FINITE-DIFFERENCE METHODS

FOR PARTIAL DIFFERENTIAL EQUATIONS

GEORGE E. FORSYTHE

PROFESSOR OF MATHEMATICS
STANFORD UNIVERSITY

WOLFGANG R. WASOW

PROFESSOR OF MATHEMATICS
UNIVERSITY OF WISCONSIN

JOHN WILEY & SONS, INC. NEW YORK · LONDON

CONTENTS

INTRODUCTION TO PARTIAL DIFFERENTIAL EQUATIONS AND COMPUTERS

1.	Remarks on the Classification of Partial Differential Equations	1
2.	Systems and Single Equations	6
3.	Properties of Digital Computing Systems	7
	3.1. Desk computation	8
	3.2. Punched-card computers	9
	3.3. Automatic digital computers	9
	3.4. Demands of partial differential equations	11
1	HYPERBOLIC EQUATIONS IN TWO INDEPENDENT VARIABLE	ES
4.	A Finite-Difference Approximation to the Equation $u_{tt} - u_{xx} = 0$	15
	4.1. Solution of the simplest initial-value problem for $u_{tt} - u_{xx} = 0$	15
	4.2. An approximating difference equation	16
	4.3. Explicit solution of the difference equation for $\lambda < 1$	19
	4.4. Solution of the difference equation by a finite Fourier series	23
	4.5. The convergence to the solution of the differential problem	24
	4.6. Stability	27
5.	Further Aspects of the Concept of Stability	29
	5.1. Definitions and simple examples	29
	5.2. Application to the wave equation	35
6.	Systems of Hyperbolic Differential Equations and Their Characteristics	38
	6.1. The normal form	38
	6.2. Examples	41
	6.3. The canonical differential system for $n=2$	42
	6.4. Remarks on the initial-value problem	44
7.	Finite-Difference Methods for Systems of Quasilinear Hyperbolic Equations	49
	7.1. Description of the procedure	49
	7.2. A general scheme for proving the convergence of difference	
	approximations	53
	7.3. The convergence of the difference scheme for hyperbolic systems	57
	7.4. Differences in a curvilinear net	61
	7.5. The round-off errors	62
8.	Integration Along Characteristics	64
	8.1. The method of Massau	64
	8.2. Quasilinear equations of order two	65
	8.3. Another integration method for n dependent variables	66
9.	Integration by Adams' Method	68

CONTENTS

10.	Shock Waves	71
	10.1. The concept of shock waves	71
	10.2. Numerical solution of problems involving shock waves	74
	10.3. Calculation of shock fronts by means of simulated viscosity terms	78
	10.4. Integration of the true equations of viscous flow	82
	10.5. The difference method of Lax	84
	2 PARABOLIC EQUATIONS	
11.	The Simplest Heat Flow Problem	88
	11.1. Preliminary remarks	88
	11.2. Solution of the initial-value problem	89
12.	The Simplest Finite-Difference Approximation	92
	12.1. The stability condition	92
	12.2. The convergence and the discretization error	95
13.	Linear Problems in a Finite Interval	98
	13.1. Differential problems	98
	13.2. A finite-difference approximation	100
	13.3. An implicit method	101
	13.4. The solution of the implicit difference equation	103
	13.5. The convergence of the implicit method	105
14.	More General Linear Parabolic Problems in Two Variables:	405
	Explicit Methods	107
	14.1. Formal explicit difference approximations	107
	14.2. Solution of nonhomogeneous linear difference problems by superposition	109
	14.3. Boundedness and stability properties of difference expressions of	109
	positive type	111
	14.4. The boundedness condition of John	113
15	Further Explicit and Implicit Methods for Linear Problems	119
	15.1. A more general approach to implicit methods	119
	15.2. Explicit methods using more than two grid lines	125
	15.3. Problems of higher order	131
16.	Other Definitions of Convergence. The Theory of Lax and Richtmyer	133
	16.1. Remarks on functional analysis	133
	16.2. Convergence and stability in the sense of Lax and Richtmyer	135
17	Nonlinear Problems	137
	17.1. Semilinear equations	137
	17.2. Examples of other parabolic problems	139
	3 ELLIPTIC EQUATIONS	
18	. Some Numerical Problems Involving Elliptic Partial Differential Equations	146
10		147
	18.1. General Laplacian boundary-value problem	148
	18.2. A water drainage problem 18.3. An oil-flow problem	150
	10.0. In on-now problem	

~	~ ~		-		
(ON	V I	+	V.	5

CONTENTS	ix
18.4. A stress problem	153
18.5. A boundary-layer problem	154
18.6. A membrane eigenvalue problem	155
18.7. A simple reactor problem	156
18.8. A biharmonic eigenvalue problem	158
18.9. Plateau's problem	158
18.10. Eigenvalue problem for the wave equation	159
19. Selected Results from the Theory of Elliptic Partial Differential Equations	159
19.1. Variational formulations	159
19.2. Variational formulation of certain eigenvalue problems	165
19.3. Self-adjointness	167
19.4. Interface conditions	170
19.5. Maximum principle	174
20. Formulating Elliptic Difference Equation Problems	175
20.1. Discretization and problems raised by it	175
20.2. The method of lines	178
20.3. Types of problems to be discretized	178 179
20.4. Irregular nets 20.5. Variational method of setting up difference equations	182
20.6. Square nets: approximating the derivatives	184
20.7. Square nets: approximating the derivatives 20.7. Square nets: approximating $L(u)$ and Δu	190
20.8. Application of the variational method to a reactor diffusion equation	195
20.9. Treatment of Dirichlet boundary conditions	198
20.10. Normal derivative boundary conditions	202
20.11. Singularities and free boundaries	204
21. Classical Theory of Solving Elliptic Difference Equations	205
21.1. The difference equations as a matrix equation	205
21.2. Elimination methods	208
21.3. Iterative methods	214
21.4. Method of simultaneous displacements; gradient method	220
21.5. Richardson's method	226
21.6. Method of successive displacements	235
21.7. Gauss-Southwell relaxation	241
22. Explicit and Implicit Overrelaxation Methods	242
22.1. The Young-Frankel theory of successive overrelaxation	242
22.2. Overrelaxation without property (A)	260
22.3. Implicit methods: overrelaxation by lines	266
22.4. Implicit alternating-direction methods	272
22.5. Summary of rates of convergence for a square	282
23. Discretization and Round-Off Errors	283
23.1. The method of Gerschgorin	283
23.2. An integral equation with a Stieltjes kernel	288
23.3. An appraisal of the solution of the integral equation	296
23.4. Appraisal of the discretization error	298
23.5. Summary of some further results concerning discretization errors	205
for linear Dirichlet problems	307
23.6. Green's function for discrete Dirichlet problems	314
23.7. Discretization error for the Neumann and third boundary-value	318

	23.8. Round-off error in solving the Dirichlet difference problem	319
	23.9. Probabilistic estimate of round-off error	326
24	The Membrane Eigenvalue Problem	329
	24.1. Introduction	329 331
	24.2. Upper bounds by difference methods 24.3. A standard L-shaped membrane	334
	24.4. Lower bounds from difference equations: Weinberger's method	336
	24.5. Asymptotic lower bounds from difference equations	340
	24.6. Proof of Theorem 24.7	343
	24.7. Experiments with L-shaped membrane	351
	24.8. Numerical solution of the finite eigenvalue problem	353
25.	Solving Elliptic Partial Difference Equations on an Automatic	
-	Digital Computer	357
	25.1. Obtaining the equations in a digital computer	357
	25.2. Obtaining the difference equations when C is curved	361
	25.3. Plans for an integrated industrial program	364
	25.4. Use of graded nets	365
	25.5. Successive overrelaxation: estimating ω	368
	25.6. Successive overrelaxation: time required	373
	25.7. Other methods for solving difference equations	375
	25.8. Solving eigenvalue problems on a computer	375
	25.9. Solving the Neumann problem on a computer	376
26	4 INITIAL-VALUE PROBLEMS IN MORE THAN TWO INDEPENDENT VARIABLES	378
20.	The Equation of Wave Propagation	378
	26.1. The differential equation 26.2. The simplest difference approximation	381

27.	Characteristics in Several Dimensions	383
28.	A Meteorological Forecast Problem	386
	28.1. Forecasting directly from the primitive equations	387
	28.2. Modified approaches to forecasting	388
	28.3. One-dimensional model	390 392
	28.4. Two-dimensional model 28.5. "Upwind" difference equations	397
	28.6. Three space dimensions	399
20		
29.	A General Discussion of the Fourier Method for Difference and Differential Equations	400
	29.1. The problem	400
	29.2. Explicit solution by infinite Fourier series	402
	29.3. Convergence of $U(x, t)$ to $u(x, t)$	405
	29.4. Stability	407
	29.5. How to test for stability and convergence	409
30.	The Method of Peaceman and Rachford	411
	30.1. General formulation	411
	30.2. Application to the equation of heat flow in two dimensions	412
Bib	liography and Author Index	415
	piect Index	433