R} = \mathbf{R}(\theta^\alpha) \quad , \quad \mathbf{D} = \mathbf{D}(\theta^\alpha) \quad , \quad [\mathbf{A}_1 \mathbf{A}_2 \mathbf{D}] > 0 \quad , \tag{3.7}$$

where

$$\mathbf{A}_\alpha = \mathbf{A}_\alpha(\theta^\gamma) = \frac{\partial \mathbf{R}}{\partial \theta^\alpha} \tag{3.8}$$

and $(3.7)_3$ ensures that $\mathbf{D}$ is nowhere tangent to the surface $\mathcal{S}_R$. If the reference configuration of $\mathcal{C}$ is specified to be the initial configuration, say at time $t = 0$, then the vector-valued functions on the right-hand sides of $(3.7)_{1,2}$ can be identified with $\hat{\mathbf{r}}(\theta^\alpha, 0)$ and $\hat{\mathbf{d}}(\theta^\alpha, 0)$, respectively. Analogously to (3.5), we set

$$\mathbf{D}_\alpha = \mathbf{A}_\alpha \quad , \quad \mathbf{D}_3 = \mathbf{D} \tag{3.9}$$

and note that the dual of (3.6) is given by

$$\mathbf{D}_i \cdot \mathbf{D}^j = \delta_i^j \quad . \tag{3.10}$$

### 3.2 Basic principles of a Cosserat surface $\mathcal{C}$.

In the development of this subsection, we follow the mode of derivation of the basic theory of a Cosserat surface employed by Naghdi (1972, Sec. 8). Let $\mathcal{P}$, bounded by a closed curve $\partial\mathcal{P}$, be a part of $\mathfrak{s}$ occupied by an arbitrary material region of $\mathcal{S}$ in the present configuration at time t and let

$$\underset{\sim}{\nu} = \nu^{\alpha}\underset{\sim}{a}_{\alpha} = \nu_{\alpha}\underset{\sim}{a}^{\alpha} \tag{3.11}$$

be the outward unit normal to $\partial\mathcal{P}$. It is convenient at this point to define certain additional quantities as follows: The mass density $\rho = \rho(\theta^{\gamma},t)$ of the surface $\mathfrak{s}$ in the present configuration; the contact force* $\underset{\sim}{n} = \underset{\sim}{n}(\theta^{\gamma},t\,;\underset{\sim}{\nu})$ and the contact director force $\underset{\sim}{m} = \underset{\sim}{m}(\theta^{\gamma},t\,;\underset{\sim}{\nu})$, each per unit length of a curve in the present configuration; the assigned force $\underset{\sim}{f} = \underset{\sim}{f}(\theta^{\gamma},t)$ and the assigned director force $\underset{\sim}{\ell} = \underset{\sim}{\ell}(\theta^{\gamma},t)$, each per unit mass of the surface $\mathfrak{s}$; the intrinsic director force $\underset{\sim}{k}$ per unit area of $\mathfrak{s}$; the inertia coefficients $y^1 = y^1(\theta^{\gamma})$ and $y^2 = y^2(\theta^{\gamma})$ which are independent of time; the specific internal energy $\varepsilon = \varepsilon(\theta^{\gamma},t)$; the heat flux $h = h(\theta^{\gamma},t\,;\underset{\sim}{\nu})$ per unit time and per unit length of a curve $\partial\mathcal{P}$; the specific heat supply $r = r(\theta^{\gamma},t)$ per unit time; and the element of area $d\sigma$ of the surface $\mathcal{P}$, and the line element $ds$ of the curve $\partial\mathcal{P}$. The assigned field $\underset{\sim}{f}$ may be regarded as representing the combined effect of (i) the stress vector on the major surfaces of the shell-like body denoted by $\underset{\sim}{f}_c$, e.g., that due to the ambient pressure of the surrounding medium, and (ii) an integrated contribution arising from the three-dimensional body force denoted by $\underset{\sim}{f}_b$, e.g., that due to gravity. A parallel statement holds for the assigned

---

*The notations for the contact force $\underset{\sim}{n}$, the contact director force $\underset{\sim}{m}$ and the surface director force $\underset{\sim}{k}$ are the same as those in Naghdi (1977), but differ from Naghdi (1972) and most of the previous papers on the subject. In fact, the vector fields $\underset{\sim}{n},\underset{\sim}{m},\underset{\sim}{k}$ of Part A of the present paper correspond, respectively, to $\underset{\sim}{N},\underset{\sim}{M},\underset{\sim}{m}$ in Naghdi (1972) and most of the previous papers on the subject. Also the notations for the inertia coefficients $y^1$ and $y^2$, which occur in (3.13)-(3.14), differ from the corresponding notations in previous papers. In most of the previous papers (for example, Green and Naghdi 1976a, Naghdi 1975a or Naghdi 1979a) the notations $k^1,k^2$ or $\alpha_1,\alpha_2$ were used in place of $y^1,y^2$.

field $\underset{\sim}{\ell}$. Similarly, the assigned heat supply r may be regarded as representing the combined effect of (i) heat supply entering the major surfaces of the shell-like body from the surrounding environment, denoted by $r_c$, and (ii) a contribution arising from the three-dimensional heat supply, denoted by $r_b$. Thus, we may write

$$\underset{\sim}{f} = \underset{\sim}{f}_b + \underset{\sim}{f}_c \quad , \quad \underset{\sim}{\ell} = \underset{\sim}{\ell}_b + \underset{\sim}{\ell}_c \quad , \quad r = r_b + r_c \quad . \tag{3.12}$$

We assume that the kinetic energy of the Cosserat surface $\mathcal{C}$ per unit area of $\mathcal{s}$ in the present configuration is given by

$$\kappa = \tfrac{1}{2}\rho(\underset{\sim}{v} \cdot \underset{\sim}{v} + 2y^1 \underset{\sim}{v} \cdot \underset{\sim}{w} + y^2 \underset{\sim}{w} \cdot \underset{\sim}{w}) \quad . \tag{3.13}$$

We further define the momentum corresponding to the velocity $\underset{\sim}{v}$ and the director momentum corresponding to the director velocity $\underset{\sim}{w}$ by

$$\frac{\partial\kappa}{\partial\underset{\sim}{v}} = \rho(\underset{\sim}{v} + y^1\underset{\sim}{w}) \quad , \quad \frac{\partial\kappa}{\partial\underset{\sim}{w}} = \rho(y^1\underset{\sim}{v} + y^2\underset{\sim}{w}) \quad . \tag{3.14}$$

Also, the physical dimensions of $\rho, \underset{\sim}{n}, \underset{\sim}{f}$ are

$$\begin{gathered} \text{phys. dim. } \rho = [ML^{-2}] \quad , \\ \text{phys. dim. } \underset{\sim}{n} = [MT^{-2}] \quad , \quad \text{phys. dim. } \underset{\sim}{f} = [LT^{-2}] \quad , \end{gathered} \tag{3.15}$$

where the symbols [L], [M] and [T] stand for the physical dimensions of length, mass and time. The dimensions of the vector fields $\underset{\sim}{m}, \underset{\sim}{\ell}$ and $\underset{\sim}{k}$ depend upon the physical dimension of** $\underset{\sim}{d}$. Here we choose $\underset{\sim}{d}$ to have the dimension of length and then $\underset{\sim}{m}, \underset{\sim}{\ell}$ will have the same physical dimensions as $\underset{\sim}{n}, \underset{\sim}{f}$ in (3.15) while $\underset{\sim}{k}$ will have the physical dimension of $[ML^{-1}T^{-2}]$.

---

**Depending on the choice of the physical dimension of $\underset{\sim}{d}$ and with reference to $\underset{\sim}{m}, \underset{\sim}{\ell}$ and $\underset{\sim}{k}$ the terminologies of the contact director couple, the assigned director couple and the intrinsic director couple, respectively, are also used in the literature. In particular, the latter terminologies are employed in Naghdi (1972), where $\underset{\sim}{d}$ is taken to be dimensionless.

In terms of the above definitions of the various field quantities and with reference to the present configuration, the conservation laws in the purely mechanical theory of a Cosserat surface $\mathcal{C}$ are*

$$
\begin{gathered}
\frac{d}{dt}\int_{\mathcal{P}} \rho\, d\sigma = 0 \quad , \\
\frac{d}{dt}\int_{\mathcal{P}} \rho(\underset{\sim}{v} + y^1 \underset{\sim}{w})d\sigma = \int_{\mathcal{P}} \rho \underset{\sim}{f}\, d\sigma + \int_{\partial\mathcal{P}} \underset{\sim}{n}\, ds \quad , \\
\frac{d}{dt}\int_{\mathcal{P}} \rho(y^1 \underset{\sim}{v} + y^2 \underset{\sim}{w})d\sigma = \int_{\mathcal{P}} (\rho\underset{\sim}{\ell} - \underset{\sim}{k})d\sigma + \int_{\partial\mathcal{P}} \underset{\sim}{m}\, ds \quad , \\
\frac{d}{dt}\int_{\mathcal{P}} \rho[\underset{\sim}{r} \times (\underset{\sim}{v} + y^1\underset{\sim}{w}) + \underset{\sim}{d} \times (y^1\underset{\sim}{v} + y^2\underset{\sim}{w})]d\sigma = \int_{\mathcal{P}} \rho(\underset{\sim}{r} \times \underset{\sim}{f} + \underset{\sim}{d} \times \underset{\sim}{\ell})d\sigma + \int_{\partial\mathcal{P}} (\underset{\sim}{r} \times \underset{\sim}{n} + \underset{\sim}{d} \times \underset{\sim}{m})ds \quad .
\end{gathered}
\tag{3.16}
$$

The first of (3.16) is a mathematical statement of the conservation of mass, the second that of the linear momentum, the third that of the director momentum and the fourth is the conservation of moment of momentum. We also record the law of conservation of energy in the form

$$
\frac{d}{dt}\int_{\mathcal{P}} \rho[\varepsilon+\kappa]d\sigma = \int_{\mathcal{P}} \rho(\underset{\sim}{f} \cdot \underset{\sim}{v} + \underset{\sim}{\ell} \cdot \underset{\sim}{w} + r)d\sigma + \int_{\partial\mathcal{P}} (\underset{\sim}{n} \cdot \underset{\sim}{v} + \underset{\sim}{m} \cdot \underset{\sim}{w} - h)ds \quad . \tag{3.17}
$$

The basic structure of $(3.16)_{1,2}$ and (3.17) and their forms are analogous to the corresponding conservation laws of the classical 3-dimensional continuum field theory. The structures of $(3.16)_3$ and $(3.16)_4$ are less obvious, but a motivation for their forms is provided by a derivation of the basic field equations for shell-like bodies obtained from the 3-dimensional equations of continuum mechanics in which the position vector $r^*$ in 3-space is approximated by an expression of the form (2.19). It should be noted here that the conservation laws (3.16)-(3.17) are consistent with the invariance conditions under superposed rigid body motions, which

---

*As the integrals on the left-hand sides of $(3.16)_{2,3,4}$ allow for coupling in inertia terms, they are slightly more general than the corresponding expressions in Naghdi (1972). The conservation laws (3.16) with coefficients $y^1 = 0$ and $y^2 = \alpha \neq 0$ reduce to those given by Eqs. (8.11) in Naghdi (1972).

ordinarily have wide acceptance in continuum mechanics. Moreover, as shown in Naghdi (1972, Sec. 8), the conservation laws $(3.16)_1$, $(3.16)_2$ and $(3.16)_4$ are equivalent to, and can be derived from the conservation of energy (3.17) and the invariance conditions under superposed rigid body motions. The conservation law $(3.16)_3$ for the director momentum must be postulated separately.

Returning to the conservation laws (3.16) and (3.17), we note that under suitable continuity assumptions the contact force $\underset{\sim}{n}$, the contact director force $\underset{\sim}{m}$ and the heat flux h can be expressed in the forms (for details see Naghdi 1972, Sec. 8):

$$\underset{\sim}{n} = \underset{\sim}{N}^{\alpha}\nu_{\alpha} \quad , \quad \underset{\sim}{m} = \underset{\sim}{M}^{\alpha}\nu_{\alpha} \quad , \quad h = q^{\alpha}\nu_{\alpha} \tag{3.18}$$

where $\underset{\sim}{N}^{\alpha}, \underset{\sim}{M}^{\alpha}$ transform as contravariant surface vectors and $q^{\alpha}$ are the contravariant components of the heat flux vector

$$\underset{\sim}{q} = q^{\alpha}\underset{\sim}{a}_{\alpha} \quad . \tag{3.19}$$

With the use of (3.18) and by usual procedures, from the conservation laws (3.15) and (3.16) follow the local field equations

$$\begin{aligned}
&\rho a^{\frac{1}{2}} = \lambda \quad \text{or} \quad \dot{\rho} + \rho \underset{\sim}{a}^{\alpha} \cdot \underset{\sim}{v}_{,\alpha} = 0 \quad , \\
&\underset{\sim}{N}^{\alpha}{}_{|\alpha} + \rho \underset{\sim}{f} = \rho(\dot{\underset{\sim}{v}} + y^{1}\dot{\underset{\sim}{w}}) \quad , \\
&\underset{\sim}{M}^{\alpha}{}_{|\alpha} + \rho \underset{\sim}{\ell} - \underset{\sim}{k} = \rho(y^{1}\dot{\underset{\sim}{v}} + y^{2}\dot{\underset{\sim}{w}}) \quad , \\
&\underset{\sim}{a}_{\alpha} \times \underset{\sim}{N}^{\alpha} + \underset{\sim}{d} \times \underset{\sim}{k} + \underset{\sim}{d}_{,\alpha} \times \underset{\sim}{M}^{\alpha} = \underset{\sim}{0}
\end{aligned} \tag{3.20}$$

and

$$\rho r - q^{\alpha}{}_{|\alpha} - \rho\dot{\varepsilon} + P = 0 \quad , \tag{3.21}$$

where

$$P = \underset{\sim}{N}^{\alpha} \cdot \underset{\sim}{v}_{,\alpha} + \underset{\sim}{k} \cdot \underset{\sim}{w} + \underset{\sim}{M}^{\alpha} \cdot \underset{\sim}{w}_{,\alpha} \tag{3.22}$$

is the mechanical power, $\lambda$ in $(3.19)_1$ is a function of $\theta^{\alpha}$ only, a comma denotes partial differentiation with respect to $\theta^{\alpha}$, a vertical line stands for covariant differentiation with respect to the metric tensor of the surface $\mathcal{S}$ and

$$a^{\frac{1}{2}} = [\underset{\sim}{a}_1 \underset{\sim}{a}_2 \underset{\sim}{a}_3] \quad . \tag{3.23}$$

3.3 <u>Hierarchical theories of Cosserat surfaces</u>

Although the theory outlined in subsection 3.2 is sufficiently general for many applications, on occasion it becomes necessary to consider Cosserat surfaces with more than one director. Therefore, we now briefly discuss the kinematics and the balance laws of Cosserat surfaces $\mathcal{C}_K$ having K $(K = 1,2,\ldots)$ directors attached to every point of a material surface $\mathcal{S}$. Thus, we admit K directors at $\underset{\sim}{r}$ denoted by $\underset{\sim}{d}_M$ $(M = 1,2,\ldots,K)$; and, instead of $(3.1)_{1,2}$ specify a motion of $\mathcal{C}_K$ by

$$\underset{\sim}{r} = \hat{\underset{\sim}{r}}(\theta^{\alpha},t) \quad , \quad \underset{\sim}{d}_M = \hat{\underset{\sim}{d}}_M(\theta^{\alpha},t) \quad . \tag{3.24}$$

The velocity vector is still given by $(3.3)_1$ but corresponding to $(3.3)_2$ we now define the director velocities

$$\underset{\sim}{w}_M = \dot{\underset{\sim}{d}}_M \tag{3.25}$$

We recall for $K = 1$ $(\mathcal{C}_1 = \mathcal{C})$, the kinetical quantities introduced in subsection 3.2 consisted of $\underset{\sim}{n},\underset{\sim}{k},\underset{\sim}{m}$ and the assigned fields $\underset{\sim}{f},\underset{\sim}{\ell}$. Keeping this in mind, for a body $\mathcal{C}_K$ we admit more general kinetical quantities and assigned fields

$$
\begin{gathered}
\underset{\sim}{n}\ ,\ \underset{\sim}{k}^N\ ,\ \underset{\sim}{m}^N\ , \\
\underset{\sim}{f}\ ,\ \underset{\sim}{\ell}^N
\end{gathered}
\tag{3.26}
$$

for $N = 1,2,\ldots,K$, and corresponding to (3.13) and $(3.14)_{1,2}$ write the more general expressions for kinetic energy of $\mathcal{C}_K$ and associated momentum and director momentum, namely

$$
\begin{gathered}
\kappa = \tfrac{1}{2}\rho \sum_{M,N=o}^{K} y^{M+N}\, \underset{\sim M}{w} \cdot \underset{\sim N}{w}\ , \quad \underset{\sim o}{w} = \underset{\sim}{v}\ , \\
\frac{\partial\kappa}{\partial \underset{\sim}{v}} = \rho(\underset{\sim}{v} + \sum_{M=1}^{K} y^{M}\underset{\sim M}{w}) = \rho \sum_{M=o}^{K} y^{M}\underset{\sim M}{w}\ , \\
\frac{\partial\kappa}{\partial \underset{\sim N}{w}} = \rho(y^{N}\underset{\sim}{v} + \sum_{M=1}^{K} y^{M+N}\underset{\sim M}{w}) = \rho \sum_{M=o}^{K} y^{M+N}\underset{\sim M}{w}\ ,
\end{gathered}
\tag{3.27}
$$

each per unit area of the surface $\mathcal{P}$. The inertia coefficients $y^{M+N}$ are functions of $\theta^\alpha$ only and satisfy the conditions

$$
y^{M+N} = y^{N+M}\ , \quad y^{M+o} = y^{o+M} = y^{M}\ , \quad y^{o} = 1\ . \tag{3.28}
$$

In the special case of $\mathcal{C}_1$ $(= \mathcal{C})$ we may use the notations

$$
\underset{\sim 1}{d} = \underset{\sim}{d}\ , \quad \underset{\sim 1}{w} = \underset{\sim}{w}\ . \tag{3.29}
$$

For a detailed statement of conservation laws appropriate for Cosserat surfaces $\mathcal{C}_K$ we refer the reader to Green and Naghdi (1976a, Sec. 2) but indicate here the structure of the corresponding local field equations. In this connection, we first note that for a purely mechanical theory by usual procedure in addition to $(3.18)_1$ we now obtain $\underset{\sim}{m}^N = \underset{\sim}{M}^{N\alpha}\nu_\alpha$. Then, the local field equations for Cosserat surface $\mathcal{C}_K$ are:

$$\overline{\lambda \dot{y}^{M}} = 0 \quad , \quad \dot{\lambda} = 0 \quad , \quad \lambda = \rho a^{\frac{1}{2}} \quad ,$$

$$\mathbf{N}^{\alpha}{}_{|\alpha} + \rho \mathbf{f} = \rho \sum_{M=0}^{K} y^{M} \dot{\mathbf{w}}_{M} \quad ,$$

$$\mathbf{M}^{N\alpha}{}_{|\alpha} + \rho \boldsymbol{\ell}^{N} - \mathbf{k}^{N} = \rho \sum_{M=0}^{K} y^{M+N} \dot{\mathbf{w}}_{M} \quad , \quad (N = 1,2,\dots,K) \quad , \tag{3.29}$$

$$\mathbf{a}_{\alpha} \times \mathbf{N}^{\alpha} + \sum_{N=1}^{K} \mathbf{d}_{N} \times \mathbf{k}^{N} + \sum_{N=1}^{K} \mathbf{d}_{N,\alpha} \times \mathbf{M}^{N\alpha} = 0 \quad .$$

Also, for Cosserat surfaces $\mathcal{C}_K$, the expression for mechanical power corresponding to (3.22) is

$$P = \mathbf{N}^{\alpha} \cdot \mathbf{v}_{,\alpha} + \sum_{N=1}^{K} \mathbf{k}^{N} \cdot \mathbf{w}_{N} + \sum_{N=1}^{K} \mathbf{M}^{N\alpha} \cdot \mathbf{w}_{N,\alpha} \quad . \tag{3.30}$$

The general development for Cosserat surfaces $\mathcal{C}_K$ outlined above is contained in a paper by Green and Naghdi (1976a, Sec. 2) which deals with application of the theory to fluid sheets and to propagation of water waves. When $K = 1$, the results in subsection 3.3 reduce to those of subsection 3.2 for a Cosserat surface $\mathcal{C}$.

## 4. Elastic shells

Within the scope of the theory of a Cosserat surface $\mathcal{C}$ outlined in Sec. 3, we discuss briefly the constitutive equations for elastic shells in the presence of finite deformation. Preliminary to the discussion that follows, we assume the existence of a strain energy or stored energy per unit mass $\psi = \psi(\theta^{\alpha}, t)$ such that $\rho\dot{\psi}$ is equal to the mechanical power defined by (3.22), i.e.,

$$P = \rho\dot{\psi} \quad . \tag{4.1}$$

In the development of nonlinear constitutive equations for elastic shells, we assume that the strain energy density $\psi$ at each material point of $\mathcal{C}$ and for all t is specified by a response function which depends on $\underset{\sim}{r}, \underset{\sim}{d}$ and their partial derivatives with respect to $\theta^{\alpha}$. But since the response function must remain unaltered under superposed rigid body translational displacement, the dependence on $\underset{\sim}{r}$ must be excluded. Thus, the constitutive assumption for the strain energy density can be written as

$$\psi = \psi'(\underset{\sim}{r}_{,\alpha}, \underset{\sim}{d}, \underset{\sim}{d}_{,\alpha} ; X) \tag{4.2}$$

and we also make similar constitutive assumptions for $\underset{\sim}{N}^{\alpha}, \underset{\sim}{k}, \underset{\sim}{M}^{\alpha}$. In these constitutive equations, which represent the mechanical response of the medium, the dependence of the response functions on the local geometrical properties of a reference state and material inhomogeneity is indicated through the argument X.

A general development of various aspects of constitutive theory of elastic shells based on assumptions of the type (4.2) or variants thereof is given in Naghdi (1972, Sec. 13). In the rest of this section, we limit the discussion to an elastic shell which is homogeneous in its reference configuration and suppose also that the dependence of the response functions on the properties of the reference state occurs through the values of the kinematical variables in the reference state (Carroll and Naghdi 1972). Then, in place of (4.2), we have

$$\psi = \bar{\psi}(\underset{\sim}{r}_{,\alpha}, \underset{\sim}{d}, \underset{\sim}{d}_{,\alpha}\,; \underset{\sim}{A}_{\alpha}, \underset{\sim}{D}, \underset{\sim}{D}_{,\alpha}) \quad , \tag{4.3}$$

with similar assumptions for $\underset{\sim}{N}^{\alpha}, \underset{\sim}{k}, \underset{\sim}{M}^{\alpha}$. After substituting (4.3) into (4.1), by usual techniques we obtain the following forms for the constitutive equations:

$$\underset{\sim}{N}^{\alpha} = \rho \frac{\partial\bar{\psi}}{\partial \underset{\sim}{r}_{,\alpha}} \quad , \quad \underset{\sim}{k} = \rho \frac{\partial\bar{\psi}}{\partial \underset{\sim}{d}} \quad , \quad \underset{\sim}{M}^{\alpha} = \rho \frac{\partial\bar{\psi}}{\partial \underset{\sim}{d}_{,\alpha}} \quad , \tag{4.4}$$

along with the restriction

$$\underset{\sim}{r}_{,\alpha} \times \frac{\partial\bar{\psi}}{\partial \underset{\sim}{r}_{,\alpha}} + \underset{\sim}{d} \times \frac{\partial\bar{\psi}}{\partial \underset{\sim}{d}} + \underset{\sim}{d}_{,\alpha} \times \frac{\partial\bar{\psi}}{\partial \underset{\sim}{d}_{,\alpha}} = 0 \quad , \tag{4.5}$$

which is obtained from the conservation of moment of momentum and which must be satisfied by the response function $\bar{\psi}$ (Naghdi 1972, Sec. 8).

We do not discuss here the reduced forms of the above constitutive equations resulting from invariance requirements under superposed rigid body motions, but for such reductions refer the reader to Naghdi (1972, Sec. 13). Just as with the equations of motion, it is necessary in applications to specific problems to obtain alternative forms of the above constitutive equations or their reduced forms in terms of tensor components. Such component forms may be expressed with respect to bases $\underset{\sim}{a}_i$, or $\underset{\sim}{d}_i$, or corresponding bases in the reference configuration. Reduced forms of (4.4) have been utilized extensively in Chapters D and E of Naghdi (1972).

## 5. Identification of the assigned fields and the inertia coefficients

The local field equations (3.20) in the mechanical theory of a Cosserat surface have the same forms as those that can be derived from the three-dimensional field equations $(2.9)_{1,2,3}$ by suitable integration between the limits $\xi_1,\xi_2$ [recall (2.17) and the definition of a shell-like body in section 2] and in terms of certain definitions for integrated mass density and resultants of stress (for details, see Naghdi 1972, Secs. 11-12 or Naghdi (1974). Similarly, the energy equation (3.21) has the same form as the one that can be derived from the energy equation in the three-dimensional theory by suitable integration between the limits $\xi_1,\xi_2$ and in terms of certain definitions for integrated internal energy density and heat flux in the three-dimensional theory as given in (Naghdi 1972). To elaborate further, we confine attention to the purely mechanical theory and recall the definitions

$$\rho a^{\frac{1}{2}} = \lambda = \int_{\xi_1}^{\xi_2} \lambda^* d\xi \quad , \quad \lambda^* = \rho^* g^{\frac{1}{2}} \quad , \tag{5.1}$$

$$\rho a^{\frac{1}{2}} k^M = \lambda k^M = \int_{\xi_1}^{\xi_2} \lambda^* \xi^M d\xi \quad , \quad (M = 1,2) \quad , \tag{5.2}$$

and the expressions

$$\lambda \underset{\sim}{f} = \rho a^{\frac{1}{2}} \underset{\sim}{f} = \int_{\xi_1}^{\xi_2} \lambda^* \underset{\sim}{f}^* d\xi + [\underset{\sim}{t} g^{\frac{1}{2}} \mathbf{f}_{(1)}(\xi)]_{\xi_1} + [\underset{\sim}{t} g^{\frac{1}{2}} \mathbf{f}_{(2)}(\xi)]_{\xi=\xi_2} \quad , \tag{5.3}$$

$$\lambda \underset{\sim}{\ell} = \rho a^{\frac{1}{2}} \underset{\sim}{\ell} = \int_{\xi_1}^{\xi_2} \lambda^* \underset{\sim}{f}^* \xi d\xi + [\underset{\sim}{t} g^{\frac{1}{2}} \xi \mathbf{f}_{(1)}(\xi)]_{\xi=\xi_1} + [\underset{\sim}{t} g^{\frac{1}{2}} \xi \mathbf{f}_{(2)}(\xi)]_{\xi=\xi_2} \quad , \tag{5.4}$$

where $\rho^*, \underset{\sim}{t}, f^*$ which occur in (5.1)-(5.4) are defined in section 2 [following (2.9)] and in order to indicate the nature of the functions $\mathbf{f}_{(\alpha)}$, $(\alpha = 1,2)$ in (5.3)-(5.4), it will suffice to record

$$\mathfrak{f}_{(1)} = [(\xi_{1,1})^2 g^{11} (\xi_{1,2})^2 g^{22} + g^{33} + 2(\xi_{1,1}\xi_{1,2}g^{12} - \xi_{1,1}g^{13} - \xi_{1,2}g^{23})]^{\frac{1}{2}} \quad , \qquad (5.5)$$

which involves the partial derivatives of $\xi_1(\theta^\alpha)$ and the components of the metric tensor in (2.3). The expression for $\mathfrak{f}_{(2)}$ can be stated analogously.

If we now adopt the approximation (2.25), then there is a 1-1 correspondence between the two-dimensional field equations that follow from the conservation laws or a Cosserat surface and those that can be derived from $(2.9)_{1,2,3}$ provided we identify $\underset{\sim}{r}$ and $\underset{\sim}{d}$ in (2.25) with $(3.1)_1$ and $(3.1)_2$, respectively, and adopt the definitions (5.1)-(5.4), as well as the definitions of the resultants mentioned above. A similar 1-1 correspondence can be shown to hold between the two-dimensional energy equation in the theory of a Cosserat surface and an integrated energy equation derived from the three-dimensional energy equation.

The various quantities in (3.12) are free to be specified in a manner which depends on the particular application in mind and, in the context of the theory of a Cosserat surface, the inertia coefficients $y^1, y^2$ and the mass density $\rho$ require constitutive equations. Indeed, $\underset{\sim}{f}_c, \underset{\sim}{\ell}_c$ and $r_c$, as well as $\underset{\sim}{f}_b, \underset{\sim}{\ell}_b$ and $r_b$, can be identified with corresponding expressions in a derivation from the three-dimensional equations (for details, see Naghdi 1972,1979a). Likewise, $\rho$ and the coefficients $y^1, y^2$ may be identified with easily accessible results from the three-dimensional theory.

In what follows, we assume that the above identifications have been made and that the quantities $\rho, y^1, y^2, \underset{\sim}{f}_b, \underset{\sim}{\ell}_b$ are known or specified. The knowledge of $\underset{\sim}{f}_c, \underset{\sim}{\ell}_c$ depends on the nature of the boundary conditions on the major surfaces of the particular shell-like body under consideration: they may be specified as known quantities on the surfaces $(2.20)_{1,2}$ or they are unknown (possibly on one of the two surfaces $(2.20)_{1,2}$ only) and must be determined as part of the solution of the problem.

## 6. Constrained theories of shells

A development of a constrained directed medium in the 3-dimensional theory, with particular reference to an incompressible liquid crystal having a single director of constant length, is contained in a paper of Green, Naghdi and Trapp (1970, Sec. 6). For a Cosserat surface with a single director, a number of constrained theories have been discussed previously. These pertain to a class of shell-like bodies for which the director is constrained to be of constant length (Green and Naghdi 1974), an incompressible Cosserat surface (Green, Laws and Naghdi 1974, Green and Naghdi 1976a) and a class of fluid sheets in which the director is constrained to remain always parallel to a fixed direction (Green and Naghdi 1977).

A special case of the constrained theory of elastic shells discussed by Green and Naghdi (1974) includes that for which the director is coincident with the unit normal* $\underset{\sim}{a}_3$ to the surface $\mathcal{s}$. This special form of the theory can be brought into 1-1 correspondence with that of a restricted theory of elastic shells given by Naghdi (1972, Secs. 10, 15), where the director is not admitted and the basic kinematical ingredients that occur in the argument of the strain energy response function are $\underset{\sim}{a}_\alpha$ and $\underset{\sim}{a}_{3,\alpha}$ (compare with (4.3)). Related developments include the construction of a theory of a deformable surface with simple force multipoles by Balaban, Green and Naghdi (1967), where the position vector $\underset{\sim}{r}$ and its first and second gradient $(\underset{\sim}{r}_{,\alpha}, \underset{\sim}{r}_{,\alpha\beta})$ are taken as the basic kinematic variables. A similar theory, but less general than that of Balaban et al. (1967), is given by Cohen and DeSilva (1966a,1968). For additional related comments see Naghdi (1972, Sec. 10).

---

*The equations resulting from such a constrained theory of elastic shells in which $\underset{\sim}{d} = \underset{\sim}{a}_3$ correspond to those which can be obtained from a derivation of shell theory under the so-called Kirchhoff-Love assumption (see Naghdi and Nordgren 1963).

In this section, we begin by considering a class of constraints which are linear relations between the kinematic variables

$$\underset{\sim}{v}_{,\alpha} \; , \; \underset{\sim}{w}_N \; , \; \underset{\sim}{w}_{N,\alpha} \qquad (N = 1,2,\ldots,K) \tag{6.1}$$

in the form (Green and Naghdi 1976a)

$$\underset{\sim}{A}^{M\alpha} \cdot \underset{\sim}{v}_{,\alpha} + \sum_{N=1}^{K} \underset{\sim}{B}^{MN} \cdot \underset{\sim}{w}_N + \sum_{N=1}^{K} \underset{\sim}{C}^{MN\alpha} \cdot \underset{\sim}{w}_{N,\alpha} = 0 \quad (M = o,2,\ldots,Q) \quad , \tag{6.2}$$

where $\underset{\sim}{A}^{M\alpha}, \underset{\sim}{B}^{MN}, \underset{\sim}{C}^{MN\alpha}$ are vector functions of $\underset{\sim}{a}_i, \underset{\sim}{d}_N, \underset{\sim}{d}_{N,\alpha}$ only and do not depend explicitly on the variables (6.1). We assume that each of the functions $\underset{\sim}{N}^{\alpha}, \underset{\sim}{k}^{N}, \underset{\sim}{M}^{N\alpha}$ are determined to within an additive constraint response so that*

$$\underset{\sim}{N}^{\alpha} = \underset{\sim}{\bar{N}}^{\alpha} + \underset{\sim}{\hat{N}}^{\alpha} \quad , \quad \underset{\sim}{k}^{N} = \underset{\sim}{\bar{k}}^{N} + \underset{\sim}{\hat{k}}^{N} \quad , \quad \underset{\sim}{M}^{N\alpha} = \underset{\sim}{\bar{M}}^{N\alpha} + \underset{\sim}{\hat{M}}^{N\alpha} \quad , \tag{6.3}$$

where $\underset{\sim}{\hat{N}}^{\alpha}, \underset{\sim}{\hat{k}}^{N}, \underset{\sim}{\hat{M}}^{N\alpha}$ are specified by constitutive equations and

$$\underset{\sim}{\bar{N}}^{\alpha} \; , \; \underset{\sim}{\bar{k}}^{N} \; , \; \underset{\sim}{\bar{M}}^{N\alpha} \quad , \tag{6.4}$$

which represent the response due to constraints are arbitrary functions of $\theta^{\alpha}$,t and are workless. Thus, recalling the expression (3.30) for mechanical power, we set

$$\underset{\sim}{\bar{N}}^{\alpha} \cdot \underset{\sim}{v}_{,\alpha} + \sum_{N=1}^{K} \underset{\sim}{\bar{k}}^{N} \cdot \underset{\sim}{w}_N + \sum_{N=1}^{K} \underset{\sim}{\bar{M}}^{N\alpha} \cdot \underset{\sim}{w}_{N,\alpha} = 0 \tag{6.5}$$

for all values of the variables (6.1) subject to the constraint conditions (6.2). It then follows that

$$\underset{\sim}{\bar{N}}^{\alpha} = - \sum_{M=o}^{Q} \underset{\sim}{A}^{M\alpha} p_M \quad , \quad \underset{\sim}{\bar{k}}^{N} = - \sum_{M=o}^{Q} \underset{\sim}{B}^{MN} p_M \quad , \quad \underset{\sim}{\bar{M}}^{N\alpha} = - \sum_{M=o}^{Q} \underset{\sim}{C}^{MN\alpha} p_M \quad , \tag{6.6}$$

---

*The development between (6.2)-(6.6) is similar to that for mechanical constraints in the 3-dimensional theory (see section 30 of Truesdell and Noll 1965). For a corresponding thermodynamical theory of a continuum in the presence of thermo-mechanical constraints see Green, Naghdi and Trapp (1970) and Green and Naghdi (1977).

where $p_M = p_M(\theta^\alpha, t)$, $(M = o, 1, \ldots, Q)$ are arbitrary functions which play the role of Lagrange multipliers.

In the rest of this section, we illustrate the nature of constrained theories with reference to two particular kinematical constraints. One of these constraints is that appropriate for incompressible media and the other pertains to a restriction on the director in the theory of a Cosserat surface $\mathcal{C}$ with a single director.

## 6.1 Incompressible Cosserat surfaces

The conditions representing approximately the (3-dimensional) incompressibility condition (2.10) may be derived with the use of the approximation $(2.22)_1$. However, in the interest of brevity, we confine attention to a special case of $(2.22)_1$ for $N = 1$ given by (2.25). Under the approximation (2.25), the base vectors are given by $\mathbf{g}_\alpha = \mathbf{a}_\alpha + \xi \mathbf{d}_{,\alpha}$, $\mathbf{g}_3 = \mathbf{d}$, where $\mathbf{a}_\alpha$ are the base vectors of the surface $\xi = 0$ calculated from (2.25). Then, the incompressibility condition (2.10) may be expressed approximately in the form

$$\frac{d}{dt}\,[\mathbf{a}_1 \mathbf{a}_2 \mathbf{d}] + \xi\,\frac{d}{dt}\,\{[\frac{\partial \mathbf{d}}{\partial \theta^1}\,\mathbf{a}_2 \mathbf{d}] + [\mathbf{a}_1\,\frac{\partial \mathbf{d}}{\partial \theta^2}\,\mathbf{d}]\} + \xi^2\,\frac{d}{dt}\,[\frac{\partial \mathbf{d}}{\partial \theta^1}\,\frac{\partial \mathbf{d}}{\partial \theta^2}\,\mathbf{d}] = 0 \tag{6.7}$$

or equivalently as

$$\begin{aligned}&[(\mathbf{d}\cdot\mathbf{a}_3)\mathbf{a}^\alpha - (\mathbf{d}\cdot\mathbf{a}^\alpha)\mathbf{a}_3 + \xi(\varepsilon^{\alpha\beta}\mathbf{d}_{,\beta}\times\mathbf{d})]\cdot\mathbf{v}_{,\alpha}\\ &+ [\mathbf{a}_3 + \xi\varepsilon^{\alpha\beta}\mathbf{a}_\alpha\times\mathbf{d}_{,\beta} + \tfrac{1}{2}\,\xi^2\varepsilon^{\alpha\beta}\mathbf{d}_{,\alpha}\times\mathbf{d}_{,\beta}]\cdot\mathbf{w} + [\xi\varepsilon^{\alpha\beta}\mathbf{a}_\beta\times\mathbf{d} + \xi^2\varepsilon^{\alpha\beta}\mathbf{d}\times\mathbf{d}_{,\beta}]\cdot\mathbf{w}_{,\alpha} = 0\end{aligned} \tag{6.8}$$

where in (6.7) and (6.8) use is made of the notation $(3.29)_{1,2}$ and $\varepsilon^{\alpha\beta}, \varepsilon_{\alpha\beta}$ denote the components of the $\varepsilon$-system in 2-space defined by

$$\begin{aligned}\varepsilon^{\alpha\beta} &= a^{-\frac{1}{2}}e^{\alpha\beta} \quad , \quad e^{11} = e^{22} = 0 \quad , \quad e^{12} = -e^{21} = 1 \quad ,\\ \varepsilon_{\alpha\beta} &= a^{\frac{1}{2}}e_{\alpha\beta} \quad , \quad e_{11} = e_{22} = 0 \quad , \quad e_{12} = -e_{21} = 1 \quad .\end{aligned} \tag{6.9}$$

We now generate two conditions representing incompressibility: One of these is obtained from integration of (6.8) with respect to $\xi$ between the limits $\xi_1, \xi_2$ and another by first multiplying (6.8) by $\xi$, neglecting terms involving $\xi^3$ and then integrating the resulting equation with respect to $\xi$ between the limits $\xi_1, \xi_2$. The resulting two conditions are:

$$\begin{aligned}
&\{\gamma^o[(\underset{\sim}{d} \cdot \underset{\sim}{a}_3)\underset{\sim}{a}^\alpha - (\underset{\sim}{d} \cdot \underset{\sim}{a}^\alpha)\underset{\sim}{a}_3] + \gamma^1 \varepsilon^{\alpha\beta}(\underset{\sim}{d}_{,\beta} \times \underset{\sim}{d})\} \cdot \underset{\sim}{v}_{,\alpha} \\
&\quad + \{\gamma^o \underset{\sim}{a}_3 + \varepsilon^{\alpha\beta}[\gamma^1(\underset{\sim}{a}_\alpha \times \underset{\sim}{d}_{,\beta}) + \tfrac{1}{2}\gamma^2(\underset{\sim}{d}_{,\alpha} \times \underset{\sim}{d}_{,\beta})\} \cdot \underset{\sim}{w} \\
&\quad + \{\varepsilon^{\alpha\beta}[\gamma^1(\underset{\sim}{a}_\beta \times \underset{\sim}{d}) + \gamma^2(\underset{\sim}{d}_{,\beta} \times \underset{\sim}{d})]\} \cdot \underset{\sim}{w}_{,\alpha} = 0 \quad , \\
&\{\gamma^1[(\underset{\sim}{d} \cdot \underset{\sim}{a}_3)\underset{\sim}{a}^\alpha - (\underset{\sim}{d} \cdot \underset{\sim}{a}^\alpha)\underset{\sim}{a}_3] + \gamma^2 \varepsilon^{\alpha\beta}(\underset{\sim}{d}_{,\beta} \times \underset{\sim}{d})\} \cdot \underset{\sim}{v}_{,\alpha} \\
&\quad + \{\gamma^1 \underset{\sim}{a}_3 + \gamma^2 \varepsilon^{\alpha\beta}(\underset{\sim}{a}_\alpha \times \underset{\sim}{d}_{,\beta})\} \cdot \underset{\sim}{w} + \gamma^2 \varepsilon^{\alpha\beta}(\underset{\sim}{a}_\beta \times \underset{\sim}{d}) \cdot \underset{\sim}{w}_{,\alpha} = 0 \quad ,
\end{aligned} \tag{6.10}$$

where the coefficients $\gamma^K$ are defined by

$$\gamma^K = \int_{\xi_1}^{\xi_2} \xi^K d\xi \quad , \quad (K = o,1,2) \quad . \tag{6.11}$$

It is perhaps interesting to observe that in special circumstances in which the quantity $\lambda^*$ in (5.1)-(5.2) is or can be approximated to be independent of $\xi$, then the coefficients $\gamma^1$ and $\gamma^2$ in (6.10) will have the same numerical values as the inertia coefficients $y^1$ and $y^2$, respectively.

For an incompressible Cosserat surface under discussion, from (6.2) the constraint conditions are

$$\begin{aligned}
&\underset{\sim}{A}^{o\alpha} \cdot \underset{\sim}{v}_{,\alpha} + \underset{\sim}{B}^{o1} \cdot \underset{\sim}{w} + \underset{\sim}{C}^{o1\alpha} \cdot \underset{\sim}{w}_{,\alpha} = 0 \quad , \\
&\underset{\sim}{A}^{1\alpha} \cdot \underset{\sim}{v}_{,\alpha} + \underset{\sim}{B}^{11} \cdot \underset{\sim}{w} + \underset{\sim}{C}^{11\alpha} \cdot \underset{\sim}{w}_{,\alpha} = 0 \quad ,
\end{aligned} \tag{6.12}$$

and the corresponding constrained response obtained from (6.6) has the form

$$\begin{aligned}
\bar{\underset{\sim}{N}}^\alpha &= -(p_o \underset{\sim}{A}^{o\alpha} + p_1 \underset{\sim}{A}^{1\alpha}) \quad , \\
\bar{\underset{\sim}{k}} &= -(p_o \underset{\sim}{B}^{o\alpha} + p_1 \underset{\sim}{B}^{11}) \quad , \\
\bar{\underset{\sim}{M}}^\alpha &= -(p_o \underset{\sim}{C}^{o1\alpha} + p_1 \underset{\sim}{C}^{11\alpha}) \quad ,
\end{aligned} \tag{6.13}$$

where $p_o, p_1$ are the Lagrange multipliers. Guided by the two conditions which follow from (6.10), we select the vectors $\underset{\sim}{A}^{o\alpha}, \underset{\sim}{B}^{o1}, \underset{\sim}{C}^{o1\alpha}, \ldots$, which occur in (6.12) to have the special values

$$
\left.\begin{aligned}
\mathbf{A}^{K\alpha} &= \text{coeff. of } \mathbf{v}_{,\alpha} \text{ in (6.10)} \\
\mathbf{B}^{K1} &= \text{coeff. of } \mathbf{w} \text{ in (6.10)} \\
\mathbf{C}^{K1\alpha} &= \text{coeff. of } \mathbf{w}_{,\alpha} \text{ in (6.10)}
\end{aligned}\right\} \quad (K = o,1) \quad . \tag{6.14}
$$

Then, it follows from (6.13) and (6.14) that the expressions for the constraint response are given by*

$$
\begin{aligned}
\bar{\mathbf{N}}^{\alpha} &= -P_o[(\mathbf{d}\cdot\mathbf{a}_3)\mathbf{a}^{\alpha} - (\mathbf{d}\cdot\mathbf{a}^{\alpha})\mathbf{a}_3] - P_1\varepsilon^{\alpha\beta}\mathbf{d}_{,\beta}\times\mathbf{d} \quad , \\
\bar{\mathbf{k}} &= -P_o\mathbf{a}_3 - P_1\varepsilon^{\alpha\beta}\mathbf{a}_{\alpha}\times\mathbf{d}_{,\beta} - \tfrac{1}{2}\gamma^2 p_o\varepsilon^{\alpha\beta}\mathbf{d}_{,\alpha}\times\mathbf{d}_{,\beta} \quad , \\
\bar{\mathbf{M}}^{\alpha} &= -P_1\varepsilon^{\alpha\beta}\mathbf{a}_{\beta}\times\mathbf{d} - \gamma^2 p_o\varepsilon^{\beta\alpha}\mathbf{d}\times\mathbf{d}_{,\beta} \quad .
\end{aligned} \tag{6.15}
$$

The arbitrary coefficient functions $P_o, P_1$ are related to the Lagrange multipliers $p_o, p_1$ and $\gamma^2 p_o$ can be expressed in terms of $P_o, P_1$ as follows:

$$
P_o = \gamma^o p_o + \gamma^1 p_1 \quad , \quad P_1 = \gamma^1 p_o + \gamma^2 p_1 \quad , \qquad p_o = \frac{\gamma^2 P_o - \gamma^1 P_1}{\gamma^o\gamma^2 - (\gamma^1)^2} \quad . \tag{6.16}
$$

In obtaining the results (6.10) and (6.15), no identification has been made between the surfaces $(3.1)_1$ in the theory of a Cosserat surface $\mathcal{C}$ and an appropriate reference surface in the (3-dimensional) shell-like body. Indeed, different values for the coefficients $\gamma^K$ in (6.10) will result depending on the choice of the identification with the surface $\xi = 0$. For example, if this surface is chosen between the major surfaces of the shell-like body in such a way that $\xi_1 = -\xi_2 = -\frac{1}{2}$, then the coefficients $\gamma^K$ in (5.8) and $P_o, P_1$ in (6.15) become

$$
\begin{gathered}
\gamma^o = 1 \quad , \quad \gamma^1 = 0 \quad , \quad \gamma^2 = \frac{1}{12} \quad , \\
P_o = p_o \quad , \quad P_1 = \frac{1}{12}\,p_1 \quad .
\end{gathered} \tag{6.17}
$$

---

*Although the expressions (6.15) have the same form as those given previously (Green and Naghdi 1976a, Eqs. (4.2)), they are not the same in view of the relations (6.16) and the incompressibility conditions (6.10) reduce to those used by Green and Naghdi (1976a, Eqs. (4.3)) for a directed fluid sheet with a single director. On the other hand, if we identify $(3.1)_1$ with the bottom surface of the shell-like body so that $\xi_1 = 0$, $\xi_2 = 1$, then the coefficients $\gamma^K$ and $P_o, P_1$ become

$$\gamma^o = 1 \quad , \quad \gamma^1 = \tfrac{1}{2} \quad , \quad \gamma^2 = \frac{1}{3} \quad ,$$
$$P_o = p_o + \tfrac{1}{2}p_1 \quad , \quad P_1 = \tfrac{1}{2}p_o + \frac{1}{3}\,p_1 \quad . \tag{6.18}$$

For a complete theory of an incompressible Cosserat surface, constitutive equations are required for the quantities $\hat{\mathbf{N}}^\alpha, \hat{\mathbf{k}}$ and $\hat{\mathbf{M}}^\alpha$ but a discussion of these can be carried out as in Sec. 4.

6.2 <u>A constrained theory with director along the normal to the surface of $\mathcal{C}$</u>

We turn now to the development of a constrained theory of a Cosserat surface in which the director is always along the normal to the material surface so that

$$\mathbf{d}\cdot\mathbf{a}_\alpha = 0 \quad , \quad \mathbf{d} = \phi\mathbf{a}_3 \quad , \quad \phi = \phi(\theta^\gamma, t) \quad . \tag{6.19}$$

Differentiating the constraint condition $(6.19)_1$ with respect to time and using $(3.3)_2$ and (3.4) we obtain

$$\mathbf{d}\cdot\mathbf{v}_{,\alpha} + \mathbf{a}_\alpha\cdot\mathbf{w} = 0 \quad , \tag{6.20}$$

which represents two constraint conditions. From $(6.19)_2$, along with the use of $(2.13)_1$ and $(2.15)_3$, follows the expression

$$\mathbf{d}_{,\alpha}\cdot\mathbf{a}_\beta = -\phi b_{\alpha\beta} \quad , \tag{6.21}$$

which is symmetric in $\alpha,\beta$. Hence

$$\varepsilon^{\alpha\beta}(\mathbf{d}_{,\alpha} \cdot \mathbf{a}_{\beta}) = 0 \quad , \tag{6.22}$$

where $\varepsilon^{\alpha\beta}$ is defined in (6.9). Differentiating (6.22) with respect to t and observing that $\dot{\varepsilon}^{\alpha\beta}(\mathbf{d}_{,\alpha} \cdot \mathbf{a}_{\beta})=0$ in view of (6.22) and the fact that $\dot{\varepsilon}^{\alpha\beta} = \dot{a}^{-\frac{1}{2}} e^{\alpha\beta}$, we also have

$$\varepsilon^{\alpha\beta}(\mathbf{d}_{,\alpha} \cdot \mathbf{v}_{,\beta} + \mathbf{a}_{\beta} \cdot \mathbf{w}_{,\alpha}) = 0 \quad , \tag{6.23}$$

as a third constraint condition.

The two conditions $(6.20)_{1,2}$ can be regarded as a special case (6.2) for $K = 1$ and with the coefficient of $\mathbf{w}_{,\alpha}$ equal to zero. Similarly, (6.23) is a special case of (6.2) for $M = o$, $K = 1$ and with the coefficient of $\mathbf{w}$ equal to zero. Thus, each of the three constraint conditions $(6.20)_{1,2}$ and (6.23) may be viewed as a special case of (6.2) with coefficient functions $\mathbf{A}^{o}, \mathbf{B}^{o1}, \mathbf{C}^{o1\alpha}, \mathbf{A}^{M\alpha}, \mathbf{B}^{M1}, \mathbf{C}^{M1\alpha}$ conveniently identified as

$$\begin{aligned} &\mathbf{A}^{o\alpha} = \varepsilon^{\beta\alpha}\mathbf{d}_{,\beta} \quad , \quad \mathbf{B}^{o} = \mathbf{o} \quad , \quad \mathbf{C}^{o1\alpha} = \varepsilon^{\alpha\beta}\mathbf{a}_{\beta} \quad , \\ &\mathbf{A}^{M\alpha} = \delta^{\alpha}_{M}\mathbf{d} \quad , \quad \mathbf{B}^{M1} = \mathbf{a}_{M} \quad , \quad \mathbf{C}^{M1\alpha} = \mathbf{o} \quad (M = 1,2) \quad . \end{aligned} \tag{6.24}$$

Now according to (6.6) and with the help of (6.24), the expressions for the constraint response are

$$\begin{aligned} &\bar{\mathbf{N}}^{\alpha} = -\,[p^{o}\varepsilon^{\beta\alpha}\mathbf{d}_{,\beta} + \sum_{M=1}^{2} p^{M}\delta^{\alpha}_{M}\mathbf{d}] = -\,[p^{o}\varepsilon^{\beta\alpha}\mathbf{d}_{,\beta} + p^{\alpha}\mathbf{d}] \quad , \\ &\bar{\mathbf{k}} = -\sum_{M=1}^{2} p^{M}\mathbf{a}_{M} = -\,p^{\alpha}\mathbf{a}_{\alpha} \quad , \quad \bar{\mathbf{M}}^{\alpha} = -\,p^{o}\varepsilon^{\alpha\beta}\mathbf{a}_{\beta} \quad , \end{aligned} \tag{6.25}$$

where $p^{o}, p^{\alpha}$ are the Lagrangian multipliers, and in line with the notation of $(3.29)_{1,2}$ and that of subsection 3.2 we have set $\mathbf{k}^{1} = \mathbf{k}$, $\mathbf{M}^{1\alpha} = \mathbf{M}^{\alpha}$.

In anticipation of the final form of the equations of the constrained theory, we could set the skew-symmetric parts of both $M^{\alpha\gamma}$ and $\hat{M}^{\alpha\gamma}$ equal to zero and thus require also the vanishing of the skew-symmetric part of $\bar{M}^{\alpha\gamma}$ which is equivalent to setting $p^{o} = 0$. However, we postpone such stipulations until later in this section, and retain in (6.25) the Lagrange multiplier $p^{o}$ which arises from the constraint condition (6.23).

Before recording the modified equations of motion appropriate for the constrained theory under discussion, we introduce the functions $S^\alpha = S^\alpha(\theta^\gamma, t)$ defined by

$$S^\alpha = -[p^\alpha + \varepsilon^{\alpha\beta} p^o_{,\beta}] \tag{6.26}$$

and note that

$$S^\alpha|_\alpha = p^\alpha|_\alpha \quad . \tag{6.27}$$

Also, for convenience, we introduce the abbreviation

$$\hat{\underset{\sim}{f}} = \underset{\sim}{f} - (\dot{\underset{\sim}{v}} + y^1 \dot{\underset{\sim}{w}}) \quad , \quad \hat{\underset{\sim}{\ell}} = \underset{\sim}{\ell} - (y^1 \dot{\underset{\sim}{v}} + y^2 \dot{\underset{\sim}{w}}) \quad . \tag{6.28}$$

Then, after substituting (6.3) and (6.25) into $(3.20)_{3,4,5}$ and making use of (6.26) and (6.27) we obtain

$$\begin{aligned} \hat{\underset{\sim}{N}}^\alpha|_\alpha + \rho \hat{\underset{\sim}{f}} &= -(\phi S^\alpha \underset{\sim}{a}_3)|_\alpha \quad , \\ \hat{\underset{\sim}{M}}^\alpha|_\alpha - \hat{\underset{\sim}{k}} + \rho \hat{\underset{\sim}{\ell}} &= S^\alpha \underset{\sim}{a}_\alpha \quad , \\ \underset{\sim}{a}_\alpha \times \hat{\underset{\sim}{N}}^\alpha + \underset{\sim}{d} \times \hat{\underset{\sim}{k}} + \underset{\sim}{d}_{,\alpha} \times \hat{\underset{\sim}{M}}^\alpha &= \underset{\sim}{0} \quad , \end{aligned} \tag{6.29}$$

as the equations of motion of the constrained theory. It should be noted that the above equations involve only two arbitrary functions of position and time related to the three multipliers $p^o, p^\alpha$ by (6.26). Moreover, $(6.29)_3$ and the normal component of $(6.29)_2$ are free from $S^\alpha$.

A further reduction of the system of equations (6.29) may be effected by eliminating $S^\alpha$ between $(6.29)_{1,2}$. For this purpose it is convenient to refer the various vector quantities in (6.29) to the base vectors $\underset{\sim}{a}_i$ and write the equations of motion in tensor components. Thus, we write*

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*As noted earlier, the order of indices in $(6.30)_{1,3}$ are opposite to those used in Naghdi (1972) and most of the earlier papers on the subject.

$$\underset{\sim}{N}^{\alpha} = N^{i\alpha}\underset{\sim}{a}_i \quad , \quad \underset{\sim}{k} = k^i\underset{\sim}{a}_i \quad , \quad \underset{\sim}{M}^{\alpha} = M^{i\alpha}\underset{\sim}{a}_i \quad , \tag{6.30}$$

$$\underset{\sim}{f} = f^i\underset{\sim}{a}_i \quad , \quad \underset{\sim}{\ell} = \ell^i\underset{\sim}{a}_i \quad , \tag{6.31}$$

with similar expressions for $\hat{\underset{\sim}{N}}^{\alpha}, \hat{\underset{\sim}{k}}, \hat{\underset{\sim}{M}}^{\alpha}, \hat{\underset{\sim}{f}}, \hat{\underset{\sim}{\ell}}$. Recalling $(6.19)_1$ and making use of formulas of the type (6.30), from the scalar product of $(6.29)_3$ with $\underset{\sim}{a}^{\beta}$ and again with $\underset{\sim}{a}^3$ we deduce

$$\begin{aligned} &\varepsilon_{\alpha\beta}(\hat{N}^{\alpha\beta} - \phi b^{\beta}_{\gamma}\hat{M}^{\alpha\gamma}) = 0 \quad , \\ &\hat{N}^{3\alpha} - \phi\hat{k}^{\alpha} - \phi b^{\alpha}_{\gamma}\hat{M}^{3\gamma} - \phi_{,\gamma}\hat{M}^{\alpha\gamma} = 0 \quad , \end{aligned} \tag{6.32}$$

where $\varepsilon^{\alpha\beta}$ is defined in (6.9).

It is instructive at this point to express the mechanical power (3.22) in terms of the tensor components (6.30). To this end, we first note from $(6.19)_1$ and $(2.13)_{1,2}$ that the tensor components of $\underset{\sim}{w}$ and $\underset{\sim}{w}_{,\alpha}$ referred to $\underset{\sim}{a}_i$ are

$$\begin{aligned} &\underset{\sim}{w} \cdot \underset{\sim}{a}_{\alpha} = -\phi(\underset{\sim}{v}_{,\alpha} \cdot \underset{\sim}{a}_3) \quad , \quad \underset{\sim}{w} \cdot \underset{\sim}{a}_3 = \dot{\phi} \\ &\underset{\sim}{w}_{,\alpha} \cdot \underset{\sim}{a}_{\beta} = -\overline{(\phi b_{\beta\alpha})}^{\cdot} + \phi b_{\alpha\gamma}(\underset{\sim}{v}_{,\beta} \cdot \underset{\sim}{a}^{\gamma}) - \phi_{,\alpha}(\underset{\sim}{v}_{,\beta} \cdot \underset{\sim}{a}_3) \quad , \\ &\underset{\sim}{w}_{,\alpha} \cdot \underset{\sim}{a}_3 = \dot{\phi}_{,\alpha} - \phi b_{\alpha\beta}(\underset{\sim}{v}_{,\gamma} \cdot \underset{\sim}{a}_3)a^{\gamma\beta} \quad . \end{aligned} \tag{6.33}$$

Then, remembering (6.4), (6.5) for $N = 1$ and the notation $(3.29)_{1,2}$, we may write (3.22) as

$$\begin{aligned} P = {} &(\hat{N}^{\beta\alpha} - \phi b^{\beta}_{\gamma}\hat{M}^{\alpha\gamma})(\underset{\sim}{v}_{,\alpha} \cdot \underset{\sim}{a}_{\beta}) + \hat{k}^3\dot{\phi} - \hat{M}^{\beta\alpha}\overline{(\phi b_{\alpha\beta})}^{\cdot} + \hat{M}^{3\alpha}\dot{\phi}_{,\alpha} \\ &+ [\hat{N}^{3\alpha} - \phi\hat{k}^{\alpha} - \hat{M}^{\alpha\beta}\phi_{,\beta} - \phi b^{\alpha}_{\beta}\hat{M}^{3\beta}](\underset{\sim}{v}_{,\alpha} \cdot \underset{\sim}{a}_3) \quad . \end{aligned}$$

Since the coefficient of $(\underset{\sim}{v}_{,\alpha} \cdot \underset{\sim}{a}_3)$ vanishes identically in view of $(6.32)_2$, the last expression reduces to

$$P = (\hat{N}^{\alpha\beta} - \phi b^{\beta}_{\gamma}\hat{M}^{\alpha\gamma})(\underset{\sim}{v}_{,\alpha} \cdot \underset{\sim}{a}_{\beta}) + \hat{k}^{3}\dot{\phi}$$

$$- \hat{M}^{\beta\alpha}\,\overline{(\phi b_{\alpha\beta})}^{\,\cdot} + \hat{M}^{3\alpha}\dot{\phi}_{,\alpha} \tag{6.34}$$

and does not involve the components $\hat{N}^{3\alpha}$ and $\hat{k}^{\alpha}$. Next, with the help of $\phi\hat{k}^{\alpha} + \phi\hat{M}^{3\gamma}b^{\alpha}_{\gamma} = \hat{N}^{3\alpha} - M^{\rho\gamma}\phi_{,\gamma}$ which follows from $(6.32)_2$, the component form of the equations of motion $(6.29)_{1,2}$ referred to $\underset{\sim}{a}^{i}$ can be written as

$$\hat{N}^{\beta\alpha}{}_{|\alpha} - b^{\beta}_{\alpha}N^{3\alpha} + \rho\hat{f}^{\beta} = 0 \quad , \quad N^{3\alpha}{}_{|\alpha} + b_{\beta\alpha}\hat{N}^{\alpha\beta} + \rho\hat{f}^{3} = 0 \quad , \tag{6.35}$$

$$(\phi M^{\beta\alpha})_{|\alpha} + \rho\hat{\ell}^{\beta} = N^{3\alpha} \quad , \quad M^{3\alpha}{}_{|\alpha} + b_{\beta\alpha}\hat{M}^{\beta\alpha} - \hat{k}^{3} + \rho\hat{\ell}^{3} = 0 \quad , \tag{6.36}$$

where in recording (6.35) and (6.36) we have also substituted $N^{3\alpha}$ for the quantity $(\hat{N}^{3\alpha} + \phi S^{\alpha})$. By substitution from $(6.36)_1$, we can now eliminate $N^{3\alpha}$ from $(6.35)_{1,2}$. In this way, the resulting two equations may be put in the form

$$\{(\hat{\underset{\sim}{N}}^{\alpha} \cdot \underset{\sim}{a}^{\beta})_{|\alpha} - b^{\beta}_{\alpha}(\phi\underset{\sim}{M}^{\gamma} \cdot \underset{\sim}{a}^{\alpha})_{|\gamma}\}\underset{\sim}{a}_{\beta} + \{(\phi\underset{\sim}{M}^{\beta} \cdot \underset{\sim}{a}^{\alpha})_{|\beta\alpha} + b_{\alpha\beta}(\hat{\underset{\sim}{N}}^{\alpha} \cdot \underset{\sim}{a}^{\beta})\}\underset{\sim}{a}_{3}$$

$$+ \{\rho\hat{\underset{\sim}{f}} + (\rho\phi\hat{\ell}^{\alpha}\underset{\sim}{a}_{3})_{|\alpha}\} = 0 \quad . \tag{6.37}$$

In a general theory of an elastic Cosserat surface (Sec. 4), constitutive equations for both the symmetric and the skew-symmetric parts of $N^{\alpha\beta}, M^{\alpha\beta}$ can be provided through the expression for mechanical power. Here, however, since $b_{\alpha\beta}$ is symmetric, the term $-\hat{M}^{\beta\alpha}\overline{(\phi b_{\alpha\beta})}^{\,\cdot}$ in (6.34) provides constitutive equations for only the symmetric part of $\hat{M}^{\alpha\beta}$. Moreover, the quantity $(\hat{N}^{\alpha\beta} - \phi b^{\beta}_{\gamma}\hat{M}^{\alpha\gamma})$ is symmetric by virtue of $(6.32)_1$ and the two differential equations resulting from (6.37) involve only the symmetric parts of $\hat{N}^{\alpha\beta}$ and $\hat{M}^{\alpha\beta}$. Thus, in line with classical results in shell theory, in order to obtain a determinate theory we now put*

$$\hat{M}^{[\alpha\beta]} = 0 \quad , \quad \hat{M}^{[\alpha\beta]} = \tfrac{1}{2}(\hat{M}^{\alpha\beta} - \hat{M}^{\alpha\beta}) \quad . \tag{6.38}$$

In summary, the relevant system of equations of the constrained theory under discussion are given by (6.37), the normal component of $(6.29)_2$, i.e., $(6.36)_2$ and the skew-symmetric part $N^{\alpha\beta}$ is determined from $(6.32)_1$. This

*Instead of introducing $(6.38)_1$, in anticipation of the fact that $\hat{M}^{[\alpha\beta]}$ makes no contribution to the mechanical power, at the outset we could have absorbed $\hat{M}^{[\alpha\beta]}$ into $\bar{M}^{[\alpha\beta]}$ or equivalently into $\underset{\sim}{\bar{M}}^{\alpha}$ in (6.25).

completes the development of the constrained theory in which the director is constrained to have the form $(6.19)_1$.

If in addition to (6.37) we also set the multiplier $p^o = 0$, then $\bar{M}^{[\alpha\beta]} = 0$ and hence the skew-symmetric $M^{[\alpha\beta]} = 0$. It then follows that

$$S^\alpha = -p^\alpha \quad , \tag{6.39}$$

$$N^{3\alpha} = N^{3\alpha} - \phi S^\alpha \quad , \quad k^\alpha = \hat{k}^\alpha - S^\alpha \tag{6.40}$$

and the relevant equations of motion of the determinate constrained theory remain as before. It is of interest to examine the reduction of the foregoing development when $\phi = 1$. In this case, we have $\underset{\sim}{d} = \underset{\sim}{a}_3$ instead of $(6.19)_1$ and the resulting equations are identical with those of a restricted theory discussed by Naghdi (1972, Secs. 10 and 15). The results with $\phi = 1$ can also be brought into correspondence with a special case of the constrained theory discussed by Green and Naghdi (1974) or those contained in the paper of Naghdi and Nordgren (1963).

The nature of the boundary conditions in the theory of a Cosserat surface $\mathcal{C}$ discussed in subsection 3.2 is clear from the expression for the rate of work $R_c$ of contact force and contact director force over the closed boundary curve $\partial\mathcal{P}$, namely

$$R_c(\mathcal{P}) = \int_{\partial\mathcal{P}} (\underset{\sim}{n} \cdot \underset{\sim}{v} + \underset{\sim}{m} \cdot \underset{\sim}{w})ds \quad . \tag{6.41}$$

However, in a constrained theory of the type discussed in subsection 6.2, the question of the boundary conditions must be reconsidered in view of the reduction in the number of differential equations*. Since the development of the reduced boundary conditions is similar to that of a restricted

*The number of the relevant scalar differential equations of the constrained theory is five as compared to the nine scalar equations in the theory of subsection 3.2.

theory (Naghdi 1972, p. 552), our discussion will be brief.

Recalling $(6.30)_{1,2,3}$ and (6.33), from (6.41) we obtain

$$R_c(\mathcal{P}) = \int_{\partial\mathcal{P}} \nu_\alpha\{(N^{\gamma\alpha} - \phi b^\gamma_\beta M^{\beta\alpha})v_\gamma + N^{3\alpha}v_3 + M^{3\alpha}\dot{\phi} - \phi M^{\gamma\alpha}v_{3,\gamma}\}ds \tag{6.42}$$

Let $\partial/\partial\nu$ stand for the directional derivative along the unit normal $\underset{\sim}{\nu}$ to the boundary curve $\partial\mathcal{P}$ and let $\partial/\partial s$ denote the directional derivative along the tangent to $\partial\mathcal{P}$. Then, provided the quantities in (6.42) are single-valued on a (sufficiently smooth) closed curve $\partial\mathcal{P}$, with the use of

$$v_{3,\gamma} = \frac{\partial v_3}{\partial\nu}\nu_\gamma - \frac{\partial v_3}{\partial s}\varepsilon_{\gamma\beta}\nu^\beta \tag{6.43}$$

and an integration by parts, (6.42) can be reduced to

$$R_c(\mathcal{P}) = \int_{\partial\mathcal{P}} \{p^\beta v_\beta + p^3 v_3 - G\frac{\partial v_3}{\partial\nu} + H\dot{\phi}\}ds \quad , \tag{6.44}$$

where

$$\begin{aligned} p^\beta &= (N^{\beta\alpha} - \phi b^\beta_\gamma M^{\gamma\alpha})\nu_\alpha \quad , \quad G = \phi M^{\gamma\alpha}\nu_\gamma\nu_\alpha = \phi M^{(\gamma\alpha)}\nu_\gamma\nu_\alpha \quad , \\ p^3 &= \phi N^{3\alpha}\nu_\alpha - \frac{\partial}{\partial s}(\phi M^{\gamma\alpha}\nu_\alpha\varepsilon_{\gamma\beta}\nu^\beta) \quad , \quad H = M^{3\alpha}\nu_\alpha \quad . \end{aligned} \tag{6.45}$$

The nature of the reduced boundary conditions of the constrained theory is now clear from (6.44) and (6.45).

## 7. Additional remarks on shells

The theory of Cosserat surfaces can easily allow for the effect of surface tension (Naghdi 1972, p. 547 and 1974) and can accommodate the specification of either tractions or displacements on major surfaces of the shell-like (or sheet-like) bodies for application to various interfacial and contact problems. Even the theory of a Cosserat surface with a single director can be used to formulate a fairly broad class of contact problems of elastic shells and plates, as discussed by Naghdi (1975a). The relevance and applicability of the basic theory of a Cosserat surface to problems of an incompressible, inviscid fluid sheet is discussed by Green, Laws and Naghdi (1974) and by Green and Naghdi (1975,1976a). The nonlinear differential equations derived in these papers include the effects of gravity and surface tension and are also valid for propagation of fairly long water waves in a stream of initial variable depth. A discussion of an incompressible viscous fluid sheet, along with further recent developments on the subject, can be found in the papers of Green and Naghdi (1976a,b;1977a;1979c). The basic theory is also applicable to problems of cell membranes, as has been emphasized by Ericksen (1979).

In the remainder of this section we briefly comment on some special cases of the general theory and also mention some recent researches which bear on the various aspects of elastic shells. Although these developments will be described mainly in the context of a mechanical theory, some recent results pertaining to thermal effects in shells are also discussed.

The well-known membrane theory of shells can be obtained as a special case of the general theory by essentially suppressing the effect of the director and corresponding kinetical variables and this is discussed briefly in Naghdi (1972, Sec. 14). A development of another special theory, known as the inextensional theory, wherein the length of each element of the surface of $\mathcal{s}$ is assumed to remain constant throughout all motions is also contained in Naghdi (1972, Sec. 14). Similarly, a nonlinear <u>restricted</u> theory of shells by direct approach, motivated mainly by the classical theory corresponding to Kirchhoff-Love theory of shells and plates, is given by Naghdi (1972, Secs. 10 and 15). Related constrained theories of an elastic Cosserat surface are already mentioned in Sec. 6 and need not be repeated here.

The nonlinear constitutive equations in Sec. 4 are valid for an elastic Cosserat surface which may be anisotropic with reference to preferred directions associated with material points of $\mathcal{S}$. A general discussion of material symmetries for shells is given by Naghdi (1972, Sec. 13). Carroll and Naghdi (1972) have subsequently examined the influence of the reference geometry on the response of elastic shells by assuming the existence of a local <u>preferred</u> state of the body and then stipulating that the influence of the reference geometry, as in (4.3), occurs through the values of the constitutive variables in the preferred state. Material symmetry restrictions for elastic shells have been discussed also, from a different point of view, by Ericksen (1972a, 1973b) who has also indicated (Ericksen 1973) a comparison with the results contained in the paper of Carroll and Naghdi (1972).

Some general aspects of wave propagation in elastic shells, based on the theory of a Cosserat surface have been discussed by Ericksen (1971). A related study on the subject, limited only to wave propagation in a surface not endowed with a director, was given earlier by Cohen and Suh (1970). The theory of small deformation superposed on a large deformation of an elastic Cosserat surface, along with related problems of stability and vibrations of initially deformed plates, is discussed by Green and Naghdi (1971). Related developments concerning plane waves and stability of elastic plates are given by Ericksen (1973c,1974). For a system of linear equations characterizing the initial mixed boundary-value problem of elastic shells, Naghdi and Trapp (1972) have obtained a uniqueness theorem without the use of definiteness assumption for the strain energy density. This result (Naghdi and Trapp 1972) holds for nonhomogeneous and anisotropic shells undergoing small motions superposed on a large deformation.

In still another study, the theory of a Cosserat surface has been employed by Naghdi (1975a) to formulate contact problems of shells and plates mentioned above. In the derivation of shell theory from the 3-dimensional equations, equations of motion in terms of resultants and detailed consideration of constitutive equations for shells are usually obtained relative to an interior surface, rather than one of the major surfaces of the shell-like body which may be the contacting surface; the interior surface ordinarily is identified with the middle surface of the shell or plate in the reference configuration. In the development of shell theory by direct approach, although the material surface of $\mathcal{S}$ may be identified with any surface of the (3-dimensional) shell-like body, nevertheless the complete discussion of constitutive equations and the identification of the inertia coefficients and the assigned fields may again require explicit use of a reference surface in the shell-like body. For certain problems it is more natural and conceptually more appealing to select one of the two major surfaces as the reference surface but then the detailed available development of the constitutive equations, as well as identification of such quantities as the inertia coefficients, have to be reconsidered relative to the new surface. This problem can be resolved by deriving appropriate transformation relations (Naghdi 1975a), which relate the kinetic variables $\underset{\sim}{n}, \underset{\sim}{k}, \underset{\sim}{m}$ (and hence the response functions) in the two formulations. The results (Naghdi 1975a) are applicable to any shell-like medium and their validity is not limited to elastic shells alone.

Controllable solutions in the theory of a Cosserat surface have been studied by Crochet and Naghdi (1969), Ericksen (1972b) and Naghdi (1975b). In a more recent study, Naghdi and Tang (1977) have discussed controllable deformations that can be maintained, in the absence of body force, in every isotropic elastic membrane by the application of edge loads and/or uniform normal surface loads on the major surfaces of the thin shell-like body. The static solutions of finitely deformed membranes, which are valid for both compressible and incompressible materials, are obtained with the use of a strain energy response function which depends on the metric tensor of the membrane it its deformed configuration. The main results are summarized by several theorems and their corollaries in accordance with three mutually exclusive cases for which the initial undeformed surface of the membrane (which may be a sector of a complete or closed surface) is, respectively, developable, spherical and a surface of variable Gaussian curvature satisfying certain differential criteria. The corresponding deformed surfaces are, respectively, a plane or a right circular cylinder, a sphere and a surface of constant mean curvature. These results are exhaustive in that they represent <u>all</u> finite deformation solutions possible in every isotropic elastic material characterized by the strain energy response mentioned above. Also discussed in the paper of Naghdi and Tang (1977) are some special cases of the general results and several families of solutions in terms of an alternative description which should be useful in application and which permit easy interpretation.

The development of the theory of Cosserat surfaces in Sec. 5 is carried out within the scope of the purely mechanical theory. In earlier work on thermo-mechanical theory of shells by direct approach (Green and Naghdi 1970, Naghdi 1972), only one temperature field was admitted and this allowed for the characterization of temperature changes along some reference surface, such as the middle surface, of the (3-dimensional) shell-like body, but not for temperature changes along the shell thickness. The latter effect has been incorporated recently by Green and Naghdi (1979a) into the thermo-mechanical theory of Cosserat surfaces, together with appropriate thermodynamical restrictions arising from the second law of thermodynamics for shells.

## 8. The basic equations in direct notation

For some purposes it is convenient to have available the basic equations for a Cosserat surface in a direct (coordinate-free) notation and this is the main purpose of the present section. As will be evident presently, the forms of the basic equations in coordinate-free notation are very similar to those of the corresponding equations in the classical 3-dimensional theory and thus may be more suitable in the discussion of general theorems or in developments which parallel those in the 3-dimensional theory.

As in the papers of Carroll and Naghdi (1972) and Naghdi (1977), we introduce the notations grad and Grad to denote the right spatial and material gradient operators, respectively, with respect to the position on the surface $\mathcal{s}$ in the current configuration and on the surface $\mathcal{S}_R$ in the reference configuration. The corresponding divergence operators will be denoted by div and Div, respectively. In particular, for a vector-valued function $\mathbf{V}(\theta^\alpha, t)$, we write*

$$\begin{aligned} \operatorname{grad} \mathbf{V} &= \mathbf{V}_{,\alpha} \otimes \mathbf{d}^\alpha \quad , \quad \operatorname{div} \mathbf{V} = \mathbf{V}_{,\alpha} \cdot \mathbf{d}^\alpha \quad , \\ \operatorname{Grad} \mathbf{V} &= \mathbf{V}_{,\alpha} \otimes \mathbf{D}^\alpha \quad , \quad \operatorname{Div} \mathbf{V} = \mathbf{V}_{,\alpha} \cdot \mathbf{d}^\alpha \quad , \end{aligned} \tag{8.1}$$

where the symbol $\otimes$ denotes the tensor product. Also, the spatial surface gradient and the spatial surface divergence operators are defined by

$$\operatorname{grad}_s V = V_{,\alpha}\, \mathbf{a}^\alpha \quad , \quad \operatorname{div}_s \mathbf{V} = \mathbf{V}_{,\alpha} \cdot \mathbf{a}^\alpha \tag{8.2}$$

*We take this opportunity to correct an error in a previous paper (Naghdi 1977). The definitions $(2.9)_{1,2}$ of Naghdi (1977) should be replaced with those in $(8.1)_{1,2}$ of the present paper with $\mathbf{d}^\alpha$ defined through (3.5). Also, the "$\widetilde{\text{div}}$" operator in $(3.10)_1$ of Naghdi (1977) should be replaced by "$\text{div}_s$" in $(8.2)_2$. The definitions $(2.9)_{3,4}$ of Naghdi (1977) remain unchanged since previously (Naghdi 1977) the director in the reference configuration was specified to have the form $\mathbf{D} = D\mathbf{A}_3$. Except for the modifications noted, all other results in the paper of Naghdi (1977) remain intact.

for all scalar-valued functions V and all vector-valued functions $\underset{\sim}{V}$.

We introduce a measure of deformation by the tensor $\underset{\sim}{F}$, namely†

$$\underset{\sim}{F} = \underset{\sim}{d}_i \otimes \underset{\sim}{D}^i = \text{Grad}\ \underset{\sim}{r} + \underset{\sim}{d}_3 \otimes \underset{\sim}{D}^3 \quad , \tag{8.3}$$

and in view of the notations (3.5) and (3.9) we observe that

$$\begin{aligned} \underset{\sim}{F}\ \underset{\sim}{D}_\alpha &= \underset{\sim}{F}\ \underset{\sim}{A}_\alpha = \underset{\sim}{a}_\alpha = \underset{\sim}{d}_\alpha \quad , \\ \underset{\sim}{F}\ \underset{\sim}{D}_3 &= \underset{\sim}{F}\ \underset{\sim}{D} = \underset{\sim}{d} = \underset{\sim}{d}_3 \quad . \end{aligned} \tag{8.4}$$

From the definition of the determinant of a second order tensor $\underset{\sim}{T}$ given by

$$\det \underset{\sim}{T}[\underset{\sim}{v}_1 \underset{\sim}{v}_2 \underset{\sim}{v}_3] = [\underset{\sim}{T}\ \underset{\sim}{v}_1, \underset{\sim}{T}\ \underset{\sim}{v}_2, \underset{\sim}{T}\ \underset{\sim}{v}_3]$$

for all arbitrary vectors $\underset{\sim}{v}_1, \underset{\sim}{v}_2, \underset{\sim}{v}_3$, and the conditions $(3.1)_3$ and $(3.7)_3$ we obtain

$$\det \underset{\sim}{F} = [\underset{\sim}{d}_1 \underset{\sim}{d}_2 \underset{\sim}{d}_3]/[\underset{\sim}{D}_1 \underset{\sim}{D}_2 \underset{\sim}{D}_3] > 0 \quad . \tag{8.5}$$

The tensor $\underset{\sim}{F}$, a linear operator on vectors in 3-space, is nonsingular; and there exists, therefore, the inverse deformation gradient tensor $\underset{\sim}{F}^{-1}$ defined by

$$\underset{\sim}{F}^{-1} = \underset{\sim}{D}_i \otimes \underset{\sim}{d}^i \quad . \tag{8.6}$$

The inverse operator $\underset{\sim}{F}^{-1}$ transforms vectors in the present configuration into vectors in the reference configuration, i.e.,

$$\underset{\sim}{F}^{-1} \underset{\sim}{d}_i = \underset{\sim}{D}_i \tag{8.7}$$

and it follows that

†This definition of $\underset{\sim}{F}$ is the same as that used by Naghdi (1977). The symbol $\underset{\sim}{F}$ in the paper of Carroll and Naghdi (1972) stands for a different quantity. The term Grad $\underset{\sim}{r}$ in (8.3) corresponds to the deformation gradient tensor $\underset{\sim}{F}$ in the paper of Carroll and Naghdi (1972).

$$\mathbf{F}^{-1}\mathbf{F} = \mathbf{F}\,\mathbf{F}^{-1} = \mathbf{I} = \mathbf{d}_i \otimes \mathbf{d}^i = \mathbf{D}_i \otimes \mathbf{D}^i \quad , \tag{8.8}$$

where $\mathbf{I}$ is the unit tensor in 3-space. We also introduce here the director gradient tensor $\mathbf{G}$ by

$$\mathbf{G} = \text{Grad}\ \mathbf{d} = \mathbf{d}_{3,\alpha} \otimes \mathbf{D}^\alpha = \mathbf{d}_{,\alpha} \otimes \mathbf{D}^\alpha \quad . \tag{8.9}$$

Recalling the definitions $(3.3)_{1,2}$ for the velocity and the director velocity and since $\dot{\mathbf{a}}_\alpha = \mathbf{v}_{,\alpha}$, we have

$$\begin{aligned} \dot{\mathbf{F}} &= \dot{\mathbf{d}}_i \otimes \mathbf{D}^i = \dot{\mathbf{d}}_\alpha \otimes \mathbf{D}^\alpha + \dot{\mathbf{d}}_3 \otimes \mathbf{D}^3 = \mathbf{v}_{,\alpha} \otimes \mathbf{D}^\alpha + \mathbf{w} \otimes \mathbf{D}^3 \quad , \\ \dot{\mathbf{G}} &= \dot{\mathbf{d}}_{3,\alpha} \otimes \mathbf{D}^\alpha = \mathbf{w}_{,\alpha} \otimes \mathbf{D}^\alpha \quad . \end{aligned} \tag{8.10}$$

Also,

$$\begin{aligned} \dot{\mathbf{F}}\,\mathbf{F}^{-1} &= \dot{\mathbf{d}}_i \otimes \mathbf{d}^i = \text{grad}\ \mathbf{v} + \mathbf{w} \otimes \mathbf{d}^3 \quad , \\ \dot{\mathbf{G}}\,\mathbf{F}^{-1} &= \mathbf{w}_{,\alpha} \otimes \mathbf{d}^\alpha = \text{grad}\ \mathbf{w} \quad . \end{aligned} \tag{8.11}$$

Having disposed of the main kinematical results in terms of the gradient tensors $\mathbf{F},\mathbf{G}$ and their rates, we now turn to kinetical quantities. The expressions corresponding to $(3.18)_{1,2}$ for the contact force $\mathbf{n}$ and the contact director $\mathbf{m}$ can now be expressed in the form[§]

$$\mathbf{n} = \mathbf{N}\,\boldsymbol{\nu} \quad , \quad \mathbf{m} = \mathbf{M}\,\boldsymbol{\nu} \ , \tag{8.12}$$

[§] The second order tensors $\mathbf{N},\mathbf{M}$ in (8.12) and their tensor components $N^{i\alpha}, M^{i\alpha}$ in (8.13) are the transpose of the corresponding quantities in Naghdi (1977). The components $N^{i\alpha}$, $M^{i\alpha}$ were used in the paper of Green, Naghdi and Wainwright (1965) but subsequently their transpose, namely $N^{\alpha i}$ and $M^{\alpha i}$, were adopted in subsequent papers so that the notation would be in agreement with that of the classical shell theory. It may be noted that in terms of the latter notation, instead of (8.12), one would have $\mathbf{n} = \mathbf{N}^T\boldsymbol{\nu}$, $\mathbf{m} = \mathbf{M}^T\boldsymbol{\nu}$, where the superposed T denotes transpose. Compare (3.6) and (3.10) of Naghdi (1977) with (8.12) and (8.15) of the present paper.

with the second order tensors $\underset{\sim}{N}, \underset{\sim}{M}$ defined by

$$\underset{\sim}{N} = \underset{\sim}{N}^{\alpha} \otimes \underset{\sim}{d}_{\alpha} = N^{i\alpha} \underset{\sim}{d}_i \otimes \underset{\sim}{d}_{\alpha} \quad , \quad \underset{\sim}{N}^{\alpha} = \underset{\sim}{N}\, \underset{\sim}{d}^{\alpha} \quad ,$$
$$\underset{\sim}{M} = \underset{\sim}{M}^{\alpha} \otimes \underset{\sim}{d}_{\alpha} = M^{i\alpha} \underset{\sim}{d}_i \otimes \underset{\sim}{d}_{\alpha} \quad , \quad \underset{\sim}{M}^{\alpha} = \underset{\sim}{M}\, \underset{\sim}{d}^{\alpha} \quad , \tag{8.13}$$

which also relate the tensors $\underset{\sim}{N}, \underset{\sim}{M}$ to $\underset{\sim}{N}^{\alpha}, \underset{\sim}{M}^{\alpha}$ in (4.8). Also, for convenience, we introduce a second order tensor $\underset{\sim}{K}$ through

$$\underset{\sim}{K} = \underset{\sim}{k} \otimes \underset{\sim}{d}_3 = k^i \underset{\sim}{d}_i \otimes \underset{\sim}{d}_3 \quad , \quad \underset{\sim}{k} = \underset{\sim}{K} \underset{\sim}{d}^3 \quad . \tag{8.14}$$

With the use of (8.13) and by usual procedures, from the conservation laws (3.16) follow the local equations

$$\dot{\rho} + \rho \operatorname{div}_s \underset{\sim}{v} = 0 \quad ,$$
$$\operatorname{div}_s \underset{\sim}{N} + \rho \underset{\sim}{f} = \rho(\dot{\underset{\sim}{v}} + y^1 \dot{\underset{\sim}{w}}) \quad ,$$
$$\operatorname{div}_s \underset{\sim}{M} + \rho \underset{\sim}{\ell} - \underset{\sim}{k} = \rho(y^1 \dot{\underset{\sim}{v}} + y^2 \dot{\underset{\sim}{w}}) \quad , \tag{8.15}$$
$$[\underset{\sim}{N} + \underset{\sim}{K} + \underset{\sim}{M}(\underset{\sim}{G}\, \underset{\sim}{F}^{-1})^T] = [\underset{\sim}{N} + \underset{\sim}{K} + \underset{\sim}{M}(\underset{\sim}{G}\, \underset{\sim}{F}^{-1})^T]^T \quad ,$$

which are equivalent to (3.20). Also, by the definition of the right divergence of a tensor field, we have

$$\operatorname{div}_s \underset{\sim}{N} = \underset{\sim}{N}^{\alpha}{}_{|\alpha} \quad , \quad \operatorname{div}_s \underset{\sim}{M} = \underset{\sim}{M}^{\alpha}{}_{|\alpha} \quad . \tag{8.16}$$

It is interesting that the last statement in (8.15) is analogous to the symmetry of the stress tensor in the 3-dimensional theory. In particular, it may be observed that $\underset{\sim}{a}_{\alpha} \times \underset{\sim}{N}^{\alpha}$, $\underset{\sim}{d} \times \underset{\sim}{k}$ and $\underset{\sim}{d}_{,\alpha} \times \underset{\sim}{M}^{\alpha}$ are, respectively, the axial vectors of $[\underset{\sim}{N} - \underset{\sim}{N}^T]$, $[\underset{\sim}{K} - \underset{\sim}{K}^T]$ and $[\underset{\sim}{M}(\underset{\sim}{G}\, \underset{\sim}{F}^{-1})^T - \underset{\sim}{M}(\underset{\sim}{G}\, \underset{\sim}{F}^{-1})]$. Furthermore, in terms of the kinetical quantities, $\underset{\sim}{N}, \underset{\sim}{M}, \underset{\sim}{K}$ in (8.12)-(8.14) and the rate quantities $(8.10)_{1,2}$, the mechanical power becomes

$$P = \operatorname{tr}\{[\dot{\underset{\sim}{F}}^T(\underset{\sim}{N} + \underset{\sim}{K}) + \dot{\underset{\sim}{G}}^T \underset{\sim}{M}](\underset{\sim}{F}^{-1})^T\} \quad . \tag{8.17}$$

With reference to constitutive equations for elastic shells, instead of the kinematical variables in section 4, we now employ the variables (8.5) and (8.9). Thus, corresponding to constitutive assumption (4.5), we now write

$$\psi = \hat{\psi}(\mathbf{F}, \mathbf{G}; {}_R\mathbf{G}) \quad , \tag{8.18}$$

where

$${}_R\mathbf{G} = \mathrm{Grad}\,\mathbf{D} = \mathbf{D}_{3,\alpha} \otimes \mathbf{D}^{\alpha} \quad , \tag{8.19}$$

along with similar assumptions for $\mathbf{N}, \mathbf{K}, \mathbf{M}$. Then, with the use of (1.1) and (8.17), by usual techniques we obtain the following alternative forms of the constitutive equations:

$$\mathbf{N} + \mathbf{K} = \rho \frac{\partial \hat{\psi}}{\partial \mathbf{F}} \mathbf{F}^T \quad , \quad \mathbf{M} = \rho \frac{\partial \hat{\psi}}{\partial \mathbf{G}} \mathbf{F}^T \quad , \tag{8.20}$$

the first of which can be resolved into

$$\mathbf{N} = \rho \frac{\partial \hat{\psi}}{\partial \mathbf{F}} (\mathbf{D}^{\alpha} \otimes \mathbf{d}_{\alpha}) \quad , \quad \mathbf{K} = \rho \frac{\partial \hat{\psi}}{\partial \mathbf{F}} (\mathbf{D}^{3} \otimes \mathbf{d}_{3}) \quad . \tag{8.21}$$

Also, the response function $\hat{\psi}$ is restricted by

$$\mathbf{S} = \mathbf{S}^T \quad ,$$

$$\mathbf{S} = \frac{\partial \hat{\psi}}{\partial \mathbf{F}} \mathbf{F}^T + \frac{\partial \hat{\psi}}{\partial \mathbf{G}} \mathbf{G}^T = \frac{\partial \hat{\psi}}{\partial \mathbf{F}} (\mathbf{D}^{\alpha} \otimes \mathbf{d}_{\alpha}) + \frac{\partial \hat{\psi}}{\partial \mathbf{F}} (\mathbf{D}^{3} \otimes \mathbf{d}_{3}) + \frac{\partial \hat{\psi}}{\partial \mathbf{G}} \mathbf{G}^T \quad . \tag{8.23}$$

## Part B

## Elastic rods: A direct formulation

In Part B (Secs. 9-13), we first summarize the main kinematics and the basic principles of the theory of Cosserat (or directed) curves and then discuss the constitutive equations for elastic rods, as well as some related aspects of the basic theory and recent developments on the subject. Although we are concerned here mainly with the purely mechanical theory involving appropriate forms of the conservation laws for mass, linear momentum, director momentum and moment of momentum, we also include a statement of the conservation of energy. The latter provides motivation in the development of certain constitutive equations, such as those for an elastic material, and in the discussion of aspects of some special solutions involving jump in energy. The contents of Part B are as follows:

## 9. The basic theory of Cosserat curves

Having defined a (three-dimensional) rod-like body in section 2, we now formally introduce a direct model for such a body. Thus, deformable media which are modelled by a material curve $\mathcal{L}$ embedded in a Euclidean 3-space, together with L ($L \geq 2$) deformable vector fields -- called directors -- attached to every point of the material curve are called Cosserat curves or directed curves and may be conveniently referred to as $\mathcal{R}_K$ ($K = 1,2,\ldots,N$). The directors which are not necessarily along the unit principal normal and the unit binormal vectors to the curve have, in particular, the property that they remain unaltered in length under superposed rigid body motions.

In the absence of the directors, we merely have a 1-dimensional material curve $\mathcal{L}$ which can serve as a model for the construction by direct approach of string theory. The relationship between the number of directors L and the number K which identifies the order of the hierarchical theory of Cosserat curves can be shown to be $L = \sum_{1}^{K}(N+1)$ so that (see Naghdi 1979b)

$$L = K(K+3)/2 \quad . \tag{9.1}$$

With $K = 1$, the directed curve is a body $\mathcal{R}_1 = \mathcal{R}$ comprising a material curve and two deformable directors attached to every point of the material curve of $\mathcal{R}$. The latter is the simplest model for the construction of a general bending theory of slender rods; and, for simplicity, we restrict attention to this particular model in most of the development[+] in section 9.

We now turn to a brief account of the basic theory of a Cosserat curve.

### 9.1 Kinematics of a Cosserat curve $\mathcal{R}$

Let the particles of the material curve $\mathcal{L}$ of $\mathcal{R}$ be identified by means of the convected coordinate $\xi$ and let the curve occupied by $\mathcal{L}$ in the present

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[+] A brief account of the more general theory for Cosserat curves is indicated at the end of this section.

configuration of $\mathcal{R}$ at time t be referred to as $\ell$. Let $\underset{\sim}{r}$ and $\underset{\sim\alpha}{d}$ $(\alpha = 1,2)$ denote the position vector of a typical point of $\ell$ and the directors at the same point, respectively, and also designate the tangent vector to the curve $\ell$ by $\underset{\sim}{a}_3$. Then, a motion of the Cosserat curve is defined by vector-valued functions which assign a position $\underset{\sim}{r}$ and a pair of directors $\underset{\sim}{d}_\alpha$ to each particle of $\mathcal{R}$ at each instant of time, i.e.**,

$$\underset{\sim}{r} = \hat{\underset{\sim}{r}}(\xi,t) \quad , \quad \underset{\sim}{d}_\alpha = \hat{\underset{\sim}{d}}_\alpha(\xi,t) \quad , \quad [\underset{\sim}{d}_1 \underset{\sim}{d}_2 \underset{\sim}{a}_3] > 0 \tag{9.2}$$

where

$$\underset{\sim}{a}_3 = \underset{\sim}{a}_3(\xi,t) = \frac{\partial \hat{\underset{\sim}{r}}}{\partial \xi} \quad . \tag{9.3}$$

The condition $(9.2)_3$ ensures that the directors $\underset{\sim}{d}_\alpha$ are nowhere tangent to $\ell$ and that $\underset{\sim}{d}_1, \underset{\sim}{d}_2$ never change their relative orientation with respect to each other and $\underset{\sim}{a}_3$. The velocity and the director velocities are defined by

$$\underset{\sim}{v} = \dot{\underset{\sim}{r}} \quad , \quad \underset{\sim}{w}_\alpha = \dot{\underset{\sim}{d}}_\alpha \quad , \tag{9.4}$$

and from (9.3) and $(9.4)_1$ we have

$$\dot{\underset{\sim}{a}}_3 = \frac{\partial \underset{\sim}{v}}{\partial \xi} \quad , \tag{9.5}$$

where a superposed dot denotes material time differentiation with respect to t holding $\xi$ fixed.

It is convenient to introduce here a slightly different notation than that adopted in a number of previous papers, e.g., Naghdi (1979a). Thus, we put

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** For convenience, we adopt the notation for $\underset{\sim}{r}$ in (2.16) and (2.30) also for the surface $(9.2)_1$. This permits an easy identification of the two curves, if desired. The choice of positive sign in $(9.2)_3$ is for definiteness. Alternatively, it will suffice to assume that $[\underset{\sim}{d}_1 \underset{\sim}{d}_2 \underset{\sim}{a}_3] \neq 0$ with the understanding that in any given motion the scalar triple product $[\underset{\sim}{d}_1 \underset{\sim}{d}_2 \underset{\sim}{a}_3]$ is either $> 0$ or $< 0$.

$$\mathbf{d}_3 = \mathbf{a}_3 \quad , \quad \mathbf{d}_i = (\mathbf{d}_\alpha, \mathbf{a}_3) \tag{9.6}$$

and observe that in view of $(9.2)_3$ and (9.6), $\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3$ are linearly independent vectors. Hence, we may introduce a set of reciprocal vectors $\mathbf{d}^i$ such that

$$\mathbf{d}_i \cdot \mathbf{d}^j = \delta_i^j \quad , \tag{9.7}$$

where $\delta_i^j$ is the Kronecker symbol in 3-space. Whenever desirable, the notations $\mathbf{d}_i = (\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3)$ and $(\mathbf{d}_\alpha, \mathbf{a}_3)$ will be used interchangeably throughout Part B depending on the particular context. Consider now a reference configuration, not necessarily the initial configuration, of the Cosserat curve $\mathcal{R}$. In the reference configuration, let the material curve of $\mathcal{R}$ be referred to by $\mathcal{L}_R$ and designate the unit principal normal, the unit binormal and the tangent vector to $\mathcal{L}_R$ by $\mathbf{A}_1, \mathbf{A}_2$ and $\mathbf{A}_3$, respectively. Further, let $\mathbf{R}$ and $\mathbf{D}_\alpha$ $(\alpha = 1,2)$ stand for the position of a typical point of $\mathcal{L}_R$ and the directors at the same point, respectively. Then, in the reference configuration we have

$$\mathbf{R} = \hat{\mathbf{R}}(\xi) \quad , \quad \mathbf{D}_\alpha = \hat{\mathbf{D}}_\alpha(\xi) \quad , \quad [\mathbf{D}_1 \mathbf{D}_2 \mathbf{A}_3] > 0 \quad , \tag{9.8}$$

where

$$\mathbf{A}_3 = \mathbf{A}_3(\xi) = \frac{\partial \hat{\mathbf{R}}}{\partial \xi} \tag{9.9}$$

and $(9.8)_3$ ensures that $\mathbf{D}_\alpha$ are nowhere tangent to the curve $\mathcal{L}_R$. If the reference configuration of $\mathcal{R}$ is specified to be the initial configuration, say at time $t = 0$, then the vector-valued functions on the right-hand sides of $(9.8)_{1,2}$ can be identified with $\hat{\mathbf{r}}(\xi,0)$ and $\hat{\mathbf{d}}_\alpha(\xi,0)$, respectively. Analogously to (9.6), we set

$$\mathbf{D}_3 = \mathbf{A}_3 \quad , \quad \mathbf{D}_i = (\mathbf{D}_\alpha, \mathbf{A}_3) \tag{9.10}$$

so that the dual of (9.7) is given by

$$\underset{\sim}{D}_i \cdot \underset{\sim}{D}^j = \delta_i^j \quad . \tag{9.11}$$

9.2 Basic principles of a Cosserat curve $\mathcal{R}$

Consider an arbitrary part of the material curve $\mathcal{I}$ in the present configuration, i.e., a part of the space curve $\ell$ bounded by $\xi = \xi_1$ and $\xi = \xi_2$ $(\xi_1 < \xi_2)$, and let

$$ds = (a_{33})^{\frac{1}{2}} d\xi \quad , \quad a_{33} = \underset{\sim}{a}_3 \cdot \underset{\sim}{a}_3 \tag{9.12}$$

be the element of the arc length of $\ell$. It is convenient at this point to define the following additional quantities: The mass density $\rho = \rho(\xi,t)$ of the curve $\ell$; the contact force* $\underset{\sim}{n} = \underset{\sim}{n}(\xi,t)$ and the contact director forces $\underset{\sim}{m}^\alpha = \underset{\sim}{m}^\alpha(\xi,t)$, each a 3-dimensional vector field in the present configuration; the assigned force $\underset{\sim}{f} = \underset{\sim}{f}(\xi,t)$ and the assigned director forces $\underset{\sim}{\ell}^\alpha = \underset{\sim}{\ell}^\alpha(\xi,t)$, each a 3-dimensional vector field and each per unit mass of the curve $\ell$; the intrinsic (curve) director forces $\underset{\sim}{k}^\alpha = \underset{\sim}{k}^\alpha(\xi,t)$ per unit length of $\ell$ which make no contribution to the supply of moment of momentum; the inertia coefficients $y^\alpha = y^\alpha(\xi)$ and $y^{\alpha\beta} = y^{\alpha\beta}(\xi)$, with $y^{\alpha\beta}$ being components of a symmetric tensor, which are independent of time; the specific internal energy $\varepsilon = \varepsilon(\xi,t)$; the specific heat supply $r = r(\xi,t)$ per unit time; and the heat flux $h = h(\xi,t)$ along $\ell$, in the direction of increasing $\xi$, per unit time. The assigned field $\underset{\sim}{f}$ represents the combined effect of (i) the stress vector on the lateral surface (2.26) of the rod-like body denoted by $\underset{\sim}{f}_c$, and (ii) an integrated contribution arising from the 3-dimensional body force denoted by $\underset{\sim}{f}_b$, e.g., that due to gravity. A parallel statement holds for the assigned fields $\underset{\sim}{\ell}^\alpha$. Similarly, the assigned heat supply $r$ represents the combined effect of (i) heat supply entering the lateral surface (2.26) of the rod-like body from the surrounding environment, denoted by $r_c$, and (ii) an integrated contribution

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*The notations for the contact force $\underset{\sim}{n}$, the contact director forces $\underset{\sim}{m}^\alpha$ and the curve director forces $\underset{\sim}{k}^\alpha$ differ from those in Green and Laws (1966), Green, Naghdi and Wenner (1974a,b), Naghdi (1979a,b) and most of the previous papers on the subject. In fact, the vector fields $\underset{\sim}{n}, \underset{\sim}{m}^\alpha, \underset{\sim}{k}^\alpha$ of Part B of this paper correspond, respectively, to $\underset{\sim}{n}, \underset{\sim}{p}^\alpha, \underset{\sim}{\pi}^\alpha$ of Green, Naghdi and Wenner (1974a,b). and most of the previous papers on the subject.

arising from the 3-dimensional heat supply denoted by $r_b$. Thus, we may write

$$\underset{\sim}{f} = \underset{\sim}{f}_b + \underset{\sim}{f}_c \quad , \quad \underset{\sim}{\ell}^\alpha = \underset{\sim}{\ell}_b^\alpha + \underset{\sim}{\ell}_c^\alpha \quad , \quad r = r_b + r_c \quad . \tag{9.13}$$

We assume that the kinetic energy of a Cosserat curve $\mathcal{R}$ per unit length of the curve $\ell$ in the present configuration is given by

$$\kappa = \tfrac{1}{2}\rho[\underset{\sim}{v} \cdot \underset{\sim}{v} + 2y^\alpha \underset{\sim}{v} \cdot \underset{\sim}{w}_\alpha + y^{\alpha\beta} \underset{\sim}{w}_\alpha \cdot \underset{\sim}{w}_\beta] \quad . \tag{9.14}$$

We further define the momentum corresponding to the velocity $\underset{\sim}{v}$ and the director momentum corresponding to the director velocities $\underset{\sim}{w}_\alpha$ by

$$\frac{\partial\kappa}{\partial\underset{\sim}{v}} = \rho(\underset{\sim}{v} + y^\alpha \underset{\sim}{w}_\alpha) \quad , \quad \frac{\partial\kappa}{\partial\underset{\sim}{w}_\alpha} = \rho(y^\alpha \underset{\sim}{v} + y^{\alpha\beta} \underset{\sim}{w}_\beta) \tag{9.15}$$

per unit length of $\ell$. Also, the physical dimensions of $\rho, \underset{\sim}{n}, f$ are

$$\begin{gathered} \text{phys. dim. } \rho = [ML^{-1}] \quad , \\ \text{phys. dim. } \underset{\sim}{n} = [MLT^{-2}] \quad , \quad \text{phys. dim. } \underset{\sim}{f} = [LT^{-2}] \quad , \end{gathered} \tag{9.16}$$

where as in section 3 the symbols [L], [M] and [T] stand for the physical dimensions of length, mass and time. The dimensions of the vector fields $\underset{\sim}{m}^\alpha, \underset{\sim}{\ell}^\alpha$ and $\underset{\sim}{k}^\alpha$ depend upon the physical dimensions of $\underset{\sim}{d}_\alpha$. Here we choose $\underset{\sim}{d}_\alpha$ to have the dimension of length. Then, $\underset{\sim}{m}^\alpha, \underset{\sim}{\ell}^\alpha$ will have the same physical dimensions as $\underset{\sim}{n}, \underset{\sim}{f}$ in (9.16) while $\underset{\sim}{k}^\alpha$ will have the physical dimension of $[ML^{-2}T^{-2}]$.

Using the above definitions of the various field quantities and th notation

$$[f(\xi,t)]_{\xi_1}^{\xi_2} = f(\xi_2,t) - f(\xi_1,t) \quad , \tag{9.17}$$

with reference to the present configuration the conservation laws for a Cosserat curve are*:

*The conservation laws (9.18) correspond to Eqs. (6.16) in the paper of Green, Naghdi and Wenner (1974b).

$$\frac{d}{dt}\int_{\xi_1}^{\xi_2} \rho \, ds = 0 \quad ,$$

$$\frac{d}{dt}\int_{\xi_1}^{\xi_2} \rho(\underset{\sim}{v} + y^{\alpha}\underset{\sim}{w}_{\alpha})ds = \int_{\xi_1}^{\xi_2} \rho \, \underset{\sim}{f} \, ds + [\underset{\sim}{n}]_{\xi_1}^{\xi_2} \quad ,$$

$$\frac{d}{dt}\int_{\xi_1}^{\xi_2} \rho(y^{\alpha}\underset{\sim}{v} + y^{\alpha\beta}\underset{\sim}{w}_{\beta})ds = \int_{\xi_1}^{\xi_2} (\rho\underset{\sim}{\ell}^{\alpha} - (a_{33})^{-\frac{1}{2}}\underset{\sim}{k}^{\alpha})ds + [\underset{\sim}{m}^{\alpha}]_{\xi_1}^{\xi_2} \quad , \tag{9.18}$$

$$\frac{d}{dt}\int_{\xi_1}^{\xi_2} \rho[\underset{\sim}{r} \times \underset{\sim}{v} + y^{\alpha}(\underset{\sim}{r} \times \underset{\sim}{w}_{\alpha} + \underset{\sim}{d}_{\alpha} \times \underset{\sim}{v}) + \underset{\sim}{d}_{\alpha} \times y^{\alpha\beta}\underset{\sim}{w}_{\beta}]ds$$
$$= \int_{\xi_1}^{\xi_2} \rho(\underset{\sim}{r} \times \underset{\sim}{f} + \underset{\sim}{d}_{\alpha} \times \underset{\sim}{\ell}^{\alpha})ds + [\underset{\sim}{r} \times \underset{\sim}{n} + \underset{\sim}{d}_{\alpha} \times \underset{\sim}{m}^{\alpha}]_{\xi_1}^{\xi_2} \quad .$$

The first of (9.18) is a statement of the conservation of mass, the second is the conservation of linear momentum, the third that of the director momentum and the fourth is the conservation of moment of momentum. We also record the law of conservation of energy in the form

$$\frac{d}{dt}\int_{\xi_1}^{\xi_2} \rho(\varepsilon + \kappa)ds = \int_{\xi_1}^{\xi_2} \rho(r + \underset{\sim}{f} \cdot \underset{\sim}{v} + \underset{\sim}{\ell}^{\alpha} \cdot \underset{\sim}{w}_{\alpha})ds + [\underset{\sim}{n} \cdot \underset{\sim}{v} + \underset{\sim}{m}^{\alpha} \cdot \underset{\sim}{w}_{\alpha} - h]_{\xi_1}^{\xi_2} \quad . \tag{9.19}$$

The basic structure of $(9.18)_{1,2}$ and (9.19) are analogous to the corresponding conservation laws of the classical 3-dimensional theory. The structure of $(9.18)_3$ and $(9.18)_4$ are less obvious, but a motivation for their forms is provided by a derivation of the basic field equations for rod-like bodies obtained from the 3-dimensional equations of continuum mechanics in which the position vector $\underset{\sim}{r}^*$ in 3-space is approximated by an expression of the form (2.30). It should be noted that the conservation laws (9.18) and (9.19) are consistent with the invariance conditions under superposed rigid body motions, which ordinarily have wide acceptance in continuum mechanics. Moreover, the conservation laws $(9.18)_1$, $(9.18)_2$ and $(9.18)_4$ are equivalent to, and can be derived from the conservation of energy (9.19) and the invariance conditions under superposed rigid body motions. The conservation law $(9.18)_3$ for the director momentum must be postulated separately.

Returning to the conservation laws, after making suitable continuity assumptions, by usual procedures from $(9.18)_{1,2,3,4}$ and (9.19) follow the local field equations

$$\lambda = \lambda(\xi) = \rho(a_{33})^{\frac{1}{2}} \quad \text{or} \quad \dot{\rho} a_{33} + \rho \underset{\sim}{a}_3 \cdot \frac{\partial \underset{\sim}{v}}{\partial \xi} = 0 \quad , \tag{9.20}$$

$$\frac{\partial \underset{\sim}{n}}{\partial \xi} + \lambda \underset{\sim}{f} = \lambda(\dot{\underset{\sim}{v}} + y^{\alpha} \dot{\underset{\sim}{w}}_{\alpha}) \quad , \tag{9.21}$$

$$\frac{\partial \underset{\sim}{m}^{\alpha}}{\partial \xi} + \lambda \underset{\sim}{\ell}^{\alpha} = \underset{\sim}{k}^{\alpha} + \lambda(y^{\alpha} \dot{\underset{\sim}{v}} + y^{\alpha\beta} \dot{\underset{\sim}{w}}_{\beta}) \quad , \tag{9.22}$$

$$\underset{\sim}{a}_3 \times \underset{\sim}{n} + \underset{\sim}{d}_{\alpha} \times \underset{\sim}{k}^{\alpha} + \frac{\partial \underset{\sim}{d}_{\alpha}}{\partial \xi} \times \underset{\sim}{m}^{\alpha} = 0 \quad , \tag{9.23}$$

and

$$\lambda r - \frac{\partial h}{\partial \xi} - \lambda \dot{\varepsilon} + P = 0 \quad , \tag{9.24}$$

where

$$P = \underset{\sim}{n} \cdot \frac{\partial \underset{\sim}{v}}{\partial \xi} + \underset{\sim}{k}^{\alpha} \cdot \underset{\sim}{w}_{\alpha} + \underset{\sim}{m}^{\alpha} \cdot \frac{\partial \underset{\sim}{w}_{\alpha}}{\partial \xi} \tag{9.25}$$

is the mechanical power.

## 9.3 Hierarchical theories of Cosserat curves

Although the theory outlined in subsection 9.2 is sufficiently general for many applications, on occasion it becomes necessary to consider a more general theory of Cosserat curves. Therefore, we now briefly discuss the kinematics and the balance laws of Cosserat curves $\mathcal{R}_K$ having L ($\geq 2$) directors attached to every point of a material line $\mathcal{L}$, the number L being given by (9.1).

Thus, instead of $(9.2)_{1,2}$, we specify a motion of $\mathcal{R}_K$ by

$$\underset{\sim}{r} = \hat{r}(\xi,t) \quad , \quad \underset{\sim}{d}_{\alpha_1\alpha_2\ldots\alpha_N} = \hat{\underset{\sim}{d}}_{\alpha_1\alpha_2\ldots\alpha_N}(\xi,t) \quad (N = 1,2,\ldots,K) \quad , \tag{9.26}$$

where the vector functions $\underset{\sim}{d}_{\alpha_1\alpha_2\ldots\alpha_N}$ are assumed to be symmetric in the indices $\alpha_1\alpha_2\ldots\alpha_N$. The velocity vector is still given by $(9.4)_1$ but corresponding to $(9.4)_2$ we now define the director velocities

$$\underset{\sim}{w}_{\alpha_1\alpha_2\ldots\alpha_N} = \dot{\underset{\sim}{d}}_{\alpha_1\alpha_2\ldots\alpha_N} \quad . \tag{9.27}$$

We recall that for $K = 1$ ($\mathcal{R}_1 = \mathcal{R}$), the kinetical quantities and the assigned fields introduced in subsection 9.2 consist of $\underset{\sim}{n}, \underset{\sim}{k}^\alpha, \underset{\sim}{m}^\alpha$ and $\underset{\sim}{f}, \underset{\sim}{\ell}^\alpha$. Keeping this in mind, for a body $\mathcal{R}_K$ we admit the more general kinetical quantities and assigned fields

$$\begin{gathered} \underset{\sim}{n} \;,\; \underset{\sim}{k}^{\alpha_1\alpha_2\ldots\alpha_N} \;,\; \underset{\sim}{m}^{\alpha_1\alpha_2\ldots\alpha_N} \quad , \\ \underset{\sim}{f} \;,\; \underset{\sim}{\ell}^{\alpha_1\alpha_2\ldots\alpha_N} \quad , \end{gathered} \tag{9.28}$$

and corresponding to (9.14) and $(9.15)_{1,2}$ write the more general expressions for kinetic energy of $\mathcal{R}_K$ and associated momentum and director momenta, namely

$$\begin{aligned} \kappa &= \tfrac{1}{2}\rho[\underset{\sim}{v}\cdot\underset{\sim}{v} + 2\sum_{N=1}^{K} y^{\alpha_1\alpha_2\ldots\alpha_N}\underset{\sim}{v}\cdot\underset{\sim}{w}_{\alpha_1\alpha_2\ldots\alpha_N} \\ &\qquad + \sum_{N=1,M=1}^{K} y^{\alpha_1\ldots\alpha_N\beta_1\ldots\beta_M}\underset{\sim}{w}_{\alpha_1\ldots\alpha_N}\cdot\underset{\sim}{w}_{\beta_1\ldots\beta_M}] \quad , \\ \frac{\partial\kappa}{\partial\underset{\sim}{v}} &= \rho[\underset{\sim}{v} + \sum_{N=1}^{K} y^{\alpha_1\ldots\alpha_N}\underset{\sim}{w}_{\alpha_1\ldots\alpha_N}] \quad , \\ \frac{\partial\kappa}{\partial\underset{\sim}{w}_{\alpha_1\ldots\alpha_N}} &= \rho[y^{\alpha_1\ldots\alpha_N}\underset{\sim}{v} + \Sigma\, y^{\alpha_1\ldots\alpha_N\beta_1\ldots\beta_M}\underset{\sim}{w}_{\beta_1\ldots\beta_M} \quad , \end{aligned} \tag{9.29}$$

each per unit length of the curve $\ell$. The inertia coefficients $y^{\alpha_1 \ldots \alpha_N}$, $y^{\alpha_1 \ldots \alpha_N \beta_1 \ldots \beta_M}$ in (9.29) are functions of $\xi$ only, $y^{\alpha_1 \ldots \alpha_N}$ are symmetric with respect to indices $\alpha_1 \ldots \alpha_N$, $y^{\alpha_1 \ldots \alpha_N \beta_1 \ldots \beta_M} = y^{\beta_1 \ldots \beta_M \alpha_1 \ldots \alpha_N}$ and are also symmetric with respect to $\alpha_1 \alpha_2 \ldots \alpha_N$ and $\beta_1 \beta_2 \ldots \beta_M$. In the special case of $K = 1$ ($\mathcal{R}_1 = \mathcal{R}$), $L = 2$, we may use the notations

$$
\begin{aligned}
\mathbf{d}_{\alpha_1} &= \mathbf{d}_{\alpha} \quad , \quad \mathbf{w}_{\alpha_1} = \mathbf{w}_{\alpha} \quad , \\
y^{\alpha_1} &= y^{\alpha} \quad , \quad y^{\alpha_1 \beta_1} = y^{\alpha\beta} \quad .
\end{aligned}
\tag{9.30}
$$

For a detailed statement of conservation laws appropriate for Cosserat curves $\mathcal{R}_K$ we refer the reader to Naghdi (1979b, Sec. 2), but indicate below the structure of the corresponding local field equations. Thus, for the purely mechanical theory of Cosserat curves $\mathcal{R}_K$, the local field equations are:

$$
\begin{aligned}
&\dot{\overline{\lambda y^{\alpha_1 \ldots \alpha_N}}} = 0 \quad (N = 1, \ldots, 2K) \quad , \quad \dot{\lambda} = 0 \quad , \quad \lambda = \lambda(\xi) = \rho a_{33}^{\frac{1}{2}} \quad , \\
&\frac{\partial \mathbf{n}}{\partial \xi} + \lambda \mathbf{f} = \mathbf{0} \quad , \\
&\frac{\partial \mathbf{m}^{\alpha_1 \ldots \alpha_N}}{\partial \xi} + \lambda \mathbf{q}^{\alpha_1 \ldots \alpha_N} = \mathbf{k}^{\alpha_1 \ldots \alpha_N} \quad (N = 1, \ldots, K) \quad , \\
&\mathbf{a}_3 \times \mathbf{n} + \sum_{N=1}^{K} \left( \mathbf{d}_{\alpha_1 \ldots \alpha_N} \times \mathbf{k}^{\alpha_1 \ldots \alpha_N} + \frac{\partial \mathbf{d}_{\alpha_1 \ldots \alpha_N}}{\partial \xi} \times \mathbf{m}^{\alpha_1 \ldots \alpha_N} \right) = \mathbf{0} \quad ,
\end{aligned}
\tag{9.31}
$$

where

$$
\mathbf{f} = \mathbf{f}_b + \mathbf{f}_c \quad , \quad \boldsymbol{\ell}^{\alpha_1 \alpha_2 \ldots \alpha_N} = \boldsymbol{\ell}_b^{\alpha_1 \alpha_2 \ldots \alpha_N} + \boldsymbol{\ell}_c^{\alpha_1 \alpha_2 \ldots \alpha_N} \quad , \tag{9.32}
$$

$$\dot{f} = \dot{f} - \dot{v} - \sum_{N=1}^{K} y^{\alpha_1 \ldots \alpha_N} \cdot \underset{\sim}{w}_{\alpha_1 \ldots \alpha_N} \quad (N = 1, \ldots, K) \quad ,$$

$$\dot{q}^{\alpha_1 \ldots \alpha_N} = \dot{\ell}^{\alpha_1 \ldots \alpha_N} - y^{\alpha_1 \ldots \alpha_N} \dot{v} - \sum_{M=1}^{K} y^{\alpha_1 \ldots \alpha_N \beta_1 \ldots \beta_M} \cdot \underset{\sim}{w}_{\beta_1 \ldots \beta_M} \quad (N = 1, \ldots K) \quad . \tag{9.33}$$

Also, for Cosserat curves $\mathcal{R}_K$, the expression for mechanical power corresponding to (9.26) is

$$P = \underset{\sim}{n} \cdot \frac{\partial \underset{\sim}{v}}{\partial \xi} + \sum_{N=1}^{K} \underset{\sim}{k}^{\alpha_1 \ldots \alpha_N} \cdot \underset{\sim}{w}_{\alpha_1 \ldots \alpha_N} + \sum_{N=1}^{K} \underset{\sim}{m}^{\alpha_1 \ldots \alpha_N} \cdot \frac{\partial \underset{\sim}{w}_{\alpha_1 \ldots \alpha_N}}{\partial \xi} \quad . \tag{9.34}$$

The general development for Cosserat curves $\mathcal{R}_K$ outlined above is contained in a paper by Naghdi (1979, Sec. 2). When $K = 1$, the results in subsection 9.3 reduce to those of subsection 9.2 for a Cosserat curve $\mathcal{R}$.

## 10. Elastic rods

Within the scope of the theory of a Cosserat curve $\mathcal{R}$ outlined in section 9, we discuss briefly the constitutive equations for elastic rod in the presence of finite deformation. As in section 4, we again suppose the existence of a strain energy or stored energy per unit mass $\psi = \psi(\xi,t)$ such that $\rho\dot{\psi}$ is equal to the mechanical power $P$ defined by (9.25), i.e.,

$$P = \rho\dot{\psi} \quad . \tag{10.1}$$

In the development of nonlinear constitutive equations for elastic rods, we assume that the strain energy density $\psi$ at each material point of $\mathcal{R}$ and for all t is specified by a response function which depends on $r, d_\alpha$ in (9.2) and their partial derivatives with respect to $\xi$. But since the response function must remain unaltered under superposed rigid body translational displacement, the dependence on r must be excluded. Thus, the constitutive assumption for the strain energy density can be written as

$$\psi = \hat{\psi}(r', d_\alpha, d_\alpha' ; X) \quad , \tag{10.2}$$

where a superposed prime stands for

$$(\ )' = \partial(\ )/\partial\xi \quad , \tag{10.3}$$

and we also make similar constitutive assumptions for $n, k', m'$ in (9.[illegible]). In these constitutive equations, which represent the mechanical response of the medium, the dependence of the response functions on the local geometrical properties of a reference state and material inhomogeneities is indicated through the argument X.

A general development of constitutive theory of elastic rods based on the assumption of the type (10.2) is contained in the paper of Green, Naghdi and Wenner (1974b). In the rest of this section, we limit the discussion to an elastic rod which is homogeneous in its reference configuration and suppose also that the dependence of the response functions on the properties of the reference state occurs through the values of the kinematical variables in the reference state. Then, in place of (10.2), we have

$$\psi = \bar{\psi}(\mathbf{r}', \mathbf{d}_\alpha, \mathbf{d}'_\alpha \,;\, \mathbf{R}', \mathbf{D}, \mathbf{D}'_\alpha) \quad , \tag{10.4}$$

with similar assumptions for $\mathbf{n}, \mathbf{k}^\alpha, \mathbf{m}^\alpha$. After substituting (10.4) into (10.1), by usual techniques we obtain the following forms for the constitutive equations:

$$\mathbf{n} = \lambda \frac{\partial\bar{\psi}}{\partial\mathbf{r}'} \quad , \quad \mathbf{k}^\alpha = \lambda \frac{\partial\bar{\psi}}{\partial\mathbf{d}_\alpha} \quad , \quad \mathbf{m}^\alpha = \lambda \frac{\partial\bar{\psi}}{\partial\mathbf{d}'_\alpha} \quad , \tag{10.5}$$

along with the restriction

$$\mathbf{d}_i \times \left[\frac{\partial\bar{\psi}}{\partial\mathbf{d}_i} + (\mathbf{d}'_\alpha \cdot \mathbf{d}^i)\, \frac{\partial\bar{\psi}}{\partial\mathbf{d}'_\alpha}\right] = 0 \quad , \tag{10.6}$$

which is obtained from the conservation of moment of momentum and which restricts the response function $\bar{\psi}$.

We do not discuss here the reduced forms of the above constitutive equations resulting from invariance requirements under superposed rigid body motions, but for such reduction refer the reader to Green, Naghdi and Wenner (1974b). Just as with the equations of motion, it is necessary in applications to specific problems to obtain alternative forms of the above constitutive equations or their reduced forms in terms of tensor components. Such component forms may be expressed with respect to bases $\mathbf{a}_i$, or $\mathbf{d}_i$, or corresponding bases in the reference configuration. Reduced forms of (10.5) are discussed in Green et al. (1974b, Sec. 7).

## 11. Identification of the assigned fields and the inertia coefficients

The local field equations in the mechanical theory of a Cosserat curve $\mathcal{R}$ have the same forms as those that can be derived from the 3-dimensional field equations $(2.9)_{1,2,3}$ by suitable integration over the cross-sectional area of the rod-like body with respect to $\theta^1$ and $\theta^2$ [recall the definition of a rod-like body at the end of section 2] and in terms of certain definitions for integrated mass density and resultants of stress (for details, see Green, Naghdi and Wenner 1974a). Similarly, the energy equation (9.24) has the same form as that which can be derived from the energy equation in the 3-dimensional theory by suitable integration over the cross-section area of the rod-like body with respect to $\theta^1$ and $\theta^2$ and in terms of certain definitions for integrated internal energy density and heat flux in the 3-dimensional theory (see Green and Naghdi 1970). To elaborate further, we confine attention to the purely mechanical theory and recall the definitions

$$\lambda = \rho a_{33}^{\frac{1}{2}} = \int_{\mathcal{A}} \lambda^* d\theta^1 d\theta^2 \quad , \quad \lambda^* = \rho^* g^{\frac{1}{2}} \quad , \tag{11.1}$$

$$\lambda y^{\alpha} = \int_{\mathcal{A}} \lambda^* \theta^{\alpha} d\theta^1 d\theta^2 \quad , \quad \lambda y^{\alpha\beta} = \int_{\mathcal{A}} \lambda^* \theta^{\alpha}\theta^{\beta} d\theta^1 d\theta^2 \quad , \tag{11.2}$$

and the expressions

$$\lambda \mathbf{f} = \int_{\mathcal{A}} \lambda^* \mathbf{f}^* d\theta^1 d\theta^2 + \int_{\partial\mathcal{A}} [d\theta^2(\mathbf{T}^1 - \grave{\lambda}^1 \mathbf{T}^3) - d\theta^1(\mathbf{T}^2 - \grave{\lambda}^2 \mathbf{T}^3)] \quad , \tag{11.3}$$

$$\lambda \mathbf{\ell}^{\alpha} = \int_{\mathcal{A}} \lambda^* \mathbf{f}^* \theta^{\alpha} d\theta^1 d\theta^2 + \int_{\partial\mathcal{A}} \theta^{\alpha} [d\theta^2(\mathbf{T}^1 - \grave{\lambda}^1 \mathbf{T}^3) - d\theta^1(\mathbf{T}^2 - \grave{\lambda}^2 \mathbf{T}^3)] \quad , \tag{11.4}$$

where $\rho^*, \mathbf{T}^i, \mathbf{f}^*$ which occur in (11.1)-(11.4) are defined in section 2 [following (2.9)], the line integrals are taken along the curve $\xi$ = const. on the material surface (2.26), $\grave{\lambda}^{\alpha} = \grave{\boldsymbol{\lambda}} \cdot \mathbf{g}^{\alpha}$ and $\grave{\boldsymbol{\lambda}} = \grave{\lambda}^{\alpha} \mathbf{g}_{\alpha} + \mathbf{g}_3$ is a vector tangential to the surface (2.26) so that $\grave{\boldsymbol{\lambda}} \cdot \boldsymbol{\nu}^* = \grave{\lambda}^{\alpha} \nu^*_{\alpha} + \nu^*_3 = 0$.

If we now adopt the approximation (2.30), then there is a 1-1 correspondence between the 1-dimensional field equations that follow from the conservation laws of a Cosserat curve and those that can be derived from the 3-dimensional equations provided we identify $\underset{\sim}{r}$ and the director $\underset{\sim}{d}_\alpha$ in (2.30) with $(9.2)_1$ and $(9.2)_2$, respectively, and adopt the definitions (11.1)-(11.4), as well as the definitions of the resultants mentioned above. A similar 1-1 correspondence can be shown to hold between the 1-dimensional energy equation in the theory of a Cosserat curve and an integrated energy equation derived from the 3-dimensional energy equation.

The various quantities in (9.13) are free to be specified in a manner which depends on the particular application in mind. Also, we remark that in the context of the theory of a Cosserat curve, the inertia coefficients $y^\alpha, y^{\alpha\beta}$ and the mass density $\rho$ require constitutive equations. Indeed, $\underset{\sim}{f}_c, \underset{\sim}{\ell}_c^\alpha$ and $r_c$, as well as $\underset{\sim}{f}_b, \underset{\sim}{\ell}_b^\alpha$ and $r_b$, can be identified with corresponding expressions in a derivation from the 3-dimensional equations indicated above (for details, see Naghdi 1979a). Likewise, $\rho$ and the coefficients $y^\alpha, y^{\alpha\beta}$ may be identified with easily accessible results from the 3-dimensional theory.

In what follows, we assume that the above identifications have been made and that the quantities $\rho, y^\alpha, y^{\alpha\beta}, \underset{\sim}{f}_b, \underset{\sim}{\ell}_b^\alpha$ are known or specified. The knowledge of $\underset{\sim}{f}_c, \underset{\sim}{\ell}_c^\alpha$ depends on the nature of the boundary conditions on the lateral surface of the particular rod-like body under consideration: they may be specified as known quantities on the surface (2.26) or they are unknown and must be determined as part of the solution of the problem.

## 12. Additional remarks on rods

Topics corresponding to those in Secs. 6 and 7 have so far received less attention in the case of rods and consequently the discussions that follow are somewhat brief. We first consider a class of constraints, apply the results to an incompressible Cosserat curve and then go on to briefly comment on some recent researches which bear on various aspects of elastic rods.

Consider a class of constraints which are linear relations between the kinematic variables

$$\mathbf{v}' \ , \ \mathbf{w}_{\alpha_1\alpha_2\ldots\alpha_N} \ , \ \mathbf{w}_{\alpha_1\alpha_2\ldots\alpha_N} \qquad (N = 1,2,\ldots,K) \ . \tag{12.1}$$

Similar to the development in Sec. 6 for shells, we consider (Q+1) constraint equations of the form[†]

$$\begin{aligned} \mathbf{A}^M \cdot \mathbf{v}' &+ \sum_{N=1}^{K} \mathbf{B}^{M\alpha_1\alpha_2\ldots\alpha_N} \cdot \mathbf{w}_{\alpha_1\alpha_2\ldots\alpha_N} \\ &+ \sum_{N=1}^{K} \mathbf{C}^{M\alpha_1\alpha_2\ldots\alpha_N} \cdot \mathbf{w}'_{\alpha_1\alpha_2\ldots\alpha_N} = \mathbf{0} \quad (M = o,1,2,\ldots,Q) \ , \end{aligned} \tag{12.2}$$

where $\mathbf{A}^M, \mathbf{B}^{M\alpha_1\alpha_2\ldots\alpha_N}, \mathbf{C}^{M\alpha_1\alpha_2\ldots\alpha_N}$ are vector functions of $\mathbf{d}_i, \mathbf{d}'_i$ only and do not depend explicitly on the variables (12.1). We assume that each of the functions $\mathbf{n}, \mathbf{k}^{\alpha_1\alpha_2\ldots\alpha_N}, \mathbf{m}^{\alpha_1\alpha_2\ldots\alpha_N}$ are determined to within an additive constraint response so that

$$\begin{gathered} \mathbf{n} = \bar{\mathbf{n}} + \hat{\mathbf{n}} \ , \\ \mathbf{k}^{\alpha_1\alpha_2\ldots\alpha_N} = \bar{\mathbf{k}}^{\alpha_1\alpha_2\ldots\alpha_N} + \hat{\mathbf{k}}^{\alpha_1\alpha_2\ldots\alpha_N} \ , \quad \mathbf{m}^{\alpha_1\alpha_2\ldots\alpha_N} = \bar{\mathbf{m}}^{\alpha_1\alpha_2\ldots\alpha_N} + \hat{\mathbf{m}}^{\alpha_1\alpha_2\ldots\alpha_N} \ , \end{gathered} \tag{12.3}$$

where

$$\hat{\mathbf{n}} \ , \ \hat{\mathbf{k}}^{\alpha_1\alpha_2\ldots\alpha_N} \ , \ \hat{\mathbf{m}}^{\alpha_1\alpha_2\ldots\alpha_N}$$

are specified by consitutive equations and

$$\bar{\mathbf{n}} \ , \ \bar{\mathbf{k}}^{\alpha_1\alpha_2\ldots\alpha_N} \ , \ \bar{\mathbf{m}}^{\alpha_1\alpha_2\ldots\alpha_N} \ , \tag{12.4}$$

which represent the response due to constraints are arbitrary functions of

---

[†] The development between (12.2)-(12.6) is similar to that for mechanical constraints in the 3-dimensional theory.

$\xi$,t and are workless. Thus, recalling the expression (9.34), we set

$$\bar{\mathbf{n}} \cdot \mathbf{v}' + \sum_{N=1}^{K} \bar{\mathbf{k}}^{\alpha_1 \ldots \alpha_N} \cdot \mathbf{w}_{\alpha_1 \ldots \alpha_N} + \sum_{N=1}^{K} \bar{\mathbf{m}}^{\alpha_1 \ldots \alpha_N} \cdot \mathbf{w}'_{\alpha_1 \ldots \alpha_N} = 0 \tag{12.5}$$

for all values of the variables (12.1) subject to the constraint conditions (12.2). It then follows that

$$\bar{\mathbf{n}} = -\sum_{M=o}^{Q} \mathbf{A}^M p_M \quad , \quad \bar{\mathbf{k}}^{\alpha_1 \ldots \alpha_N} = -\sum_{M=1}^{Q} \mathbf{B}^{M\alpha_1 \ldots \alpha_N} p_M \quad ,$$
$$\bar{\mathbf{m}}^{\alpha_1 \ldots \alpha_N} = -\sum_{M=1}^{Q} \mathbf{C}^{M\alpha_1 \ldots \alpha_N} p_M \quad , \tag{12.6}$$

where $p_M = p_M(\xi,t)$ are arbitrary functions which play the role of Lagrange multipliers.

We consider now an incompressible Cosserat curve $\mathcal{R}$ with two directors within the scope of the above constrained theory. As in the case of an incompressible shell-like body discussed in Sec. 6, the conditions representing approximately the (3-dimensional) incompressibility condition (2.10) may be derived with the use of approximation (2.27) for $N = 1$ given by (2.30). Under this approximation (2.30), the base vectors are given by $\mathbf{g}_\alpha = \mathbf{d}_\alpha$, $\mathbf{g}_3 = \mathbf{a}_3 + \theta^\alpha \mathbf{d}'_\alpha$, where $\mathbf{a}_3$ is the tangent vector to the curve $\theta^\alpha = 0$ and a superposed prime is defined by (10.3). Then, from the incompressibility condition $(2.10)_1$ we obtain an approximate expression as a linear function of $\theta^1, \theta^2$ in the form

$$\frac{d}{dt}\,[\mathbf{d}_1 \mathbf{d}_2 \mathbf{a}_3] + \theta^\alpha \frac{d}{dt}\,[\mathbf{d}_1 \mathbf{d}_2 \mathbf{d}'_\alpha] = 0 \quad , \tag{12.7}$$

or equivalently as

$$\mathbf{d}^3 \cdot \mathbf{v}' + [\mathbf{d}^\alpha + \theta^\beta (\mathbf{d}'_\beta \cdot \mathbf{d}^3)\mathbf{d}^\alpha - \theta^\beta (\mathbf{d}'_\beta \cdot \mathbf{d}^\alpha)\mathbf{d}^3] \cdot \mathbf{w}_\alpha$$
$$+ \theta^\alpha \mathbf{d}^3 \cdot \mathbf{w}'_\alpha = 0 \quad , \tag{12.8}$$

where in (12.7) and (12.8) use is made of the notation $(9.6)_{1,2}$ and (9.7). We now generate three conditions representing incompressibility: One of these is obtained from integration of (12.8) with respect to $\theta^1,\theta^2$ over the cross-section of the rod-like body and the other two are obtained by first multiplying (12.8) by $\theta^\lambda$ $(\lambda = 1,2)$ and then integrating the resulting equation with respect to $\theta^1,\theta^2$ over the cross-section of the rod-like body. These three conditions can be written as

$$\gamma^o \mathbf{d}^3 \cdot \mathbf{v}' + \{\gamma^o \mathbf{d}^\alpha + \gamma^\beta[(\mathbf{d}_\beta' \cdot \mathbf{d}^3)\mathbf{d}^\alpha - (\mathbf{d}_\beta' \cdot \mathbf{d}^\alpha)\mathbf{d}^3]\} \cdot \mathbf{w}_\alpha + \gamma^\alpha \mathbf{d}^3 \cdot \mathbf{w}_\alpha' = 0 \quad , \tag{12.9}$$

$$\gamma^\lambda \mathbf{d}^3 \cdot \mathbf{v}' + \{\gamma^\lambda \mathbf{d}^\alpha + \gamma^{\lambda\beta}[(\mathbf{d}_\beta' \cdot \mathbf{d}^3)\mathbf{d}^\alpha - (\mathbf{d}_\beta' \cdot \mathbf{d}^\alpha)\mathbf{d}^3]\} \cdot \mathbf{w}_\alpha + \gamma^{\lambda\alpha} \mathbf{d}^3 \cdot \mathbf{w}_\alpha' = 0 \quad , \tag{12.10}$$

where

$$\gamma^o = \int_{\mathcal{A}} d\theta^1 d\theta^2 \quad , \quad \gamma^\alpha = \int_{\mathcal{A}} \theta^\alpha d\theta^1 d\theta^2 \quad , \quad \gamma^{\alpha\beta} = \int_{\mathcal{A}} \theta^\alpha \theta^\beta d\theta^1 d\theta^2 \quad . \tag{12.11}$$

For an incompressible Cosserat curve under discussion, from (12.2) the constraint conditions are

$$\mathbf{A}^M \cdot \mathbf{v}' + \mathbf{B}^{M\alpha} \cdot \mathbf{w}_\alpha + \mathbf{C}^{M\alpha} \cdot \mathbf{w}_\alpha' = 0 \qquad (M = o,1,2) \tag{12.12}$$

and the constrained response obtained from (12.6) has the form

$$\begin{aligned} \bar{\mathbf{n}} &= -(p_o \mathbf{A}^o + p_1 \mathbf{A}^1 + p_2 \mathbf{A}^2) \quad , \\ \bar{\mathbf{k}}^\alpha &= -p_o \mathbf{B}^{o\alpha} + p_1 \mathbf{B}^{1\alpha} + p_2 \mathbf{B}^{2\alpha} \quad , \\ \bar{\mathbf{m}}^\alpha &= -(p_o \mathbf{C}^{o\alpha} + p_1 \mathbf{C}^{1\alpha} + p_2 \mathbf{C}^{2\alpha}) \quad , \end{aligned} \tag{12.13}$$

where $p_o, p_1, p_2$ are the Lagrange multipliers. Guided by the three conditions (12.9) and (12.10) for $\lambda = 1,2$, we select the vector-valued functions $\mathbf{A}^M, \mathbf{B}^{M\alpha}$, $\mathbf{C}^{M\alpha}$ in (12.12) and (12.13) to have the special values

$$\left.\begin{aligned} \underset{\sim}{A}^{M} &= \text{coeff. of } \underset{\sim}{v}' \text{ in (12.9) and (12.10)} \\ \underset{\sim}{B}^{M\alpha} &= \text{coeff. of } \underset{\sim}{w}_{\alpha} \text{ in (12.9) and (12.10)} \\ \underset{\sim}{C}^{M\alpha} &= \text{coeff. of } \underset{\sim}{w}'_{\alpha} \text{ in (12.9) and (12.10)} \end{aligned}\right\} (M = o,1,2) \quad . \tag{12.11}$$

Then, it follows from (12.13) and (12.4) that the expressions for the constraint response are given by

$$\begin{aligned} \bar{\underset{\sim}{n}} &= - P_o \underset{\sim}{d}^3 \quad , \\ \bar{\underset{\sim}{k}}^{\alpha} &= - P_o \underset{\sim}{d}^{\alpha} - P_1^{\beta}[(\underset{\sim}{d}'_{\beta} \cdot \underset{\sim}{d}^3)\underset{\sim}{d}^{\alpha} - (\underset{\sim}{d}_{\beta} \cdot \underset{\sim}{d}^{\alpha})\underset{\sim}{d}^3] \quad , \\ \bar{\underset{\sim}{m}}^{\alpha} &= - P_1^{\alpha} \underset{\sim}{d}^3 \quad , \end{aligned} \tag{12.15}$$

where the arbitrary coefficients $P_o, P_1^{\alpha}$ are related to the Lagrange multipliers by

$$P_o = \gamma^o p_o + \gamma^{\alpha} p_{\alpha} \quad , \quad P_1^{\alpha} = \gamma^{\alpha} p_o + \gamma^{\alpha\beta} p_{\beta} \quad . \tag{12.16}$$

The applicability of the theory of Cosserat curves is not limited to only elastic rods but in fact can be applied also to problems of fluid jets. These developments, which pertain to both inviscid and viscous jets, have been discussed in the papers of Green and Laws (1968), Green (1975, 1976) and Naghdi (1979b).

A constrained theory of a Cosserat curve with two directors is discussed by Green and Laws (1973) and includes as a special case results corresponding to those of the Bernoulli-Euler beam theory. The theory of small deformation superposed on a large deformation of an elastic Cosserat curve, together with a discussion of stability problems of rods, is given by Green, Knops and Laws (1968) and some simpler problems in the context of the nonlinear theory of rods are discussed by Ericksen (1970).

The development of the theory of Cosserat curves in Sec. 9 is carried out within the scope of the purely mechanical theory. In earlier work on the thermo-mechanical theory of rods by direct approach (Green and Naghdi 1970), only one temperature field was admitted and this allowed for the characterization of the temperature changes along some reference curve such as the central line of a rod in the (3-dimensional) rod-like body, but not for temperature changes in the cross-section of the rod. The latter effect has been incorporated recently by Green and Naghdi (1979b) into the thermo-mechanical theory of Cosserat curves, together with appropriate thermodynamical restrictions arising from the second law of thermodynamics for rods.

## 13. The basic equations for rods in direct notation

In parallel to the development of section 8 for shells, for some purposes it is convenient to have available the basic equations of a Cosserat curve in a direct (coordinate-free) notation and this is the main purpose of the present section. Just as in the case of shells, we shall see that the basic equations for rods in coordinate-free notation are very similar to those of the corresponding equations in the 3-dimensional theory and thus may be more suitable in the discussion of general theorems or in the developments which parallel those in the 3-dimensional theory.

We introduce the notations grad and Grad to denote the right spatial and material gradient operators, respectively, with respect to the position on the curve c in the current configuration and on the curve $\mathcal{C}_R$ in the reference configuration. The corresponding divergence operators will be denoted by div and Div, respectively. In particular, for a vector-valued function $\mathbf{V}(\xi,t)$, we write*

$$
\begin{aligned}
\operatorname{grad} \mathbf{V} &= \mathbf{V}' \otimes \mathbf{d}^3 \quad , \quad \operatorname{div} \mathbf{V} = \mathbf{V}' \cdot \mathbf{d}^3 \quad , \\
\operatorname{Grad} \mathbf{V} &= \mathbf{V}' \otimes \mathbf{D}^3 \quad , \quad \operatorname{Div} \mathbf{V} = \mathbf{V}' \cdot \mathbf{D}^3 \quad ,
\end{aligned}
\tag{13.1}
$$

where a prime denotes partial differentiation with respect to $\xi$ and the symbol $\otimes$ denotes tensor product. Also, the spatial curve gradient operator is defined by

$$
\operatorname{grad}_c V = V' \mathbf{a}^3 \quad , \tag{13.2}
$$

for all scalar-valued functions $V(\xi,t)$.

As in section 8, we introduce a measure of deformation by the tensor $\mathbf{F}$,

---

*It is clear that the notations grad, Grad, div and Div in this section stand for operators with respect to position on the curve c and need not be confused with the similar notations in section 8 for surface operators.

namely

$$\underset{\sim}{F} = \underset{\sim}{d}_i \otimes \underset{\sim}{D}^i = \text{Grad}\ \underset{\sim}{r} + \underset{\sim}{d}_\alpha \otimes \underset{\sim}{D}^\alpha \quad , \tag{13.3}$$

and in view of the notations (9.6) and (9.10) we observe that

$$\underset{\sim}{F}\underset{\sim}{D}_3 = \underset{\sim}{d}_3 = \underset{\sim}{a}_3 \quad , \quad \underset{\sim}{F}\underset{\sim}{D}_\alpha = \underset{\sim}{d}_\alpha \quad . \tag{13.4}$$

From the definition of the determinant of a second order tensor used in section 8 [following (8.4)] and the conditions $(9.2)_3$ and $(9.8)_3$, we obtain

$$\det \underset{\sim}{F} = [\underset{\sim}{d}_1\underset{\sim}{d}_2\underset{\sim}{d}_3]/[\underset{\sim}{D}_1\underset{\sim}{D}_2\underset{\sim}{D}_3] > 0 \quad . \tag{13.5}$$

The tensor $\underset{\sim}{F}$, a linear operator on vectors in 3-space, is nonsingular; and there exists, therefore, the inverse deformation gradient $\underset{\sim}{F}^{-1}$ defined by

$$\underset{\sim}{F}^{-1} = \underset{\sim}{D}_i \otimes \underset{\sim}{d}^i \quad . \tag{13.6}$$

The inverse operator $\underset{\sim}{F}^{-1}$ transforms vectors in the present configuration into vectors in the reference configuration, i.e.,

$$\underset{\sim}{F}^{-1}\underset{\sim}{d}_i = \underset{\sim}{D}_i \tag{13.7}$$

and it follows that

$$\underset{\sim}{F}^{-1}\underset{\sim}{F} = \underset{\sim}{F}\,\underset{\sim}{F}^{-1} = \underset{\sim}{I} = \underset{\sim}{d}_i \otimes \underset{\sim}{d}^i = \underset{\sim}{D}_i \otimes \underset{\sim}{D}^i \quad , \tag{13.8}$$

where $\underset{\sim}{I}$ is the unit tensor in 3-space. We also introduce here the gradient of the directors by

$$\underset{\sim}{G}_\alpha = \text{Grad}\ \underset{\sim}{d}_\alpha = \underset{\sim}{d}_\alpha' \otimes \underset{\sim}{D}^3 \quad . \tag{13.9}$$

Recalling the definitions $(9.4)_{1,2}$ for the velocity and the director velocities, as well as (9.5), we have

$$\dot{\mathbf{F}} = \dot{\mathbf{d}}_i \otimes \mathbf{D}^i = \dot{\mathbf{d}}_3 \otimes \mathbf{D}^3 + \dot{\mathbf{d}}_\alpha \otimes \mathbf{D}^\alpha = \mathbf{v}' \otimes \mathbf{D}^3 + \mathbf{w}_\alpha \otimes \mathbf{D}^\alpha \ ,$$
$$\dot{\mathbf{G}}_\alpha = \dot{\mathbf{d}}_\alpha' \otimes \mathbf{D}^3 = \mathbf{w}_\alpha' \otimes \mathbf{D}^\alpha \ . \tag{13.10}$$

Also

$$\dot{\mathbf{F}}\mathbf{F}^{-1} = \dot{\mathbf{d}}_3 \otimes \mathbf{d}^3 + \dot{\mathbf{d}}_\alpha \otimes \mathbf{d}^\alpha = \operatorname{grad} \mathbf{v} + \mathbf{w}_\alpha \otimes \mathbf{d}^\alpha \ ,$$
$$\dot{\mathbf{G}}_\alpha \mathbf{F}^{-1} = \mathbf{w}_\alpha' \otimes \mathbf{d}^3 = \operatorname{grad} \mathbf{w}_\alpha \ . \tag{13.11}$$

The formulas (13.3)-(13.10) represent the main kinematical results in terms of the gradient tensors $\mathbf{F}, \mathbf{G}_\alpha$ and their rates. We now turn to kinetical quantities and note that the contact force $\mathbf{n}$ and the contact director force $\mathbf{m}^\alpha$, as linear functions of $\mathbf{d}^3$, can be expressed in the form

$$\mathbf{n} = d_{33}^{1/2}\, \mathbf{N}\, \mathbf{d}^3 \quad , \quad \mathbf{m}^\alpha = d_{33}^{1/2}\, \mathbf{M}^\alpha \mathbf{d}^3 \ , \tag{13.12}$$

with the second order tensors $\mathbf{N}, \mathbf{M}^\alpha$ defined by

$$d_{33}^{1/2}\, \mathbf{N} = \mathbf{n} \otimes \mathbf{d}_3 = n^i \mathbf{d}_i \otimes \mathbf{d}_3 \ ,$$
$$d_{33}^{1/2}\, \mathbf{M}^\alpha = \mathbf{m}^\alpha \otimes \mathbf{d}_3 = m^{i\alpha} \mathbf{d}_i \otimes \mathbf{d}_3 \ , \tag{13.13}$$

where

$$n^i = \mathbf{n} \cdot \mathbf{d}^i \quad , \quad m^{i\alpha} = \mathbf{m}^\alpha \cdot \mathbf{d}^i \ . \tag{13.14}$$

Also, it is convenient to introduce a tensor $\mathbf{K}$ through

$$\mathbf{k}^\alpha = d_{33}^{1/2}\, \mathbf{K}\, \mathbf{d}^\alpha \ ,$$
$$d_{33}^{1/2}\, \mathbf{K} = \mathbf{k}^\alpha \otimes \mathbf{d}_\alpha = k^{i\alpha} \mathbf{d}_i \otimes \mathbf{d}_\alpha \ , \tag{13.15}$$

where

$$k^{i\alpha} = \mathbf{k}^{\alpha} \cdot \mathbf{d}^{i} \quad . \tag{13.16}$$

Before proceeding further, we recall the divergence of a second order tensor field T is defined by

$$\mathbf{c} \cdot \operatorname{div} \mathbf{T} = \operatorname{div}(\mathbf{T}^{T}\mathbf{c})$$

for all constant vector $\mathbf{c}$, where superscript T denotes transpose. Applying the above definition to the tensor N in $(13.12)_1$, and recalling $(9.12)_1$, we obtain

$$\begin{aligned}\operatorname{div}_c(\mathbf{N}^{T}\mathbf{c}) &= \operatorname{div}_c[(\mathbf{n} \cdot \mathbf{c})\ \mathbf{d}_3/d_{33}^{1/2}] \\ &= [(\mathbf{n} \cdot \mathbf{c})\mathbf{d}_3/d_{33}^{1/2}]' \cdot \mathbf{a}^{3} = \frac{\partial \mathbf{n}}{\partial s} \cdot \mathbf{c} \quad ,\end{aligned}$$

with a similar result for the tensor $\mathbf{M}$ . Thus, we have

$$\operatorname{div}_c \mathbf{N} = \frac{\partial \mathbf{n}}{\partial s} \quad , \quad \operatorname{div}_c \mathbf{M}^{\alpha} = \frac{\partial \mathbf{m}^{\alpha}}{\partial s} \quad . \tag{13.17}$$

With the use of (13.1), the kinematical results (13.8)-(13.9), and (13.17) from the conservation laws (9.18) follow the local field equations

$$\begin{aligned}&\dot{\rho} + \rho \operatorname{div}_c \mathbf{v} = 0 \quad , \\ &\operatorname{div}_c \mathbf{N} + \rho \mathbf{f} = \rho(\dot{\mathbf{v}} + y^{\alpha}\dot{\mathbf{w}}_{\alpha}) \quad , \\ &\operatorname{div}_c \mathbf{M}^{\alpha} + \rho \boldsymbol{\ell}^{\alpha} - \mathbf{k}^{\alpha} = \rho(y^{\alpha}\dot{\mathbf{v}} + y^{\alpha\beta}\dot{\mathbf{w}}_{\beta}) \quad , \\ &[\mathbf{N} + \mathbf{K} + \mathbf{M}^{\alpha}(\mathbf{G}_{\alpha}\mathbf{F}^{-1})^{T}] = [\mathbf{N} + \mathbf{K} + \mathbf{M}^{\alpha}(\mathbf{G}_{\alpha}\mathbf{F}^{-1})^{T}]^{T} \quad .\end{aligned} \tag{13.18}$$

As in corresponding results in section 8, the last statement in (13.18) is similar to the symmetry of the stress tensor in the 3-dimensional theory. In particular, it may be observed that $\mathbf{a}_3 \times \mathbf{n}$, $\mathbf{d} \times \mathbf{k}$ and $\mathbf{d}_{\alpha}' \times \mathbf{m}^{\alpha}$ are, respectively, the axial vectors of $a_{33}^{1/2}[\mathbf{N}-\mathbf{N}^{T}]$, $a_{33}^{1/2}[\mathbf{K}-\mathbf{K}^{T}]$ and $a_{33}^{1/2}[\mathbf{M}^{\alpha}(\mathbf{G}_{\alpha}\mathbf{F}^{-1})^{T} - (\mathbf{G}_{\alpha}\mathbf{F}^{-1})(\mathbf{M}^{\alpha})^{T}]$.

Furthermore, in terms of the kinetical quantities $\underset{\sim}{N}, \underset{\sim}{M}^{\alpha}, \underset{\sim}{K}$ in (13.12) and (13.15)$_1$ and the rate quantities (13.10)$_{1,2}$, the mechanical power becomes

$$a_{33}^{-1/2} P = \operatorname{tr}\{[\dot{\underset{\sim}{F}}^T(\underset{\sim}{N}+\underset{\sim}{K}) + \dot{\underset{\sim}{G}}_\alpha^T \underset{\sim}{M}^\alpha](\underset{\sim}{F}^{-1})^T\} \quad . \tag{13.19}$$

With reference to constitutive equations for elastic rods, instead of the kinematic variables used in section 10, we now employ the variables (13.5) and (13.9). Thus, corresponding to the constitutive assumption (10.1), we now write

$$\psi = \tilde{\psi}(\underset{\sim}{F}, \underset{\sim}{G}_\alpha ; {}_R\underset{\sim}{G}_\alpha) \quad , \tag{13.20}$$

where

$${}_R\underset{\sim}{G}_\alpha = \operatorname{Grad} \underset{\sim}{D}_\alpha = \underset{\sim}{D}_\alpha' \otimes \underset{\sim}{D}^3 \quad , \tag{13.21}$$

along with similar assumptions for $\underset{\sim}{N}, \underset{\sim}{K}, \underset{\sim}{M}^\alpha$. Then, with the use of (10.1), (13.19) and (13.20), by usual technique we obtain the following alternative forms of the constitutive equations:

$$\underset{\sim}{N} + \underset{\sim}{K} = \rho \frac{\partial\tilde{\psi}}{\partial \underset{\sim}{F}} \underset{\sim}{F}^T \quad , \quad \underset{\sim}{M}^\alpha = \rho \frac{\partial\tilde{\psi}}{\partial \underset{\sim}{G}_\alpha} (\underset{\sim}{D}^3 \otimes \underset{\sim}{d}_3) \quad , \tag{13.22}$$

the first of which can be resolved into

$$\underset{\sim}{N} = \rho \frac{\partial\tilde{\psi}}{\partial \underset{\sim}{F}} (\underset{\sim}{D}^3 \otimes \underset{\sim}{d}_3) \quad , \quad \underset{\sim}{K} = \rho \frac{\partial\tilde{\psi}}{\partial \underset{\sim}{F}} (\underset{\sim}{D}^\alpha \otimes \underset{\sim}{d}_\alpha) \quad . \tag{13.23}$$

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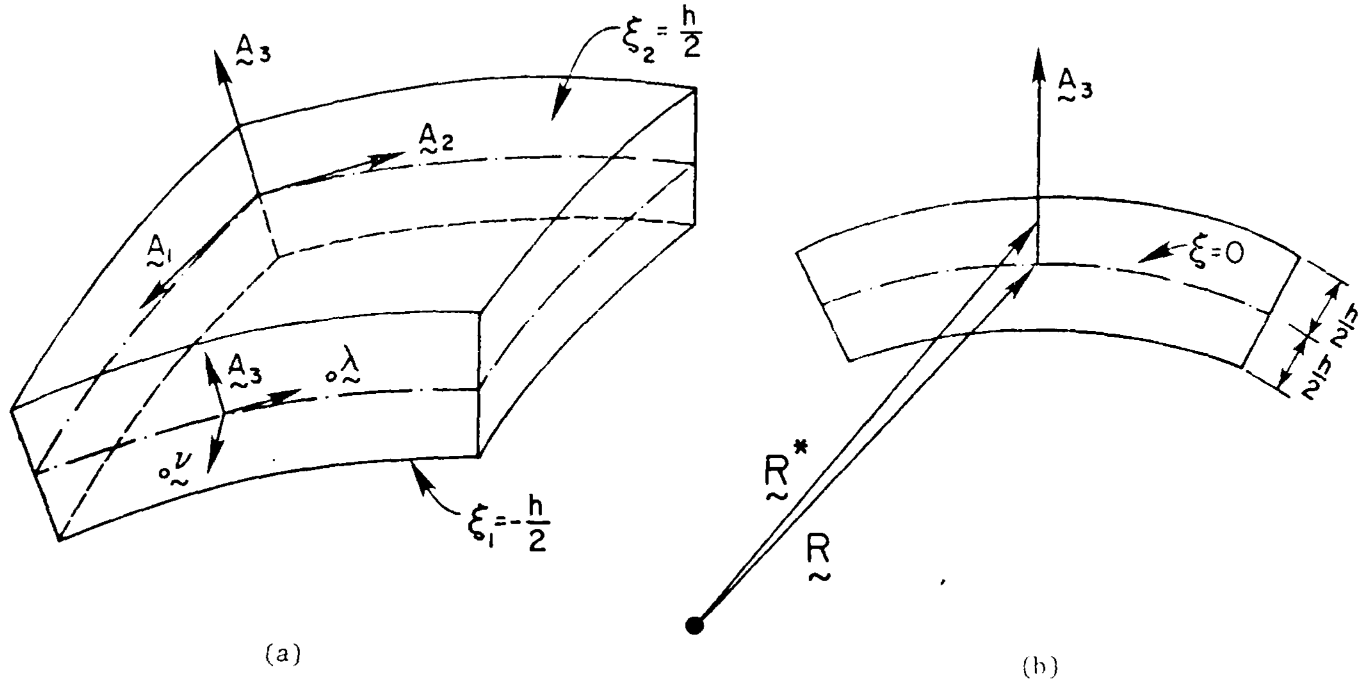
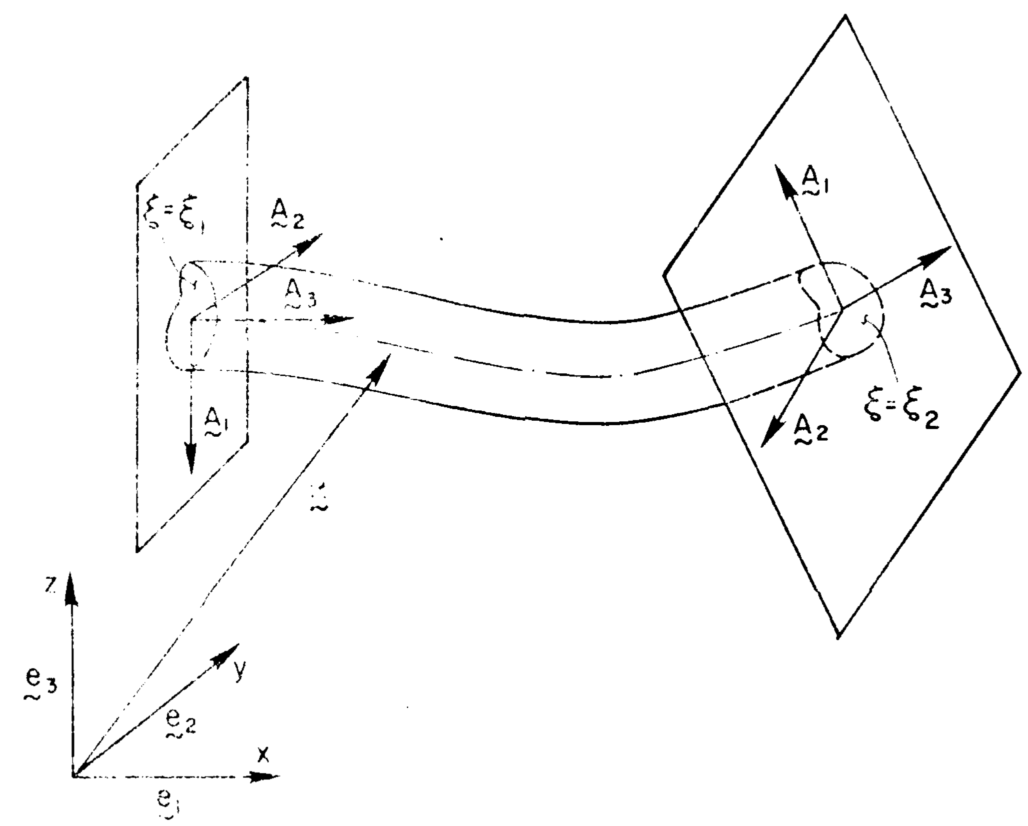



Fig. 1 (a) Element of a shell-like body in a reference configuration showing the middle surface $\xi = 0$ and the major surfaces $\xi = \pm(h/2)$, h being the shell thickness; and (b) the position vector $R$ from a fixed origin to a point on the middle surface $\xi = 0$ and the position vector $R^*$ to any point of the region of space occupied by the shell in the reference configuration. Also shown are the base vectors $A_\alpha$ on the middle surface, the unit normal $A_3$ to the middle surface and the right-handed triad $\{{}_o\nu, {}_o\lambda, A_3\}$ with ${}_o\nu$ and ${}_o\lambda$ being, respectively, the unit normal to a curve on the middle surface and the unit tangent to a curve on the middle surface.




[illegible] body in a reference configuration showing the line of [illegible] with position vector $\underset{\sim}{R}$ (referred to rectangular Cartesian [illegible] and the end normal cross-sections $\xi = \xi_1$, $\xi = \xi_2$. Also [illegible] principal normal $A_1$, the unit binormal $\underset{\sim}{A}_2$ and [illegible] vector $A_3$ to the curve with position vector R.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