KYUNGPOOK Math. J. 48(2008), 37-43

Finite Dimension in Associative Rings

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Abstract. The aim of the present paper is to introduce the concept "Finite dimension" in the theory of associative rings R with respect to two sided ideals. We obtain that if R has finite dimension on two sided ideals, then there exist uniform ideals $U_1, U_2, \cdots, U_n$ of R whose sum is direct and essential in R. The number n is independent of the choice of the uniform ideals U_i and 'n' is called the dimension of R.

1. Introduction

The dimension of a vector space is defined as the number of elements in the basis. One can define a basis of a vector space as a maximal set of linearly independent vectors or a minimal set of vectors which span the space. The former when generalized to modules over rings become the concept of Goldie dimension. Goldie proved a structure theorem for modules which states that "a module with finite Goldie dimension (FGD, in short) contains a finite number of uniform submodules $U_1, U_2, \cdots, U_n$ whose sum is direct and essential in M". The number n obtained here is independent of the choice of $U_1, U_2, \cdots, U_n$ and it is called as Goldie dimension of M. The concept Goldie dimension in Modules was studied by several authors

Received July 6, 2006.
2000 Mathematics Subject Classification: 16B99, 16D25, 16D70, 16P99.
Key words and phrases: ring, two sided ideal, essential ideal, uniform ideal.

like Reddy, Satyanarayana, Syam Prasad, Nagaraju (refer [4], [5], [6]).

If we consider ring as a module over itself, then the existing literature tells about dimension theory for ideals (i.e., two sided ideals) in case of commutative rings; and left (or right) ideals in case of associative (but not commutative) rings. So at present we can understand the structure theorem for associative rings in terms of one sided ideals only (that is, if R has FGD with respect to left (right) ideals, then there exist n uniform left (or right) ideals of R whose sum is direct and essential in R). This result cannot say about the structure theorem for associative rings in terms of two sided ideals. So to fill the gap, we prove the structure theorem for associative rings with respect to two sided ideals.

Throughout the paper R denotes an associative ring (need not be commutative). The paper is divided into three sections. In Secton-2 we introduce and study the concepts: complement, essential with respect to two sided ideals of R. In Section-3, we introduce the concept: uniform ideal and study few fundamental results which are useful in Section-4. In Section-4, we introduce the concept "finite dimension". We obtain an equivalent condition for an associative ring R to have finite dimension, which is used in the later part. Finally, we prove the main theorem: If an ideal H has FDIR, then there exist uniform ideals $U_1, U_2, \cdots, U_n$ of R whose sum is direct and essential in H. The number is independent of the choice of the uniform ideals U_i, 1 ≤ i ≤ n. This number n is called the dimension of H and we denote it by dim H.

Let R be a fixed (not necessarily commutative) ring. We write K ⊴ R to denote 'K is an ideal of R'. We use the term "ideal" for "two sided ideal". The ideal generated by an element a ∈ R is denoted by $< a >$. We do not include the proofs of some results when they are easy or straight forward verification.

2. Essential ideals

We start this section with the following definition.

Definition 2.1. Let I, J be two ideals of R such that I ⊆ J.
(i) We say that I is essential (or ideal essential) in J if it satisfies the following condition: K ⊴ R, K ⊆ J, I ∩ K = (0) imply K = (0).
(ii) If I is essential in J and I ≠ J, then we say that J is a proper essential extension of I. If I is essential in J, then we denote this fact by I $\leq_e$ J.

Definition 2.2. If K ⊴ R, A ⊴ R and K is a maximal element in {I / I ⊴ R, I ∩ A = (0)}, then we say that K is a complement of A (or a complement in R).

Note 2.3. Let I and J be ideals of R.
(i) I $\leq_e$ J ⇔ I ∩ K = (0), K ⊴ R ⇒ J ∩ K = (0).
(ii) B is a complement in R ⇔ there exists an ideal A of R such that B ∩ A = (0) and $K^1 \cap A \neq (0)$ for any ideal K^1 of R with B ⊊ K^1. In this case B + A $\leq_e$ R.

[*Verification*: Suppose B is a complement in R ⇒ B is complement of an ideal A of R ⇒ B is a maximal ideal of R with respect to the property B ∩ A=(0). Suppose

that $K^1 \trianglelefteq$ R with B $\subsetneq K^1$. If $K^1 \cap$ A = (0), then since B is maximal with respect to B $\cap$ A=(0) we get K^1 = B, a contradiction. Therefore $K^1 \cap A \neq$(0).The converse is clear.
Now we show that B + A $\leq_e$ R. Let J $\trianglelefteq$ R such that (B + A)$\cap$ J = (0). Let x $\in$ (B + J) $\cap$ A $\Rightarrow$ x = b + j $\in$ A where b $\in$ B and j $\in$ J $\Rightarrow$ b - x = -j $\in$ (B + A) $\cap$ J = (0) $\Rightarrow$ j = 0 $\Rightarrow$ x = b $\in$ A $\cap$ B = (0). Therefore (B + J) $\cap$ A = (0). Since B is maximal with respect to the property that B $\cap$ A = (0), we have that B + J = B. Now J $\subseteq$ B $\subseteq$ B + A $\Rightarrow$ J = J $\cap$ (B + A) = (0). Thus B + A $\leq_e$ R].

(iii) If A $\cap$ B = (0), and C is an ideal of R which is maximal with respect to the property C $\supseteq$ A and C $\cap$ B = (0), then C $\oplus$ B is essential in R. Moreover, C is a complement of B containing A. [Verification: Follows from (ii)].

Result 2.4. (i) *The intersection of finite number of essential ideals is essential;*
(ii) *If I, J, K are ideals of R such that* $I \leq_e J$*, and* $J \leq_e K$*, then* $I \leq_e K$*;*
(iii) $I \leq_e J \Rightarrow I \cap K \leq_e J \cap K$*;*
(iv) *If* $I \subseteq J \subseteq K$*, then* $I \leq_e K$ *if and only if* $I \leq_e J$*, and* $J \leq_e K$*; and*
(v) *If* R_1*,* R_2 *are two rings, f:* $R_1 \rightarrow R_2$ *is a ring isomorphism, and A is an ideal of* R_1*, then* $A \leq_e R_1 \Leftrightarrow f(A) \leq_e R_2$*.*

Proof. Proof for (i) to (iv) is a straight forward verification.
(v) Suppose A $\leq_e R_1$. We have to show that f(A) $\leq_e R_2$. Let B be an ideal of R_2 such that f(A) $\cap$ B = (0). Now A $\cap$ $f^{-1}(B)$ = (0). Since A is essential in R_1, we have $f^{-1}(B) = (0)$ and so B = $f(f^{-1}(B))$ = (0). Thus f(A)$\leq_e R_2$. If f(A) $\leq_e R_2$. The other part is similar. □

Note 2.5(Refer page 158 of [1])**.** If R is a ring, a $\in$ R, then $< a >$ = {ra + as + na + $\sum_{i=1}^{k} r_i a s_i$ / k $\in$ Z, n $\in$ Z, k $\geq$ 0, r, s, s_i, r_i $\in$ R }.

Remark 2.6. If a, b $\in$ R and x $\in$ $< a >$, then there exists y $\in$ $< b >$ such that x + y $\in$ $< a + b >$.

[*Verification*: Since x $\in$ $< a >$, by above Note 2.5, it follows that x = ra + as + na + $\sum_{i=1}^{k} r_i a s_i$. If y = rb + bs + nb + $\sum_{i=1}^{k} r_i b s_i \in < b >$, then x + y = r(a + b) + (a + b)s + n(a + b) + $\sum_{i=1}^{k} r_i (a+b) s_i \in < a + b >$.]

Lemma 2.7. (i) L_1*,* L_2*,* K_1*,* K_2 *are ideals of R such that* $L_i \subseteq K_i$ *for i = 1, 2 and* $K_1 \cap K_2 = (0)$*. Then* $L_1 \leq_e K_1$ *and* $L_2 \leq_e K_2 \Leftrightarrow L_1 + L_2 \leq_e K_1 + K_2$*; and*
(ii) *Let* $K_1, K_2, \cdots, K_t$*,* $L_1, L_2, \cdots, L_t$ *are ideals of R such that the sum* $K_1 + K_2 + \cdots + K_t$ *is direct and* $L_i \subseteq K_i$ *for* $1 \leq i \leq t$*. Then* $L_1 + L_2 + \cdots + L_t \leq_e K_1 + K_2 + \cdots + K_t \Leftrightarrow L_i \leq_e K_i$ *for* $1 \leq i \leq t$*.*

Proof. (i) Assume that $L_1 \leq_e K_1$ and $L_2 \leq_e K_2$. Write $A_1 = L_1 + K_2$ and $A_2 = K_1 + L_2$. We show that $A_1 \leq_e K_1 + K_2$. Let 0 $\neq$ a $\in K_1 + K_2$. Then a = $a_1 + a_2$ for some $a_1 \in K_1$, $a_2 \in K_2$. If $a_1 = 0$, then a $\in A_1$ and hence $< a > \cap A_1 \neq (0)$. If $a_1 \neq 0$, then since $L_1 \leq_e K_1$ and $0 \neq < a_1 > \subseteq K_1$ there exists $0 \neq x_1 \in < a_1 > \cap L_1$. By Remark 2.6, there exists $x_2 \in < a_2 >$ such that $x_1 + x_2 \in < a_1 + a_2 >$.

Since $x_1 \neq 0$, $0 \neq x_1 + x_2 \in \ < a_1 + a_2 > \cap A_1 = \ < a > \cap A_1$. Thus $A_1 \leq_e K_1 + K_2$. Similarly $A_2 \leq_e K_1 + K_2$. Since $L_1 + L_2 = A_1 \cap A_2$ by Result 2.4(i), it follows that $L_1 + L_2 = A_1 \cap A_2 \leq_e K_1 + K_2$.
Converse: Suppose that $L_1 + L_2 \leq_e K_1 + K_2$. To show $L_1 \leq_e K_1$, take $(0) \neq$ A $\trianglelefteq$ R such that A $\subseteq K_1$ and $L_1 \cap$ A = (0). Now x $\in$ A $\cap$ $(L_1 + L_2) \Rightarrow$ x $\in$ A and x $= l_1 + l_2$ for some $l_1 \in L_1$ and $l_2 \in L_2 \Rightarrow -l_1$ + x $= l_2 \in (L_1 + A) \cap L_2 \subseteq K_1 \cap K_2 = (0)$, $l_2 = 0 \Rightarrow$ x $= l_1 \in L_1 \cap$ A = (0). Therefore A $\cap$ $(L_1 + L_2) = (0)$. Since $L_1 + L_2 \leq_e K_1 + K_2$ we have that A = (0). Thus $L_1 \leq_e K_1$. In a similar way, we can show $L_2 \leq_e K_2$. The rest follows by using (i) and Mathematical induction on t. □

Note 2.8. Consider ideals A, B, C of R as in Note 2.3 (ii) and (iii). Here A $\oplus$ B $\leq_e$ R and A $\oplus$ B $\subseteq$ C $\oplus$ B $\subseteq$ R. Using Result 2.4 (iv), we get that A $\oplus$ B $\leq_e$ C $\oplus$ B. By Lemma 2.7, it follows that A $\leq_e$ C. Note that C is a complement ideal which is also an essential extension of A.

3. Uniform ideals

Definition 3.1. A non-zero ideal I of R is said to be uniform if (0) $\neq$ J $\trianglelefteq$ R, and J $\subseteq$ I $\Rightarrow$ J $\leq_e$ I.

Note 3.2. Let R_1, R_2 be two rings and f: $R_1 \to R_2$ is a ring isomorphism. I, J $\trianglelefteq$ R_1. Then $f^{-1}(I) \cap f^{-1}(J) = f^{-1}(I \cap J)$.

Theorem 3.3. (i) *I is an uniform ideal* $\Leftrightarrow L \trianglelefteq R,\ K \trianglelefteq R,\ L \subseteq I,\ K \subseteq I,\ L \cap K = (0) \Rightarrow L = (0)$ *or* $K = (0)$.
(ii) *Let* R_1 *and* R_2 *be two rings and f:* $R_1 \to R_2$ *be ring isomorphism. If U is ideal of* R_1*, then U is uniform in* $R_1 \Leftrightarrow f(U)$ *is uniform in* R_2.
(iii) *Let H and K be two ideals of R such that* $H \cap K = (0)$*. For an ideal U of R contained in H, we have that U is uniform* $\Leftrightarrow (U + K)/K$ *is uniform in R/K.*
(iv) *If U and K are two ideals of R such that* $U \cap K = (0)$*, then U is uniform in R* $\Leftrightarrow (U + K)/K$ *is uniform in R/K.*

Proof. (i) Let I be an uniform ideal of R. Suppose L $\trianglelefteq$ R, K $\trianglelefteq$ R, L $\subseteq$ I, K $\subseteq$ I, L $\cap$ K=(0). Suppose L $\neq$ (0). By our supposition, we have that L $\leq_e$ I. Since L $\leq_e$ I and L $\cap$ K = (0) by Note 2.3 (i), we have that K $\cap$ I = (0). Since K $\subseteq$ I it follows that K = K $\cap$ I = (0). The other part of (i) is straight forward verification.
(ii) is direct verification.
(iii) Define $f : H \to (H+K)/K$ by $f(h) = h+K$. Then f is a ring isomorphism. By (ii), we get that U is uniform in H $\Leftrightarrow f(U) = (U+K)/K$ is uniform in $(H+K)/K$.
(iv) follows from (iii). □

Remark 3.4. Let K be an uniform ideal of R and $L \trianglelefteq R$ such that $L \subseteq K$. Then either $L = (0)$ or L is uniform.

[*Verification*: Suppose $L \neq (0)$. Let $A \trianglelefteq R,\ B \trianglelefteq R$ such that $A \subseteq L,\ B \subseteq L$ and $A \cap B = (0)$. Since $A,\ B \subseteq L \subseteq K,\ A \cap B = (0)$ and K is uniform, it follows that $A = (0)$ or $B = (0)$. This shows that L is uniform.]

4. Associative rings with finite dimension

Definition 4.1. (i) We say that R has Finite Dimension on Ideals (FDI, in short) if R do not contain infinite number of non-zero ideals of R whose sum is direct.
(ii) Let $(0) \neq K \trianglelefteq R$. We say that K has finite dimension on ideals of R (FDIR, in short) if K does not contain an infinite number of non-zero ideals of R whose sum is direct. It is clear that if R has FDI, then every ideal K of R has FDIR.

Theorem 4.2. *K has FDIR $\Leftrightarrow$ for any strictly increasing sequence H_1, H_2, $\cdots$ of ideals of R contained in K, there is an integer i such that $H_k \leq_e H_{k+1}$ for every k $\geq$ i.*

Proof. Suppose K has FDIR. Take a strictly increasing sequence $H_1 \subseteq H_2 \cdots$ of ideals of R contained in K. In a contrary way, suppose that for every integer i there exists k $\geq$ i such that H_k is not essential in H_{k+1} $\cdots$ $\cdots$ (i)
Take i = 1. Then there exists $k_1 \geq 1$ such that H_{K_1} is not essential in H_{K_1+1} . Write $i_2 = k_1 + 1$. Then by (i), there exists $k_2 \geq i_2$ such that H_{K_2} is not essential in H_{K_2+1} . Note that $k_2 \geq k_1 + 1$. If we continue this process, then we get a subsequence $\{ H_{K_i}\}_{i=1}^{\infty}$ of $\{ H_i\}_{i=1}^{\infty}$ such that H_{K_i} is not essential in H_{K_i+1} and $k_{i+1} \geq k_i + 1$. Since the sequence $H_1 \subseteq H_2 \cdots$ is increasing we have that $H_{K_{i+1}} \supseteq H_{K_i+1}$. Since H_{K_i} is not essential in H_{K_i+1} and $H_{K_i+1} \subseteq H_{K_{i+1}}$ we have that H_{K_i} is not essential in $H_{K_{i+1}}$. Thus we got a subsequence $\{ H_{K_i}\}_{i=1}^{\infty}$ of $\{ H_i\}_{i=1}^{\infty}$ such that H_{K_i} is not essential in $H_{K_{i+1}}$ for all i. Write $B_i = H_{K_i}$ for i ≥ 1. Now $\{B_i\}_{i=1}^{\infty}$ is an increasing sequence of ideals of R contained in K such that B_i is not essential in B_{i+1}. Now for each i there exists a non-zero ideal A_i of R contained in K such that $A_i \subseteq B_{i+1}$ and $B_i \cap A_i = (0)$. Now we verify that the sum $\sum_{i=1}^{\infty} A_i$ is a direct sum of infinite number of non-zero ideals $\{A_i\}_{i=1}^{\infty}$. Let $x_1 \in A_{i_1}$, $x_2 \in A_{i_2}, \cdots, x_n \in A_{i_n}$ such that $x_1 + x_2 + \cdots + x_n = 0$. With out loss of generality, we may assume that $i_1 < i_2 < \cdots < i_n$. Also we can suppose that $x_n \neq 0$.
Now $A_{i_1} \subseteq B_{i_1+1} \subseteq \cdots \subseteq B_{i_n}$, $A_{i_2} \subseteq B_{i_2+1} \subseteq \cdots \subseteq B_{i_n}, \cdots, A_{i_{n-1}} \subseteq B_{i_{n-1}+1} \subseteq \cdots \subseteq B_{i_n}$
$x_1 + x_2 + \cdots + x_{n-1} \in A_{i_1} + A_{i_2} + \cdots + A_{i_{n-1}} \subseteq B_{i_n}$.
Now $(A_{i_1} + A_{i_2} + \cdots + A_{i_{n-1}}) \cap A_{i_n} \subseteq B_{i_n} \cap A_{i_n} = (0) \Rightarrow x_1 + x_2 + \cdots + x_{n-1} = -x_n \in (A_{i_1} + A_{i_2} + \cdots + A_{i_{n-1}}) \cap A_{i_n} = (0) \Rightarrow x_n = 0$, a contradiction. Hence the sum $\sum_{i=1}^{\infty} A_i$ is a direct sum of infinite number of non-zero ideals of R contained in K, a contradiction to (i). This completes the proof for (i) $\Rightarrow$ (ii).
(ii) $\Rightarrow$ (i): Suppose (ii). We have to verify that K does not contain a direct sum of infinite number of non-zero of ideals of R. In a contrary way, suppose that K contains a direct sum of infinite number of non-zero of ideals $\{I_i\}_{i=1}^{\infty}$. Write $J_n = I_1 + I_2 + \cdots + I_n$ for n ≥ 1. Then $J_1 \subseteq J_2 \subseteq \cdots$. Since $(0) \neq I_n \subseteq J_{n+1}$ and $J_n \cap I_{n+1} = (0)$, it follows that J_n is not essential in J_{n+1}. This is true for all n $\geq$ 1. Thus we arrived at a strictly increasing sequence $J_1 \subseteq J_2 \subseteq \cdots$ of ideals of R contained in K such that J_i is not essential in J_{i+1} for i ≥ 1, a contradiction to the

assumed condition (ii). This completes the proof. □

Lemma 4.3. *If R has FDI and* $(0) \neq K \trianglelefteq R$*, then K contains an uniform ideal of R.*

Proof. In a contrary way, suppose K contains no uniform ideal of R. Then K is not uniform ideal of R. So there exist $(0) \neq K_1 \trianglelefteq$ R and $(0) \neq L_1 \trianglelefteq$ R such that $K_1 \cap L_1 = (0)$, and $K_1 + L_1 \subseteq$ K (by Theorem 3.3(i)). Now L_1 is not uniform and so there exist $(0) \neq K_2 \trianglelefteq$ R, $(0) \neq L_2 \trianglelefteq$ R such that $K_2 \cap L_2 = (0)$ and $K_2 + L_2 \subseteq L_1$. If we continue this process, we get two infinite sequences $\{K_i\}_{i=1}^{\infty}$, $\{L_i\}_{i=1}^{\infty}$ of non-zero ideals of R such that $K_i \cap L_i = (0)$ for each i and $K_i + L_i \subseteq L_{i-1}$ for i ≥ 2. Also note that $L_1 \supseteq L_2 \supseteq \cdots$. The sum $\sum_{i=1}^{\infty} K_i$ is an infinite direct sum of non-zero ideals of R. [*Verification*: In contrary way, suppose that there exist non-zero elements $x_1 \in K_{i_1}$, $x_2 \in K_{i_2}$, $\cdots$, $x_n \in K_{i_n}$ such that $x_1+x_2+\cdots+x_n = 0$ where $i_1 < i_2 < \cdots < i_n$. $x_n \in K_{i_n} \subseteq L_{(i_n)-1} \subseteq \cdots \subseteq L_{i_1}$, $x_{n-1} \in K_{i_{n-1}} \subseteq L_{(i_{n-1})-1} \subseteq \cdots \subseteq L_{i_1}$, $x_2 \in K_{i_2} \subseteq L_{(i_2)-1} \subseteq L_{i_1}$. Now $x_2 + x_3 + \cdots + x_n \in L_{i_1}$. Since $x_1 \in K_{i_1}$ and $x_2+x_3+\cdots+x_n \in L_{i_1}$, it follows that $-x_1 = x_2+\cdots+x_n \in K_{i_1} \cap L_{i_1} = (0)$ and so $x_1 = 0$, a contradiction]. This is a contradiction to the fact that R has FDI. This completes the proof. □

Theorem 4.4. *Suppose* $0 \neq H \trianglelefteq R$ *and H has FDIR. Then the following conditions hold.*
(i) (*Existence*) *There exist uniform ideals* $U_1, U_2, \cdots, U_n$ *of R whose sum is direct and essential in H;*
(ii) *If* V_i*,* $1 \leq i \leq k$ *are uniform ideals of R, whose sum is direct and essential in H, then* $k \leq n$*.*
(iii) (*Uniqueness*) *If* V_i*,* $1 \leq i \leq k$ *are uniform ideals of R whose sum is direct and essential in H, then* $k = n$*.*

Proof. (i) Suppose H has FDIR. In a contrary way, suppose that for any finite number of uniform ideals U_i, 1 $\leq$ i $\leq$ n whose sum is direct, the sum $\sum U_i$ is not essential in H. By Lemma 4.3, H contains an uniform ideal U_1. Then U_1 is not essential in H. So there exists $0 \neq H_1 \trianglelefteq$ R such that $H_1 \subseteq$ H with $U_1 \cap H_1 = (0)$. Again by using Lemma 4.3, we conclude that H_1 contains a uniform ideal U_2. Now the sum $U_1 + U_2$ is a direct sum of two uniform ideals. So $U_1 + U_2$ is not essential in H. This means, there exists $(0) \neq H_2 \trianglelefteq$ R, $H_2 \subseteq$ H such that $(U_1 + U_2) \cap H_2 = (0)$. Again by using Lemma 4.3, we get an uniform ideal $U_3 \subseteq H_2$. Now the sum $U_1 + U_2 + U_3$ is direct. If we continue this process, we get an infinite strictly increasing chain $U_1 \subset (U_1 + U_2) \subset (U_1 + U_2 + U_3) \cdots$ of ideals of R such that $U_1 \oplus U_2 \oplus \cdots \oplus U_s$ is not essential in $U_1 \oplus U_2 \oplus \cdots \oplus U_s \oplus U_{s+1}$ for all s ≥ 1. By Theorem 4.2, it follows that H has no FDIR, a contradiction to our assumption. Hence there exist uniform ideals U_i, 1 $\leq$ i $\leq$ n in R whose sum $U_1 + U_2 + \cdots + U_n$ is direct and essential in H.
(ii) Suppose V_i, 1 $\leq$ i $\leq$ k are uniform ideals of R whose sum is direct and $V_1 + V_2 + \cdots + V_k \subseteq$ H. Write $K_1 = V_2 \oplus V_3 \oplus \cdots \oplus V_k$. Since K_1 is not essential in H, there exists i (1 $\leq$ i $\leq$ n) such that $K_1 \cap U_i = (0)$. Without loss of generality, we may

assume that $K_1 \cap U_1 = (0)$. Note that the sum $U_1 + V_2 + \cdots + V_k$ is direct. Write $K_2 = U_1 \oplus V_3 \oplus \cdots \oplus V_k$. Since K_2 is not essential in H there exists an i $(2 \leq i \leq n)$ such that $K_2 \cap U_i = (0)$. We may suppose that $K_2 \cap U_2 = (0)$. Now the sum $U_1 + U_2 + V_3 + \cdots + V_k$ is direct. If we continue this process, we can replace each V_j $(1 \leq j \leq k)$ by some U_i $(1 \leq i \leq n)$. From this discussion, we can conclude $k \leq n$.
(iii) In (ii) we verified that $k \leq n$. Similarly, we can verify that $n \leq k$. Hence $k = n$. □

As a consequence of this result we have the following Corollary.

Corollary 4.5. *If R is a ring with FDI, then the following* (i) - (ii) *are true:*
(i) (*Existence*) *There exist uniform ideals* $U_1, U_2, \cdots U_n$ *in R whose sum is direct and essential in R;*
(ii) (*Uniqueness*) *If* V_i*,* $1 \leq i \leq k$*, are uniform ideals of R whose sum is direct and essential in R, then* $k = n$*.*

Definition 4.6. The number n of the above Theorem is independent of the choice of the uniform ideals. This number n is called the dimension of H, and is denoted by dim H.

Acknowledgment. The first and second authors are thankful to the UGC (New Delhi) for its financial assistance under the grant No. F. 8-8/2004 (SR), dated 29-12-2003.

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