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FINITE-DIMENSIONAL APPROXIMATIONS OF UNSTABLE INFINITE-DIMENSIONAL SYSTEMS*

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Abstract. This paper studies approximation of possibly unstable linear time-invariant infinite-dimensional systems. The system transfer function is assumed to be continuous on the imaginary axis with finitely many poles in the open right half plane. A unified approach is proposed for rational approximations of such infinite-dimensional systems. A procedure is developed for constructing a sequence of finite-dimensional approximants, which converges to the given model in the L_∞ norm under a mild frequency domain condition. It is noted that the proposed technique uses only the FFT and singular value decomposition algorithms for obtaining the approximations. Numerical examples are included to illustrate the proposed method.

Key words. finite-dimensional approximations, infinite-dimensional systems, optimal Hankel approximation, balanced realization, discrete Fourier transform

AMS(MOS) subject classifications. 93C25, 93B15, 41A65, 41A20

1. Introduction. Since it is difficult to deal with infinite-dimensional systems directly, often a finite-dimensional approximate model is sought. The problem of approximating infinite-dimensional systems with finite-dimensional ones has been addressed by many authors in both the time domain [2], [8], [14], and the frequency domain [10]-[12], [17], [18], [21], [25], [26]. In this paper, we consider the approximation of possibly *unstable* linear time-invariant infinite-dimensional systems in the *frequency domain.* It is assumed that the transfer function $T(s)$ of the given system is continuous on the imaginary axis, including infinity and has only *finitely* many poles in the open right half plane. The objective is to seek a rational approximant $T_r(s)$, having the same number of unstable poles as $T(s)$, such that $\|T - T_r\|_\infty$ is suitably small. The motivation for this problem comes from feedback design considerations. For example, it follows from Curtain and Glover [5] and Chen and Desoer [4] that a controller stabilizes $T_r(s)$ also stabilizes $T(s)$, provided that $T(s)$ and $T_r(s)$ have the same number of poles in the right half plane and $\|T - T_r\|_\infty$ is suitably small.

One approach for the approximation of such unstable infinite-dimensional systems is to first separate the (finite-dimensional) unstable part by partial fraction expansion, and then consider the approximation of the stable part of system [5], [21]. Although partial fraction expansion is very effective for extracting the unstable part of the given infinite-dimensional system, it requires computation of the right half plane poles of the system. In this paper, we propose an alternative technique for the approximation of unstable infinite-dimensional systems. This work extends to unstable systems

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certain approximation techniques developed in [11], [17], [18], [25] for stable systems. An important feature of this proposed technique is the unification of approximations of both the stable and the unstable parts of $T(s)$ within a single algorithm. It will be shown that under certain mild conditions, the resulting approximants $T_r(s)$, which have same number of unstable poles as $T(s)$, converge to $T(s)$ in the L_∞-norm. Moreover, the proposed approximation technique uses only the FFT and singular value decomposition algorithms. Therefore, we expect our method to be preferable from the computational point of view. It should be noted that the fast Fourier transform technique has been used in many problems in the literature on computational complex analysis. See, for example, the survey paper by Henrici [13] and the references therein. Our work shows that these ideas are also very useful in system approximation problems, and lead to concrete convergence results as well as L_∞-error bounds. Also, the resulting algorithms are computationally very efficient.

It is also noted that other frequency domain approximation techniques such as those developed in [10], [12], [21] might be applicable to the approximation problem considered in this paper. However, we believe that our algorithm is attractive from a computational point of view as compared to some of these algorithms. Also, the extensive work in the Padé approximation literature is potentially applicable to the present problem. However, in this case, convergence and error analysis in the L_∞ norm remains a topic for future research in the context of our problem. Finally, the work of Trefethen [23] is also of interest for our problem. The Caratheodory–Fejer (CF) method proposed in [23] is considered to be very effective for frequency domain approximation [12]. However, it has been recently pointed out by Saff and Totik [22] that the CF method does not always provide a better approximation than partial sums of Fourier series, and there exist functions whose Fourier series converges uniformly but the approximant obtained using the CF method diverges. We would like to emphasize that this should not be taken to imply that the CF method is inferior in comparison to Fourier series. As indicated in [22], the CF method is superior to the partial sum of Fourier series for those functions that are sufficiently smooth.

We believe that the technique proposed in the present paper offers an effective alternative to the techniques that could be derived from the references cited above. Preliminary analysis appears to imply that all these techniques may have different domains of applicability. A comparative study of all these algorithms remains a subject for future work.

The paper is organized as follows. A preliminary result will be presented first for discrete-time systems in §2, which will be used to establish the main result of this paper in §3. Two numerical examples will be given in §4 to illustrate the approximation technique.

2. A preliminary result. Before studying the approximation of unstable infinite-dimensional systems, we will first establish a simple result that will be useful in the next section. Let $G(z)$ be the transfer function of a given linear, time-invariant, finite-dimensional, exponentially stable, discrete time system of McMillan degree n. Suppose that $G(z)$ is given by

$$G(z) = \sum_{k=1}^{\infty} g_k z^{-k}, \quad \text{with } g_k \in \mathcal{R}^{p\times m}. \tag{2.1}$$

Define the partial summation

$$S_N(z) := \sum_{k=1}^{N} g_k z^{-k}. \tag{2.2}$$

A simple (possibly nonminimal) realization for $S_N(z)$ is given by

$$A_N = \begin{bmatrix} 0 & 0 & .. & .. & 0 \\ I_m & 0 & 0 & .. & 0 \\ 0 & I_m & 0 & .. & .. \\ .. & .. & .. & .. & .. \\ 0 & .. & .. & I_m & 0 \end{bmatrix}, \quad B_N = \begin{bmatrix} I_m \\ 0 \\ .. \\ .. \\ 0 \end{bmatrix}, \quad C_N^T = \begin{bmatrix} g_1^T \\ .. \\ .. \\ .. \\ g_N^T \end{bmatrix}. \tag{2.3}$$

Since the above realization is controllable, an input normal realization [8], [10] of $S_N(z)$ (which has properties similar to balanced realization) can be easily found by solving two Lyapunov equations. In fact, with the realization as in (2.3), a much simpler algorithm can be used to compute a similarity transformation T (which is a unitary matrix) using only one singular value decomposition (see [11] for more details) such that

$$(A_b, B_b, C_b) = (TA_NT^T, TB_N, C_NT^T) \tag{2.4}$$

is an input normal realization of $S_N(z)$. Now (for $N \geq n$) an approximant $G_n^N(z)$ of degree n can be obtained by direct truncation of the input normal realization (A_b, B_b, C_b) as follows:

$$A_n^N = [I_n \quad 0]\, A_b \begin{bmatrix} I_n \\ 0 \end{bmatrix}, \qquad B_n^N = [I_n \quad 0]\, B_b, \qquad C_n^N = C_b \begin{bmatrix} I_n \\ 0 \end{bmatrix}. \tag{2.5}$$

It is noted that the McMillan degree of $G_n^N(z)$ may be smaller than n. However, if the McMillan degree of S_N is no smaller than n, then the McMillan degree of G_n^N is exactly n. With $G_n^N(z) := C_n^N(zI - A_n^N)^{-1}B_n^N$ as described above, we have the following result.

THEOREM 2.1. *Let $G(z)$ be given as in* (2.1). *Suppose that $G(z)$ is exponentially stable and has McMillan degree n. Then, $\lim_{N\to\infty} \|G - G_n^N\|_\infty = 0$, where G_n^N is obtained from* (2.2)–(2.5).

Proof. It is easy to show that (see also [10]), for $k > n$,

$$\sigma_k(S_N) \leq \sum_{k=1}^{\infty} \sigma_{max}(g_{N+k}), \tag{2.6}$$

where $\sigma_k(S_N)$ is the kth Hankel singular value (in descending order) of $S_N(z)$ and $\sigma_{max}(g_{N+k})$ is the maximum singular value of $p \times m$ matrix g_{N+k}. Hence,

$$\sum_{n+1}^{N} \sigma_k(S_N) \leq (N-n) \sum_{k=1}^{\infty} \sigma_{max}(g_{N+k}). \tag{2.7}$$

Since G_n^N is obtained from (2.2)–(2.5), using the results from Glover [9], Enns [6], and Al-Saggaf and Franklin [1], it is easy to see that

$$\|S_N - G_n^N\|_\infty \le 2 \sum_{k=n+1}^{N} \sigma_k(S_N). \tag{2.8}$$

By the triangle inequality and the fact that $\|G - S_N\|_\infty \le \sum_{k=1}^{\infty} \sigma_{max}(g_{N+k})$, it follows from (2.6)–(2.8) that

$$\|G - G_n^N\|_\infty \le \|G - S_N\|_\infty + \|S_N - G_n^N\|_\infty \le 2(N - n + \tfrac{1}{2}) \sum_{k=1}^{\infty} \sigma_{\max}(g_{N+k}). \tag{2.9}$$

By the hypothesis, $G(z)$ admits a minimal realization (A, B, C) where $A \in \mathcal{R}^{n\times n}$, $B \in \mathcal{R}^{n\times m}$ and $C \in \mathcal{R}^{p\times n}$ such that $G(z) = C(zI - A)^{-1}B$ and the spectral radius of A, $\rho(A) < 1$. Therefore,

$$\sigma_{\max}(g_k) \le \alpha\rho(A)^k \tag{2.10}$$

for some $\alpha > 0$. Hence, the error estimate in (2.9) can then be bounded as

$$\|G - G_n^N\|_\infty \le 2(N - n + \tfrac{1}{2}) \frac{\alpha\rho(A)^N}{1 - \rho(A)}. \tag{2.11}$$

The condition $\rho(A) < 1$ guarantees that $\lim_{N\to\infty} \|G - G_n^N\|_\infty = 0$.

Note that the approximant $G_n^N(z)$ can also be obtained from nth-order optimal Hankel approximation of $S_N(z)$ as in (2.2) for which Theorem 2.1 is still true (see Glover [9] and Kung and Lin [16]). However, as N becomes large, the computational burden associated with the Hankel approximation technique would be significantly higher compared to the input-normal-realization-based direct truncation technique. Finally, although the error bound in (2.11) is conservative, it does indicate that the convergence depends directly on the value of $\rho(A)$.

3. Main result. In this section, we consider the approximation of unstable, continuous-time infinite-dimensional system $T(s)$. Let $\mathcal{H}$ and $\mathcal{D}$ denote the open right half plane and the open unit disc, respectively. It is assumed that $T(s) \in L_\infty$ and has only finitely many poles in $\mathcal{H}$.

As mentioned earlier, one technique for obtaining finite-dimensional approximations is to use the partial fraction expansion to decompose $T(s) = T_s(s) + T_u(s)$ with $T_s(s)$ and $T_u(-s)$ both analytic in $\mathcal{H}$. Consequently, much of the existing research work concentrates only on the approximation of stable part $T_s(s) = T(s) - T_u(s)$. From the computational point of view, it is preferable to avoid the partial fraction decomposition.

In the rest of this section, we develop a new technique to obtain rational approximations for possibly unstable systems. The transfer function of the given continuous-time infinite-dimensional system is first transformed to a function on the unit circle by means of a bilinear transformation. This transformation preserves the L_∞ norm as well as Hankel singular values [9]. The rational approximant is then obtained by using the FFT and singular value decomposition algorithms. The resulting approximant is then transformed back to obtain an approximation of the original system by means of the inverse bilinear transformation.

Define the bilinear transformation

$$s := \lambda \frac{1-z}{1+z} \quad \text{or} \quad z := \frac{\lambda - s}{\lambda + s}, \tag{3.1}$$

which is a conformal mapping from $\mathcal{H}$ to $\mathcal{D}$. Next, define

$$F(z) := T\left(\lambda \frac{1-z}{1+z}\right). \tag{3.2}$$

Then, $F(z) = \sum_{k=-\infty}^{\infty} f_k z^k = F_s(z) + F_u(z)$, (which converges in the L_2-sense) with

$$F_u(z) = \sum_{k=1}^{\infty} f_{-k} z^{-k} \text{ and } F_s(z) = \sum_{k=0}^{\infty} f_k z^k. \tag{3.3}$$

Clearly, $F_u(z)$ and $T_u(s)$ have the same McMillan degree which by assumption is finite. Furthermore, since the bilinear transformation does not change the Hankel singular values of the original transfer function as shown in [9], the Hankel singular values of $F_u(z)$ are exactly the same as those of $T_u(s)$. Therefore, if the sequence $\{f_{-k}\}_{k=1}^{\infty}$ is known precisely, $F_u(z)$ can be reconstructed using a number of different techniques from the realization theory literature. A problem arises, since we would like to avoid computing the sequence $\{f_k\}$ exactly. Let us define a $2M$-point inverse discrete Fourier transform as follows to compute $\{f_k\}$ approximately:

$$f_M(k) = \frac{1}{2M} \sum_{r=-M}^{M-1} F(W_{2M}^r) W_{2M}^{-rk}, \qquad k = -M, -M+1, \cdots, M-1, \tag{3.4}$$

where $W_{2M} = e^{j\pi/M}$. The sequence $\{f_M(k)\}$ can then be used as an approximation for $\{f_k\}$.

The DFT-based approximation has been studied in [11] for stable infinite-dimensional systems, and the convergence, as well as the error bounds, are established for a class of infinite-dimensional systems. Here, we concentrate on the approximation of the unstable part of the system and obtain some similar results. We first state a lemma based on which the main result of the paper will be obtained.

LEMMA 3.1. *Let $F(z)$ be defined as in* (3.2) *and let $F_u(z)$ be of finite McMillan degree. Suppose that $dF(e^{j\omega})/de^{j\omega} \in L_2[0, 2\pi]$. Then,*

(i) $\{\|f_k\|\} \in \ell^1$, *(that is,* $\sum_{k=-\infty}^{\infty} \|f_k\| < \infty$*) and*

(ii) $f_M(k) = \sum_{L=-\infty}^{\infty} f_{2LM+k}$,

where f_k and $f_M(k)$ are defined by (3.3) *and* (3.4), *respectively.*

This result is quite well known. See, for example, [15], [13].

Note that since $F_u(z)$ is analytic on unit circle and its McMillan degree is finite, the condition $dF(e^{j\omega})/de^{j\omega} \in L_2[0, 2\pi]$ is, in fact, equivalent to $dF_s(e^{j\omega})/de^{j\omega} \in$

$L_2[0, 2\pi]$. Hence, $F(z)$ (or $F_s(z)$) is continuous on the unit circle. By (3.2), the continuity of $F(z)$ on unit circle is equivalent to the continuity of $T(s)$ on the extended $j\omega$ axis. Therefore the hypothesis in Lemma 3.1 implies that the transfer function $T(s)$ admits rational approximants that converge in L_∞-norm to T. Our results in this paper, in fact, give a constructive procedure for obtaining such approximants.

THEOREM 3.2. *Let $F(z)$ be defined as in (3.2) and $F_u(z)$ have McMillan degree n. Define $S_N^M(z) := \sum_{k=1}^N f_M(-k)z^{-k}$, with $f_M(k)$ as in (3.4), $N > n$. Suppose that $dF(e^{j\omega})/de^{j\omega} \in L_2[0, 2\pi]$. Then*

$$\lim_{\sqrt{M} \geq N \to \infty} \|F_u - F_{u;n}^{M;N}\|_\infty = 0, \tag{3.5}$$

where $F_{u;n}^{M;N}(z)$ is an nth-order approximant of $S_N^M(z)$ obtained using the balanced truncation scheme described in §2 (or the optimal Hankel approximation method).

Proof. By the triangle inequality,

$$\|F_u - F_{u;n}^{M;N}\|_\infty \leq \|F_u - S_N\|_\infty + \|S_N - S_N^M\|_\infty + \|S_N^M - F_{u;n}^{M;N}\|_\infty, \tag{3.6}$$

where $S_N(z) = \sum_{k=1}^N f_{-k}z^{-k}$. We will show that the three terms on the right-hand side of (3.6) approach zero as $\sqrt{M} \geq N \to \infty$.

Indeed, because $F_u(z)$ is rational and has all its poles in $\mathcal{D}$, it is easy to see that

$$\lim_{N \to \infty} \|F_u - S_N\|_\infty = 0. \tag{3.7}$$

Furthermore, since $\|z^{-k}\|_\infty = 1$, we have that $\|S_N - S_N^M\|_\infty \leq \sum_{k=1}^N \sigma_{\max}(f_M(-k) - f_{-k})$. Lemma 3.1 implies that

$$f_\Delta(k) = f_M(k) - f_k = \sum_{L \neq 0} f_{2LM+k}, \tag{3.8}$$

where the summation is with respect to L and k is fixed. Hence,

$$\|S_N - S_N^M\|_\infty \leq \sum_{k=1}^N \sigma_{\max}(f_\Delta(k)) \leq \sum_{k=1}^\infty \{\sigma_{\max}(f_{M+k-1}) + \sigma_{\max}(f_{-M-k})\} \to 0 \tag{3.9}$$

as $M \to \infty$. Finally, the third term on the right-hand side of (3.6) is bounded by

$$\|S_N^M - F_{u;n}^{M;N}\|_\infty \leq \beta \sum_{i=n+1}^N \sigma_i(S_N^M), \tag{3.10}$$

where $\beta = 1$ if $F_{u;n}^{M;N}$ is obtained from optimal Hankel approximation of S_N^M [9] and $\beta = 2$, if $F_{u;n}^{M,N}$ is obtained from the reconstruction scheme in §2 or the balanced realization technique for S_N^M [1]. Now

$$S_N^M(z) = \sum_{k=1}^N f_M(-k)z^{-k} = S_N(z) + \sum_{k=1}^N f_\Delta(-k)z^{-k}. \tag{3.11}$$

By singular value perturbation results and Theorem 2.1, we get

$$\begin{aligned}\sigma_i(S_N^M) &\le \sigma_i(S_N) + \sum_{k=1}^{N} \sigma_{\max}(f_\Delta(-k)) \\ &\le \sum_{k=1}^{\infty} \sigma_{\max}(f_{-N-k}) + \sum_{k=1}^{N}\sum_{L\neq 0} \sigma_{\max}(f_{2LM-k}),\end{aligned} \tag{3.12}$$

whenever $i > n$. Note that in deriving the above inequality, (2.6) is used with g_k replaced by f_{-k}. Taking sum of σ_i for $n+1 \le i \le N$ above, we get

$$\|S_N^M - F_{u;n}^{M;N}\|_\infty \le \beta(N-n)\sum_{k=1}^{\infty}\{2\sigma_{\max}(f_{-N-k}) + \sigma_{\max}(f_{M+k})\}. \tag{3.13}$$

Since the derivative of $F(z)$ is absolutely square-integrable on the unit circle and its unstable part is finite-dimensional, the derivative of $F_s(z)$ is also absolutely square-integrable on the unit circle, which implies that

$$f_{M+k} = \frac{\hat{f}_{M+k}}{M+k} \quad \text{and} \quad \sum_{k=1}^{\infty} \sigma_{\max}(\hat{f}_k)^2 < \infty, \tag{3.14}$$

where $\hat{f}_k$ is the kth Fourier series coefficient of $dF(z)/dz$. Moreover, since the unstable part has only n poles on open unit disc, $\sigma_{\max}(f_{-k}) \le \alpha_o \rho_o^k$ for some $\alpha_o > 0$ and $\rho_o < 1$, where $k > 0$. Therefore, using the Schwarz inequality, we have

$$\|S_N^M - F_{u;n}^{M;N}\|_\infty \le 2\beta(N-n)\frac{\alpha_o\rho_o^N}{1-\rho_o} + \frac{\beta(N-n)}{\sqrt{M}}\sqrt{\sum_{k=1}^{\infty}\sigma_{\max}(\hat{f}_{M+k})^2} \to 0, \tag{3.15}$$

as $\sqrt{M} \ge N \to \infty$. The proof is now complete using (3.6)–(3.9) and (3.15).

Remark 3.1. It is important to note that n, the McMillan degree of $F_u(z)$, may not be known in advance. However, from Theorem 3.2 (also (3.12)), the first n Hankel singular values of $S_N^M(z)$ converge to the true Hankel singular values of $F_u(z)$ and the rest of the Hankel singular values converge to zero as $(M, N) \to (\infty, \infty)$, with $M > N$. Therefore, as M, N are both large, a gap between $\sigma_n(S_N^M)$ and $\sigma_{n+1}(S_N^M)$ would be significant if $\sigma_n(F_u)$ is not too small. In this case, the McMillan degree of $F_u(z)$ can also be identified in the approximation process.

Since $F(z)$ is given by (3.2), the frequency domain condition $dF(e^{j\omega})/de^{j\omega} \in L_2[0, 2\pi]$ is, in fact, equivalent to $(\lambda - j\omega)(dT(j\omega)/dj\omega) \in L_2[-\infty, \infty]$ (see [24]). This condition is difficult to verify in general. However, for a class of time delay systems, we can state the following.

PROPOSITION 3.3. *Let $T(s)$ be a transfer function of the form*

$$T(s) = \frac{\sum_{k=o}^{m} Q_k(s)\exp(-h_k s)}{s^n + \sum_{k=0}^{n} p_k(s)\exp(-\tau_k s)}, \tag{3.16}$$

where $p_k(s)$ is a scalar polynomial of s, $Q_k(s)$ is a polynomial matrix of size $m \times r$, and $0 \le h_0 \le h_1 \le \cdots \le h_m, \tau_k > 0$ for $0 \le k \le n$. Let $d_k = \deg(Q_k(s))$ and

$\delta_k = \deg(p_k(s))$. *It is assumed that* $d_k < n$ *and* $\delta_k < n$. *Then* $(\lambda - j\omega)(dT(j\omega)/dj\omega) \in L_2[-\infty, \infty]$, *if the following statements hold:* (1) $T(s)$ *is continuous on imaginary axis; and* (2) (i) $d_k \le n - 1$, *if* $h_k = 0$; (ii) $d_k < n - 1$, *if* $h_k \ne 0$.

We omit the proof of the above proposition, as it is an easy extension of a result in [11]. To conclude our results, we summarize the following algorithm for rational approximation of unstable part of the given infinite-dimensional system.

Algorithm 3.1 (Rational approximation).
Step 1: For a given unstable infinite-dimensional transfer function $T(s)$, verify first if $(\lambda - j\omega)(dT(j\omega)/dj\omega) \in L_2[-\infty, \infty]$ and choose $\lambda > 0$ to find $F(z)$ as in (3.2);
Step 2: Use $2M$-point inverse FFT algorithm to compute $f_M(k)$ as defined in (3.4);
Step 3: Compute Hankel singular values of S_N^M as defined in (3.11) with $N^2 \le M$, N large enough, and estimate n: the number of unstable poles of $T(s)$;
Step 4: Apply (2.3)–(2.5) to $S_N^M(z)$ to obtain $F_{u;n}^{M;N}(z) = C_n^N(zI - A_n^N)^{-1}B_n^N$;
Step 5: Use bilinear transform (3.1) to obtain an approximation for the unstable part of $T(s)$.
End

As discussed earlier, the hypothesis in Theorem 3.2 (which is same as in Lemma 3.1) implies that $dF_s(e^{j\omega})/de^{j\omega} \in L_2[0, 2\pi]$. The approximation of such $F_s(z)$ using $\{f_M(k)\}_{k=0}^{\infty}$ has been studied previously in [11], where some convergence results and the error bounds have been established. Hence, we will only briefly describe the approximation of the stable part of $F(z)$ below.

Define the partial summation as the approximant of the stable part

$$S^{M;L}(z) = \sum_{k=0}^{L} f_M(k) z^k = J_L + H_L(z^{-1}I - F_L)^{-1}G_L, \tag{3.17}$$

where the realization (F_L, G_L, H_L, J_L) is similar to (2.3). Therefore, the above realization can be easily converted into an input normal realization by computing only one singular value decomposition. The rational approximant of the stable part of McMillan degree no larger than ℓ can then be obtained by direct truncation as in (2.5), which is denoted as

$$F_{s;\ell}^{M;L}(z) = J_\ell^L + H_\ell^L(z^{-1}I - F_\ell^L)^{-1}G_\ell^L. \tag{3.18}$$

It has been established in [11] that if the conditions in Theorem 3.2 are true, then

$$\lim_{M \ge L \ge \ell \to \infty} F_{s;l}^{M;L}(z) = F_s(z), \quad \forall z \in \mathcal{D}. \tag{3.19}$$

The procedure described above is similar to the approximation of unstable part except that N is replaced by L, and n is replaced by ℓ, and while n is kept fixed, $\ell \to \infty$. Therefore, the approximation of both stable and unstable part of $T(s)$ can be handled with Algorithm 3.1. The final approximant of $F(z)$ can then be obtained as

$$F_r(z) = F_{s;\ell}^{M;L}(z) + F_{u;n}^{M;N}(z). \tag{3.20}$$

Finally, the finite-dimensional approximation given by $T_r(s) := F_r((\lambda - s)/(\lambda + s))$ is an approximation of $T(s)$. With these definitions, we have the main result of the paper.

THEOREM 3.4. *Let $T(s)$ be the transfer function of a given infinite-dimensional system having finitely many poles on open right half plane. Assume that $T(s)$ is continuous on extended $j\omega$-axis and $(\lambda - j\omega)dT(j\omega)/dj\omega \in L_2(-\infty, \infty)$. Then, with $T_r(s)$ obtained from the above approximation procedure,* $\lim \|T - T_r\|_\infty = 0$, *as* $M \geq L \geq \ell \to \infty$ *and* $\sqrt{M} \geq N \to \infty$.

It is noted that the bilinear transform does not change the L_∞-norm of the transfer function and thus $\|T - T_r\|_\infty = \|F - F_r\|_\infty$.

Remark 3.2. We would like to indicate further that as the approximate model is used for feedback control system design, the following condition should be satisfied to ensure the existence of stabilizing feedback compensator [5]:

$$\|T - T_r\|_\infty < \sigma_n(F_{u;n}^{M;N}) = \sigma_{\min}(F_{u;n}^{M;N}). \tag{3.21}$$

This is due to the fact that the bilinear transform does not change the Hankel singular values.

4. Illustrative examples. To illustrate the approximation technique proposed in this paper, two examples are presented below.

Example 4.1. Consider the following transfer function:

$$T(s) = \frac{20(6e^{-2s} + 2e^{-s} - 6)}{6s^2 + (6e^{-2s} - 2e^{-s} - 66)s - (2e^{-3s} + 30e^{-2s} - 12e^{-s} - 180)}. \tag{4.1}$$

It is easy to verify that the above $T(s)$ is continuous on imaginary axis. Furthermore, the convergence condition in Theorem 3.2 is true in light of Proposition 3.3. If the partial fraction is used for approximation, the poles of $T(s)$ as well as the residues must be computed, which, computationally, is very demanding. Using Algorithm 3.1, we applied bilinear transform (3.1) with $\lambda = 10$. A 2048-point FFT program is used to compute the sequence $\{f_M(k)\}$ as defined in (3.4). The partial sum $S_N^M(z)$ is obtained with $N = 9$. The gap between the second and third Hankel singular values was significant, suggesting that the number of unstable poles of $T(s)$ is $n = 2$. The approximate unstable part of the system is finally obtained as

$$\hat{T}_u(s) = \frac{-0.0448(s + 439.34)}{(s - 5.0035)(s - 5.9981)}. \tag{4.2}$$

It is noted that the exact unstable poles are 5.002224 and 5.999994, which are very close to the poles of $\hat{T}_u(s)$ above.

We would like to mention that this particular transfer function has a very rich frequency response (see the dash line curve in Fig. 4.1). Hence, it is not easy to find a simple finite-dimensional approximation for this transfer function. We have also used Algorithm 3.1 (with necessary modification) for approximation of the stable part with $L = 45$ and $\ell = 15$. The approximant $\hat{T}(s)$ of degree 15 is obtained. By setting $T_r(s) = \hat{T}_s(s) + \hat{T}_u(s)$ as the approximant, a satisfactory result is achieved. The magnitude frequency response of the approximant $T_r(s)$ is plotted in solid line in Fig. 4.1. It is seen that the frequency response of $T_r(s)$ matches very well with that of $T(s)$ for the first seven peaks. The frequency response of the error function can be found in Fig. 4.2. Note that

$$\|T - T_r\|_\infty = 0.0437 < \sigma_{\min}(\hat{T}_u) = 0.0689. \tag{4.3}$$

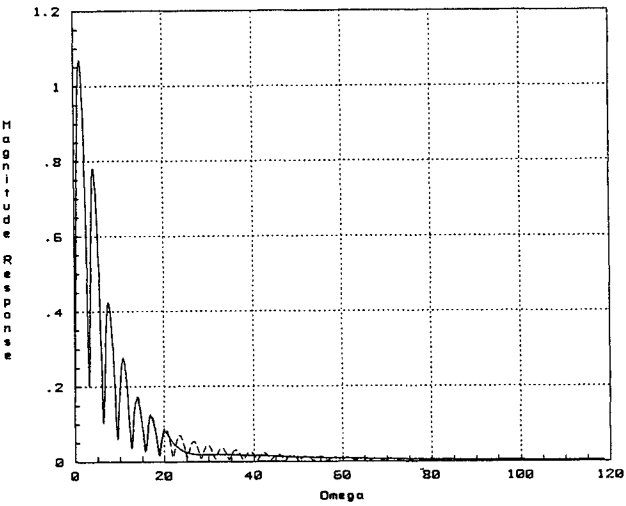

FIG. 4.1. *Frequency response of* $\hat{T}(s)$ *and* $T(s)$.

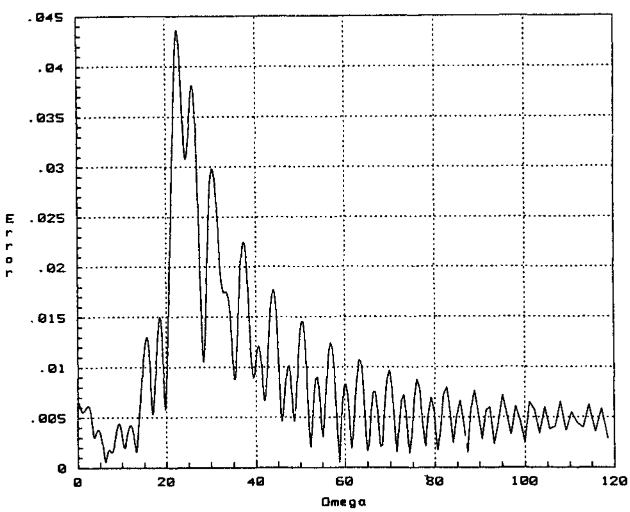

FIG. 4.2. *Frequency response of* $\hat{T}(s) - T(s)$.

Therefore, the existence of feedback compensators, which stabilizes both $T_r(s)$ and $T(s)$, is guaranteed. The approximant $T_r(s)$ in fractional form is listed in Table 4.1.

Example 4.2. Consider the system described by delay-differential equation

$$\begin{aligned}\dot{x}(t) &= A_1x(t) + A_2x(t-1) + Bu(t)\\ &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & -1 & 1 \\ 0 & 1 & -1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} x(t-1) + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} u(t).\end{aligned} \tag{4.4}$$

The above system was used in Fiagbedzi and Pearson [7], where stabilization

TABLE 4.1
Coefficients of $\hat{T}_s(s)$ in Example 4.1

Power of s	Numerator coefficients	Denominator coefficients
17	−4.674443306368986e−03	1.000000000000000e+00
16	4.239380836236473e−01	2.344791598941533e+01
15	−4.521836609299845e+01	1.291295941920766e+03
14	3.466157426093782e+02	2.081800085936672e+04
13	−6.582229625369409e+04	5.962725161939924e+05
12	9.097441642147570e+04	6.002018359817575e+06
11	−3.713504005871573e+07	1.216507871727509e+08
10	6.293813076095961e+06	5.190579448022859e+08
9	−9.927606421876093e+09	1.075305465480490e+10
8	−6.799049187753367e+08	−3.125834632061094e+10
7	−1.313153757820815e+12	3.399723344157356e+11
6	−1.029096097997340e+11	−5.810536260868385e+12
5	−8.195766310713688e+13	6.376021158219976e+12
4	−3.100690880350008e+12	−1.624831770313712e+14
3	−2.059046683673155e+15	5.092921660281645e+14
2	−5.551124575148588e+13	3.600520359638086e+14
1	−1.354380608580234e+16	8.398888911991564e+15
0	2.229241917067798e+15	9.157772117540838e+15

with state feedback was investigated. The synthesis technique proposed in [7] involves the computation of undesirable modes of the system. The state feedback law is then designed to shift the undesirable modes to the left of $\mathrm{Re}(s) = \nu_o$, where $\nu_o < 0$ represents the stability margin of the closed-loop system. For the above system, $\nu_o = -1$ was chosen in [7]. We demonstrate that the proposed approximation technique can also be used to compute the poles to the right side of $\mathrm{Re}(s) = \nu_o$.

First, we choose $C = B^T$, so that

$$T(s) = C(sI - A)^{-1}B = \frac{s^3 + (1 + e^{-s})s^2 + (1 + 2e^{-s})s + e^{-s}}{s^4 + (1 + e^{-s})s^3 + 2(1 + e^{-s})s^2 + (1 + 2e^{-s})s + 2e^{-s}}, \tag{4.5}$$

where $A = A_1 + A_2e^{-s}$. It is not difficult to show that with $C = B^T$, the system is both controllable and observable. Furthermore, using Proposition 3.3, the convergence conditions as in Theorem 3.2 are satisfied. Next, we take $\tilde{s} = s + 1$ and determine the unstable part of $T_a(\tilde{s}) = C(\tilde{s}I - A - I)^{-1}B$. Using Algorithm 3.1, we select $\lambda = 1$ for bilinear transform and use a 2048-point inverse FFT to compute $\{f_M(k)\}$. Since the fifth Hankel singular value of $S_{15}^{1024}(z)$ is very small, it is clear that the number of unstable poles of $T_a(\tilde{s})$ is $n = 4$. With the model reduction scheme described in §2, we obtain a fourth-order approximant $F_{u;n}^{M;N}(z)$. Based on the rational approximant of unstable part, we finally computed the poles of $T(s)$ on the right of $Re(s) = -1$ approximately, as below:

$$\{-0.186364675 \pm j0.91770066797;\ 0.11438695855 \pm j1.517680152\}.$$

The above poles are very close to the ones in [7], which are computed using the algorithm in Manitius et al. [19] (within 11-digit of exact poles).

Remark 4.1. It should be emphasized that in the above computation, we used only an inverse FFT and a singular value decomposition program, whereas the algorithm in [19] involves searching for poles on each rectangular region of the s-plane by computing Cauchy index with contour integral and then applying the numerical procedure to find the roots of the exponential polynomial in that particular region. The proposed method, therefore, provides significant computational saving.

Remark 4.2. The selection of parameter λ in bilinear transformation in (3.1) is important in computing the approximant. Extensive experimental experience shows that selecting λ as the bandwidth of $T(s)$ often yields better numerical results.

5. Concluding remarks. In this paper, we proposed a systematic procedure for rational approximation of unstable infinite-dimensional systems with finitely many right half plane poles (McMillan degree n). Convergence results for rational approximation from truncated Fourier series expansion of the given system transfer function were established. A computational procedure using FFT and singular value decomposition algorithms was outlined. The proposed technique is numerically more reliable and computationally more efficient than the existing procedures to achieve the same objective. Numerical examples illustrated the performance of the proposed technique.

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