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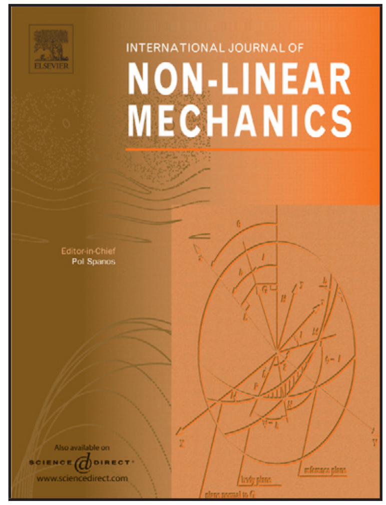

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PII: S0020-7462(07)00068-6
DOI: doi:10.1016/j.ijnonlinmec.2007.02.015
Reference: NLM 1353

To appear in: *International Journal of Non-Linear Mechanics*

Received date: 23 October 2006
Revised date: 25 February 2007
Accepted date: 25 February 2007

Cite this article as: A.J. M. Spencer and K.P. Soldatos, Finite deformations of fibre-reinforced elastic solids with fibre bending stiffness, *International Journal of Non-Linear Mechanics* (2007), doi:10.1016/j.ijnonlinmec.2007.02.015

SES/Rivlin Special Issue

FINITE DEFORMATIONS OF FIBRE-REINFORCED ELASTIC SOLIDS WITH FIBRE BENDING STIFFNESS

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In memory of Ronald Rivlin

Abstract.

In the conventional theory of finite deformations of fibre-reinforced elastic solids it is assumed that the strain-energy is an isotropic invariant function of the deformation and a unit vector **A** that defines the fibre direction and is convected with the material. This leads to a constitutive equation that involves no natural length . To incorporate fibre bending stiffness into a continuum theory, we make the more general assumption that the strain-energy depends on deformation, fibre direction, and the gradients of the fibre direction in the deformed configuration. The resulting extended theory requires, in general, a non-symmetric stress and the couple stress. The constitutive equations for stress and couple-stress are formulated in a general way, and specialized to the case in which dependence on the fibre direction gradients is restricted to dependence on their directional derivatives in the fibre direction. This is further specialized to the case of plane strain, and finite pure bending of a thick plate is solved as an example. We also formulate and develop the linearized theory in which the stress and couple-stress are linear functions of the first and second spacial derivatives of the displacement. In this case for the symmetric part of the stress we recover the standard equations of transversely isotropic linear elasticity, with five elastic moduli, and find that, in the most general case, a further seven moduli are required to characterize the couple-stress.

1. Introduction

The continuum theory of finite deformations of elastic materials reinforced by cords or fibres was initiated by Adkins and Rivlin [1] and further developed by Adkins [2-4]. Initially they assumed that the reinforcing cords lay in discrete surfaces, but they also considered the case in which the fibres are continuously distributed through the bulk of the material. These developments are described in Green and Adkins [5].

A slightly different approach was followed by Spencer [6]. In this theory the fibres are characterized by a unit vector field that defines the fibre direction and is convected with the material. This vector is treated as a constitutive variable with appropriate invariance properties. From this follows a formulation that is free from dependence on the choice of any special coordinate system, and does not restrict the reinforcement to any special geometrical arrangement. This approach has close connections with the theory of anisotropic tensor representations based on the use of structural tensors that was initiated by Boehler [7] and developed and extended by Zheng [8]. The fibre vector formulation has been applied to many kinds of material behaviour. Particular applications to the theory of finite elastic deformations are in Spencer [6, 9] and Rivlin [10]. Recently the theory has been applied extensively to the analysis of biological materials (see, for example, Holzapfel and Ogden [11]). The theory is outlined in Sections 2 and 3.

In all of this work there is an assumption, either explicit or implicit, that the reinforcing fibres are perfectly flexible. This assumption is a valid approximation in many cases of interest, but is not invariably applicable. A consequence of it is that the constitutive equations contain no parameter with the dimensions of length. Consequently the theory cannot account for any size effects, such as those due to fibre diameter or fibre spacing. In Section 4 we show by an illustrative example that in general size effects are present, and that the theory that assumes perfect fibre flexibility is a limiting case. To incorporate fibre bending stiffness into a continuum theory, in Section 5 we make the constitutive assumption that the elastic strain-energy depends not only on the deformation and the fibre vectors, but also on the space derivatives of the deformed fibre vector, subject to appropriate invariance requirements. This leads to a theory that has some similarities to the theory of liquid crystals, although there are important differences. The extended theory requires the inclusion of couple stress and non-symmetric stress. The constitutive equations for the stress and couple-stress are formulated in Section 5. In Section 6 we specialize to a simpler theory in which dependence on the fibre vector gradients is restricted to dependence only on the directional derivative in the fibre direction of the fibre vector. In Section 7 we consider plane strain deformations, and in Section 8 we apply the theory to the problem of pure bending of a rectangular block into a sector of a circular cylinder. Finally in Section 9 we develop the linear theory for the case of small displacement gradients. It is shown that in this case, in addition to the usual five elastic moduli required to describe transversely isotropic linear elasticity, an additional seven moduli are required for specification of the couple-stress, but a single one of these suffices for the linearized version of the restrited theory that is described in Section 6.

The theory of elastic materials with couple-stress has been examined in detail by Toupin [12] and Mindlin and Tiersten [13]. These authors also give references to earlier work in this area. We also mention that Hilgers and Pipkin [14] formulated a theory of large deformations of elastic membranes with bending stiffness which has some connections to the theory presented here.

2. Notation and general theory

All vector and tensor quantities are referred to a rectangular Cartesian coordinate system $Ox_1x_2x_3$. In the conventional notation, a typical particle is initially at a position $\mathbf{X}$, with coordinates X_R, in a reference configuration, and moves to the position $\mathbf{x}$, with coordinates x_i, in the deformed configuration. The deformation is described by equations

$$\mathbf{x} = \mathbf{x}(\mathbf{X}) \quad \text{or} \quad x_i = x_i(X_R). \tag{2.1}$$

The deformation gradient tensor $\mathbf{F}$ has components F_{iR}, where

$$F_{iR} = \partial x_i / \partial X_R. \tag{2.2}$$

If the material is incompressible, then det $\mathbf{F} = 1$. The deformation tensors $\mathbf{C}$ and $\mathbf{B}$ and their components C_{RS}, B_{ij}, are defined as

$$\mathbf{C} = \mathbf{F}^T\mathbf{F}, \quad \mathbf{B} = \mathbf{F}\mathbf{F}^T, \quad C_{RS} = \frac{\partial x_i}{\partial X_R}\frac{\partial x_i}{\partial X_S}, \quad B_{ij} = \frac{\partial x_i}{\partial X_R}\frac{\partial x_j}{\partial X_R}. \tag{2.3}$$

For a material reinforced by a single family of fibres, the fibre direction is defined

by a unit vector field $\mathbf{A}(\mathbf{X})$ in the reference configuration, and a unit vector field $\mathbf{a}(\mathbf{x})$ in the deformed configuration. The fibres are convected with the material, and thus

$$\lambda\mathbf{a} = \mathbf{F}\mathbf{A}, \quad \lambda a_i = F_{iR}A_R, \quad \lambda^2 = A_R A_S C_{RS}, \tag{2.4}$$

where λ denotes the stretch in the fibre direction. If the fibres are inextensible, then

$$\lambda = 1, \quad A_R A_S C_{RS} = 1. \tag{2.5}$$

The Cauchy stress tensor is denoted as σ, and its components as σ_{ij}. If the material is incompressible, then σ includes a reaction stress $-p\mathbf{I}$,where p represents a pressure. If it is inextensible in the fibre direction, the reaction stress has the form $T\mathbf{a}\otimes\mathbf{a}$, where T represents a tension in the fibre direction.

3. Finite deformations of a fibre-reinforced elastic solid without bending stiffness

In this section we give a brief summary of the theory of finite deformations of fibre-reinforced elastic materials as formulated by Spencer [6, 9]. We consider an elastic solid with a strain-energy function $W = W(F_{iR})$, or in the reduced form, $W = W(C_{RS})$. Then the constitutive equation for the stress is

$$\sigma_{ij} = \frac{\rho}{\rho_0} F_{jR} \frac{\partial W}{\partial F_{iR}}, \tag{3.1}$$

or, in the reduced form

$$\sigma_{ij} = \frac{\rho}{\rho_0} F_{iR}F_{jS} \left(\frac{\partial W}{\partial C_{RS}} + \frac{\partial W}{\partial C_{SR}} \right), \tag{3.2}$$

where ρ and ρ_0 denote densities in the deformed and reference configurations respectively. If the material is reinforced by a single family of fibres, characterized by the unit vector $\mathbf{A}$ in the reference configuration (or, more generally, is locally transversely isotropic with preferred direction $\mathbf{A}$), then

$$W = W(\mathbf{C}, \mathbf{A}), \quad \text{with} \qquad W = W(\mathbf{C}, \mathbf{A}) = W(\mathbf{QCQ}^T, \mathbf{QA}), \tag{3.3}$$

for any orthogonal tensor $\mathbf{Q}$. If the sense of $\mathbf{A}$ is not significant, then W is even in the components of $\mathbf{A}$, and W can be expressed as a function of $\mathbf{C}$ and $\mathbf{A} \otimes \mathbf{A}$. It follows that W can be expressed as a function of the invariants

$$\begin{aligned} &I_1 = \text{tr } \mathbf{C} = \text{tr } \mathbf{B}, \quad I_2 = \frac{1}{2}\left\{(\text{tr } \mathbf{C})^2 - \text{tr } \mathbf{C}^2\right\} = \frac{1}{2}\left\{(\text{tr } \mathbf{B})^2 - \text{tr } \mathbf{B}^2\right\}, \\ &I_3 = \det \mathbf{C} = \det \mathbf{B} = \left(\frac{\rho_0}{\rho}\right)^2, \\ &I_4 = \mathbf{ACA} = \lambda^2, \quad I_5 = \mathbf{AC}^2\mathbf{A} = \mathbf{aBa}. \end{aligned} \tag{3.4}$$

Then it follows that

$$\begin{aligned} \sigma = \frac{2\rho}{\rho_0}\mathbf{F}\{ &(W_1 + I_1W_2 + I_2W_3)\,\mathbf{I} - (W_2 + I_1W_3)\,\mathbf{C} + W_3\mathbf{C}^2 \\ &+ W_4\mathbf{A}\otimes\mathbf{A} + W_5\,(\mathbf{A}\otimes\mathbf{CA} + \mathbf{CA}\otimes\mathbf{A})\}\mathbf{F}^T, \end{aligned} \tag{3.5}$$

or equivalently

$$\begin{aligned} \sigma = \frac{2\rho}{\rho_0}\{ &(I_2W_2 + I_3W_3)\,\mathbf{I} + W_1\mathbf{B} - I_3W_2\mathbf{B}^{-1} \\ &+ I_4W_4\mathbf{a}\otimes\mathbf{a} + I_4W_5\,(\mathbf{a}\otimes\mathbf{Ba} + \mathbf{Ba}\otimes\mathbf{a})\}, \end{aligned} \tag{3.6}$$

where $W_\alpha = \partial W/\partial I_\alpha$. If the material is incompressible, $I_3 = 1$, and then

$$\sigma = 2\left\{W_1\mathbf{B} - W_2\mathbf{B}^{-1} + I_4W_4\mathbf{a}\otimes\mathbf{a} + I_4W_5\,(\mathbf{a}\otimes\mathbf{Ba} + \mathbf{Ba}\otimes\mathbf{a})\right\} - p\mathbf{I}. \tag{3.7}$$

If also the material is inextensible in the $\mathbf{A}$ direction, then $I_4 = 1$, and

$$\sigma = 2\left\{W_1\mathbf{B} - W_2\mathbf{B}^{-1} + W_5\,(\mathbf{a}\otimes\mathbf{Ba} + \mathbf{Ba}\otimes\mathbf{a})\right\} - p\mathbf{I} + T\mathbf{a}\otimes\mathbf{a}. \tag{3.8}$$

4. Micromechanical considerations

The constitutive equation (3.6) is homogeneous in $\mathbf{X}$ and does not include any natural length scale, and so can not account for any size effects, such as effects of fibre diameter or fibre spacing. In fact there are experimental and theoretical reasons to expect size effects to be observed in real fibre composites.

To illustrate the theoretical basis for expecting size effects, it is sufficient to consider pure bending in plane strain of a linear elastic plate with variable Young's modulus. In terms of rectangular Cartesian coordinates (x, y, z), suppose the middle surface of the plate lies in the plane $y = 0$, and the lateral surfaces are $y = \pm h$. Also suppose the plate undergoes a pure bending deformation, as in the elementary Euler-Bernoulli bending theory, so that the displacement in the x direction is

$$u = \frac{xy}{R}, \tag{4.1}$$

where R is the radius of curvature of the deformed plate. Further, suppose that the extension modulus E in the x direction depends on y, so that $E = E(y)$, with mean value E_0, where

$$E_0 = \frac{1}{2h}\int_{-h}^{h} E(y)dy. \tag{4.2}$$

Then the stress component σ_{xx} is

$$\sigma_{xx} = E(y)\frac{\partial u}{\partial x} = E(y)\frac{y}{R}, \tag{4.3}$$

and the bending moment applied to a section $x = const.$, per unit length in the z direction, is

$$M_z = \int_{-h}^{h} \sigma_{xx} y dy = \frac{1}{R}\int_{-h}^{h} E(y)y^2 dy, \tag{4.4}$$

and so the bending stiffness B is

$$B = M_z R = \int_{-h}^{h} E(y)y^2 dy. \tag{4.5}$$

On the other hand, if the extension modulus has the constant value E_0 the bending stiffness is

$$B_0 = \int_{-h}^{h} E_0 y^2 dy = \frac{2}{3}E_0 h^3. \tag{4.6}$$

Clearly, in general B and B_0 are not the same. To take a simple example for illustration, let

$$E(y) = E_0 - E_1 \cos\frac{\pi y}{d}, \quad \text{where} \;\; d = h/N, \tag{4.7}$$

and N is an odd integer. Thus d is a measure of the scale of the inhomogeneity of the material. Then from (4.5)

$$B = B_0 + \frac{4d^2hE_1}{\pi^2} = B_0 + B_1\frac{d^2}{h^2}, \quad B_1 = \frac{6E_1}{E_0\pi^2}, \tag{4.8}$$

and so the bending stiffness differs from B_0 by a term of order $(d/h)^2$. Similar results (with a slightly different value of B_1) are obtained if we consider that the plate is composed of alternate layers of materials with extension moduli $E_0 \pm E_1$ respectively.

In the theory of fibre-reinforced materials proposed by Adkins and Rivlin [1] it was explicitly assumed that the fibres are infinitesimally thin, and thus infinitely flexible, with zero bending stiffness for an individual fibre (represented in the theory by a mathematical curve). The same assumption is made, explicitly or implicitly, in the subsequent literature. It plainly corresponds to the limit $d/h \to 0$ in the above illustration. In order to relax this assumption, whilst remaining within a continuum theory, it is necessary to introduce some length scale into the theory, which effectively endows the fibres with bending stiffness. This is the purpose of the present paper.

5. Fibres with bending stiffness

A natural way to incorporate bending stiffness into the theory described in Section 3 is to assume that the strain-energy density W depends not only on the right Cauchy-Green deformation tensor $\mathbf{C}$ and the fibre direction $\mathbf{A}$, but also on the gradients of the deformed fibre vectors. This means, for example, that the fibre curvature is included. Dependence on the gradient of $\mathbf{a}$ implies that the stress σ is not necessarily symmetric and that in general a couple stress tensor $\mathbf{m}$, with Cartesian components m_{ij}, is present. The resulting theory in many respects resembles the continuum theory of liquid crystals (see, for example Stewart [15]) and also the theory of nematic elastomers, but there are important differences from these theories. For example, in the present theory the trajectories of the fibre direction $\mathbf{a}$ are material curves, but in liquid crystal theory the directors do not, in general, define material curves. We emphasize that, because the fibres are assumed to be embedded in the matrix, the fibre direction vector components are not variables to be determined independently of the deformation, but are given by (2.4). Hence the essential problem is to determine the position vector $\mathbf{x}$. We do not enter into a detailed consideration of boundary conditions, but note the discussion of boundary conditions for couple-stress in linear elasticity by Mindlin and Tiersten [13].

The equilibrium equations for the stress and couple-stress, neglecting body forces and body couples, are

$$\frac{\partial \sigma_{ji}}{\partial x_j} = 0, \quad \frac{\partial m_{ji}}{\partial x_j} + e_{ijk}\sigma_{jk} = 0, \tag{5.1}$$

where e_{ijk} is the third-order alternating tensor. In accordance with the usual definitions of Cauchy stress and couple-stress, if S is a surface with unit normal $\mathbf{n}$, then the components t_i of the traction vector $\mathbf{t}$ and l_i of the moment $\mathbf{l}$ per unit area applied to S are given by

$$t_i = \sigma_{ji} n_j, \quad l_i = m_{ji} n_j. \tag{5.2}$$

We denote by d_{ij} and ω_{ij} the components of the rate-of deformation and spin tensors $\mathbf{d}$ and ω respectively, and by ω_i the components of the spin vector $\frac{1}{2}\nabla \times \mathbf{v}$. Thus

$$d_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right), \quad \omega_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right), \quad \frac{\partial v_i}{\partial x_j} = d_{ij} + \omega_{ij},$$
$$\omega_i = \frac{1}{2} e_{ijk} \frac{\partial v_k}{\partial x_j} = -\frac{1}{2} e_{ijk} \omega_{jk}, \quad \omega_{jk} = -e_{ijk}\omega_i. \tag{5.3}$$

We also note that it can be shown [6] that the material derivative $\dot{\mathbf{a}}$ of the fibre vector $\mathbf{a}$ has components

$$\dot{a}_i = (\delta_{ij} - a_i a_j)\, a_k \frac{\partial v_j}{\partial x_k}. \tag{5.4}$$

To proceed, we have to formulate constitutive equations for the symmetric part of the stress and for the couple stress. The anti-symmetric part of the stress is then given by the equilibrium equation (5.1). For quasi-static deformations, and neglecting body forces and moments, (there is no difficulty in incorporating inertia and body forces and couples, but this has no effect on the final results) we propose the usual energy balance equation for an arbitrary volume V in the deformed configuration. Thus if V has surface S, then

$$\frac{D}{Dt}\int_V W\, dV = \int_S \{\mathbf{t}\cdot\mathbf{v} + \mathbf{l}\cdot\omega\} dS, \tag{5.5}$$

where the components t_i of $\mathbf{t}$ and l_i of $\mathbf{l}$ are given by (5.2). By applying the divergence theorem and Reynold's Transport Theorem in the conventional way, it follows that

$$\frac{\rho}{\rho_0}\dot{W} = \sigma_{ji}\frac{\partial v_i}{\partial x_j} + m_{ji}\frac{\partial \omega_i}{\partial x_j} + v_i \frac{\partial \sigma_{ji}}{\partial x_j} + \omega_i \frac{\partial m_{ji}}{\partial x_j}, \tag{5.6}$$

and hence, using (5.1), (5.2) and (5.3)

$$\begin{aligned}\frac{\rho}{\rho_0}\dot{W} &= \sigma_{ji}\frac{\partial v_i}{\partial x_j} + m_{ji}\frac{\partial \omega_i}{\partial x_j} - e_{ijk}\omega_i\sigma_{jk} \\ &= \sigma_{ji}(d_{ij} + \omega_{ij}) + m_{ji}\frac{\partial \omega_i}{\partial x_j} + \omega_{jk}\sigma_{jk} \\ &= \sigma_{ji} d_{ij} + m_{ji}\frac{\partial \omega_i}{\partial x_j}.\end{aligned} \tag{5.7}$$

We make the constitutive assumption that W depends, in addition to the displacement gradients F_{iR} and $\mathbf{A}$, on the gradients of the deformed fibre vectors. However rather than including dependence on the gradients $\partial a_i/\partial X_R$, it is more convenient to introduce a vector $\mathbf{b}$, with Cartesian components b_i, such that

$$\mathbf{b} = \lambda \mathbf{a}, \quad b_i = \lambda a_i = A_R \frac{\partial x_i}{\partial X_R} = F_{iR} A_R, \tag{5.8}$$

and to assume that W depends instead on the gradients $\partial b_i/\partial X_R$. Since $\lambda^2 = A_R A_S F_{iR} F_{iS}$, dependence on $\mathbf{F}, \mathbf{a}$ and $\mathbf{A}$ is equivalent to dependence on $\mathbf{F}, \mathbf{b}$ and $\mathbf{A}$. The advantage of using $\mathbf{b}$ rather than $\mathbf{a}$ is that

$$\overset{\bullet}{b}_i = A_R \frac{\partial v_i}{\partial X_R} = A_R \frac{\partial v_i}{\partial x_j} \frac{\partial x_j}{\partial X_R} = b_j \frac{\partial v_i}{\partial x_j}, \tag{5.9}$$

which is a simpler form than (5.4). Therefore we postulate that

$$W = W\left(F_{iR}, G_{iR}, A_R\right), \quad \text{or} \quad W = W(\mathbf{F}, \mathbf{G}, \mathbf{A}), \tag{5.10}$$

where

$$F_{iR} = \frac{\partial x_i}{\partial X_R}, \quad G_{iR} = \frac{\partial b_i}{\partial X_R}. \tag{5.11}$$

Therefore

$$\dot{W} = \frac{\partial W}{\partial F_{iR}} \dot{F}_{iR} + \frac{\partial W}{\partial G_{iR}} \dot{G}_{iR} = \frac{\partial W}{\partial F_{iR}} \frac{\partial v_i}{\partial X_R} + \frac{\partial W}{\partial G_{iR}} \frac{\partial \dot{b}_i}{\partial X_R}. \tag{5.12}$$

Hence

$$\dot{W} = \frac{\partial W}{\partial F_{iR}} \frac{\partial x_j}{\partial X_R} \frac{\partial v_i}{\partial x_j} + \frac{\partial W}{\partial G_{iR}} \frac{\partial x_j}{\partial X_R} \frac{\partial \dot{b}_i}{\partial x_j} = F_{jR} \left\{ \frac{\partial W}{\partial F_{iR}} \frac{\partial v_i}{\partial x_j} + \frac{\partial W}{\partial G_{iR}} \frac{\partial \dot{b}_i}{\partial x_j} \right\}. \tag{5.13}$$

From (5.9)

$$F_{jR} \frac{\partial \dot{b}_i}{\partial x_j} = \frac{\partial x_j}{\partial X_R} \frac{\partial b_k}{\partial x_j} \frac{\partial v_i}{\partial x_k} + F_{jR} b_k \frac{\partial^2 v_i}{\partial x_j \partial x_k} = G_{kR} \frac{\partial v_i}{\partial x_k} + F_{jR} b_k \frac{\partial^2 v_i}{\partial x_j \partial x_k}, \tag{5.14}$$

and so from (5.12) and (5.13)

$$\overset{\bullet}{W} = \left(F_{jR} \frac{\partial W}{\partial F_{iR}} + G_{jR} \frac{\partial W}{\partial G_{iR}} \right) (d_{ij} + \omega_{ij}) + F_{jR} \frac{\partial W}{\partial G_{iR}} b_k \frac{\partial^2 v_i}{\partial x_j \partial x_k}. \tag{5.15}$$

We now denote the components of the symmetric and antisymmetric parts of σ as $\sigma_{(ij)}$ and $\sigma_{[ij]}$ respectively, so that

$$\sigma_{(ij)} = \frac{1}{2}(\sigma_{ij} + \sigma_{ji}), \quad \sigma_{[ij]} = \frac{1}{2}(\sigma_{ij} - \sigma_{ji}), \quad \sigma_{ij} = \sigma_{(ij)} + \sigma_{[ij]}, \tag{5.16}$$

and note that

$$\sigma_{(ij)}\omega_{ij} = 0, \quad \sigma_{[ij]}d_{ij} = 0. \tag{5.17}$$

Hence, by comparing (5.7) and (5.15), we obtain

$$\left\{\sigma_{(ij)} - \frac{\rho}{\rho_0}\left(F_{jR}\frac{\partial W}{\partial F_{iR}} + G_{jR}\frac{\partial W}{\partial G_{iR}}\right)\right\} d_{ij} - \frac{\rho}{\rho_0}\left(F_{jR}\frac{\partial W}{\partial F_{iR}} + G_{jR}\frac{\partial W}{\partial G_{iR}}\right)\omega_{ij} + m_{ji}\frac{\partial \omega_i}{\partial x_j} - \frac{\rho}{\rho_0}F_{jR}\frac{\partial W}{\partial G_{iR}}b_k\frac{\partial^2 v_i}{\partial x_j \partial x_k} = 0. \tag{5.18}$$

Since d_{ij} and ω_{ij} are arbitrary, it follows that

$$\sigma_{(ij)} = \frac{\rho}{\rho_0}\left(F_{jR}\frac{\partial W}{\partial F_{iR}} + G_{jR}\frac{\partial W}{\partial G_{iR}}\right), \tag{5.19}$$

and that the coefficient of ω_{ij} in (5.18) is symmetric with respect to interchanges of i and j, thus

$$F_{jR}\frac{\partial W}{\partial F_{iR}} + G_{jR}\frac{\partial W}{\partial G_{iR}} = F_{iR}\frac{\partial W}{\partial F_{jR}} + G_{iR}\frac{\partial W}{\partial G_{jR}}. \tag{5.20}$$

Equation (5.19) is the constitutive equation for the symmetric part of the stress σ; (5.20) is a restriction on the admissible forms of W, the validity of which is confirmed below. There now remains from (5.18)

$$m_{ji}\frac{\partial \omega_i}{\partial x_j} - \frac{\rho}{\rho_0}F_{jR}\frac{\partial W}{\partial G_{iR}}b_k\frac{\partial^2 v_i}{\partial x_j \partial x_k} = 0, \tag{5.21}$$

or equivalently, using (5.3),

$$\left(-\frac{1}{2}e_{pik}m_{jp} - \frac{\rho}{\rho_0}F_{jR}\frac{\partial W}{\partial G_{iR}}b_k\right)\frac{\partial^2 v_i}{\partial x_j \partial x_k} = 0. \tag{5.22}$$

It follows that the symmetric part (with respect to the indices j and k) of the bracketed term in (5.22) must be zero, and therefore

$$-\frac{1}{2}\left(e_{pik}m_{jp} + e_{pij}m_{kp}\right) = \frac{\rho}{\rho_0}\frac{\partial W}{\partial G_{iR}}\left(F_{jR}b_k + F_{kR}b_j\right). \tag{5.23}$$

By multiplying each side of (5.23) by e_{rik} and using the $\epsilon - \delta$ identities, there follows

$$2\delta_{pr}m_{jp} + (\delta_{pr}\delta_{kj} - \delta_{rj}\delta_{kp})m_{kp} = -2e_{rik}\frac{\rho}{\rho_0}\frac{\partial W}{\partial G_{iR}}\left(F_{jR}b_k + F_{kR}b_j\right), \tag{5.24}$$

and hence

$$3m_{jr} - m_{kk}\delta_{rj} = -2e_{rik}\frac{\rho}{\rho_0}\frac{\partial W}{\partial G_{iR}}\left(F_{jR}b_k + F_{kR}b_j\right) \tag{5.25}$$

which is a constitutive equation for the couple stress m_{ij}.

If we set $r = j$ in (5.25), then each side reduces to zero, and so the spherical part m_{kk} of m_{ij} is indeterminate. This is consistent with the observation that if m_{ij} is decomposed into its spherical and deviatoric parts

$$m_{jr} = \overline{m}_{jr} + \frac{1}{3} m_{kk} \delta_{rj}, \tag{5.26}$$

then, because $\partial \omega_i / \partial x_i = 0$, m_{kk} makes no contribution to the energy balance equation (5.7). This indeterminacy in the couple-stress is not specific to fibre-reinforced materials, but is a general result in couple-stress theory. It is discussed at length in Toupin [12] and Mindlin and Tiersten [13]. Using (5.26) we can write (5.25) as

$$\overline{m}_{jr} = -\frac{2}{3} e_{rik} \frac{\rho}{\rho_0} \frac{\partial W}{\partial G_{iR}} \left(F_{jR} b_k + F_{kR} b_j \right), \quad \overline{m}_{kk} = 0. \tag{5.27}$$

Clearly, if $r \neq j$, then $\overline{m}_{jr} = m_{jr}$.

Invariance under the superposed rigid rotation $\mathbf{x} \to \mathbf{Q}\mathbf{x}$ requires that

$$W(\mathbf{F}, \mathbf{G}, \mathbf{A}) = W(\mathbf{QF}, \mathbf{QG}, \mathbf{A}), \tag{5.28}$$

for any orthogonal tensor $\mathbf{Q}$. It follows that W depends on the scalar products of the vectors with components (for each fixed R) F_{iR} and G_{iR}, and therefore W can be expressed as a function of the tensors

$$\mathbf{C} = \mathbf{F}^T\mathbf{F}, \quad \mathbf{\Gamma} = \mathbf{G}^T\mathbf{G}, \quad \mathbf{\Lambda} = \mathbf{F}^T\mathbf{G}, \quad \mathbf{\Lambda}^T = \mathbf{G}^T\mathbf{F} \tag{5.29}$$

and the vector $\mathbf{A}$, where $\mathbf{C}, \mathbf{\Gamma}, \mathbf{\Lambda}, \mathbf{A}$ have components, respectively

$$\begin{aligned} C_{RS} &= \frac{\partial x_i}{\partial X_R} \frac{\partial x_i}{\partial X_S} = F_{iR} F_{iS}, \quad \Gamma_{RS} = \frac{\partial b_i}{\partial X_R} \frac{\partial b_i}{\partial X_S} = G_{iR} G_{iS}, \\ \Lambda_{RS} &= \frac{\partial x_i}{\partial X_R} \frac{\partial b_i}{\partial X_S} = F_{iR} G_{iS}, \quad \text{and} \quad A_R. \end{aligned} \tag{5.30}$$

However, from (5.29)

$$\mathbf{\Gamma} = \mathbf{\Lambda}^T \mathbf{C}^{-1} \mathbf{\Lambda}, \quad \mathbf{C} = \mathbf{\Lambda} \mathbf{\Gamma}^{-1} \mathbf{\Lambda}^T, \tag{5.31}$$

and, by the Cayley-Hamilton Theorem for $\mathbf{C}$

$$I_3 \mathbf{C}^{-1} = \mathbf{C}^2 - I_1 \mathbf{C} + I_2 \mathbf{I}. \tag{5.32}$$

Hence $\mathbf{\Gamma}$ can be expressed in terms of $\mathbf{\Lambda}, \mathbf{C}$ and invariants of $\mathbf{C}$, and therefore W can be expressed as a function of these quantities. Invariance under rigid rotations of the undeformed body then requires that

$$W(\mathbf{C}, \mathbf{\Lambda}, \mathbf{A}) = W(\mathbf{QCQ}^T, \mathbf{Q\Lambda Q}^T, \mathbf{QA}), \tag{5.33}$$

so that W can be expressed as an isotropic invariant of $\mathbf{C}, \mathbf{\Lambda}, \mathbf{A}$. If the sense of the fibres is not significant, then W must also be even in the components of $\mathbf{A}$ and even in the components of $\mathbf{\Lambda}$. In this case dependence on the vector $\mathbf{A}$ can be replaced by dependence on the tensor $\mathbf{A} \otimes \mathbf{A}$, but we do not impose this restriction at this stage.

Since W depends on $\mathbf{F}$ and $\mathbf{G}$ only through the tensors $\mathbf{C}$ and $\mathbf{\Lambda}$, we have

$$\begin{aligned}
\frac{\partial W}{\partial F_{iR}} &= \frac{\partial W}{\partial C_{PQ}}\frac{\partial C_{PQ}}{\partial F_{iR}} + \frac{\partial W}{\partial \Lambda_{PQ}}\frac{\partial \Lambda_{PQ}}{\partial F_{iR}}, \\
\frac{\partial W}{\partial G_{iR}} &= \frac{\partial W}{\partial C_{PQ}}\frac{\partial C_{PQ}}{\partial G_{iR}} + \frac{\partial W}{\partial \Lambda_{PQ}}\frac{\partial \Lambda_{PQ}}{\partial G_{iR}},
\end{aligned} \tag{5.34}$$

and, since $C_{PQ} = F_{kP}F_{kQ}$ and $\Lambda_{PQ} = F_{kP}G_{kQ}$

$$\begin{aligned}
&\frac{\partial C_{PQ}}{\partial F_{iR}} = \delta_{ik}\delta_{PR}F_{kQ} + \delta_{ik}\delta_{QR}F_{kP} = F_{iQ}\delta_{PR} + F_{iP}\delta_{QR}, \quad \frac{\partial C_{PQ}}{\partial G_{iR}} = 0, \\
&\frac{\partial \Lambda_{PQ}}{\partial F_{iR}} = \delta_{ik}\delta_{RP}G_{kQ} = G_{iQ}\delta_{RP}, \quad \frac{\partial \Lambda_{PQ}}{\partial G_{iR}} = \delta_{ik}\delta_{RQ}F_{kP} = F_{iP}\delta_{RQ}.
\end{aligned} \tag{5.35}$$

Hence from (5.34), using (5.35)

$$\begin{aligned}
F_{jR}\frac{\partial W}{\partial F_{iR}} &= F_{jR}\left\{\frac{\partial W}{\partial C_{PQ}}\frac{\partial C_{PQ}}{\partial F_{iR}} + \frac{\partial W}{\partial \Lambda_{PQ}}\frac{\partial \Lambda_{PQ}}{\partial F_{iR}}\right\} \\
&= F_{jR}\left\{(F_{iQ}\delta_{PR} + F_{iP}\delta_{QR})\frac{\partial W}{\partial C_{PQ}} + G_{iQ}\delta_{RP}\frac{\partial W}{\partial \Lambda_{PQ}}\right\} \\
&= F_{jR}F_{iP}\left(\frac{\partial W}{\partial C_{PR}} + \frac{\partial W}{\partial C_{RP}}\right) + F_{jR}G_{iP}\frac{\partial W}{\partial \Lambda_{RP}}, \\
G_{jR}\frac{\partial W}{G_{iR}} &= G_{jR}\frac{\partial W}{\partial \Lambda_{PQ}}\frac{\partial \Lambda_{PQ}}{\partial G_{iR}} = G_{jR}F_{iP}\frac{\partial W}{\partial \Lambda_{PR}}, \\
F_{jR}\frac{\partial W}{G_{iR}} &= F_{jR}\frac{\partial W}{\partial \Lambda_{PQ}}\frac{\partial \Lambda_{PQ}}{\partial G_{iR}} = F_{jR}F_{iP}\frac{\partial W}{\partial \Lambda_{PR}}.
\end{aligned} \tag{5.36}$$

Hence from (5.36)

$$F_{jR}\frac{\partial W}{\partial F_{iR}} + G_{jR}\frac{\partial W}{\partial G_{iR}} = F_{jR}F_{iP}\left(\frac{\partial W}{\partial C_{PR}} + \frac{\partial W}{\partial C_{RP}}\right) + (F_{jR}G_{iP} + F_{iR}G_{jP})\frac{\partial W}{\partial \Lambda_{RP}}, \tag{5.37}$$

from which (5.20) follows immediately. Hence (5.19) and (5.27) can now be expressed (with some renaming of indices) as

$$\begin{aligned}
\sigma_{(ij)} &= \frac{\rho}{\rho_0}\left\{F_{iR}F_{jS}\left(\frac{\partial W}{\partial C_{RS}} + \frac{\partial W}{\partial C_{SR}}\right) + (G_{iR}F_{jS} + G_{jR}F_{iS})\frac{\partial W}{\partial \Lambda_{SR}}\right\}, \\
\overline{m}_{ji} &= \frac{2}{3}e_{ikm}\frac{\rho}{\rho_0}\frac{\partial W}{\partial \Lambda_{PR}}F_{mP}(F_{jR}b_k + F_{kR}b_j).
\end{aligned} \tag{5.38}$$

If the couple-stress is non-zero, then in general the stress must have an antisymmetric part in order to preserve moment equilibrium of an element of the body. This antisymmetric part of the stress is given by the equilibrium equation (5.1_2).

The strain-energy W is an isotropic invariant of the tensors $\mathbf{C}, \mathbf{\Lambda}$ and the vector $\mathbf{A}$. Canonical forms for these invariants are known and can be read from tables (for example, Zheng [8, Table 1]). A list of the invariants is given in the Appendix. This list contains thirty-three independent invariants which in the general case clearly leads to excessively complicated constitutive equations. In order to progress, therefore, it is necessary to make further simplifying assumptions. There are several plausible ways in which this may be done; for example by considering only restricted classes of deformations, as in the plane strain theory discussed in Section 7, or by adopting the linearized theory which is described in Section 9. Another possibility, not pursued here, is to restrict W to be at most quadratic in the gradients of the fibre vector, which is analogous to the assumption usually made in liquid crystal theory. This assumption has the consequence that the equations then contain a single parameter with the dimensions of length, which can be interpreted as a characteristic fibre radius of curvature. This quadratic formulation can be interpreted as the first approximation to the general theory when the characteristic fibre radius of curvature is large compared to the dimensions that characterize the microstructure of the material (for example fibre diameters or fibre spacings).

In appropriate cases, some simplification can be achieved by introducing the kinematic constraints of incompressibility and/or fibre inextensibility. Another simplified theory is described in the next section.

6. Dependence on fibre curvature

In this section it is assumed that, rather than general dependence on the gradients of $\mathbf{b}$, the strain energy depends on the gradients of $\mathbf{b}$ only through the directional derivative of the fibre vector in the fibre direction; that is, essentially, on the curvature of the fibres. In doing this, we exclude effects due to fibre 'splay' and fibre 'twist', both of which feature in liquid crystal theory, but it is plausible that in fibre composite solids the major factor is fibre curvature.

Accordingly we make the initial assumption that the strain-energy depends on the deformation gradients $\partial x_i/\partial X_R$, the directional derivatives $A_R \partial b_i/\partial X_R$, and the initial fibre direction vector $\mathbf{A}$. Invariance under a superposed rigid rotation $\mathbf{x} \to \mathbf{Qx}$ of the deformed body requires that W can be expressed as a function of the scalar products, formed by contracting on the index i, of the vectors $\partial x_i/\partial X_R = F_{iR}$, and $A_R \partial b_i/\partial X_R = G_{iR} A_R = \kappa_i$. These scalar products are

$$C_{RS} = F_{iR} F_{iS}, \quad K_R = \kappa_i F_{iR} = A_S \frac{\partial x_i}{\partial X_R} \frac{\partial b_i}{\partial X_S} = \Lambda_{RS} A_S,$$
$$\kappa^2 = \kappa_i \kappa_i = A_R A_S \frac{\partial b_i}{\partial X_R} \frac{\partial b_i}{\partial X_S} = A_R A_S \Gamma_{RS}. \tag{6.1}$$

Then invariance under rotations of the undeformed body requires that W is an isotropic invariant of the tensor $\mathbf{C}$ (components C_{RS}), the vectors $\mathbf{K}$ (components K_R) and $\mathbf{A}$, and the scalar κ^2. It follows from tables of invariants that W can be expressed as a function of

$$\begin{aligned}
J_1 &= I_1 = \text{tr }\mathbf{C} = \text{tr }\mathbf{B},\\
J_2 &= I_2 = \frac{1}{2}\left\{(\text{tr }\mathbf{C})^2 - \text{tr }\mathbf{C}^2\right\} = \frac{1}{2}\left\{(\text{tr }\mathbf{B})^2 - \text{tr }\mathbf{B}^2\right\},\\
J_3 &= I_3 = \det \mathbf{C} = \det \mathbf{B},\\
J_4 &= I_4 = \mathbf{ACA} = \mathbf{b} \bullet \mathbf{b},\\
J_5 &= I_5 = \mathbf{AC}^2\mathbf{A} = \mathbf{bBb},\\
J_6 &= \mathbf{K} \bullet \mathbf{K} = \mathbf{A}\boldsymbol{\Lambda}^T\boldsymbol{\Lambda}\mathbf{A} = \mathbf{b}\beta^T\mathbf{B}\beta\mathbf{b},\\
J_7 &= \mathbf{KCK} = \mathbf{A}\boldsymbol{\Lambda}^T\mathbf{C}\boldsymbol{\Lambda}\mathbf{A} = \mathbf{b}\beta^T\mathbf{B}^2\beta\mathbf{b},\\
J_8 &= \mathbf{KC}^2\mathbf{K} = \mathbf{A}\boldsymbol{\Lambda}^T\mathbf{C}^2\boldsymbol{\Lambda}\mathbf{A} = \mathbf{b}\beta^T\mathbf{B}^3\beta\mathbf{b},\\
J_9 &= \mathbf{A} \bullet \mathbf{K} = \mathbf{A}\boldsymbol{\Lambda}\mathbf{A} = \mathbf{b}\beta\mathbf{b},\\
J_{10} &= \mathbf{ACK} = \mathbf{AC}\boldsymbol{\Lambda}\mathbf{A} = \mathbf{bB}\beta\mathbf{b},\\
J_{11} &= \mathbf{AC}^2\mathbf{K} = \mathbf{AC}^2\boldsymbol{\Lambda}\mathbf{A} = \mathbf{bB}^2\beta\mathbf{b},
\end{aligned} \tag{6.2}$$

where

$$\beta_{ij} = \frac{\partial b_i}{\partial x_j}, \quad \beta\mathbf{F} = \mathbf{G} = \mathbf{F}^{-T}\boldsymbol{\Lambda}. \tag{6.3}$$

The invariant $\kappa^2 = \mathbf{A}\boldsymbol{\Gamma}\mathbf{A} = \mathbf{A}\boldsymbol{\Lambda}^T\mathbf{C}^{-1}\boldsymbol{\Lambda}\mathbf{A} = \mathbf{b}\beta^T\beta\mathbf{b}$ can be expressed as a linear combination of J_6, J_7 and J_8 by the Cayley-Hamilton theorem for $\mathbf{B}$, and so is omitted from the list.

From (6.2) there follow

$$\begin{aligned}
&\partial J_1/\partial\mathbf{C} = \mathbf{I}, \quad \partial J_2/\partial\mathbf{C} = I_1\mathbf{I} - \mathbf{C}, \quad \partial J_3/\partial\mathbf{C} = I_2\mathbf{I} - I_1\mathbf{C} + \mathbf{C}^2,\\
&\partial J_4/\partial\mathbf{C} = \mathbf{A}\otimes\mathbf{A}, \quad \partial J_5/\partial\mathbf{C} = \mathbf{A}\otimes(\mathbf{CA}) + (\mathbf{CA})\otimes\mathbf{A},\\
&\partial J_6/\partial\mathbf{C} = \mathbf{0}, \qquad \partial J_7/\partial\mathbf{C} = (\boldsymbol{\Lambda}\mathbf{A})\otimes(\boldsymbol{\Lambda}\mathbf{A}), \quad \partial J_8/\partial\mathbf{C} = (\boldsymbol{\Lambda}\mathbf{A})\otimes(\mathbf{C}\boldsymbol{\Lambda}\mathbf{A}),\\
&\partial J_9/\partial\mathbf{C} = \mathbf{0}, \qquad \partial J_{10}/\partial\mathbf{C} = \mathbf{A}\otimes(\boldsymbol{\Lambda}\mathbf{A}), \quad \partial J_{11}/\partial\mathbf{C} = \mathbf{A}\otimes(\mathbf{C}\boldsymbol{\Lambda}\mathbf{A}),\\
&\partial J_1/\partial\boldsymbol{\Lambda} = \mathbf{0}, \quad \partial J_2/\partial\boldsymbol{\Lambda} = \mathbf{0}, \quad \partial J_3/\partial\boldsymbol{\Lambda} = \mathbf{0}, \quad \partial J_4/\partial\boldsymbol{\Lambda} = \mathbf{0}, \quad \partial J_5/\partial\boldsymbol{\Lambda} = \mathbf{0},\\
&\partial J_6/\partial\boldsymbol{\Lambda} = 2\boldsymbol{\Lambda}\mathbf{A}\otimes\mathbf{A}, \quad \partial J_7/\partial\boldsymbol{\Lambda} = 2(\mathbf{C}\boldsymbol{\Lambda}\mathbf{A})\otimes\mathbf{A}, \quad \partial J_8/\partial\boldsymbol{\Lambda} = 2\left(\mathbf{C}^2\boldsymbol{\Lambda}\mathbf{A}\right)\otimes\mathbf{A},\\
&\partial J_9/\partial\boldsymbol{\Lambda} = \mathbf{A}\otimes\mathbf{A}, \quad \partial J_{10}/\partial\boldsymbol{\Lambda} = (\mathbf{CA})\otimes\mathbf{A}, \quad \partial J_{11}/\partial\boldsymbol{\Lambda} = \left(\mathbf{C}^2\mathbf{A}\right)\otimes\mathbf{A},
\end{aligned} \tag{6.4}$$

where $\partial J_\alpha/\partial\mathbf{C}$ denotes the tensor whose (R, S) component is $\partial J_\alpha/\partial C_{RS}$, and similarly with $\partial J_\alpha/\partial\boldsymbol{\Lambda}$. From these results it is straightforward to write down the constitutive equations for the case in which W is a function of $J_1 - J_{11}$, but in this generality the resulting equations are still too complicated for practical applications. For further simplification, in the next section we consider plane strain deformations.

7. Plane strain

In this section, Greek indices take the values $1, 2$. For plane strain deformations with fibres lying in the (X_1, X_2) planes

$$x_\alpha = x_a(X_\Gamma), \quad b_\alpha = b_a(X_\Gamma), \quad x_3 = X_3, \quad \mathbf{A} = (A_1, A_2, 0)^T.$$

Thus

$$\mathbf{F} = \begin{bmatrix} \widehat{\mathbf{F}} & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} \widehat{\mathbf{G}} & \mathbf{0} \\ \mathbf{0} & 0 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} \widehat{\mathbf{C}} & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix}, \quad \mathbf{\Lambda} = \begin{bmatrix} \widehat{\mathbf{\Lambda}} & \mathbf{0} \\ \mathbf{0} & 0 \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} \widehat{\mathbf{A}} \\ 0 \end{bmatrix},$$

$$\widehat{F}_{\alpha\Gamma} = \frac{\partial x_\alpha}{\partial X_\Gamma}, \quad \widehat{G}_{\alpha\Gamma} = \frac{\partial b_\alpha}{\partial X_\Gamma}, \quad \widehat{C}_{\Lambda\Gamma} = \frac{\partial x_\alpha}{\partial X_\Lambda}\frac{\partial x_\alpha}{\partial X_\Gamma}, \quad \widehat{\Lambda}_{\Lambda\Gamma} = \frac{\partial x_\alpha}{\partial X_\Lambda}\frac{\partial b_\alpha}{\partial X_\Gamma}, \quad \widehat{\mathbf{A}} = (A_1, A_2)^T. \tag{7.2}$$

The constitutive equations (5.38) reduce to

$$\sigma_{(\alpha\beta)} = \frac{\rho}{\rho_0}\left\{\widehat{F}_{\alpha\Gamma}\widehat{F}_{\beta\Delta}\left(\frac{\partial W}{\partial \widehat{C}_{\Gamma\Delta}} + \frac{\partial W}{\partial \widehat{C}_{\Delta\Gamma}}\right) + \left(\widehat{F}_{\alpha\Gamma}\widehat{G}_{\beta\Delta} + \widehat{F}_{\beta\Gamma}\widehat{G}_{\alpha\Delta}\right)\frac{\partial W}{\partial \widehat{\Lambda}_{\Gamma\Delta}}\right\},$$
$$m_{\alpha 3} = \frac{2}{3}\epsilon_{\beta\gamma}\frac{\rho}{\rho_0}\frac{\partial W}{\partial \widehat{\Lambda}_{\Delta\Gamma}}\widehat{F}_{\gamma\Delta}\left(\widehat{F}_{\alpha\Gamma}\widehat{b}_\beta + \widehat{F}_{\beta\Gamma}\widehat{b}_\alpha\right),$$
$$m_{3\alpha} = 0, \quad m_{12} = 0, \quad m_{21} = 0, \quad \overline{m}_{11} = 0, \quad \overline{m}_{22} = 0, \quad \overline{m}_{33} = 0, \tag{7.3}$$

(and hence $m_{11} = m_{22} = m_{33}$) where

$$\epsilon_{\alpha\beta} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{7.4}$$

In direct notation, $(7.3)_{1,2}$ may be written as

$$\frac{1}{2}\left(\sigma + \sigma^T\right) = \frac{\rho}{\rho_0}\left\{\widehat{\mathbf{F}}\left(\frac{\partial W}{\partial \widehat{\mathbf{C}}} + \frac{\partial W}{\partial \widehat{\mathbf{C}}^T}\right)\widehat{\mathbf{F}}^T + \widehat{\mathbf{F}}\frac{\partial W}{\partial \widehat{\mathbf{\Lambda}}}\widehat{\mathbf{G}}^T + \widehat{\mathbf{G}}\frac{\partial W}{\partial \widehat{\mathbf{\Lambda}}^T}\widehat{\mathbf{F}}^T\right\}, \tag{7.5}$$

$$\widehat{\mathbf{m}} = \frac{2\rho}{3\rho_0}\left\{\left[\mathrm{tr}\ \epsilon\widehat{\mathbf{F}}\frac{\partial W}{\partial \widehat{\mathbf{\Lambda}}}\widehat{\mathbf{F}}^T\right]\widehat{\mathbf{b}} - \left[\widehat{\mathbf{F}}\frac{\partial W}{\partial \widehat{\mathbf{\Lambda}}^T}\widehat{\mathbf{F}}^T\right]\left(\epsilon\widehat{\mathbf{b}}\right)\right\}, \tag{7.6}$$

where

$$\widehat{\mathbf{m}} = \begin{bmatrix} m_{13} \\ m_{23} \end{bmatrix} = \begin{bmatrix} \overline{m}_{13} \\ \overline{m}_{23} \end{bmatrix}, \quad \epsilon = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \tag{7.7}$$

$\partial W / \partial\widehat{\mathbf{C}}$ is the tensor whose (Δ, Γ) component is $\partial W / \partial \widehat{C}_{\Delta\Gamma}$, and similarly for $\partial W / \partial\widehat{\mathbf{\Lambda}}$.

It follows that now W is an isotropic invariant function of $\widehat{\mathbf{C}}$,$\widehat{\mathbf{\Lambda}}$,$\widehat{\mathbf{A}}$, or, if dependence is only on the fibre curvature,

$$W = W(\widehat{\mathbf{C}},\widehat{\mathbf{K}},\widehat{\mathbf{A}}), \quad \widehat{\mathbf{K}}=\widehat{\mathbf{\Lambda}}\widehat{\mathbf{A}}. \tag{7.8}$$

In the latter case, from tables of invariants (for example, Zheng [8, Table 4] it follows that W can be expressed as a function of

$$\begin{aligned}
&\widehat{I}_1 = \text{tr } \widehat{\mathbf{C}}, \quad \widehat{I}_2 = \det \widehat{\mathbf{C}}, \quad \widehat{I}_3 = \widehat{\mathbf{A}}\widehat{\mathbf{C}}\widehat{\mathbf{A}}, \quad \widehat{I}_4 = \widehat{\mathbf{K}} \bullet \widehat{\mathbf{K}} = \widehat{\mathbf{A}}\widehat{\mathbf{\Lambda}}^T\widehat{\mathbf{\Lambda}}\widehat{\mathbf{A}}, \\
&\widetilde{I}_5 = \widehat{\mathbf{K}}\widehat{\mathbf{C}}\widehat{\mathbf{K}} = \widehat{\mathbf{A}}\widehat{\mathbf{\Lambda}}^T\widehat{\mathbf{C}}\widehat{\mathbf{\Lambda}}\widehat{\mathbf{A}}, \quad K_1 = \widehat{\mathbf{A}} \bullet \widehat{\mathbf{K}} = \widehat{\mathbf{A}}\widehat{\mathbf{\Lambda}}\widehat{\mathbf{A}}, \quad K_2 = \widehat{\mathbf{A}}\widehat{\mathbf{C}}\widehat{\mathbf{K}} = \widehat{\mathbf{A}}\widehat{\mathbf{C}}\widehat{\mathbf{\Lambda}}\widehat{\mathbf{A}}.
\end{aligned} \tag{7..9}$$

However, it can be shown by the generalized Cayley-Hamilton theorem in two dimensions that $\widetilde{I}_5$ can be expressed in terms of $\widehat{I}_1, \widehat{I}_2, \widehat{I}_3, \widehat{I}_4, K_1$ and K_2, and so $\widetilde{I}_5$ may be omitted from the list. Moreover, if W is required to be even in $\widehat{\mathbf{\Lambda}}$, then K_1 and K_2 can appear only through their squares and product, and therefore W is expressible as a function of

$$\widehat{I}_1, \quad \widehat{I}_2, \quad \widehat{I}_3, \quad \widehat{I}_4, \quad \widehat{I}_5 = K_1^2, \quad \widehat{I}_6 = K_2^2, \quad \widehat{I}_7 = K_1 K_2. \tag{7.10}$$

From (7.9),

$$\begin{aligned}
&\partial\widehat{I}_1/\partial\widehat{\mathbf{C}} = \mathbf{I}, \quad \partial\widehat{I}_2/\partial\widehat{\mathbf{C}} = -\widehat{\mathbf{C}} + \widehat{I}_1\mathbf{I}, \quad \partial\widehat{I}_3/\partial\widehat{\mathbf{C}} = \widehat{\mathbf{A}}\otimes\widehat{\mathbf{A}}, \quad \partial\widehat{I}_4/\partial\widehat{\mathbf{C}} = \mathbf{0}, \\
&\partial K_1/\partial\widehat{\mathbf{C}} = \mathbf{0}, \quad \partial K_2/\partial\widehat{\mathbf{C}} = \widehat{\mathbf{A}} \otimes \left(\widehat{\mathbf{\Lambda}}\widehat{\mathbf{A}}\right), \\
&\partial\widehat{I}_1/\partial\widehat{\mathbf{\Lambda}} = \mathbf{0}, \quad \partial\widehat{I}_2/\partial\widehat{\mathbf{\Lambda}} = \mathbf{0}, \quad \partial\widehat{I}_3/\partial\widehat{\mathbf{\Lambda}} = \mathbf{0}, \quad \partial\widehat{I}_4/\partial\widehat{\mathbf{\Lambda}} = 2\left(\widehat{\mathbf{\Lambda}}\widehat{\mathbf{A}}\right) \otimes \widehat{\mathbf{A}}, \\
&\partial K_1/\partial\widehat{\mathbf{\Lambda}} = \widehat{\mathbf{A}}\otimes\widehat{\mathbf{A}}, \quad \partial K_2/\partial\widehat{\mathbf{\Lambda}} = \left(\widehat{\mathbf{C}}\widehat{\mathbf{A}}\right)\otimes\widehat{\mathbf{A}}.
\end{aligned} \tag{7.11}$$

It follows that

$$\begin{aligned}
\frac{\partial W}{\partial \widehat{\mathbf{C}}} &= \sum_{\alpha=1}^{7} W_\alpha \frac{\partial \widehat{I}_\alpha}{\partial \widehat{\mathbf{C}}} = \left(W_1 + \widehat{I}_1 W_2\right)\mathbf{I} - W_2\widehat{\mathbf{C}} + W_3\widehat{\mathbf{A}}\otimes\widehat{\mathbf{A}} + (2K_2W_6 + K_1W_7)\,\widehat{\mathbf{A}} \otimes \left(\widehat{\mathbf{\Lambda}}\widehat{\mathbf{A}}\right), \\
\frac{\partial W}{\partial \widehat{\mathbf{\Lambda}}} &= \sum_{\alpha=1}^{7} W_\alpha \frac{\partial \widehat{I}_\alpha}{\partial \widehat{\mathbf{\Lambda}}} = 2W_4\left(\widehat{\mathbf{\Lambda}}\widehat{\mathbf{A}}\right) \otimes \widehat{\mathbf{A}} + (2K_1W_5 + K_2W_7)\widehat{\mathbf{A}}\otimes\widehat{\mathbf{A}} \\
&\quad + (2K_2W_6 + K_1W_7)\left(\widehat{\mathbf{C}}\widehat{\mathbf{A}}\right) \otimes \widehat{\mathbf{A}},
\end{aligned} \tag{7.12}$$

where W_α denotes $\partial W/\partial \widehat{I}_a$. Hence, from (7.5) and (7.6)

$$\begin{aligned}
\frac{1}{2}\left(\sigma + \sigma^T\right) = \frac{\rho}{\rho_0}[2&\left(W_1 + \widehat{I}_1 W_2\right)\widehat{\mathbf{B}} - 2W_2\widehat{\mathbf{B}}^2 + 2W_3\widehat{\mathbf{b}}\otimes\widehat{\mathbf{b}} + 2W_4\left\{\widehat{\kappa} \otimes \left(\widehat{\mathbf{B}}\widehat{\kappa}\right) + \left(\widehat{\mathbf{B}}\widehat{\kappa}\right) \otimes \widehat{\kappa}\right\} \\
&+ (2K_1W_5 + K_2W_7)\left(\widehat{\mathbf{b}} \otimes \widehat{\kappa} + \widehat{\kappa} \otimes \widehat{\mathbf{b}}\right) \\
&+ (2K_2W_6 + K_1W_7)\left\{\widehat{\mathbf{b}} \otimes \left(\widehat{\mathbf{B}}\widehat{\kappa}\right) + \left(\widehat{\mathbf{B}}\widehat{\kappa}\right) \otimes \widehat{\mathbf{b}} + \left(\widehat{\mathbf{B}}\widehat{\mathbf{b}}\right) \otimes \widehat{\kappa} + \widehat{\kappa} \otimes \left(\widehat{\mathbf{B}}\widehat{\mathbf{b}}\right)\right\}],
\end{aligned} \tag{7.13}$$

$$\widehat{\mathbf{m}} = \frac{2}{3}\frac{\rho}{\rho_0}[2W_4\left\{\mathrm{tr}\ \left[\left(\epsilon\widehat{\mathbf{B}}\widehat{\kappa}\right)\otimes\widehat{\mathbf{b}}\right]\ \widehat{\mathbf{b}} - \left[\widehat{\mathbf{b}}\otimes\left(\widehat{\mathbf{B}}\widehat{\kappa}\right)\right]\epsilon\widehat{\mathbf{b}}\right\}$$
$$+ (2K_2W_6 + K_1W_7)\left\{\mathrm{tr}\ \left[\left(\epsilon\widehat{\mathbf{B}}\widehat{\mathbf{b}}\right)\otimes\widehat{\mathbf{b}}\right]\ \widehat{\mathbf{b}} - \left[\widehat{\mathbf{b}}\otimes\left(\widehat{\mathbf{B}}\widehat{\mathbf{b}}\right)\right]\epsilon\widehat{\mathbf{b}}\right\}], \tag{7.14}$$

where

$$\widehat{\mathbf{B}} = \widehat{\mathbf{F}}\widehat{\mathbf{F}}^T, \quad \widehat{\mathbf{b}} = \widehat{\mathbf{F}}\widehat{\mathbf{A}}, \quad \widehat{\kappa} = \widehat{\mathbf{G}}\widehat{\mathbf{A}}, \quad \widehat{\kappa}_\alpha = \frac{\partial\widehat{b}_\alpha}{\partial X_\Gamma}A_\Gamma = \frac{\partial\widehat{b}_\alpha}{\partial x_\beta}\frac{\partial x_\beta}{\partial X_\Gamma}A_\gamma = \frac{\partial\widehat{b}_\alpha}{\partial x_\beta}\widehat{b}_\beta. \tag{7.15}$$

The first three terms in (7.13) form the usual expression for the symmetric Cauchy stress in a transversely isotropic elastic solid in plane strain. The remaining terms in (7.13) arise from the bending stiffness. To within a length scaling factor, $\widehat{\kappa}$ represents the fibre curvature. We also note that

$$\widehat{I}_4 = \widehat{\kappa}\widehat{\mathbf{B}}\widehat{\kappa}, \quad K_1 = \widehat{\mathbf{b}}\bullet\widehat{\kappa}, \quad K_2 = \widehat{\mathbf{b}}\widehat{\mathbf{B}}\widehat{\kappa}. \tag{7.16}$$

By using the Cayley-Hamilton theorem for $\widehat{\mathbf{B}}$, (7.13) may be written as

$$\frac{1}{2}\left(\sigma+\sigma^T\right) = \frac{\rho}{\rho_0}[-2W_2I_2\mathbf{I} + 2W_1\widehat{\mathbf{B}} + 2W_3\widehat{\mathbf{b}}\otimes\widehat{\mathbf{b}} + 2W_4\left\{\widehat{\kappa}\otimes\left(\widehat{\mathbf{B}}\widehat{\kappa}\right) + \left(\widehat{\mathbf{B}}\widehat{\kappa}\right)\otimes\widehat{\kappa}\right\}$$
$$+ (2K_1W_5 + K_2W_7)\left(\widehat{\mathbf{b}}\otimes\widehat{\kappa} + \widehat{\kappa}\otimes\widehat{\mathbf{b}}\right)$$
$$+ (2K_2W_6 + K_1W_7)\left\{\widehat{\mathbf{b}}\otimes\widehat{\mathbf{B}}\widehat{\kappa} + \widehat{\mathbf{B}}\widehat{\kappa}\otimes\widehat{\mathbf{b}} + \widehat{\mathbf{B}}\widehat{\mathbf{b}}\otimes\widehat{\kappa} + \widehat{\kappa}\otimes\widehat{\mathbf{B}}\widehat{\mathbf{b}}\right\}], \tag{7.17}$$

or, in the case of an incompressible material

$$\frac{1}{2}\left(\sigma+\sigma^T\right) = -p\mathbf{I} + 2W_1\widehat{\mathbf{B}} + 2W_3\widehat{\mathbf{b}}\otimes\widehat{\mathbf{b}} + 2W_4\left\{\widehat{\kappa}\otimes\left(\widehat{\mathbf{B}}\widehat{\kappa}\right) + \left(\widehat{\mathbf{B}}\widehat{\kappa}\right)\otimes\widehat{\kappa}\right\}$$
$$+ (2K_1W_5 + K_2W_7)\left(\widehat{\mathbf{b}}\otimes\widehat{\kappa} + \widehat{\kappa}\otimes\widehat{\mathbf{b}}\right)$$
$$+ (2K_2W_6 + K_1W_7)\left\{\widehat{\mathbf{b}}\otimes\left(\widehat{\mathbf{B}}\widehat{\kappa}\right) + \left(\widehat{\mathbf{B}}\widehat{\kappa}\right)\otimes\widehat{\mathbf{b}} + \left(\widehat{\mathbf{B}}\widehat{\mathbf{b}}\right)\otimes\widehat{\kappa} + \widehat{\kappa}\otimes\left(\widehat{\mathbf{B}}\widehat{\mathbf{b}}\right)\right\}, \tag{7.18}$$

where p represents a hydrostatic pressure.

The antisymmetric part of the stress, that in the presence of a couple-stress is required to preserve local moment equilibrium, is given by the equations of equilibrium. If it is assumed that in plane strain $m_{\alpha\beta} = 0$, and that $m_{3\alpha}$ and m_{33} are independent of x_3, these reduce to

$$\sigma_{21} - \sigma_{12} = \frac{\partial m_{13}}{\partial x_1} + \frac{\partial m_{23}}{\partial x_2}. \tag{7.19}$$

8. Pure bending

As an example, consider the problem of pure bending in plane strain, in which a rectangular slab of incompressible material is bent into a sector of a circular cylinder. For finite deformations of an isotropic elastic material, this is one of the classic solutions obtained by Rivlin [16,17], and the extension to transversely isotropic material is described in Green and Adkins [5, Chapter 2]. Suppose that initially the fibres lie in straight lines parallel to the X_2 axis, so that $\widehat{\mathbf{A}} = (0, 1)^T$, and in the deformed body they form concentric circular arcs. Thus a particle at (X_1, X_2) in the reference configuration moves to position (referred to plane polar coordinates) (r, θ) in the deformed configuration, where

$$r = r(X_1), \quad \theta = \theta(X_2). \tag{8.1}$$

In (r, θ) coordinates

$$\widehat{\mathbf{F}} = \begin{bmatrix} r'(X_1) & 0 \\ 0 & r\theta'(X_2) \end{bmatrix}, \tag{8.2}$$

where primes denote derivatives with respect to the stated arguments. The incompressibility condition gives

$$\det \widehat{\mathbf{F}} = r'(X_1) r\theta'(X_2) = r\frac{dr}{dX_1}\frac{d\theta}{dX_2} = 1. \tag{8.3}$$

The relevant solution is

$$r\frac{dr}{dX_1} = \lambda, \quad \frac{d\theta}{dX_2} = \frac{1}{\lambda}, \tag{8.4}$$

where λ is constant. Hence

$$r^2 - a^2 = 2\lambda X_1, \quad \theta = \frac{X_2}{\lambda}, \tag{8.5}$$

where the plane $X_2 = 0$ becomes the plane $\theta = 0$, and the plane $X_1 = 0$ deforms into the circular cylindrical surface $r = a$. If initially the block is bounded by planes $X_1 = 0$ and $X_1 = B$, then the surface $X_1 = B$ deforms into the circular cylindrical surface $r = b$, where

$$b^2 - a^2 = 2\lambda B. \tag{8.6}$$

We now have

$$\widehat{\mathbf{F}} = \begin{bmatrix} \lambda r^{-1} & 0 \\ 0 & \lambda^{-1} r \end{bmatrix}. \tag{8.7}$$

Since $\widehat{\mathbf{A}} = (0, 1)^T$, it follows that, in plane polar coordinates

$$\widehat{\mathbf{b}} = \begin{bmatrix} 0 \\ \lambda^{-1} r \end{bmatrix}, \quad \widehat{\kappa} = \begin{bmatrix} -r\lambda^{-2} \\ 0 \end{bmatrix},$$
$$\widehat{\mathbf{B}} = \begin{bmatrix} \lambda^2 r^{-2} & 0 \\ 0 & \lambda^{-2} r^2 \end{bmatrix}, \quad \widehat{\mathbf{G}} = \begin{bmatrix} 0 & -\lambda^{-2} r \\ r^{-1} & 0 \end{bmatrix}, \quad \widehat{\mathbf{\Lambda}} = \begin{bmatrix} 0 & -\lambda^{-1} \\ \lambda^{-1} & 0 \end{bmatrix},$$
$$\widehat{I}_1 = \lambda^2 r^{-2} + \lambda^{-2} r^2, \qquad \widehat{I}_2 = 1, \quad \widehat{I}_3 = \lambda^{-2} r^2, \quad \widehat{I}_4 = \lambda^{-2}, \quad K_1 = 0, \quad K_2 = 0. \tag{8.8}$$

It follows from (7.13) that

$$\begin{aligned} \sigma_{rr} &= -p + 2W_1 \lambda^2 r^{-2} + 4W_4 \lambda^{-2}, \\ \sigma_{\theta\theta} &= -p + 2(W_1 + W_3)\lambda^{-2} r^2 - 4W_4 \lambda^{-2}, \\ \sigma_{r\theta} + \sigma_{\theta r} &= 0. \end{aligned} \tag{8.9}$$

If $W_4 = 0$, this reduces to the symmetric stress for bending of a transversely isotropic elastic material without bending stiffness, as given (in different notation) in Green and Adkins [5]. For the material with bending stiffness, (8.9) includes the additional terms in W_4, and (7.14) gives the couple stress components

$$m_{r3} = 0, \quad m_{\theta 3} = \frac{8}{3} W_4 \lambda^{-2} r. \tag{8.10}$$

We assume that in plane strain deformations m_{33} is independent of x_3. Then the antisymmetric part of the stress follows from the equilibrium equation (5.1) as

$$\sigma_{\theta r} - \sigma_{r\theta} = \frac{\partial m_{\theta 3}}{r \partial \theta} + \frac{\partial m_{r3}}{\partial r} = 0, \tag{8.11}$$

so for this deformation the stress is symmetric. This symmetry occurs because

in pure bending the deformation and stress are independent of θ. This result can be likened to the situation in classical beam theory of pure bending of an Euler-Bernoulli beam, in which the shear force (analogous to $\sigma_{\theta r} - \sigma_{r\theta}$) is zero when the bending moment (analogous to $m_{\theta 3}$) is constant along the beam. We note that $m_{\theta 3}$ is proportional to the magnitude of κ, which is a measure of the curvature of the fibres. The bending moment M, and normal force N, per unit length in the X_3 direction, and applied to a section $\theta = const.$, are

$$M = \int_a^b (r\sigma_{\theta\theta} + m_{\theta 3})\, dr, \quad N = \int_a^b \sigma_{\theta\theta} dr. \tag{8.12}$$

The tractions on the curved surfaces of the block can be made zero by appropriate choices of p and λ.

9. Small displacement gradients - linearized theory

We define the displacement as $\mathbf{u} = \mathbf{x} - \mathbf{X}$. Then

$$F_{iR} = \delta_{iR} + \frac{\partial u_i}{\partial X_R}, \tag{9.1}$$

and so

$$b_i = \left(\delta_{iR} + \frac{\partial u_i}{\partial X_R}\right) A_R = A_i + \frac{\partial u_i}{\partial X_R} A_R. \tag{9.2}$$

For simplicity we assume that initially the fibres are straight, so that $\mathbf{A}$ is constant. Then

$$G_{iR} = \frac{\partial b_i}{\partial X_R} = \frac{\partial^2 u_i}{\partial X_R \partial X_P} A_P, \quad \Lambda_{RS} = \frac{\partial x_i}{\partial X_R}\frac{\partial b_i}{\partial X_S} = \left(\delta_{iR} + \frac{\partial u_i}{\partial X_R}\right)\frac{\partial^2 u_i}{\partial X_S \partial X_P} A_P.$$

To formulate the linear theory, we suppose that all partial derivatives of u_i exist and are of order of magnitude $O(e)$, where e is much smaller than one. Then from (9.2)

$$b_i - A_i = O(e).$$

and it follows that the gradients $\partial b_i / \partial X_R$ are $O(e)$.

We consider that W depends on the 33 invariants $I_1 - I_{33}$ listed in (A2) in the Appendix, but for the linear theory we suppose W to be quadratic in the derivatives of u_i, so that we disregard terms in W that are $O(e^3)$ or of higher order. For the purposes of this section it is convenient to replace the invariants $I_1 - I_5$ by an equivalent set $J_1 - J_5$, where

$$\begin{aligned} &J_1 = I_1 - 3, \quad J_2 = I_2^2 - 2I_1 - 2I_2 + 3, \\ &J_3 = I_1^3 - 3I_1^2 - 3I_1 I_2 + 3I_1 + 3I_3 + 6I_2 - 3, \quad J_4 = I_4 - 1, \quad J_5 = I_5 - 2I_4 + 1. \end{aligned} \tag{9.3}$$

By inserting (9.1) into (3.4) and (9.3) and then retaining only the lowest order terms, this set of modified invariants reduces to

$$\begin{aligned} &J_1 = 2\frac{\partial u_R}{\partial X_R} + O\left(e^2\right), \quad J_2 = 2\left(\frac{\partial u_R}{\partial X_S}\frac{\partial u_S}{\partial X_R} + \frac{\partial u_R}{\partial X_S}\frac{\partial u_R}{\partial X_S}\right) + O(e^3), \quad J_3 = O(e^3), \\ &J_4 = 2\frac{\partial u_R}{\partial X_S} A_R A_S + O\left(e^2\right), \quad J_5 = 2\left(\frac{\partial u_R}{\partial X_P}\frac{\partial u_S}{\partial X_P} + 2\frac{\partial u_R}{\partial X_P}\frac{\partial u_P}{\partial X_S} + \frac{\partial u_P}{\partial X_R}\frac{\partial u_P}{\partial X_S}\right) A_R A_S + O(e^3), \end{aligned} \tag{9.4}$$

so that of these only J_1, J_2, J_4, J_5 can contribute to a first-order elasticity theory. Similarly, not all of the remaining invariants (A2) can contribute towards the development of a first-order elasticity theory, because we may discard those with order of magnitude higher than $O(e^2)$. From (A2), (9.1) and (9.2) we find that

$$
\begin{aligned}
I_6 = I_{10} = I_{11} &= \frac{\partial^2 u_R}{\partial X_R \partial X_S} A_S + O(e^2),\\
I_7 = I_{12} = I_{13} &= \frac{1}{2}\frac{\partial^2 u_R}{\partial X_S \partial X_Q}\left(\frac{\partial^2 u_R}{\partial X_S \partial X_P} + \frac{\partial^2 u_S}{\partial X_R \partial X_P}\right) A_P A_Q + O\left(e^3\right),\\
I_8 = I_{14} = I_{15} &= \frac{1}{2}\frac{\partial^2 u_R}{\partial X_S \partial X_P}\left(\frac{\partial^2 u_R}{\partial X_S \partial X_Q} - \frac{\partial^2 u_S}{\partial X_R \partial X_Q}\right) A_P A_Q + O\left(e^3\right),\\
I_9 = I_{16} = I_{17} &= I_{26} = I_{28} = I_{30} = I_{31} = I_{32} = O(e^3),\\
I_{18} = I_{29} = I_{33} &= O(e^4),\\
I_{19} = O(e^6), &\quad I_{20} = I_{23} = \frac{\partial^2 u_R}{\partial X_S \partial X_P} A_R A_S A_P + O(e^2),\\
I_{21} + I_{22} &= \frac{\partial^2 u_R}{\partial X_S \partial X_Q}\frac{\partial^2 u_S}{\partial X_P \partial X_M} A_R A_P A_Q A_M + O(e^3),\\
I_{21} - I_{22} &= \frac{1}{2}\left(\frac{\partial^2 u_R}{\partial X_S \partial X_Q}\frac{\partial^2 u_P}{\partial X_S \partial X_N} + \frac{\partial^2 u_S}{\partial X_R \partial X_Q}\frac{\partial^2 u_S}{\partial X_P \partial X_M}\right) A_R A_P A_Q A_M + O(e^3),\\
I_{24} = I_{25} &= \frac{1}{2}\left(\frac{\partial u_R}{\partial X_P} + \frac{\partial u_P}{\partial X_R}\right)\left(\frac{\partial^2 u_P}{\partial X_S \partial X_Q} - \frac{\partial^2 u_S}{\partial X_P \partial X_Q}\right) A_R A_S A_Q + O(e^3),\\
I_{27} &= \frac{1}{4}\left(2\frac{\partial^2 u_R}{\partial X_M \partial X_Q}\frac{\partial^2 u_R}{\partial X_N \partial X_P} - \frac{\partial^2 u_M}{\partial X_R \partial X_Q}\frac{\partial^2 u_N}{\partial X_R \partial X_P}\right) A_P A_Q A_M A_N + O(e^3).
\end{aligned}
\tag{9.5}
$$

It follows that if W is a function of the invariants (A2), a quadratic function of the derivatives of u_i, and even in the components of $\mathbf{A}$, then it can be expressed as

$$
\begin{aligned}
W = {} & \widetilde{a}_1 J_1^2 + \widetilde{a}_2 J_2 + \widetilde{a}_3 J_1 J_4 + \widetilde{a}_4 J_4^2 + \widetilde{a}_5 J_5 + \\
& + \widetilde{b}_1 I_6^2 + \widetilde{b}_2 I_7 + \widetilde{b}_3 I_8 + \widetilde{b}_4 I_{20}^2 + \widetilde{b}_5 I_{21} + \widetilde{b}_6 I_{22} + \widetilde{b}_7 I_{27} + \widetilde{b}_8 I_6 I_{20},
\end{aligned}
\tag{9.6}
$$

where $\widetilde{a}_\alpha$ and $\widetilde{b}_\alpha$ are coefficients. I_{24} and I_{25} are excluded because they are of odd degree in the components of $\mathbf{A}$. The coefficients denoted $\widetilde{a}_\alpha$ are associated with the invariants and products of invariants met in linear transverse isotropic elasticity theory, while those denoted $\widetilde{b}_\alpha$ are associated with invariants and products of invariants related to fibre resistance in bending.

Constitutive equations. For convenience we rename the invariants in (9.6) as

$$
(J_1, J_2, J_4, J_5, I_6, I_7, I_8, I_{20}, I_{21}, I_{22}, I_{27}) = (\widetilde{I}_1, \widetilde{I}_2, \widetilde{I}_3, \widetilde{I}_4, \widetilde{I}_5, \widetilde{I}_6, \widetilde{I}_7, \widetilde{I}_8, \widetilde{I}_9, \widetilde{I}_{10}, \widetilde{I}_{11}),
\tag{9.7}
$$

so that (9.6) becomes

$$W = \widetilde{a}_1\widetilde{I}_1^2 + \widetilde{a}_2\widetilde{I}_2 + \widetilde{a}_3\widetilde{I}_1\widetilde{I}_3 + \widetilde{a}_4 I_3^2 + \widetilde{a}_5\widetilde{I}_4 + \widetilde{b}_1\widetilde{I}_5^2 + \widetilde{b}_2\widetilde{I}_6 + \widetilde{b}_3\widetilde{I}_7 + \widetilde{b}_4\widetilde{I}_8^2 + \widetilde{b}_5\widetilde{I}_9 + \widetilde{b}_6\widetilde{I}_{10} + \widetilde{b}_7\widetilde{I}_{11} + \widetilde{b}_8\widetilde{I}_5\widetilde{I}_8. \tag{9.8}$$

For this form of W, we have $W = O(e^2)$. Since $\rho/\rho_0 = 1 + O(e)$, $F_{iR} = \delta_{iR} + O(e)$, $G_{iR} = O(e)$, it follows from (5.38) that the symmetric stress and couple-stress are now given to $O(e)$ in the linearized theory by

$$\sigma_{(ij)} = \frac{\partial W}{\partial C_{ij}} + \frac{\partial W}{\partial C_{ji}} = \sum_{n=1}^{4} \frac{\partial W}{\partial \widetilde{I}_n}\left(\frac{\partial \widetilde{I}_n}{\partial C_{ij}} + \frac{\partial \widetilde{I}_n}{\partial C_{ji}}\right),$$

$$m_{ji} - \frac{1}{3}m_{kk}\delta_{ij} = \frac{2}{3}e_{ikm}\left(\frac{\partial W}{\partial \Lambda_{mj}}b_k + \frac{\partial W}{\partial \Lambda_{mk}}b_j\right) = \frac{2}{3}e_{ikm}\sum_{n=5}^{11}\frac{\partial W}{\partial \widetilde{I}_n}\left(\frac{\partial \widetilde{I}_n}{\partial \Lambda_{mj}}b_k + \frac{\partial \widetilde{I}_n}{\partial \Lambda_{mk}}b_j\right). \tag{9.9}$$

Moreover

$$\frac{\partial \widetilde{I}_1}{\partial C_{ij}} = \delta_{ij}, \quad \frac{\partial \widetilde{I}_2}{\partial C_{ij}} = C_{ij} = \delta_{ij} + \frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} + O(e^2), \quad \frac{\partial \widetilde{I}_3}{\partial C_{ij}} = A_i A_j,$$

$$\frac{\partial \widetilde{I}_4}{\partial C_{ij}} = A_i C_{jk} A_k = \delta_{ij} + A_i A_k \frac{\partial u_j}{\partial X_k} + O(e^2), \quad \frac{\partial \widetilde{I}_5}{\partial C_{ij}} = 0,$$

$$\frac{\partial \widetilde{I}_6}{\partial C_{ij}} = 0, \quad \frac{\partial \widetilde{I}_7}{\partial C_{ij}} = 0, \quad \frac{\partial \widetilde{I}_8}{\partial C_{ij}} = 0, \quad \frac{\partial \widetilde{I}_9}{\partial C_{ij}} = 0, \quad \frac{\partial \widetilde{I}_{10}}{\partial C_{ij}} = 0, \quad \frac{\partial \widetilde{I}_{11}}{\partial C_{ij}} = 0,$$

$$\frac{\partial \widetilde{I}_1}{\partial \Lambda_{kj}} = 0, \quad \frac{\partial \widetilde{I}_2}{\partial \Lambda_{kj}} = 0, \quad \frac{\partial \widetilde{I}_3}{\partial \Lambda_{kj}} = 0, \quad \frac{\partial \widetilde{I}_4}{\partial \Lambda_{kj}} = 0,$$

$$\frac{\partial \widetilde{I}_5}{\partial \Lambda_{kj}} = \delta_{kj}, \quad \frac{\partial \widetilde{I}_6}{\partial \Lambda_{kj}} = \Lambda_{kj} + \Lambda_{jk} = \left(\frac{\partial^2 u_j}{\partial X_k \partial X_l} + \frac{\partial^2 u_k}{\partial X_j \partial X_l}\right) A_l + O(e^2),$$

$$\frac{\partial \widetilde{I}_7}{\partial \Lambda_{kj}} = \Lambda_{kj} - \Lambda_{jk} = -\left(\frac{\partial^2 u_j}{\partial X_k \partial X_l} - \frac{\partial^2 u_k}{\partial X_j \partial X_l}\right) A_l + O(e^2), \quad \frac{\partial \widetilde{I}_8}{\partial \Lambda_{kj}} = A_k A_j,$$

$$\frac{\partial \widetilde{I}_9}{\partial \Lambda_{kj}} = \frac{1}{2}\left[A_k(\Lambda_{jm} + \Lambda_{mj}) + A_j(\Lambda_{km} + \Lambda_{mk})\right] A_m = \left(\frac{\partial e_{kq}}{\partial X_m}A_j + \frac{\partial e_{jq}}{\partial X_m}A_k\right) A_q A_m + O\left(e^2\right),$$

$$\begin{aligned}\frac{\partial \widetilde{I}_{10}}{\partial \Lambda_{kj}} &= \frac{1}{2}\left[A_k(\Lambda_{jm} - \Lambda_{mj}) - A_j(\Lambda_{km} - \Lambda_{mk})\right] A_m \\ &= \frac{1}{2}\left[\left(\frac{\partial^2 u_j}{\partial X_q \partial X_m} - \frac{\partial^2 u_q}{\partial X_j \partial X_m}\right) A_k - \left(\frac{\partial^2 u_k}{\partial X_q \partial X_m} - \frac{\partial^2 u_q}{\partial X_k \partial X_m}\right) A_j\right] A_q A_m + O(e^2),\end{aligned}$$

$$\begin{aligned}\frac{\partial \widetilde{I}_{11}}{\partial \Lambda_{kj}} &= \frac{1}{2}\left[A_k(\Lambda_{jm} - \Lambda_{mj}) + A_j(\Lambda_{km} - \Lambda_{mk})\right] A_m \\ &= \frac{1}{2}\left(\frac{\partial^2 u_k}{\partial X_q \partial X_m}A_j - \frac{\partial^2 u_q}{\partial X_j \partial X_m}A_k\right) A_q A_m + O\left(e^2\right).\end{aligned} \tag{9.10}$$

By inserting (9.8) and (9.10) in (9.9), and retaining only leading order terms, we obtain the linearized constitutive equations in the form

$$\sigma_{(ij)} = 4(2\widetilde{a}_1 e_{RR} + \widetilde{a}_3 e_{RQ} A_R A_Q)\delta_{ij} + 8\widetilde{a}_2 e_{ij} + 2(2\widetilde{a}_3 e_{RR} + 4\widetilde{a}_4 e_{RQ} A_R A_Q) A_i A_j + 4\widetilde{a}_5 (e_{ik} A_k A_j + e_{jk} A_k A_i), \tag{9.11}$$

$$\begin{aligned} m_{ji} - \frac{1}{3} m_{kk}\delta_{ij} = & -\frac{2}{3} e_{ijM}\left(2\widetilde{b}_1 \frac{\partial^2 u_R}{\partial X_R \partial X_Q} + \widetilde{b}_8 \frac{\partial^2 u_P}{\partial X_N \partial X_Q} A_P A_N\right) A_Q A_M \\ & - \frac{2}{3} e_{ikM}(\widetilde{b}_2 - \widetilde{b}_3)\left(\frac{\partial^2 u_k}{\partial X_j \partial X_Q} A_Q A_M + \frac{\partial^2 u_k}{\partial X_M \partial X_Q} A_Q A_j\right) \\ & - \frac{2}{3} e_{ikM}(\widetilde{b}_2 + \widetilde{b}_3)\left(\frac{\partial^2 u_j}{\partial X_k \partial X_Q} A_Q A_M + \frac{\partial^2 u_M}{\partial X_k \partial X_Q} A_Q A_j\right) \\ & - \frac{2}{3} e_{ikM}\left((\widetilde{b}_5 - \widetilde{b}_6 + \widetilde{b}_7)\frac{\partial^2 u_k}{\partial X_P \partial X_Q} + (\widetilde{b}_5 + \widetilde{b}_6)\frac{\partial^2 u_Q}{\partial X_k \partial X_P}\right) A_Q A_P A_j A_M \\ & - \frac{1}{3} e_{ikM}\left((\widetilde{b}_5 - \widetilde{b}_6 - \widetilde{b}_7)\frac{\partial^2 u_Q}{\partial X_M \partial X_P} + (\widetilde{b}_5 + \widetilde{b}_6)\frac{\partial^2 u_M}{\partial X_Q \partial X_P}\right) A_Q A_P A_j A_k, \end{aligned} \tag{9.12}$$

where e_{ij} denote the components of the infinitesimal strain tensor

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right). \tag{9.13}$$

The expressions (9.11) are essentially the constitutive equations of transverse isotropic linear elasticity. Comparison with the corresponding constitutive equations (6.44) of [9] gives

$$8\widetilde{a}_1 = \lambda, \quad 4\widetilde{a}_2 = \mu_T, \quad 4\widetilde{a}_3 = \alpha, \quad 8\widetilde{a}_4 = \beta, \quad 2\widetilde{a}_5 = \mu_L - \mu_T, \tag{9.14}$$

where the moduli appearing in the right hand sides are elastic moduli employed in transverse isotropic linear elasticity. It is of interest that the constitutive equation for $\sigma_{(ij)}$ involves only the coefficients $\widetilde{a}_\alpha$, and that for m_{ij} involves only the coefficients $\widetilde{b}_\alpha$, so that in the linear theory the constitutive equations for $\sigma_{(ij)}$ and m_{ij} (and hence for $\sigma_{[ij]}$) are uncoupled. However there is coupling through the equilibrium equations, which involve both the symmetric and the anti-symmetric parts of the stress tensor. We also note that the coefficient $\widetilde{b}_4$ does not appear in (9.12), because by the properties of the alternating tensor

$$e_{ikm} F_{jR} F_{kS} \frac{\partial W}{\partial(\widetilde{I}_8^2)} \frac{\partial(\widetilde{I}_8^2)}{\partial \Lambda_{SR}} b_m = 2 e_{ikm} F_{jR} F_{kS} \widetilde{I}_8 \frac{\partial W}{\partial(\widetilde{I}_8^2)} A_S A_R b_m = 2 e_{ikm} \widetilde{I}_8 \frac{\partial W}{\partial(\widetilde{I}_8^2)} b_j b_k b_m = 0.$$

This result obtains for finite as well as for infinitesimal deformations. To leading order in e, the invariant $\widetilde{I}_8$ represents the directional derivative in the fibre direction of the fibre stretch.

Fibres aligned along the x_1-direction. If the family of the straight fibres is aligned along the x_1-direction, then,

$$\mathbf{A} = (1,0,0)^T, \tag{9.15}$$

and the constitutive equations (9.11) and (9.12) reduce to

$$\begin{aligned}
\sigma_{(ij)} &= 4(2\widetilde{a}_1 e_{rr} + \widetilde{a}_3 e_{11})\delta_{ij} + 8\widetilde{a}_2 e_{ij} + 4(\widetilde{a}_3 e_{rr} + 2\widetilde{a}_4 e_{11})\delta_{1i}\delta_{1j} + 4\widetilde{a}_5(e_{i1}\delta_{j1} + e_{j1}\delta_{i1}),\\
m_{ji} - \frac{1}{3}m_{kk}\delta_{ij} &= -\frac{2}{3}e_{ij1}\left[2\widetilde{b}_1\frac{\partial^2 u_R}{\partial X_R \partial X_1} + \widetilde{b}_8\frac{\partial^2 u_1}{\partial X_1^2}\right]\\
&\quad - \frac{2}{3}e_{ik1}\left[(\widetilde{b}_2 - \widetilde{b}_3)\frac{\partial^2 u_k}{\partial X_j \partial X_1} + (\widetilde{b}_2 + \widetilde{b}_3)\frac{\partial^2 u_j}{\partial X_k \partial X_1}\right]\\
&\quad - \frac{2}{3}e_{ikm}\delta_{j1}\left[(\widetilde{b}_2 - \widetilde{b}_3)\frac{\partial^2 u_k}{\partial X_m \partial X_1} + (\widetilde{b}_2 + \widetilde{b}_3)\frac{\partial^2 u_m}{\partial X_k \partial X_1}\right]\\
&\quad - \frac{1}{3}e_{ik1}\delta_{j1}\left[\left(\widetilde{b}_5 - 3\widetilde{b}_6 + 2\widetilde{b}_7\right)\frac{\partial^2 u_k}{\partial X_1^2} + \left(\widetilde{b}_5 + 3\widetilde{b}_6 + \widetilde{b}_7\right)\frac{\partial^2 u_1}{\partial X_k \partial X_1}\right].
\end{aligned} \tag{9.16}$$

It follows that $m_{21} = m_{31} = 0$, and hence the couple-stress tensor possesses only seven non-zero components. The moment equilibrium equations (5.1_2) reduce to

$$\sigma_{[sr]} = \frac{1}{2}e_{irs}\frac{\partial m_{ji}}{\partial x_j}, \tag{9.17}$$

which can be expressed in the form

$$\begin{aligned}
\sigma_{[sr]} - \frac{1}{2}e_{jrs}\frac{\partial m_{kk}}{\partial X_j} &= \frac{2}{3}\widetilde{b}_3\left(\frac{\partial^3 u_s}{\partial X_1^2 \partial X_r} - \frac{\partial^3 u_r}{\partial X_1^2 \partial X_s}\right)\\
&\quad + \frac{1}{3}(\widetilde{b}_2 - \widetilde{b}_3)\left(\delta_{s1}\frac{\partial^3 u_r}{\partial X_m \partial X_m \partial X_1} - \delta_{r1}\frac{\partial^3 u_s}{\partial X_m \partial X_m \partial X_1}\right)\\
&\quad + \frac{1}{3}(2\widetilde{b}_1 + \widetilde{b}_2 + \widetilde{b}_3)\left(\delta_{s1}\frac{\partial^3 u_m}{\partial X_m \partial X_r \partial X_1} - \delta_{r1}\frac{\partial^3 u_m}{\partial X_m \partial X_s \partial X_1}\right)\\
&\quad + \frac{1}{3}\left(\widetilde{b}_5 - 3\widetilde{b}_6 + 2\widetilde{b}_7\right)\left(\delta_{s1}\frac{\partial^3 u_r}{\partial X_1^3} - \delta_{r1}\frac{\partial^3 u_s}{\partial X_1^3}\right)\\
&\quad + \frac{1}{3}\left(\widetilde{b}_5 + 3\widetilde{b}_6 + \widetilde{b}_7 + \widetilde{b}_8\right)\left(\delta_{s1}\frac{\partial^3 u_1}{\partial X_1^2 \partial X_r} - \delta_{r1}\frac{\partial^3 u_1}{\partial X_1^2 \partial X_s}\right).
\end{aligned} \tag{9.18}$$

It is of interest also to present the constitutive equations (9.16) in their matrix forms,

$$\begin{bmatrix} \sigma_{(11)} \\ \sigma_{(22)} \\ \sigma_{(33)} \\ \sigma_{(23)} \\ \sigma_{(31)} \\ \sigma_{(12)} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{12} & 0 & 0 & 0 \\ c_{12} & c_{22} & c_{23} & 0 & 0 & 0 \\ c_{12} & c_{23} & c_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}(c_{22}-c_{23}) & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{66} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{bmatrix} \begin{bmatrix} e_{11} \\ e_{22} \\ e_{33} \\ 2e_{23} \\ 2e_{31} \\ 2e_{12} \end{bmatrix}, \tag{9.19}$$

$$\begin{bmatrix} 2m_{11} - m_{22} - m_{33} \\ -m_{11} + 2m_{22} - m_{33} \\ -m_{11} - m_{22} + 2m_{33} \end{bmatrix} = \begin{bmatrix} \widetilde{b}_3 & 0 \\ -\frac{1}{2}\widetilde{b}_3 & -\widetilde{b}_2 \\ -\frac{1}{2}\widetilde{b}_3 & \widetilde{b}_2 \end{bmatrix} \begin{bmatrix} \partial^2 u_2/\partial x_3 \partial x_1 - \partial^2 u_3/\partial x_2 \partial x_1 \\ \partial e_{23}/\partial x_1 \end{bmatrix}, \tag{9.20}$$

$$\begin{bmatrix} -m_{32} \\ m_{23} \\ -m_{12} \\ m_{13} \end{bmatrix} = \begin{bmatrix} d_{11} & d_{22} & d_{33} & 0 & 0 & 0 & 0 \\ d_{11} & d_{33} & d_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & d_{31} & d_{13} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & d_{31} & d_{13} \end{bmatrix} \begin{bmatrix} \partial e_{11}/\partial x_1 \\ \partial e_{22}/\partial x_1 \\ \partial e_{33}/\partial x_1 \\ \partial^2 u_3/\partial x_1^2 \\ \partial e_{11}/\partial x_3 \\ \partial^2 u_2/\partial x_1^2 \\ \partial e_{11}/\partial x_2 \end{bmatrix}. \tag{9.21}$$

Here (9.19) is essentially the form of generalized Hooke's law for classical transversely isotropic elasticity and contains five independent elastic moduli c_{ij} which are related to $\widetilde{a}_1....\widetilde{a}_5$.To describe the couple-stress there appear seven additional independent elastic moduli which comprise $\widetilde{b}_2, \widetilde{b}_3$, and five moduli d_{ij} which are related to $\widetilde{b}_1....\widetilde{b}_3$, $\widetilde{b}_5....\widetilde{b}_8$ as

$$d_{11} = \frac{2}{3}\left(2\widetilde{b}_1 + \widetilde{b}_8\right), \quad d_{22} = \frac{4}{3}\left(\widetilde{b}_1 + \widetilde{b}_2\right), \qquad d_{33} = \frac{4}{3}\widetilde{b}_1,$$
$$d_{13} = \frac{1}{3}\left(2\widetilde{b}_2 - 2\widetilde{b}_3 + \widetilde{b}_5 + 3\widetilde{b}_6 + 2\widetilde{b}_7\right), \quad d_{31} = \frac{1}{3}\left(2\widetilde{b}_2 + 2\widetilde{b}_3 + \widetilde{b}_5 - 3\widetilde{b}_6 + 2\widetilde{b}_7\right). \tag{9.22}$$

The form of (9.20) and (9.21) shows that the seven couple-stress components may be split into three groups each one of which interacts independently with a set of strain gradients. The couple-stresses that appear in (9.20) correspond loosely to the so-called "twist" mode met in the mechanics of liquid crystals. We note that in (9.20) the combination $m_{11} + m_{22} + m_{33}$ is indeterminate, as was discussed in Section 5. The first pair of couple stresses in (9.21) correspond to the "splay" mode for liquid crystals, while the second pair correspond to the "bending" mode. Notably, the case of inextensible fibres ($e_{11} = 0$) requires the introduction of only six non-zero curvature strains, and their interpretation then resembles more closely the interpretation of their counterparts in liquid crystal mechanics.

For the linearized version of the restricted theory described in Section 7, in which the strain-energy depends only on the deformation and the fibre curvature, we find that

$$\widetilde{b}_2 = 0, \quad \widetilde{b}_3 = 0, \quad d_{11} = 0, \quad d_{22} = 0, \quad d_{33} = 0, \quad d_{13} = 0, \quad d_{31} = \frac{2}{3}\left(2\widetilde{b}_5 + \widetilde{b}_7\right), \tag{9.23}$$

and only one additional modulus d_{31} is required in this case.

Plane Strain with fibres aligned in the x_1 direction. In the plane strain state

$$u_1 = u_1(x_1, x_2), \quad u_2 = u_2(x_1, x_2), \quad u_3 = 0,$$

with $\mathbf{A} = (1, 0, 0)^T$, the constitutive equations (9.19) for the symmetric part of the stress tensor become

$$\begin{bmatrix} \sigma_{(11)} \\ \sigma_{(22)} \\ \sigma_{(33)} \\ \sigma_{(12)} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & 0 \\ c_{12} & c_{22} & 0 \\ c_{12} & c_{23} & 0 \\ 0 & 0 & c_{66} \end{bmatrix} \begin{bmatrix} e_{11} \\ e_{22} \\ 2e_{12} \end{bmatrix}, \tag{9.24}$$

and, for the antisymmetric part of σ

$$\sigma_{[21]} = c_1 \frac{\partial^2 e_{22}}{\partial x_1 \partial x_2} + c_2 \frac{\partial^2 e_{11}}{\partial x_1 \partial x_2} + c_3 \frac{\partial^3 u_2}{\partial x_1^3}, \tag{9.25}$$

where c_1, c_2, c_3 are related to $\widetilde{b}_1 .... \widetilde{b}_3, \widetilde{b}_5 .... \widetilde{b}_8$, so that the total number of the independent elastic moduli involved in the plane strain version of the theory reduces to eight. Moreover c_{23} appears only in the expression for $\sigma_{(33)}$, which is not involved in the equations of equilibrium. In the plane strain case the indeterminacy in the couple-stress has no effect, because $\partial m_{kk}/\partial x_3$ (required for the derivation of (9.25)) is zero.

The third of the equilibrium equations $(5.1)_1$ is trivially satisfied, while the first and the second of these equations are

$$\begin{aligned} \frac{\partial \sigma_{(11)}}{\partial x_1} + \frac{\partial(\sigma_{(12)} + \sigma_{[21]})}{\partial x_2} &= 0, \\ \frac{\partial(\sigma_{(12)} - \sigma_{[21]})}{\partial x_1} + \frac{\partial \sigma_{(22)}}{\partial x_2} &= 0. \end{aligned} \tag{9.26}$$

Then inserting the constitutive equations (9.24) and (9.25) into the equilibrium equations (9.26) gives the Navier-type partial differential equations

$$\begin{aligned} c_{11} \frac{\partial^2 u_1}{\partial x_1^2} + (c_{12} + c_{66}) \frac{\partial^2 u_2}{\partial x_1 \partial x_2} + c_{66} \frac{\partial^2 u_1}{\partial x_2^2} + c_2 \frac{\partial^4 u_1}{\partial x_1^2 \partial x_2^2} + c_3 \frac{\partial^4 u_2}{\partial x_1^3 \partial x_2} + c_1 \frac{\partial^4 u_2}{\partial x_1 \partial x_2^3} &= 0, \\ c_{66} \frac{\partial^2 u_2}{\partial x_1^2} + (c_{12} + c_{66}) \frac{\partial^2 u_1}{\partial x_1 \partial x_2} + c_{22} \frac{\partial^2 u_2}{\partial x_2^2} - c_1 \frac{\partial^4 u_2}{\partial x_1^2 \partial x_2^2} - c_2 \frac{\partial^4 u_1}{\partial x_1^3 \partial x_2} - c_3 \frac{\partial^4 u_2}{\partial x_1^4} &= 0. \end{aligned} \tag{9.27}$$

We note that these equations admit the usual separable (trigonometric-type) form of solutions for cylindrical bending problems of simply supported rectangular plates.

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Appendix

It is assumed that W is an isotropic invariant of the tensors $\mathbf{C}, \mathbf{\Lambda}$ and the vector $\mathbf{A}$. We denote by Λ_s and Λ_a the symmetric and antisymmetric parts respectively of Λ, so that

$$\begin{aligned} \mathbf{\Lambda} = \mathbf{\Lambda}_s + \mathbf{\Lambda}_a, \quad \mathbf{\Lambda}^T = \mathbf{\Lambda}_s - \mathbf{\Lambda}_a, \\ 2\mathbf{\Lambda}_s = \mathbf{\Lambda} + \mathbf{\Lambda}^T, \quad 2\mathbf{\Lambda}_a = \mathbf{\Lambda} - \mathbf{\Lambda}^T. \end{aligned} \tag{A1}$$

Then a complete list of isotropic invariants is (Zheng [8])

$$\begin{aligned} &I_1 = \text{tr } \mathbf{C}, \quad I_2 = \frac{1}{2}\left\{(\text{tr } \mathbf{C})^2 - \text{tr } \mathbf{C}^2\right\}, \quad I_3 = \det \mathbf{C}, \\ &I_4 = \mathbf{ACA}, \quad I_5 = \mathbf{AC}^2\mathbf{A}, \\ &I_6 = \text{tr } \mathbf{\Lambda}_s = \text{tr } \mathbf{\Lambda}, \quad I_7 = \text{tr } \mathbf{\Lambda}_s^2, \quad I_8 = \text{tr } \mathbf{\Lambda}_a^2, \quad I_9 = \text{tr } \mathbf{\Lambda}_s^3, \\ &I_{10} = \text{tr } \mathbf{C\Lambda}_s = \text{tr } \mathbf{C\Lambda}, \quad I_{11} = \text{tr } \mathbf{C}^2\mathbf{\Lambda}_s = \text{tr } \mathbf{C}^2\mathbf{\Lambda}, \\ &I_{12} = \text{tr } \mathbf{C\Lambda}_s^2, \quad I_{13} = \text{tr } \mathbf{C}^2\mathbf{\Lambda}_s^2, \\ &I_{14} = \text{tr } \mathbf{C\Lambda}_a^2, \quad I_{15} = \text{tr } \mathbf{C}^2\mathbf{\Lambda}_a^2, \quad I_{16} = \text{tr } \mathbf{C}^2\mathbf{\Lambda}_a^2\mathbf{C\Lambda}_a, \\ &I_{17} = \text{tr } \mathbf{\Lambda}_s\mathbf{\Lambda}_a^2, \quad I_{18} = \text{tr } \mathbf{\Lambda}_s^2\mathbf{\Lambda}_a^2, \quad I_{19} = \text{tr } \mathbf{\Lambda}_s^2\mathbf{\Lambda}_a^2\mathbf{\Lambda}_s\mathbf{\Lambda}_a, \\ &I_{20} = \mathbf{A\Lambda}_s\mathbf{A} = \mathbf{A\Lambda A}, \quad I_{21} = \mathbf{A\Lambda}_s^2\mathbf{A}, \quad I_{22} = \mathbf{A\Lambda}_a^2\mathbf{A}, \\ &I_{23} = \mathbf{AC\Lambda}_s\mathbf{A}, \quad I_{24} = \mathbf{AC\Lambda}_a\mathbf{A}, \quad I_{25} = \mathbf{AC}^2\mathbf{\Lambda}_a\mathbf{A}, \quad I_{26} = \mathbf{A\Lambda}_a\mathbf{C\Lambda}_a^2\mathbf{A}, \\ &I_{27} = \mathbf{A\Lambda}_s\mathbf{\Lambda}_a\mathbf{A}, \quad I_{28} = \mathbf{A\Lambda}_s^2\mathbf{\Lambda}_a\mathbf{A}, \quad I_{29} = \mathbf{A\Lambda}_a\mathbf{\Lambda}_s\mathbf{\Lambda}_a^2\mathbf{A}. \\ &I_{30} = \text{tr } \mathbf{C\Lambda}_s\mathbf{\Lambda}_a, \quad I_{31} = \text{tr } \mathbf{C}^2\mathbf{\Lambda}_s\mathbf{\Lambda}_a, \quad I_{32} = \text{tr } \mathbf{C\Lambda}_s^2\mathbf{\Lambda}_a, \quad I_{33} = \text{tr } \mathbf{C\Lambda}_a^2\mathbf{\Lambda}_s\mathbf{\Lambda}_a. \end{aligned} \tag{A2}$$

This set is complete, but may include redundant elements. The list is not unique, and for applications it may be convenient to replace some invariants by other equivalent invariants; for example, instead of I_{21}, I_{22} and I_{27} we can use the equivalent set

$$\mathbf{A\Lambda}^2\mathbf{A} = (I_{21}+I_{22}), \quad \mathbf{A\Lambda\Lambda}^T\mathbf{A} = (I_{21}-I_{22})-2I_{27}, \quad \mathbf{A\Lambda}^T\mathbf{\Lambda A} = (I_{21}-I_{22})+2I_{27}.$$

If the sense of the fibres is not significant, then W has to be even in $\mathbf{\Lambda}$. Consequently, if W has to be even in $\mathbf{\Lambda}$, the invariants $I_6, I_9, I_{10}, I_{11}, I_{16}, I_{17}, I_{20}, I_{23}, I_{24}, I_{25}, I_{26}, I_{28}, I_{32}$ can occur in W only through their squares and product in pairs.

In terms of the invariants (A2), the constitutive equations (5.38) take the form

$$\begin{aligned} \sigma_{(ij)} &= \frac{\rho}{\rho_0}\sum_\alpha \frac{\partial W}{\partial I_\alpha}\left\{F_{iR}F_{jS}\left(\frac{\partial I_\alpha}{\partial C_{RS}} + \frac{\partial I_\alpha}{\partial C_{SR}}\right) + (G_{iR}F_{jS} + G_{jS}F_{iR})\frac{\partial I_\alpha}{\partial \Lambda_{SR}}\right\}, \\ \overline{m}_{ji} &= \frac{2}{3}e_{ikm}\frac{\rho}{\rho_0}\frac{\partial W}{\partial I_\alpha}\frac{\partial I_\alpha}{\partial \Lambda_{PR}}F_{mP}\left(F_{jR}b_k + F_{kR}b_j\right). \end{aligned} \tag{A3}$$

This clearly leads to lengthy expressions for $\sigma_{(ij)}$ and $\overline{m}_{ji}$