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FINITE-DIMENSIONAL ATTRACTOR FOR THE VISCOUS CAHN-HILLIARD EQUATION IN AN UNBOUNDED DOMAIN

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Abstract. We consider the viscous Cahn-Hilliard equation in an infinite domain. Due to the noncompactness of operators, we use weighted Sobolev spaces to prove that the semigroup generated by this equation has the global attractor which has finite Hausdorff dimension.

1. Introduction. Many equations arising from mechanics and physics possess a global attractor, which is a compact invariant set which uniformly attracts the trajectories as time goes to infinity, and thus appears as a suitable object for the study of the asymptotic behaviour of the system. An important issue is then to study the dimension, in the sense of the Hausdorff or fractal dimension, of the global attractor. A finite bound of the dimension of the attractor means that the system has an asymptotic behaviour determined by a finite number of degrees of freedom, indeed a remarkable improvement compared to the a priori infinite-dimensional dynamics (see [5] and [15]).

For the equations on bounded domains, the known constructions of global attractors make use of some compactness properties in an essential manner, and more specifically of the compact embedding of H^{m_1} into H^{m_2}, when $m_1 > m_2$. Such properties are no longer valid for equations on unbounded domains, and it is thus more difficult to develop a general theory of existence of the global attractor in this case. A possibility then consists of working in weighted Sobolev spaces (see [1], [3], [4], [6], and [10]).

In this article, we study, on an unbounded domain, the existence of global attractors and their Hausdorff dimensions for the viscous Cahn-Hilliard equation of the form

$$\partial_t(u - \beta\Delta u) + \nu\Delta^2 u - \Delta(f(u) + \lambda_0 u + g) = 0, \tag{1.1}$$

where $\beta, \nu > 0$, $\lambda_0 \geq 0$, and f and g are given functions and $u = u(x,t)$ is the unknown function. Such equations, where f is the derivative of some double-well potential F, are

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generalizations of the Cahn-Hilliard equation which is very important in material science and models the qualitative behaviours of two phase systems (see [7], [8], [9], [11], [12], [13] and [14]).

The layout of this paper is as follows. In Section 2, we set the problem. In Section 3 and 4, we obtain some results, as existence of solutions, in unweighted and in weighted spaces, respectively. Section 5 is devoted to the study of the existence of the global attractor. Finally, in Section 6, we prove that the Hausdorff dimension of the global attractor is finite.

Throughout this paper, the same letter C and c (and sometimes c_i, $i = 0, 1, 2, ...$) shall denote positive constants that may change from line to line.

2. Setting of the problem. For the sake of simplicity, we take, in this section, $\lambda_0 = 0$ and $g = 0$, and consider the following system:

$$\begin{cases} \partial_t(u - \beta\Delta u) + \nu\Delta^2 u - \Delta f(u) = 0 \quad \text{in} \quad \Omega \times \mathbb{R}^+, \\ u|_{t=0} = u_0 \quad \text{in } \Omega, \\ u = \Delta u = 0 \quad \text{on } \partial\Omega \times \mathbb{R}^+, \end{cases} \tag{2.1}$$

where $\Omega \subset \mathbb{R}^2$ is defined by the inequalities

$$b_1(x_1) \leq x_2 \leq b_2(x_1), \quad x_1 \in \mathbb{R}, \tag{2.2}$$

and where b_1 and b_2 are twice continuously differentiable functions bounded over the entire axis

$$\begin{cases} -M \leq b_1(x_1) \leq b_2(x_1) \leq M, \\ |b_i'(x_1) + b_i''(x_1)| \leq c, \quad i = 1, 2. \end{cases} \tag{2.3}$$

All the results of this paper are valid for any open set Ω of $\mathbb{R}^d$, $d = 2$ or 3, regular and bounded at least in one direction. That is, Ω is included in the set limited by two hyperplanes orthogonal to that direction. The nonlinear term $f(u)$ is supposed to satisfy the following conditions:

$$\begin{cases} f(u)u \geq 0, \\ f(0) = 0, \quad f'(0) = 0, \quad f'(u) \geq -c, \\ |f'(u) - f'(v)| \leq c|u - v|^{\alpha_0}(1 + |u| + |v|)^{q_0}, \\ |f(u)| \leq c_1|u|^{1+\alpha_1}(1 + |u|)^{q_1}, \end{cases} \tag{2.4}$$

where $\alpha_0, \alpha_1 \geq 1$, $q_0, q_1 > 0$ are arbitrary when $d = 2$ and

$$q_0 + \alpha_0 \leq 2, \quad q_1 + \alpha_1 \leq 2, \tag{2.5}$$

when $d = 3$. For instance, the function $f(u) = u^5 - \sigma u^3$, $\sigma > 0$, satisfies (2.4) for $|u| \geq \sqrt{\sigma}$. We denote by $\|.\|$ and $(.,.)$ the usual norm and inner product of $L^2(\Omega)$, respectively, and set $H = L^2(\Omega)$, $H_1 = H_0^1(\Omega)$, $H_2 = H^2(\Omega) \cap H_0^1(\Omega)$ and $H_3 = \{v \in H^3(\Omega) \cap H_0^1(\Omega), \Delta v \in H_0^1(\Omega)\}$. The first eigenvalue λ_1 of the operator $A = -\Delta : D(A) \to H$, with $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$, is positive, that is,

$$\inf\{(-\Delta v, v), \quad v \in D(A), \quad \|v\| = 1\} = \lambda_1 > 0. \tag{2.6}$$

In the case $b_1 = -M$, $b_2 = M$, we have $\lambda_1 = (\frac{\pi}{2M})^2$ (see [3]).

REMARK 2.1. For $v \in L^2(\Omega)$, the solution ξ (denoted by $N(v)$) of the Dirichlet problem

$$\begin{cases} -\Delta\xi = v & \text{in } \Omega, \\ \xi = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.7}$$

is such that $\xi \in D(A)$. Furthermore, the norm $\|\Delta\xi\|$ is equivalent on $D(A)$ to the canonical norm $\|\xi\|_{H^2(\Omega)}$ (see [1] and [2]).

If we denote $\|q\|_{-1} = \|\nabla N(q)\|$, then there exists $c_1, c_2 > 0$ such that

$$c_1\|q\|_{-1} \le \|q\| \le c_2\|\nabla q\|, \quad \forall q \in H_0^1(\Omega). \tag{2.8}$$

We now endow H, H_1, H_2 and H_3 with the norms $\|q\|_0 = (\|q\|_{-1}^2 + \beta\|q\|^2)^{\frac{1}{2}}$, $\|q\|_1 = (\|q\|^2 + \beta\|\nabla q\|^2)^{\frac{1}{2}}$, $\|q\|_2 = (\|\nabla q\|^2 + \beta\|\Delta q\|^2)^{\frac{1}{2}}$ and $\|q\|_3 = \|\nabla\Delta q\|$, respectively. These norms are equivalent to the usual $L^2(\Omega)$, $H^1(\Omega)$, $H^2(\Omega)$ and $H^3(\Omega)$ norms, respectively.

We finally denote

$$\begin{cases} \|u\|_{0,\gamma}^2 = \int_\Omega |u|^2(1+|x|^2)^\gamma \, dx, \\ \|u\|_{l,\gamma}^2 = \sum_{|\alpha|\le l} \|\partial^\alpha u\|_{0,\gamma}^2, \quad l = 1, 2, \end{cases} \tag{2.9}$$

where $\gamma > 0$ and $\partial^\alpha = \frac{\partial^{\alpha_1}}{\partial {x_1}^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial {x_2}^{\alpha_2}}$ with $|\alpha| = \alpha_1 + \alpha_2$, $(\alpha_1, \alpha_2) \in \mathbb{N}^2$. The space $H_{l,\gamma}(\Omega)$ is the set of u such that the norm $\|u\|_{l,\gamma}$ is finite. The space $H_{l,0}(\Omega)$ is denoted by $H^l(\Omega)$.

3. Estimates in unweighted spaces. In order to obtain the existence of solution results for the system (2.1)-(2.5), we introduce a weak variational formulation of the problem:
Find $u : [0;T] \to H_2$ such that $u(0) = u_0$, and for a.e. $t \in [0,T]$, $\forall T > 0$,

$$\frac{d}{dt}[(u,v) + \beta(\nabla u, \nabla v)] + \nu(\Delta u, \Delta v) - (f(u), \Delta v) = 0, \quad \forall v \in H_2. \tag{3.1}$$

THEOREM 3.1. Let $u_0 \in H_1$. Then, there exists a unique function u solution of (3.1) such that $u \in L^\infty(0,T;H_1) \cap L^2(0,T;H_2)$ and $\partial_t u \in L^2(0,T;H)$. Furthermore, if $u_0 \in H_2$, then $u \in L^\infty(0,T;H_2) \cap L^2(0,T;H_3)$ and $\partial_t u \in L^2(0,T;H_1)$.

Proof. Using Galerkin approximation arguments, we prove, for each $n \in \mathbb{N}$, the existence of a solution u_n of (3.1) in the truncated domain $\Omega_n = \Omega \cap (]-n,n[\times[-M,M])$. For $u_0 \in H_1$, we easily obtain, noting that the constant c_2 in (2.8) is chosen to depend on M only, that there exists a constant $C > 0$ independent of n such that

$$\int_{\Omega_n} (|u_n|^2 + \beta|\nabla u_n|^2)\, dx + c\int_{\Omega_n\times]0,T[} |\Delta u_n|^2 \, dxdt \le C. \tag{3.2}$$

Equation (2.1) restricted to Ω_n can be written as

$$\partial_t u_n = L^{-1}\Delta(-\nu\Delta u_n + f(u_n)), \tag{3.3}$$

where $L = I - \beta\Delta$ is an isomorphism from $H_0^1(\Omega)$ onto $H^{-1}(\Omega)$, and can be viewed alternately as an unbounded operator on $L^2(\Omega)$ with domain $D(L) = H^2(\Omega) \cap H_0^1(\Omega)$. Furthermore, $\|Lq\|$ is a norm which is equivalent on $D(L)$ to the norm $\|\Delta q\|$, and therefore, $\|L^{-1}\Delta q\|$ is a norm which is equivalent on $L^2(\Omega)$ to the usual $L^2(\Omega)$ norm (see [1] and [2]). Therefore, constants in inequalities deduced from the respective equivalence of these norms restricted to Ω_n can be chosen to depend on M only, but not on n. From (3.3), we have

$$\begin{aligned}\int_{\Omega_n\times]0,T[}|\partial_t u_n|^2\,dxdt &\le C\int_{\Omega_n\times]0,T[}(|\Delta u_n|^2+|f(u_n)|^2)\,dxdt\\ &\le C(1+\int_{\Omega_n\times]0,T[}|u_n|^{2+2\alpha_1}(1+|u_n|^{2q_1})\,dxdt)\\ &\le C(1+T(\|u_n\|^r_{H^1(\Omega_n)}+1))\le C,\end{aligned}\tag{3.4}$$

where $r>0$, C is chosen to depend on T, M, $\|u_0\|_1$ only, but not on n. We further note that Sobolev embeddings are valid on the whole domain Ω even when $\Omega=\mathbb{R}^d$, so, in the deduced inequalities, we can choose constants independently of Ω_n (see [2] and [15]). In (3.4), we used the continuous embedding $H^1(\Omega_n)\hookrightarrow L^q(\Omega_n)$, $\forall q\ge 1$, and the fact that there exists a constant $c>0$ independent of n such that $\|u_n\|^2_{H^1(\Omega_n)}\le c\int_{\Omega_n}(|u_n|^2+\beta|\nabla u_n|^2)\,dx$. We then deduce the existence of solutions u_n in Ω_n and a subsequence (which we still denote by $\{u_n\}_n$) which converges to u as $n\to+\infty$ and u is a solution of (3.1). We remark that, here, due to the noncompactness of operators, A^{-1} for instance, we are not able to apply classical methods used to directly obtain a solution of (3.1).

Let u_1 and u_2 be two solutions of (3.1) with the same initial data. Setting $w=u_1-u_2$, we have $w(0)=0$ and

$$\tfrac{1}{2}\tfrac{d}{dt}\|w\|_0^2+\nu\|\nabla w\|^2=-(f(u_1)-f(u_2),w).\tag{3.5}$$

Noting that $(f(u_1)-f(u_2),w)\ge -c\|w\|^2$, we obtain

$$\tfrac{d}{dt}\|w\|_0^2\le c\|w\|_0^2,\tag{3.6}$$

hence the uniqueness of solution.

Now, let u_0 in H_2 and multiply (2.1) by $v=-\Delta u_n$. Integrating over Ω_n, we obtain

$$\begin{aligned}\tfrac{1}{2}\tfrac{d}{dt}(\int_{\Omega_n}(|\nabla u_n|^2+\beta|\Delta u_n|^2)\,dx)+\nu\int_{\Omega_n}|\nabla\Delta u_n|^2\,dx\\ =\int_{\Omega_n}\nabla f(u_n)\nabla\Delta u_n\,dx.\end{aligned}\tag{3.7}$$

We have

$$\begin{aligned}|\int_{\Omega_n}\nabla f(u_n)\nabla\Delta u_n\,dx| &\le \|f'(u_n)\nabla u_n\|_{L^2(\Omega_n)}\|\nabla\Delta u_n\|_{L^2(\Omega_n)}\\ &\le c\|f'(u_n)\nabla u_n\|^2_{L^2(\Omega_n)}+\tfrac{\nu}{2}\|\nabla\Delta u_n\|^2_{L^2(\Omega_n)}\end{aligned}\tag{3.8}$$

and

$$\begin{aligned}\|f'(u_n)\nabla u_n\|^2_{L^2(\Omega_n)} &=\int_{\Omega_n}|f'(u_n)|^2|\nabla u_n|^2\,dx\\ &\le c\int_{\Omega_n}|u_n|^{2\alpha_0}(1+|u_n|^{2q_0})|\nabla u_n|^2\,dx.\end{aligned}\tag{3.9}$$

Since $d\le 3$, we have

$$\begin{aligned}\int_{\Omega_n}|u_n|^{2\alpha_0}(1+|u_n|^{2q_0})|\nabla u_n|^2\,dx &\le c(\|u_n\|^{2\alpha_0}_{L^{8\alpha_0}(\Omega_n)}\\ &+\|u_n\|^{2(\alpha_0+q_0)}_{L^{8(\alpha_0+q_0)}(\Omega_n)})\|\nabla u_n\|_{L^4(\Omega_n)}\|\nabla u_n\|_{L^2(\Omega_n)},\end{aligned}\tag{3.10}$$

and, using Sobolev embeddings, we obtain

$$\begin{aligned}\int_{\Omega_n}|u_n|^{2\alpha_0}(1+|u_n|^{2q_0})|\nabla u_n|^2\,dx &\le c\|u_n\|^{2\alpha_0}_{H^1(\Omega_n)}\\ &\times(1+\|u_n\|^{2q_0}_{H^1(\Omega_n)})\|\Delta u_n\|^2_{L^2(\Omega_n)}.\end{aligned}\tag{3.11}$$

We finally obtain the estimate

$$\int_{\Omega_n}(|\nabla u_n|^2+\beta|\Delta u_n|^2)\,dx+c\int_{\Omega_n\times]0,T[}|\nabla\Delta u_n|^2\,dxdt\le C,\tag{3.12}$$

where C is chosen to depend on $\|u_0\|_1$, $\|u_0\|_2$, T, M only, but not on n; hence the result, passing to the limit.

4. Estimates in weight multipliers. Let $\varphi(x) = \varphi(x, \epsilon, \rho, \gamma)$ be a function of the variable $x \in \Omega$, which depends on parameters ϵ, ρ, γ and satisfies the following conditions:

$$\begin{cases} \varphi(x, \epsilon, \rho, \gamma) \geq 1, \quad \varphi(x, \epsilon, \rho, \gamma) = \varphi(\epsilon x, 1, \rho, 1)^{\gamma}, \\ \varphi(x, 1, \rho, \gamma) \quad \text{does not depend on } \rho \quad \text{if} \quad |x| \leq \rho, \\ \varphi(x, 1, \rho, \gamma) = \varphi(\rho + 1, 1, \rho, \gamma) \quad \text{as} \quad |x| \geq \rho + 1, \\ |\partial^{\alpha} \varphi(x, \epsilon, \rho, \gamma)| \leq C \epsilon^{|\alpha|} \varphi(x, \epsilon, \rho, \gamma) \quad \text{for} \quad |\alpha| \leq 2, \\ \varphi(x, \epsilon, \rho_1, \gamma) \geq \varphi(x, \epsilon, \rho_2, \gamma) \quad \text{as} \quad \rho_1 \geq \rho_2 \geq 1, \quad \gamma \geq 0, \\ \lim_{\rho \to \infty} \varphi(x, 1, \rho, \gamma) = (1 + |x|^2)^{\frac{\gamma}{2}} = \phi(x). \end{cases} \tag{4.1}$$

Let $\varphi = \varphi(x, \epsilon, \rho, 2\gamma)$ and $\psi = \varphi^{\frac{1}{2}}$.

Remark 4.1. In [3], A.V. Babin gives an example of a function satisfying all the above conditions.

The following propositions and their proofs can be found in [3].

Proposition 4.1. If $u \in H^1(\Omega)$, then

$$|\|\psi \nabla u\| - \|\nabla(\psi u)\|| \leq C\epsilon \|\psi u\|. \tag{4.2}$$

If $u \in H^2(\Omega)$, then

$$|\|\psi \Delta u\| - \|\Delta(\psi u)\|| \leq C\epsilon \|\psi(|u| + |\nabla u|)\|. \tag{4.3}$$

Proposition 4.2. If $u \in H^1(\Omega)$ such that $u|_{\partial\Omega} = 0$, then

$$2\|\psi \nabla u\|^2 \geq \lambda_1 \|\psi u\|^2. \tag{4.4}$$

Proposition 4.3. For ϵ sufficiently small and $u \in H^2(\Omega)$ such that $u|_{\partial\Omega} = 0$, we have

$$\|\psi \nabla u\| \leq 2{\lambda_1}^{-1/2} \|\psi \Delta u\|. \tag{4.5}$$

Proposition 4.4. We have

$$\begin{cases} \|u\|_{1,\gamma} \leq c\|\psi \nabla u\|, \quad \forall u \in H_{1,\gamma}(\Omega), \quad u|_{\partial\Omega} = 0, \\ \|u\|_{2,\gamma} \leq c\|\psi \Delta u\|, \quad \forall u \in H_{2,\gamma}(\Omega), \quad u|_{\partial\Omega} = 0. \end{cases} \tag{4.6}$$

Lemma 4.1. Let $\gamma > 0$ and $u_0 \in H_2 \cap H_{1,\gamma}(\Omega)$. Then, the solution $u(t)$ of (2.1)-(2.5) satisfies the estimates

$$\begin{cases} \|u(t)\|_{1,\gamma}^2 \leq C, \quad 0 \leq t \leq T, \\ \int_0^T \|u(t)\|_{2,\gamma}^2 \, dt \leq C, \quad \forall T > 0. \end{cases} \tag{4.7}$$

Proof. For the solution u of (2.1)-(2.5), equation (2.1) is equivalent to

$$\partial_t(N(u) + \beta u) - \nu \Delta u + f(u) = 0. \tag{4.8}$$

We multiply (4.8) by $\varphi(-\Delta u + \epsilon \partial_t u)$. We omit the details here (see for instance [5]). We treat only the term

$$|(f(u), \varphi(-\Delta u + \epsilon \partial_t u))| \leq \|\psi f(u)\| \|\psi \Delta u\| + \epsilon \|\psi f(u)\| \|\psi \partial_t u\| \tag{4.9}$$

and

$$\begin{aligned}\|\psi f(u)\|^2 &= \int_\Omega |f(u)|^2\varphi\,dx \le c\int_\Omega |u|^{2+2\alpha_1}(1+|u|^{2q_1})\varphi\,dx \\ &\le c\|u\|^{2\alpha_1}_{L^\infty(\Omega)}(1+\|u\|^{2q_1}_{L^\infty(\Omega)})\|\psi u\|^2 \le c\|u\|^{2\alpha_1}_{H^2(\Omega)} \\ &\quad\times(1+\|u\|^{2q_1}_{H^2(\Omega)})\|\psi u\|^2 \le c\|\psi u\|^2.\end{aligned} \tag{4.10}$$

We finally obtain an estimate of the form

$$\begin{aligned}&\tfrac{1}{2}\tfrac{d}{dt}(\|\psi u\|^2+\beta\|\psi\nabla u\|^2)+\nu\|\psi\Delta u\|^2+\beta\epsilon\|\psi\partial_t u\|^2 \\ &\le C\epsilon^2\|\psi u\|\,\|\psi\Delta u\|+C\epsilon\|\psi\partial_t u\|\,\|\psi u\|+C\epsilon^2\|\psi\partial_t u\|^2 \\ &\quad+C\beta\|\psi\partial_t u\|\,\|\psi\nabla u\|+C\epsilon\nu\|\psi\nabla u\|\,\|\psi\partial_t u\| \\ &\quad+C\|\psi u\|\,\|\psi\Delta u\|+C\epsilon\|\psi u\|\,\|\psi\partial_t u\|.\end{aligned} \tag{4.11}$$

For ϵ sufficiently small, we obtain, using Young's inequality and Propositions 4.1 through 4.3, the estimate

$$\begin{aligned}&\tfrac{1}{2}\tfrac{d}{dt}(\|\psi u\|^2+\beta\|\psi\nabla u\|^2)+\nu\|\psi\Delta u\|^2+\beta\epsilon\|\psi\partial_t u\|^2 \\ &\le \tfrac{\nu}{10}\|\psi\Delta u\|^2+\tfrac{\beta\epsilon}{10}\|\psi\partial_t u\|^2+\tfrac{\nu}{10}\|\psi\Delta u\|^2+\tfrac{\beta\epsilon}{10}\|\psi\partial_t u\|^2 \\ &\quad+\tfrac{\beta\epsilon}{10}\|\psi\partial_t u\|^2+c_1\|\psi\nabla u\|^2+\tfrac{\nu}{10}\|\psi\Delta u\|^2+\tfrac{\beta\epsilon}{10}\|\psi\partial_t u\|^2 \\ &\quad+c_2\|\psi u\|^2+\tfrac{\nu}{10}\|\psi\Delta u\|^2+\tfrac{\nu}{10}\|\psi\Delta u\|^2+\tfrac{\beta\epsilon}{10}\|\psi\partial_t u\|^2,\end{aligned} \tag{4.12}$$

and therefore

$$\begin{aligned}&\tfrac{d}{dt}(\|\psi u\|^2+\beta\|\psi\nabla u\|^2)+\nu\|\psi\Delta u\|^2+\beta\epsilon\|\psi\partial_t u\|^2 \\ &\le c_1\|\psi u\|^2+c_2\|\psi\nabla u\|^2.\end{aligned} \tag{4.13}$$

There exists $\delta>0$ such that

$$\begin{aligned}&\tfrac{d}{dt}(\|\psi u\|^2+\beta\|\psi\nabla u\|^2)+\nu\|\psi\Delta u\|^2+\beta\epsilon\|\psi\partial_t u\|^2 \\ &\le \delta(\|\psi u\|^2+\beta\|\psi\nabla u\|^2),\end{aligned} \tag{4.14}$$

hence the result.

THEOREM 4.1. Let $\gamma>0$ and $u_0\in H_2\cap H_{1,\gamma}(\Omega)$. Then, $u\in L^\infty(0,T;H_{1,\gamma}(\Omega))\cap L^2(0,T;H_{2,\gamma}(\Omega))$ and $\partial_t u\in L^2(0,T;H_{0,\gamma}(\Omega))$, $\forall T>0$.

Proof. We integrate (4.14) and pass to the limit on ρ using Fatou's lemma.

We set $H(\gamma)=H_2\cap H_{1,\gamma}(\Omega)$. Thanks to Theorem 4.1, we can define the semigroup $\{S_t\}$, $S_t: H(\gamma)\to H(\gamma)$, $u_0\mapsto u(t)$. We have the following results.

LEMMA 4.2. Operators $\{S_t\}$ are bounded in $H_{1,\gamma}(\Omega)$,

$$\|S_t u_0\|_{1,\gamma}\le c(\|u_0\|_{1,\gamma},T), \quad \text{as} \quad 0\le t\le T. \tag{4.15}$$

Proof. It follows from (4.14).

REMARK 4.2. For sufficiently smooth u, we have $\partial_t N(u)\in H^2(\Omega)$ and then

$$\partial_t N(u)=N(\partial_t u), \tag{4.16}$$

and therefore, $\partial_t N(u)|_{\partial\Omega}=0$ (see [2]).

LEMMA 4.3. Operators $\{S_t\}$ are bounded from $H(\gamma)$ into $H_{2,\gamma}(\Omega)$ as $t\ge 0$.

Proof. On differentiating (4.8) with respect to t and multiplying by $\varphi t\partial_t(N(u)+\beta u)$, we obtain using estimate

$$|(\partial_t u f'(u), \varphi t\partial_t(N(u)+\beta u))| \leq ct\|\psi\partial_t(N(u)+\beta u)\|\,\|\psi\partial_t u\| \tag{4.17}$$

that

$$\begin{aligned}\tfrac{1}{2}\tfrac{d}{dt}(t\|\psi\partial_t(N(u)+\beta u)\|^2)+t\|\psi\partial_t u\|^2 &\leq \tfrac{1}{2}\|\psi\partial_t(N(u)+\beta u)\|^2\\ &+C\epsilon^2 t\|\psi\partial_t u\|\,\|\psi N(\partial_t u)\|+C\epsilon t\|\psi\partial_t u\|\,\|\psi\nabla N(\partial_t u)\|\\ &+\beta\epsilon t\|\psi\partial_t u\|^2+t\|\psi\partial_t(N(u)+\beta u)\|\,\|\psi\partial_t u\|.\end{aligned} \tag{4.18}$$

For ϵ sufficiently small and using the estimates

$$\|\psi N(\partial_t u)\| \leq c_1\|\psi\nabla N(\partial_t u)\| \text{ and } \|\psi\nabla N(\partial_t u)\| \leq c_2\|\psi\partial_t u\|, \tag{4.19}$$

which follow from (4.4) and (4.5), we obtain

$$\begin{aligned}\tfrac{1}{2}\tfrac{d}{dt}(t\|\psi\partial_t(N(u)+\beta u)\|^2)+t\|\psi\partial_t u\|^2 &\leq c(1+t)\|\psi\partial_t(N(u)+\beta u)\|^2\\ &+\tfrac{t}{8}\|\psi\partial_t u\|^2+\tfrac{t}{8}\|\psi\partial_t u\|^2+\tfrac{t}{8}\|\psi\partial_t u\|^2+\tfrac{t}{8}\|\psi\partial_t u\|^2,\end{aligned} \tag{4.20}$$

and therefore

$$\tfrac{d}{dt}(t\|\psi\partial_t(N(u)+\beta u)\|^2) \leq c(1+t)\|\psi\partial_t(N(u)+\beta u)\|^2, \tag{4.21}$$

hence

$$\begin{aligned}\tau\|\psi\partial_t(N(u)+\beta u)\|^2 &\leq c(1+\tau)\textstyle\int_0^\tau \|\psi\partial_t(N(u)+\beta u)\|^2\,dt\\ &\leq c(1+\tau)\textstyle\int_0^\tau(\nu^2\|\psi\Delta u\|^2+\|\psi f(u)\|^2)\,dt.\end{aligned} \tag{4.22}$$

We pass to the limit on ρ and obtain

$$\tau\|\phi\partial_t(N(u)+\beta u)\|^2 \leq C(T,\|u(0)\|_{1,\gamma}). \tag{4.23}$$

We multiply (4.8) by $\varphi t\Delta u$ and obtain

$$\tau\nu\|\psi\Delta u\|^2 \leq \tau\|\psi\partial_t(N(u)+\beta u)\|^2+\tau C\|\psi u\|^2. \tag{4.24}$$

We pass to the limit on ρ and using (4.6) we obtain that

$$\tau\|u(\tau)\|_{2,\gamma}^2 \leq C(T,\|u(0)\|_{1,\gamma}), \quad \forall\tau\in[0,T]. \tag{4.25}$$

THEOREM 4.2. The semigroup $\{S_t\}$ has an absorbing set in $H(\gamma)$ which is invariant and bounded in $H_{2,\gamma}(\Omega)$.

Proof. We first prove the existence of an absorbing set bounded in $H_{1,\gamma}(\Omega)$. We multiply (4.8) by $\varphi(u+\epsilon\partial_t u)$ and use (4.19) and the estimates

$$\|\psi N(u)\| \leq c_1\|\psi\nabla N(u)\| \text{ and } \|\psi\nabla N(u)\| \leq c_2\|\psi u\|, \tag{4.26}$$

which follow from (4.4) and (4.5). We find

$$\begin{aligned}\tfrac{1}{2}\tfrac{d}{dt}(\|\psi\nabla N(u)\|^2+\nu\epsilon\|\psi\nabla u\|^2+\beta\|\psi u\|^2)+\nu\|\psi\nabla u\|^2+\beta\epsilon\|\psi\partial_t u\|^2&\\ \leq C\epsilon\|\psi\partial_t u\|\,\|\psi\nabla u\|+C\epsilon^2\|\psi\partial_t u\|^2+C\epsilon\nu\|\psi\nabla u\|^2&\\ +C\epsilon^2\nu\|\psi\partial_t u\|\,\|\psi\nabla u\|+C\epsilon\|\psi\nabla u\|\,\|\psi\partial_t u\|.&\end{aligned} \tag{4.27}$$

For ϵ sufficiently small, we have

$$\begin{aligned}\tfrac{1}{2}\tfrac{d}{dt}(\|\psi\nabla N(u)\|^2+\nu\epsilon\|\psi\nabla u\|^2+\beta\|\psi u\|^2)+\nu\|\psi\nabla u\|^2+\beta\epsilon\|\psi\partial_t u\|^2&\\ \leq \tfrac{\beta\epsilon}{8}\|\psi\partial_t u\|^2+\tfrac{\nu}{8}\|\psi\nabla u\|^2+\tfrac{\beta\epsilon}{8}\|\psi\partial_t u\|^2+\tfrac{\nu}{8}\|\psi\nabla u\|^2&\\ +\tfrac{\beta\epsilon}{8}\|\psi\partial_t u\|^2+\tfrac{\nu}{8}\|\psi\nabla u\|^2+\tfrac{\beta\epsilon}{8}\|\psi\partial_t u\|^2+\tfrac{\nu}{8}\|\psi\nabla u\|^2,&\end{aligned} \tag{4.28}$$

and therefore,

$$\tfrac{d}{dt}(\|\psi\nabla N(u)\|^2+\nu\epsilon\|\psi\nabla u\|^2+\beta\|\psi u\|^2)+\nu\|\psi\nabla u\|^2\leq 0. \tag{4.29}$$

There exists $\eta>0$ such that

$$\nu\|\psi\nabla u\|^2\geq \eta(\|\psi\nabla N(u)\|^2+\nu\epsilon\|\psi\nabla u\|^2+\beta\|\psi u\|^2). \tag{4.30}$$

We then deduce after passing to the limit on ρ that

$$\|u(t)\|^2_{1,\gamma}\leq c\|u_0\|^2_{1,\gamma}e^{-\eta t},\quad \forall t>0. \tag{4.31}$$

We deduce the existence of a bounded absorbing set B_0 for $\{S_t\}$ in $H(\gamma)$. We denote by $B_1=S_{t_1}B_0$ for some $t_1>0$ such that $S_tB_0\subset B_0$, $\forall t\geq t_1$ and $B_2=\bigcup_{t\geq t_1}S_tB_1$. The set B_1 is a bounded absorbing set and B_2 is a bounded absorbing and invariant set in $H(\gamma)$ which is bounded in $H_{2,\gamma}(\Omega)$ (which follows from Lemma 4.3).

5. Existence of the global attractor.

Let E be a Banach space.

Definition 5.1. The set $\mathcal{A}$ is called an $(E,E)-$attractor of the semigroup $\{S_t\}$ if

1. $\mathcal{A}$ is compact in E.
2. $\mathcal{A}$ is strictly invariant, that is, $S_t\mathcal{A}=\mathcal{A}$, $\forall t\geq 0$.
3. $\mathcal{A}$ is an attracting set for $\{S_t\}$ in the following sense: for any bounded set $B\subset E$ the Hausdorff distance

$$\mathrm{dist}_E(S_tB,\mathcal{A})\to 0\quad\text{as}\quad t\to\infty.$$

Theorem 5.1. We assume that the operators $\{S_t\}$ are continuous on E for any $t\geq 0$ and there exists a compact subset $K\subset E$ having the following attracting property: for any bounded set $B\subset E$

$$\mathrm{dist}_E(S_tB,K)\to 0\quad\text{as}\quad t\to\infty. \tag{5.1}$$

Then, the semigroup $\{S_t\}$ has an $(E,E)-$attractor $\mathcal{A}\subset K$.

We want to prove the existence of a compact set B_4 of $H(\gamma)$ which is bounded in $H_{2,\gamma}(\Omega)$ and satisfying (5.1).

Any solution u of (2.1) can be decomposed into the sum

$$u=v+w, \tag{5.2}$$

where v is a solution of the linear problem

$$\begin{cases}\partial_t(v-\beta\Delta v)+\nu\Delta^2v=0,\\ v|_{t=0}=v_0=u_0,\\ v|_{\partial\Omega}=\Delta v|_{\partial\Omega}=0,\end{cases} \tag{5.3}$$

and w is a solution of

$$\begin{cases}\partial_t(w-\beta\Delta w)+\nu\Delta^2w-\Delta f(u)=0,\\ w|_{t=0}=w_0=0,\\ w|_{\partial\Omega}=\Delta w|_{\partial\Omega}=0.\end{cases} \tag{5.4}$$

LEMMA 5.1. Let $u_0 \in B_0$. Then, the solution v of (5.3) satisfies the inequality

$$\|v(t)\|_{1,\gamma}^2 \leq Ce^{-\mu t}, \quad \forall t \geq 0, \tag{5.5}$$

where $\mu > 0$.

Proof. We multiply $(-\Delta)^{-1}$ of (5.3) by $\varphi(v+\partial_t v)$ and integrate with respect to x and pass to the limit on ρ.

LEMMA 5.2. Let w be a solution of (5.4) where $u(t) \in B_2$. Let $\alpha \leq \alpha_1$ (α_1 of (2.4)–(2.5)). Then,

$$\left\{ \begin{array}{l} \|w(t)\|_{2,\gamma} \leq C, \quad \forall t \geq 0, \\ \|w(t)\|_{1,(1+\alpha)\gamma} \leq C, \quad \forall t \geq 0. \end{array} \right. \tag{5.6}$$

Proof. The first estimate of (5.6) follows from the fact that u is bounded in $H_{2,\gamma}(\Omega)$. We multiply $(-\Delta)^{-1}$ of (5.4) by $\varphi^{1+\alpha}(w + \partial_t w)$ and obtain an inequality of the form

$$\begin{array}{l} \frac{d}{dt}(\|\psi^{1+\alpha}\nabla N(w)\|^2 + \beta\|\psi^{1+\alpha}w\|^2 + \nu\|\psi^{1+\alpha}\nabla w\|^2) \\ +\beta\|\psi^{1+\alpha}\partial_t w\|^2 + \nu\|\psi^{1+\alpha}\nabla w\|^2 \leq C\|\psi^{1+\alpha}f(u)\|^2. \end{array} \tag{5.7}$$

There exists $\eta > 0$ such that

$$\begin{array}{l} \frac{d}{dt}(\|\psi^{1+\alpha}\nabla N(w)\|^2 + \beta\|\psi^{1+\alpha}w\|^2 + \nu\|\psi^{1+\alpha}\nabla w\|^2) + \eta(\|\psi^{1+\alpha}\nabla N(w)\|^2 \\ +\beta\|\psi^{1+\alpha}w\|^2 + \nu\|\psi^{1+\alpha}\nabla w\|^2) \leq C\|\psi^{1+\alpha}f(u)\|^2. \end{array} \tag{5.8}$$

To deduce the last estimate of (5.6) it is sufficient to prove that $\|\psi^{1+\alpha}f(u)\|$ is bounded. We have

$$\begin{array}{l} \|\psi^{1+\alpha}f(u)\|^2 \leq C\int_\Omega |u|^{2+2\alpha_1}(1+|u|^{2q_1})\varphi^{1+\alpha}\,dx \\ \leq C\int_\Omega |\psi u|^{2+2\alpha_1}\,dx + C\int_\Omega |\psi u|^{2+2\alpha_1+2q_1}\,dx \\ \leq C(\|\psi u\|_{H^2(\Omega)}^r + 1), \end{array} \tag{5.9}$$

where $r > 0$, noting the continuous embeddings $H^2(\Omega) \subset L^q(\Omega)$, $\forall q \geq 2$. Passing to the limit $\rho \to +\infty$ after integrating with respect to t, and noting that $\|\phi u\|_{H^2(\Omega)} \leq c\|u\|_{2,\gamma}$ and the boundedness of $u(t)$ in $H_{2,\gamma}(\Omega)$, we deduce the result.

LEMMA 5.3. Operators $\{S_t\}$ are continuous from $H(\gamma)$ into $H(\gamma)$.

Proof. Let u_{01} and u_{02} be two initial values of problem (2.1)-(2.5). We denote by $u_1(t) = S_t u_{01}$, $u_2(t) = S_t u_{02}$ and $s(t) = u_1(t) - u_2(t)$. The following equation is satisfied:

$$\partial_t(N(s) + \beta s) - \nu\Delta s + f(u_1) - f(u_2) = 0. \tag{5.10}$$

We multiply (5.10) by $\varphi(s + \partial_t s)$ and obtain after calculations

$$\|s(t)\|_{1,\gamma} \leq Ce^{\eta t}\|s(0)\|_{1,\gamma}, \tag{5.11}$$

with $\eta > 0$, and therefore

$$\|u_1(t) - u_2(t)\|_{1,\gamma} \leq Ce^{\eta t}\|u_{01} - u_{02}\|_{1,\gamma}, \tag{5.12}$$

hence the result.

LEMMA 5.4. A set, which is bounded in $H_{2,\gamma}(\Omega)$ and in $H_{1,\beta}(\Omega)$, $\beta > \gamma > 0$, is precompact in $H_{1,\gamma}(\Omega)$.

Proof. See the proof of Lemma 4.5 in [3].

THEOREM 5.2. The semigroup $\{S_t\}$, $S_t : H(\gamma) \to H(\gamma)$, generated by (2.1)-(2.5), has a $(H(\gamma), H(\gamma))$-attractor $\mathcal{A}(\gamma)$, bounded in $H_{2,\gamma}(\Omega)$ and in $H((1+\alpha)\gamma)$ as $\alpha \leq \alpha_1$.

Proof. We denote by B_3 the set

$$B_3 = \{y \in H(\gamma),\ y = w(t, u_0),\ \ t \geq 0,\ \ u_0 \in B_2\},$$

which is the union of all the values $w(t)$ of solution w of (5.4) for all $t \geq 0$ and all $u_0 \in B_2$. The set B_3 is bounded in $H_{2,\gamma}(\Omega)$ and in $H_{1,(1+\alpha)\gamma}(\Omega)$, since $u(t) \in B_2$. Such a set is precompact in $H_{1,\gamma}(\Omega)$. We denote by B_4 the closure of B_3 in $H_{1,\gamma}(\Omega)$. The set B_4 is compact in $H_{1,\gamma}(\Omega)$, and for all $u(t) \in B_2$, we have $w(t) \in B_4$ and

$$\operatorname{dist}_{H_{1,\gamma}(\Omega)}(u(t), B_4) \leq Ce^{-\mu t},\ \ \mu > 0. \tag{5.13}$$

Since the operators $\{S_t\}$ are continuous on $H(\gamma)$, we apply Theorem 5.1.

A careful examination of the previous study shows that Theorem 5.2 is applicable to the following system:

$$\begin{cases} \partial_t(u - \beta\Delta u) + \nu\Delta^2 u - \Delta(f(u) + \lambda_0 u + g) = 0, \\ u|_{t=0} = u_0, \\ u|_{\partial\Omega} = \Delta u|_{\partial\Omega} = 0, \end{cases} \tag{5.14}$$

where $\lambda_0 > 0$ and $g \in H^2(\Omega)$.

Here, any solution of (5.14) can be decomposed into the sum $u = v + w + z$, where v is a solution of the linear problem

$$\begin{cases} \partial_t(v - \beta\Delta v) + \nu\Delta^2 v - \lambda_0\Delta v = 0, \\ v|_{t=0} = v_0 = u_0, \\ v|_{\partial\Omega} = \Delta v|_{\partial\Omega} = 0, \end{cases} \tag{5.15}$$

w is a solution of

$$\begin{cases} \partial_t(w - \beta\Delta w) + \nu\Delta^2 w - \Delta(f(u) + \lambda_0 w) = 0, \\ w|_{t=0} = w_0 = 0, \\ w|_{\partial\Omega} = \Delta w|_{\partial\Omega} = 0, \end{cases} \tag{5.16}$$

and z is a solution of the stationary problem

$$\begin{cases} \nu\Delta^2 z - \Delta(\lambda_0 z + g) = 0, \\ z|_{\partial\Omega} = \Delta z|_{\partial\Omega} = 0. \end{cases} \tag{5.17}$$

We have the following result.

THEOREM 5.3. Let $g \in H_{2,\gamma}(\Omega)$. The semigroup $\{S_t\}$, $\ S_t : H(\gamma) \to H(\gamma)$ generated by (5.14), (2.2)-(2.5) has a $(H(\gamma), H(\gamma))$-attractor $\mathcal{A}(\gamma)$, bounded in $H_{2,\gamma}(\Omega)$. This attractor can be represented in the form

$$\mathcal{A}(\gamma) = z + \mathcal{A}_1, \tag{5.18}$$

where z is a solution of (5.17) and the set $\mathcal{A}_1$ is bounded in $H_{1,(1+\alpha)\gamma}(\Omega)$ as $\alpha \leq \alpha_1$.

Proof. We denote $B_5 = z + B_4$. We have $u = v + z + w$ and $z + w \in B_5$ as $t \geq 0$, $v \in u - B_5$ and v satisfies (5.5); therefore the result.

Corollary 5.1. Let $g \in H_{2,\gamma_0}(\Omega)$, $0 < \gamma \le \gamma_0$. Then, the semigroup $\{S_t\}$ generated by (5.14), (2.2)-(2.5) has a $(H(\gamma), H(\gamma))$-attractor $\mathcal{A}(\gamma)$ as in Theorem 5.3.

Proof. Let $g \in H_{2,\gamma_0}(\Omega)$, $0 < \gamma \le \gamma_0$. Then, $g \in H_{2,\gamma}(\Omega)$, and we apply Theorem 5.3.

Theorem 5.4. Let $g \in H_{2,\gamma_0}(\Omega)$, $\gamma_0 > 0$, $0 < \gamma \le \gamma_0$. Then, the attractors $\mathcal{A}(\gamma)$ of semigroups $\{S_t\}$ acting in $H(\gamma)$ coincide with $\mathcal{A}(\gamma_0)$.

Proof. See Theorem 4.2 in [3].

6. Finite Hausdorff dimension of the attractor. We consider the $(H(\gamma), H(\gamma))$-attractor $\mathcal{A}$ of the semigroup $\{S_t\}$ acting in $H(\gamma) \subset H$ obtained in Theorem 5.3. We prove the following result when $d = 2$ (a similar result is available for $d = 3$).

Theorem 6.1. Suppose $g \in H_{2,\gamma}(\Omega)$, $\gamma > 0$, and f satisfies the conditions of Therorem 5.3. Then, the following statements are valid:

1. If $|f'(u)| \le C|u|^{\alpha_0} C_1(u)$, then
$$\dim_H \mathcal{A} \le C\nu^{-1}\lambda_0^{-3-2/\alpha_0}\|g\|. \tag{6.1}$$
2. If $-f'(u) \le c|u|$, then
$$\dim_H \mathcal{A} \le C\nu^{-1}\lambda_0^{-3}, \tag{6.2}$$

where $\dim_H \mathcal{A}$ is the Hausdorff dimension of $\mathcal{A}$ in the topology of H.

Proof. Due to the equivalence, first, of equations (5.14) and
$$\partial_t(N(u) + \beta u) - \nu\Delta u + f(u) + \lambda_0 u + g = 0 \tag{6.3}$$
on $H(\gamma)$ and, secondly, of norms $\|u\|_0$ and $\|u\|$ on H, the differential $S_t'(u_0)v_0 = v(t)$ of operator S_t acting on a function v_0 is the solution of the equation in variations
$$\partial_t(N(v) + \beta v) = \nu\Delta v - f'(u(t))v - \lambda_0 v, \tag{6.4}$$
where $u(t) = S_t(u_0)$, $u_0 \in \mathcal{A}$. The operator $S_t'(u_0) : H \to H$ generated by (6.4) is the differential of S_t on $\mathcal{A}$ at the point u_0 (see [6]). We then apply Theorem 3.2 of [6].

References

[1] F. Abergel, *Existence and finite dimensionality of the global attractor for evolution equations on unbounded domains*, J. Diff. Eq. **83**(1990), 85-108. MR1031379 (90m:58121)

[2] S. Agmon, A. Douglis and L. Niremberg, *Estimates near the boundary for solutions of partial differential equation satisfying general boundary conditions I,II*, Comm. Pure Appl. Math. **12**(1959), 623-727; **17**(1964), 35-92. MR0125307 (23:A2610); 0162050 (28:5252)

[3] A.V. Babin, *The attractor of a Navier-Stokes system in an unbounded channel-like domain,* J. Dyn. Diff. Eq. **4**(4)(1992), 555-584. MR1187223 (94e:35101)

[4] A. Babin and B. Nicolaenko, *Exponential attractors of reaction-diffusion systems in an unbounded domain,* J. Dyn. Diff. Eq. **7**(4)(1995), 567-589. MR1362671 (96j:35120)

[5] A.V. Babin and M.I. Vishik, *Attractors of Evolution Equations,* North-Holland, Amsterdam, London, New York, Tokyo (1992). MR1156492 (93d:58090)

[6] A.V. Babin and M.I. Vishik, *Attractors of partial differential evolution equations in an unbounded domain,* Proc. Royal Soc. Edimburgh **116A**(1990), 221-243. MR1084733 (91m:35106)

[7] A. Bonfoh and A. Miranville, *On Cahn-Hilliard-Gurtin equations* (Proceedings of the 3rd World Congress of Nonlinear Analysts), Nonlinear Anal. **47**(2001), 3455-3466. MR1979242 (2004h:74065)

[8] J.W. Cahn, *On spinodal decomposition,* Acta Metall. **9**(1961), 795-801.

[9] J.W. Cahn and J.E. Hilliard, *Free energy of a non-uniform system I. Interfacial free energy,* J. Chem. Phys. **2**(1958), 258-267.

[10] M. Efendiev and A. Miranville, *Finite-dimensional attractors for a reaction-diffusion equation in* $\mathbf{R}^n$ *with a strong nonlinearity,* Disc. Cont. Dyn. Sys. **5**(2)(1999), 399-424. MR1665748 (99m:35117)

[11] M. Efendiev, A. Miranville and S. Zelik, *Exponential attractors for a singularly perturbed Cahn-Hilliard system,* Math. Nachrichten **272**(1)(2004), 11-31. MR2079758 (2005h:37195)

[12] M. Gurtin, *Generalized Ginzburg-Landau and Cahn-Hilliard equations based on a microforce balance*, Physica D **92**(1996), 178-192. MR387065 (98m:73009)

[13] A. Miranville, A. Piétrus and J.M. Rakotoson, *Dynamical aspect of a generalized Cahn-Hilliard equation based on a microforce balance,* Asymptotic Anal. **16**(1998), 315-345. MR1612825 (99c:35095)

[14] A. Miranville, *Some generalizations of the Cahn-Hilliard equation,* Asymptotic Anal. **22**(2000), 235-259. MR1753766 (2001b:35153)

[15] R. Temam, *Infinite Dimensional Dynamical Systems in Mechanics and Physics*, 2nd Edition, Springer-Verlag, Berlin, Heidelberg, New York, 1997. MR1441312 (98b:58056)