

# **Finite Difference Methods for Ordinary and Partial Differential Equations**

**Steady-State and Time-Dependent Problems**

**Randall J. LeVeque**

University of Washington  
Seattle, Washington

# Contents

Preface	xiii
<b>I Boundary Value Problems and Iterative Methods</b>	<b>1</b>
<b>1 Finite Difference Approximations</b>	<b>3</b>
1.1 Truncation errors . . . . .	5
1.2 Deriving finite difference approximations . . . . .	7
1.3 Second order derivatives . . . . .	8
1.4 Higher order derivatives . . . . .	9
1.5 A general approach to deriving the coefficients . . . . .	10
<b>2 Steady States and Boundary Value Problems</b>	<b>13</b>
2.1 The heat equation . . . . .	13
2.2 Boundary conditions . . . . .	14
2.3 The steady-state problem . . . . .	14
2.4 A simple finite difference method . . . . .	15
2.5 Local truncation error . . . . .	17
2.6 Global error . . . . .	18
2.7 Stability . . . . .	18
2.8 Consistency . . . . .	19
2.9 Convergence . . . . .	19
2.10 Stability in the 2-norm . . . . .	20
2.11 Green's functions and max-norm stability . . . . .	22
2.12 Neumann boundary conditions . . . . .	29
2.13 Existence and uniqueness . . . . .	32
2.14 Ordering the unknowns and equations . . . . .	34
2.15 A general linear second order equation . . . . .	35
2.16 Nonlinear equations . . . . .	37
2.16.1 Discretization of the nonlinear boundary value problem .	38
2.16.2 Nonuniqueness . . . . .	40
2.16.3 Accuracy on nonlinear equations . . . . .	41
2.17 Singular perturbations and boundary layers . . . . .	43
2.17.1 Interior layers . . . . .	46

2.18	Nonuniform grids . . . . .	49
2.18.1	Adaptive mesh selection . . . . .	51
2.19	Continuation methods . . . . .	52
2.20	Higher order methods . . . . .	52
2.20.1	Fourth order differencing . . . . .	52
2.20.2	Extrapolation methods . . . . .	53
2.20.3	Deferred corrections . . . . .	54
2.21	Spectral methods . . . . .	55
<b>3</b>	<b>Elliptic Equations</b>	<b>59</b>
3.1	Steady-state heat conduction . . . . .	59
3.2	The 5-point stencil for the Laplacian . . . . .	60
3.3	Ordering the unknowns and equations . . . . .	61
3.4	Accuracy and stability . . . . .	63
3.5	The 9-point Laplacian . . . . .	64
3.6	Other elliptic equations . . . . .	66
3.7	Solving the linear system . . . . .	66
3.7.1	Sparse storage in MATLAB . . . . .	68
<b>4</b>	<b>Iterative Methods for Sparse Linear Systems</b>	<b>69</b>
4.1	Jacobi and Gauss–Seidel . . . . .	69
4.2	Analysis of matrix splitting methods . . . . .	71
4.2.1	Rate of convergence . . . . .	74
4.2.2	Successive overrelaxation . . . . .	76
4.3	Descent methods and conjugate gradients . . . . .	78
4.3.1	The method of steepest descent . . . . .	79
4.3.2	The $A$ -conjugate search direction . . . . .	83
4.3.3	The conjugate-gradient algorithm . . . . .	86
4.3.4	Convergence of conjugate gradient . . . . .	88
4.3.5	Preconditioners . . . . .	93
4.3.6	Incomplete Cholesky and ILU preconditioners . . . . .	96
4.4	The Arnoldi process and GMRES algorithm . . . . .	96
4.4.1	Krylov methods based on three term recurrences . . . . .	99
4.4.2	Other applications of Arnoldi . . . . .	100
4.5	Newton–Krylov methods for nonlinear problems . . . . .	101
4.6	Multigrid methods . . . . .	103
4.6.1	Slow convergence of Jacobi . . . . .	103
4.6.2	The multigrid approach . . . . .	106
<b>II</b>	<b>Initial Value Problems</b>	<b>111</b>
<b>5</b>	<b>The Initial Value Problem for Ordinary Differential Equations</b>	<b>113</b>
5.1	Linear ordinary differential equations . . . . .	114
5.1.1	Duhamel’s principle . . . . .	115
5.2	Lipschitz continuity . . . . .	116

---

5.2.1	Existence and uniqueness of solutions . . . . .	116
5.2.2	Systems of equations . . . . .	117
5.2.3	Significance of the Lipschitz constant . . . . .	118
5.2.4	Limitations . . . . .	119
5.3	Some basic numerical methods . . . . .	120
5.4	Truncation errors . . . . .	121
5.5	One-step errors . . . . .	122
5.6	Taylor series methods . . . . .	123
5.7	Runge–Kutta methods . . . . .	124
5.7.1	Embedded methods and error estimation . . . . .	128
5.8	One-step versus multistep methods . . . . .	130
5.9	Linear multistep methods . . . . .	131
5.9.1	Local truncation error . . . . .	132
5.9.2	Characteristic polynomials . . . . .	133
5.9.3	Starting values . . . . .	134
5.9.4	Predictor–corrector methods . . . . .	135
<b>6</b>	<b>Zero-Stability and Convergence for Initial Value Problems</b>	<b>137</b>
6.1	Convergence . . . . .	137
6.2	The test problem . . . . .	138
6.3	One-step methods . . . . .	138
6.3.1	Euler’s method on linear problems . . . . .	138
6.3.2	Relation to stability for boundary value problems . . . . .	140
6.3.3	Euler’s method on nonlinear problems . . . . .	141
6.3.4	General one-step methods . . . . .	142
6.4	Zero-stability of linear multistep methods . . . . .	143
6.4.1	Solving linear difference equations . . . . .	144
<b>7</b>	<b>Absolute Stability for Ordinary Differential Equations</b>	<b>149</b>
7.1	Unstable computations with a zero-stable method . . . . .	149
7.2	Absolute stability . . . . .	151
7.3	Stability regions for linear multistep methods . . . . .	153
7.4	Systems of ordinary differential equations . . . . .	156
7.4.1	Chemical kinetics . . . . .	157
7.4.2	Linear systems . . . . .	158
7.4.3	Nonlinear systems . . . . .	160
7.5	Practical choice of step size . . . . .	161
7.6	Plotting stability regions . . . . .	162
7.6.1	The boundary locus method for linear multistep methods .	162
7.6.2	Plotting stability regions of one-step methods . . . . .	163
7.7	Relative stability regions and order stars . . . . .	164
<b>8</b>	<b>Stiff Ordinary Differential Equations</b>	<b>167</b>
8.1	Numerical difficulties . . . . .	168
8.2	Characterizations of stiffness . . . . .	169
8.3	Numerical methods for stiff problems . . . . .	170

8.3.1	A-stability and A( $\alpha$ )-stability . . . . .	171
8.3.2	L-stability . . . . .	171
8.4	BDF methods . . . . .	173
8.5	The TR-BDF2 method . . . . .	175
8.6	Runge–Kutta–Chebyshev explicit methods . . . . .	175
<b>9</b>	<b>Diffusion Equations and Parabolic Problems</b>	<b>181</b>
9.1	Local truncation errors and order of accuracy . . . . .	183
9.2	Method of lines discretizations . . . . .	184
9.3	Stability theory . . . . .	186
9.4	Stiffness of the heat equation . . . . .	186
9.5	Convergence . . . . .	189
9.5.1	PDE versus ODE stability theory . . . . .	191
9.6	Von Neumann analysis . . . . .	192
9.7	Multidimensional problems . . . . .	195
9.8	The locally one-dimensional method . . . . .	197
9.8.1	Boundary conditions . . . . .	198
9.8.2	The alternating direction implicit method . . . . .	199
9.9	Other discretizations . . . . .	200
<b>10</b>	<b>Advection Equations and Hyperbolic Systems</b>	<b>201</b>
10.1	Advection . . . . .	201
10.2	Method of lines discretization . . . . .	203
10.2.1	Forward Euler time discretization . . . . .	204
10.2.2	Leapfrog . . . . .	205
10.2.3	Lax–Friedrichs . . . . .	206
10.3	The Lax–Wendroff method . . . . .	207
10.3.1	Stability analysis . . . . .	209
10.4	Upwind methods . . . . .	210
10.4.1	Stability analysis . . . . .	211
10.4.2	The Beam–Warming method . . . . .	212
10.5	Von Neumann analysis . . . . .	212
10.6	Characteristic tracing and interpolation . . . . .	214
10.7	The Courant–Friedrichs–Lewy condition . . . . .	215
10.8	Some numerical results . . . . .	218
10.9	Modified equations . . . . .	218
10.10	Hyperbolic systems . . . . .	224
10.10.1	Characteristic variables . . . . .	224
10.11	Numerical methods for hyperbolic systems . . . . .	225
10.12	Initial boundary value problems . . . . .	226
10.12.1	Analysis of upwind on the initial boundary value problem	226
10.12.2	Outflow boundary conditions . . . . .	228
10.13	Other discretizations . . . . .	230
<b>11</b>	<b>Mixed Equations</b>	<b>233</b>
11.1	Some examples . . . . .	233

---

11.2	Fully coupled method of lines . . . . .	235
11.3	Fully coupled Taylor series methods . . . . .	236
11.4	Fractional step methods . . . . .	237
11.5	Implicit-explicit methods . . . . .	239
11.6	Exponential time differencing methods . . . . .	240
11.6.1	Implementing exponential time differencing methods .	241
<b>III</b>	<b>Appendices</b>	<b>243</b>
<b>A</b>	<b>Measuring Errors</b>	<b>245</b>
A.1	Errors in a scalar value . . . . .	245
A.1.1	Absolute error . . . . .	245
A.1.2	Relative error . . . . .	246
A.2	“Big-oh” and “little-oh” notation . . . . .	247
A.3	Errors in vectors . . . . .	248
A.3.1	Norm equivalence . . . . .	249
A.3.2	Matrix norms . . . . .	250
A.4	Errors in functions . . . . .	250
A.5	Errors in grid functions . . . . .	251
A.5.1	Norm equivalence . . . . .	252
A.6	Estimating errors in numerical solutions . . . . .	254
A.6.1	Estimates from the true solution . . . . .	255
A.6.2	Estimates from a fine-grid solution . . . . .	256
A.6.3	Estimates from coarser solutions . . . . .	256
<b>B</b>	<b>Polynomial Interpolation and Orthogonal Polynomials</b>	<b>259</b>
B.1	The general interpolation problem . . . . .	259
B.2	Polynomial interpolation . . . . .	260
B.2.1	Monomial basis . . . . .	260
B.2.2	Lagrange basis . . . . .	260
B.2.3	Newton form . . . . .	260
B.2.4	Error in polynomial interpolation . . . . .	262
B.3	Orthogonal polynomials . . . . .	262
B.3.1	Legendre polynomials . . . . .	264
B.3.2	Chebyshev polynomials . . . . .	265
<b>C</b>	<b>Eigenvalues and Inner-Product Norms</b>	<b>269</b>
C.1	Similarity transformations . . . . .	270
C.2	Diagonalizable matrices . . . . .	271
C.3	The Jordan canonical form . . . . .	271
C.4	Symmetric and Hermitian matrices . . . . .	273
C.5	Skew-symmetric and skew-Hermitian matrices . . . . .	274
C.6	Normal matrices . . . . .	274
C.7	Toeplitz and circulant matrices . . . . .	275
C.8	The Gershgorin theorem . . . . .	277

C.9	Inner-product norms . . . . .	279
C.10	Other inner-product norms . . . . .	281
<b>D</b>	<b>Matrix Powers and Exponentials</b>	<b>285</b>
D.1	The resolvent . . . . .	286
D.2	Powers of matrices . . . . .	286
D.2.1	Solving linear difference equations . . . . .	290
D.2.2	Resolvent estimates . . . . .	291
D.3	Matrix exponentials . . . . .	293
D.3.1	Solving linear differential equations . . . . .	296
D.4	Nonnormal matrices . . . . .	296
D.4.1	Matrix powers . . . . .	297
D.4.2	Matrix exponentials . . . . .	299
D.5	Pseudospectra . . . . .	302
D.5.1	Nonnormality of a Jordan block . . . . .	304
D.6	Stable families of matrices and the Kreiss matrix theorem . . . . .	304
D.7	Variable coefficient problems . . . . .	307
<b>E</b>	<b>Partial Differential Equations</b>	<b>311</b>
E.1	Classification of differential equations . . . . .	311
E.1.1	Second order equations . . . . .	311
E.1.2	Elliptic equations . . . . .	312
E.1.3	Parabolic equations . . . . .	313
E.1.4	Hyperbolic equations . . . . .	313
E.2	Derivation of partial differential equations from conservation principles	314
E.2.1	Advection . . . . .	315
E.2.2	Diffusion . . . . .	316
E.2.3	Source terms . . . . .	317
E.2.4	Reaction-diffusion equations . . . . .	317
E.3	Fourier analysis of linear partial differential equations . . . . .	317
E.3.1	Fourier transforms . . . . .	318
E.3.2	The advection equation . . . . .	318
E.3.3	The heat equation . . . . .	320
E.3.4	The backward heat equation . . . . .	322
E.3.5	More general parabolic equations . . . . .	322
E.3.6	Dispersive waves . . . . .	323
E.3.7	Even- versus odd-order derivatives . . . . .	324
E.3.8	The Schrödinger equation . . . . .	324
E.3.9	The dispersion relation . . . . .	325
E.3.10	Wave packets . . . . .	327
<b>Bibliography</b>		<b>329</b>
<b>Index</b>		<b>337</b>