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FINITE DIFFERENCE METHODS FOR COMPUTING EIGENVALUES OF FOURTH ORDER BOUNDARY VALUE PROBLEMS

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ABSTRACT. This brief report describes some new finite difference methods of order 2 and 4 for computing eigenvalues of a two point boundary value problem associated with a fourth order linear differential equation $y^{(4)} + (p(x) - \lambda q(x))y = 0$. These methods are derived from the formula

$$h^4 y_i^{(4)} = (\delta^4 - \frac{1}{6}\delta^6 + \frac{7}{240}\delta^8 - \ldots.)y_i \quad .$$

Numerical results are included to demonstrate practical usefulness of our methods.

KEY WORDS AND PHRASES. Central difference formula, Finite difference methods, Generalized symmetric eigenvalue problem, Positive definite matrix, Two-point boundary value problem.
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1. INTRODUCTION

We shall consider the fourth order linear differential equation

$$y^{(4)} + [p(x) - \lambda q(x)]y = 0 \tag{1.1}$$

associated with one of the following pairs of homogeneous boundary conditions:

$$\begin{aligned}
&\text{(a)} \quad y(a) = y'(a) = y(b) = y'(b) = 0 \\
&\text{(b)} \quad y(a) = y''(a) = y(b) = y''(b) = 0 \\
&\text{(c)} \quad y(a) = y'(a) = y''(b) = y'''(b) = 0
\end{aligned} \tag{1.2}$$

Such boundary value problems occur frequently in applied mathematics, engineering and modern physics, see [1,2,3]. In (1.1), the functions $p(x)$, $q(x) \in C[a,b]$ and they satisfy the conditions

$$p(x) \geq 0 \ , \ q(x) > 0 \ , \ x \in [a,b] \quad . \tag{1.3}$$

Recently, Chawla and Katti [4] have developed a finite difference method of order 2 for computing approximate values of λ for a boundary value problem (1.1)-(1.2a). For the same problem, a fourth order method was developed by Chawla [5] which leads to a generalized seven-band symmetric matrix eigenvalue problem.

The purpose of this note is to present some new finite difference methods for computing approximate values of λ for the boundary value problems (1.1)-(1.2b) and (1.1)-(1.2c). These methods lead to generalized five-band and seven-band symmetric matrix eigenvalue problem and provide $0(h^2)$ and $0(h^4)$ -convergent approximations for the eigenvalues.

2. A SECOND ORDER METHOD FOR COMPUTING λ FOR (1.1) and (1.2b)

For a positive integer $N \geq 4$, let $h = (b-a)/(N+1)$ and $x_i = a + ih$, $i = 0(1)N+1$. We shall designate $y_i = y(x_i)$, $p_i = p(x_i)$ and $q_i = q(x_i)$. The boundary value problem (1.1) and (1.2b) is discretized by the difference equations

$$
\begin{aligned}
&\text{(a)}\quad -2y_0 + 5y_1 - 4y_2 + y_3 = -h^2 y_0'' + h^4[-\tfrac{1}{12}\, y^{(4)} + y_1^{(4)}] + t_1 , \\
&\text{(b)}\quad \delta^4 y_i = h^4 y_1^{(4)} + t_i , \; i = 2(1)N-1 , \\
&\text{(c)}\quad y_{N-2} - 4y_{N-1} + 5y_N - 2y_{N+1} = -h^2 y_{N+1}'' + h^4[y_N^{(4)} - \tfrac{1}{12}\, y_{N+1}^{(4)}] + t_N .
\end{aligned}
\tag{2.1}
$$

Note that the truncation errors t_i, $i = 1(1)N$, are

$$
t_i = \begin{cases} \frac{59}{360} h^6 y^{(6)}(\xi_1) , & \xi_1 \in (a, x_3) \\ \frac{1}{6} h^6 y^{(6)}(\xi_i) , & \xi_i \in (x_{i-2}, x_{i+2}) , \; i = 2(1)N-1 \\ \frac{59}{360} h^6 y^{(6)}(\xi_N) , & \xi_N \in (x_{N-2}, b) . \end{cases}
\tag{2.2}
$$

The formula 2.1(b) is obtained from the well-known central difference formula

$$
h^4 y_i^{(4)} = [\delta^4 - \tfrac{1}{6}\delta^6 + \tfrac{7}{240}\delta^8 - \tfrac{41}{7560}\delta^8 + \ldots] y_i , \tag{2.3}
$$

$$(i = 2,3,4,\ldots) .$$

The discretizations 2.1(a) and 2.1(c) are introduced so that the resulting coefficient matrix in (2.1) is a five-band symmetric matrix. The system of linear equations (2.1) can be written in matrix form

$$
(J^2 + h^4 P)Y = \lambda h^4 QY + t , \tag{2.4}
$$

where J^2 is a symmetric five-band matrix and $J = (j_{rs})$ is a tridiagonal matrix such that

$$
j_{rs} = \begin{cases} 2, & r = s \\ -1, & |r-s| = 1 \\ 0, & \text{otherwise.} \end{cases}
\tag{2.5}
$$

The matrices P and Q are diagonal matrices

$$P = \text{diag}[p_1\; p_2\; \ldots\; p_N] , \quad Q = \text{diag}[q_1\; q_2\; \ldots\; q_N]$$

and

$$Y = [y_1\; y_2\; \ldots\; y_N]^T , \quad t = [t_1\; t_2\; \ldots\; t_N]^T .$$

Thus, our method for computing approximations Λ for λ of (1.1)-(1.2b) can be expressed as a generalized five-band symmetric matrix eigenvalue problem

$$
(J^2 + h^4 P)\tilde{Y} = \Lambda h^4 Q\tilde{Y} . \tag{2.6}
$$

In fact, the matrix J^2 is a positive definite matrix and hence for any step-size $h > 0$, the approximations Λ for λ by (2.6) are real and positive for all $p(x) \geq 0$ and $q(x) > 0$. That our method provides $O(h^2)$ convervent approximations Λ for λ can be established following Grigorieff [6]. We omit the details of convergence proof for brevity.

3. A FOURTH ORDER METHOD

The boundary value problem (1.1)-(1.2b) is now discretized by the following difference equations

$$\begin{aligned}
&\text{(a)}\quad -17y_0 + 44y_1 - 38y_2 + 12y_3 - y_4 = -8h^2y_0'' h^4[\tfrac{1}{3} y_0^{(4)} + 6y_1^{(4)}] + \tfrac{13}{90} h^6y_0^{(6)} + \ldots \\
&\text{(b)}\quad 10y_0 - 38y_1 + 56y_2 - 39y_3 + 12y_4 - y_5 = h^2y_0'' \\
&\qquad\qquad + h^4[\tfrac{1}{12} y_0^{(4)} + 6y_2^{(4)}] + \tfrac{1}{360} h^6y_0^{(6)} + \ldots, \\
&\text{(c)}\quad (6\delta^4 - \delta^6)y_i = 6h^4y_i^{(4)} - \tfrac{7}{40} h^8y_i^{(8)} + \ldots,\ i = 3(1)N-2\ ; \qquad (3.1) \\
&\text{(d)}\quad -y_{N-4} + 12y_{N-3} - 39y_{N-2} + 56y_{N-1} - 38y_N + 10y_{N+1} \\
&\qquad\qquad = h^2y_{N+1}'' + h^4[6y_{N-1}^{(4)} + \tfrac{1}{12} y_{N+1}^{(4)}] + \tfrac{1}{360} h^6y_N^{(6)} + \ldots, \\
&\text{(e)}\quad -y_{N-3} + 12y_{N-2} - 38y_{N-1} + 44y_N - 17y_{N+1} = -8h^2y_{N+1}'' \\
&\qquad\qquad + h^4[6v_N^{(4)} + \tfrac{1}{3} y_{N+1}^{(4)}] + \tfrac{13}{90} h^6y_N^{(6)} + \ldots.
\end{aligned}$$

As in [7], the derivation of (3.1c) is immediate from (2.3), on truncating the infinite series on the right of equality sign after the two terms. The additional difference equations (3.1a, b, d, e) are chosen so that the resulting matrix associated with the system of linear equations in (3.1) is a seven-band symmetric matrix. It turns out that our method for computing approximations Λ for λ of (1.1)-(1.2b) can be expressed as a generalised seven-band symmetric matrix eigenvalue problem

$$[(6J^2 + J^3) + 6h^4P]\tilde{Y} = 6\Lambda h^4Q\tilde{Y} \quad . \qquad (3.2)$$

The matrix $6J^2 + J^3$ is a positive definite matrix and hence for any step-size $h > 0$, the approximations Λ for λ by (3.2) are real and positive for all $p(x) \geq 0$ and $q(x) > 0$. As before, it follows from the results of Grigorieff (1975) that our present method provides $O(h^4)$ -convergent approximations Λ for λ .

4. METHODS FOR COMPUTING λ FOR (1.1)-(1.2c)

For this section, let $h = (b-a)/N$ and $x_i = a + ih$, $i = 0(1)N$. The boundary value problem (1.1)-(1.2c) is discretized by the following scheme

$$\begin{aligned}
&\text{(a)}\quad -4y_0 + 7y_1 - 4y_2 + y_3 = 2hy_0' + h^4y_1^{(4)} + (\tfrac{h^3}{3} y_0''' + \ldots)\ , \\
&\text{(b)}\quad \delta^4y_i = h^4y_i^{(4)} + \tfrac{1}{6} h^4y^{(6)}(\xi_i)\ ,\ \xi_i \in (x_{i-2}, x_{i+2})\ , \\
&\qquad i = 2(1)N-2\ , \qquad (4.1) \\
&\text{(c)}\quad y_{N-3} - 4y_{N-2} + 5y_{N-1} - 2y_N = -h^2y_N'' + \tfrac{11}{12} h^4y_{N-1}^{(4)} \\
&\qquad\qquad + (\tfrac{1}{12} h^5y_N^{(5)} + \ldots)\ , \\
&\text{(d)}\quad y_{N-2} - 2y_{N-1} + y_N = h^2y_N'' - h^3y_N''' + \tfrac{7}{12} h^4y_N^{(4)} + (\tfrac{1}{4} h^5y_N^{(5)} + \ldots)\ .
\end{aligned}$$

This system can be written in matrix form

$$(A + h^4P)y = \lambda h^4Qy + t \tag{4.2}$$

and our method for computing approximation Λ for λ of (1.1)-(1.2c) can be written as a generalized five-band symmetric matrix eigenvalue problem

$$(A + h^4P)\tilde{y} = \Lambda h^4Q\tilde{y} \tag{4.3}$$

where

$$A = \begin{bmatrix} 7 & -4 & 1 & & & & & & \\ -4 & 6 & -4 & 1 & & & & & \\ 1 & -4 & 6 & -4 & 1 & & & & \\ 0 & 1 & -4 & 6 & -4 & 1 & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \\ & & & 1 & -4 & 6 & -4 & 1 \\ & & & & 1 & -4 & 5 & -2 \\ & & & & & 1 & -2 & 1 \end{bmatrix} \tag{4.4}$$

It can be established (see Appendix A) that the matrix A is a positive definite matrix and the approximations Λ for λ by (4.3) are $0(h^2)$ -convergent.

A third order method is obtained on discretizing (1.1)-(1.2c) by the difference equations

$$\begin{aligned}
&\text{(a)} \quad -45y_0 + 76y_1 - 42y_2 + 12y_3 - y_4 = 24hy_0' \\
&\qquad + h^4(-y_0^{(4)} + 6y_1^{(4)}) - \frac{4}{5}h^5y_0^{(5)} - (\frac{1}{6}h^6y_0^{(6)} + \ldots) , \\
&\text{(b)} \quad \frac{27}{2}y_0 - 42y_1 + \frac{113}{2}y_2 - 39y_3 + 12y_4 - y_5 = -3hy_0' \\
&\qquad + h^4(\frac{1}{4}y_0^{(4)} + 6y_2^{(4)}) + \frac{1}{10}h^5y_0^{(5)} + (\frac{1}{24}h^6y_0^{(6)} + \ldots) , \\
&\text{(c)} \quad (6\delta^4 - \delta^6)y_i = 6h^4y_i^{(4)} - (\frac{7}{40}h^8y_i^{(8)} + \ldots) , \ i = 3(1)N-3 , \\
&\text{(d)} \quad -y_{N-5} + 12y_{N-4} - 39y_{N-3} + 54y_{N-2} - 34y_{N-1} + 8y_N = -h^2y_N'' \\
&\qquad + 2h^3y_N''' + \frac{59}{12}h^4y_{N-2}^{(4)} + (-\frac{5}{3}h^5y_N^{(5)} + \ldots) , \\
&\text{(e)} \quad -y_{N-4} + 12y_{N-3} - 34y_{N-2} + 36y_{N-1} - 13y_N = -4h^2y_N'' \\
&\qquad -4h^3y_N''' + \frac{26}{3}h^4y_{N-1}^{(4)} + (\frac{5}{3}h^5y_N^{(5)} + \ldots) , \\
&\text{(f)} \quad -y_{N-3} + 8y_{N-2} - 13y_{N-1} + 6y_N = 5h^2y_N'' - 4y_N''' \\
&\qquad + \frac{17}{12}h^4y_N^{(4)} - \frac{23}{72}h^6y^{(6)}(\xi_N) , \ \xi_N \in (x_{N-3}, b) .
\end{aligned} \tag{4.5}$$

This third order method gives rise to a generalized seven-band symmetric matrix eigenvalue problem

$$(B + h^4P)\tilde{y} = \Lambda h^4Q\tilde{y} \tag{4.6}$$

where

$$B = \begin{bmatrix} 76 & -42 & 12 & -1 & & & & & & \\ -42 & \frac{113}{2} & -39 & 12 & -1 & & & & & \\ 12 & -39 & 56 & -39 & 12 & -1 & & & & \\ -1 & 12 & -39 & 56 & -39 & 12 & -1 & & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & & \\ & & & -1 & 12 & -39 & 54 & -34 & 8 \\ & & & & -1 & 12 & -34 & 36 & -13 \\ & & & & & -1 & 8 & -13 & 6 \end{bmatrix} \tag{4.7}$$

5. NUMERICAL ILLUSTRATIONS

To illustrate our methods for order 2 and 4 for the approximation λ of (1.1)-(1.2b), we consider the eigenvalue problem

$$\begin{cases} y^{(4)} - \dfrac{\lambda}{(1+x)^4}\, y = 0 \ , \\ y(0) = y''(0) = y(b) = y''(b) = 0 \ . \end{cases} \tag{5.1}$$

The smallest eigenvalues $\lambda(b)$ for $b = 1,2$ are

$$\lambda(1) = 416.324,564,86...$$

and $\lambda(2) = 646.269,207,...$ respectively. We computed approximations $\Lambda(1)$ and $\Lambda(2)$ by our methods (2.1) and (3.1) applied to the problem (5.1) for $h = 2^{-m}$, $m = 3(1)6$. The corresponding errors $\left|1 - \frac{\lambda(b)}{\Lambda(b)}\right|$ are shown in Table I. It is easily verified that our methods based on finite difference approximations (2.1) and (3.1) do provide $0(h^2)$ and $0(h)^4$- convergent approximations for the smallest eigenvalue of (5.1).

TABLE I

b	h	Error $\left\|1 - \frac{\lambda(b)}{\Lambda(b)}\right\|$ based on the method (2.1)	(3.1)
1	1/8	2.68-2*	1.09-4
	1/16	6.76-3	8.34-6
	1/32	1.69-3	5.46-7
	1/64	4.23-4	3.29-8
2	1/8	2.61-2	2.50-3
	1/16	7.13-3	1.70-4
	1/32	1.82-3	1.08-5
	1/64	4.58-4	6.39-7

*We write 2.68-2 for 2.68×10^{-2} .

We now illustrate our methods for the approximation of λ of (1.1)-(1.2c) by approximating the value of the smallest eigenvalue λ_1 satisfying

$$\begin{gathered} y^{(4)} + \left((1+x^2) - \frac{\lambda}{(1+x)^4} \right) y = 0 \\ y(0) = y'(0) = y''(1) = y'''(1) = 0 \ , \end{gathered} \tag{5.2}$$

where $\lambda_1 = 135.320,349,281,57....$ We list the errors $\left|1 - \frac{\lambda_1}{\Lambda_1}\right|$ for $h = 2^{-m}$, $m = 3,4,5,6$. It is readily verified that the relative errors $\left|\frac{\lambda_1}{\Lambda_1}\right|$ based on the finite difference scheme (4.1) are $O(h^2)$- convergent and likewise the relative errors based on the scheme (4.5) are $O(h^3)$- convergent.

TABLE II

h	Error $\left|1 - \frac{\lambda_1}{\Lambda_1}\right|$ based on methods	
	(4.1)	(4.5)
1/8	1.40-2	2.80-3
1/16	3.41-3	3.50-4
1/32	8.41-4	4.31-5
1/64	2.08-4	4.78-6
1/128	5.13-5	

APPENDIX A

It is well known that the tridiagonal matrix J introduced in (2.4) is a positive definite matrix. it follows that the matrices J^2 and $(6J^2 + J^3)$ introduced in equations (2.6) and (3.2) respectively are also positive definite matrices. In order to establish that the real symmetric matrix A given by (4.4) is a positive definite matrix, it suffices to prove that the $(N-1)$ principal minors

$$1, \begin{vmatrix} 5 & -2 \\ -2 & 1 \end{vmatrix}, \begin{vmatrix} 6 & -4 & 1 \\ -4 & 5 & -2 \\ 1 & -2 & 1 \end{vmatrix}, \ldots, \begin{vmatrix} 6 & -4 & 1 & & & & \\ -4 & 6 & -4 & 1 & & & \\ 1 & -4 & 6 & -4 & 1 & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & & 1 & -4 & 6 & -4 & 1 \\ & & & 1 & -4 & 5 & -2 \\ & & & & 1 & -2 & 1 \end{vmatrix}$$

are each equal to 1 and $|A| = 2$

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