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# Finite-dimensional Banach spaces with symmetry constant of order $\sqrt{n}$

by

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**Abstract.** It is proved that there exists an absolute constant $c > 0$ such that for every positive integer $n$ there is an $n$-dimensional Banach space $X_n$ with symmetry constant $s(X_n) \geq c\sqrt{n}$.

We shall only consider finite-dimensional Banach spaces over the real field. The complex case can be treated after obvious modifications in exactly the same way.

Given a finite-dimensional Banach space $X$, let $\mathscr{G}(X)$ denote the set of all compact groups of linear isomorphisms of $X$ with trivial commutator (i.e., all groups $G$ of linear isomorphisms of $X$ with the property that if a linear operator $T\colon X \to X$ commutes with every element of the group $G$ then $T = \lambda \operatorname{Id}_X$ for some $\lambda \in \boldsymbol{R}$). Define

$$s(X, G) = \max\{\|T\|_{X \to X}\colon T \in G\}.$$

The symmetry constant $s(X)$ of the space $X$ is defined by

$$s(X) = \inf\{s(X, G)\colon G \in \mathscr{G}(X)\}.$$

It is easy to see that $s(\cdot)$ is a Lipschitz 1 function with respect to the Banach–Mazur distance and that $s(Y) = 1$ for every finite-dimensional Banach space with 1-symmetric basis. Therefore $s(X) \leq \sqrt{n}$ for every $n$-dimensional Banach space $X$. On the other hand, all known examples of finite-dimensional Banach spaces for which the symmetry constant has been computed have shown the order of growth to be not bigger than the fourth root of the dimension. In [2] Garling and Gordon conjectured that

$$s_n = \sup\{s(X)\colon \dim X = n\} = O(n^{1/4}).$$

The aim of this note is to disprove this conjecture. More precisely we shall prove the following

THEOREM. $s_n = O(\sqrt{n})$.

Because of the trivial estimate $s_n \leq \sqrt{n}$ it will be enough to prove

THEOREM 1. *There is a constant $c > 0$ such that for every $n \in N$ there is an $n$-dimensional Banach space $X_n$ with*

$$s(X_n) \geqslant c\sqrt{n}.$$

As often happens, we are unable to determine spaces with this property; instead, for every $n \in N$ we construct a class of "random" $n$-dimensional Banach spaces with the property that for the "vast majority" of the spaces in this class the desired estimate holds. For this, we use the spaces introduced by Gluskin, [3], to prove that the Banach–Mazur distance between certain $n$-dimensional Banach spaces is of order $n$. Similar spaces have been used by Gluskin [4] and Szarek [7] to construct finite-dimensional spaces with the "worst possible" Schauder basis constant. Similar examples of a "random approach" to finite-dimensional Banach spaces can also be found in Figiel, Johnson [6] and in Figiel, Kwapień, Pełczyński [5]. Our proof consists of two basic arguments. The first one is a kind of a "subspace mixing" property for groups $G$ in $\mathscr{G}(\boldsymbol{R}^n)$ and the second is a "small perturbation" of the improved version of Gluskin's argument, [3], due to Szarek [7].

**1. Notation.** We shall use the strandard notation. For every $n \in N$, let $\{e_i: 1 \leqslant i \leqslant n\}$ denote the standard unit vector basis in $\boldsymbol{R}^n$ and for $x, y \in \boldsymbol{R}^n$ let $(x, y)$ and $|x|$ denote the standard scalar product and the standard norm on $\boldsymbol{R}^n$. For $n \in N$, let $K_n = \{x \in \boldsymbol{R}^n: |x| \leqslant 1\}$ and $S_n = \{x \in \boldsymbol{R}^n: |x| = 1\}$. We shall denote $n$-dimensional volume in $\boldsymbol{R}^n$ by $\mathrm{vol}_n$ and normalized Lebesgue surface measure on $S_n$ by $\mu_n$.

For all $n \in N$, we define

$$\mathscr{A}_n = \{(x_1, x_2, \ldots, x_{20n}): x_i \in S_n, 1 \leqslant i \leqslant 20n\} = \prod_{i=1}^{20n} S_n,$$

and we let $\boldsymbol{P}_n$ denote the product measure of $20n$ copies of $\mu_n$. If $A = (x_1, x_2, \ldots, x_{20n}) \in \mathscr{A}_n$ we define $\|\cdot\|_A$ to be the norm on $\boldsymbol{R}^n$ with the unit ball

$$\tilde{A} = \mathrm{abs\,conv}\{e_1, e_2, \ldots, e_n, x_1, x_2, \ldots, x_{20n}\};$$

in the sequel we shall also denote the Banach space $(\boldsymbol{R}^n, \|\cdot\|_A)$ by $A$. Note that $\boldsymbol{P}_n$ induces a normalized measure on the set of Banach spaces $\mathscr{A}_n$ in a natural way; this will also be denoted by $\boldsymbol{P}_n$.

For every $n \in N$ and every $A, B \in \mathscr{A}_n$ we shall mean by $A \times B$ the $l_2$-product of spaces $A$ and $B$ (i.e., the Banach space $(\boldsymbol{R}^{2n}, \|\cdot\|_{A\times B})$, where for $(x, y) \in \boldsymbol{R}^n \times \boldsymbol{R}^n = \boldsymbol{R}^{2n}$

$$\|(x, y)\|_{A\times B} = (\|x\|_A^2 + \|y\|_B^2)^{1/2}.$$

Also, we let

$$E_1^n = \{x \in \boldsymbol{R}^{2n}: x \in \mathrm{lin}\{e_1, e_2, \ldots, e_n\}\}$$



and

$$E_2^n = \{x \in \boldsymbol{R}^{2n}: x \in \mathrm{lin}\{e_{n+1}, e_{n+2}, \ldots, e_{2n}\}\}$$

and we let $P_i^n$ be the orthogonal projection on $E_i^n$ for $i = 1, 2$.

We shall deal mainly with linear operators in $\boldsymbol{R}^n$. The set of all such operators will be denoted by $L(\boldsymbol{R}^n)$. If $T \in L(\boldsymbol{R}^n)$ and $\|\cdot\|_A$, $\|\cdot\|_B$ are two norms on $\boldsymbol{R}^n$, then $\|T\|_{A\to B}$ will denote the norm of $T$ regarded as an operator from $(\boldsymbol{R}^n, \|\cdot\|_A)$ into $(\boldsymbol{R}^n, \|\cdot\|_B)$. In what follows, operators from $E_i^n$ into $E_j^n$, $i, j = 1, 2$, will be identified in a natural way with operators in $L(\boldsymbol{R}^n)$.

We shall prove the following theorem which easily implies Theorem 1.

THEOREM 2. *There is an absolute constant $c > 0$ such that*

$$\boldsymbol{P}_n \times \boldsymbol{P}_n \{(A, B) \in \mathscr{A}_n \times \mathscr{A}_n: s(A \times B) < c\sqrt{n}\} \underset{n\to\infty}{\longrightarrow} 0$$

From now on, in order to simplify the notation, we shall assume that $n = 10k$ for $k = 1, 2, \ldots$ and we shall say that an operator $T \in L(\boldsymbol{R}^n)$ is *thick* iff $|Tx| \geqslant \frac{1}{4}|x|$ for every $x$ in some $k$-dimensional subspace of $\boldsymbol{R}^n$.

**2. Some properties of $\mathscr{G}^n = \mathscr{G}(\boldsymbol{R}^n)$.** In the sequel we shall need the following easily verified properties of $\mathscr{G}^n$:

(*) For every $G \in \mathscr{G}^n$ and every $U \in L(\boldsymbol{R}^n)$ we have

$$\int_G T^{-1} U T \, dh_G(T) = \lambda \mathrm{Id} \quad \text{with} \quad \lambda = \frac{\mathrm{tr}(U)}{n}$$

where $h_G$ denotes the Haar measure on $G$.

(**) For every $G \in \mathscr{G}(\boldsymbol{R}^n)$ there is a unique ellipsoid $\varepsilon_G$ of the smallest volume containing the set

$$\mathrm{conv} \bigcup_{T\in G} T(K_n),$$

and this is $G$-invariant (i.e., $T(\varepsilon_G) = \varepsilon_G$ for every $T \in G$). In other words $G$ can be considered as a group of isometries of $(\boldsymbol{R}^n, \|\ \|_G)$ where $\|\ \|_G$ is the hilbertian norm defined by the scalar product $\langle , \rangle_G$ induced by the ellipsoid $\varepsilon_G$. In particular, if $\varepsilon_G = K_n$, then $G$ is a group of isometries of $(\boldsymbol{R}^n, |\ |)$. The set of all such groups will be denoted by $\mathscr{G}_0(\boldsymbol{R}^n)$.

LEMMA 1. *For every pair $F_1$, $F_2$ of $5k$-dimensional subspaces of $\boldsymbol{R}^{2n}$ and every $G \in \mathscr{G}_0(\boldsymbol{R}^{2n})$ there is a $T_0$ in $G$ such that*

$$|P_{F_2} T_0 x| \geqslant \tfrac{1}{4}|x|$$

*for every $x$ in some $k$-dimensional subspace $\tilde{F}_1$ of $F_1$, where $P_{F_2}$ denotes the orthogonal projection onto $F_2$.*

Proof. Let $P = P_{F_2}$ and $\mu$ be the normalized Lebesgue surface measure on $S_{F_1} = \{x \in F_1: |x| = 1\}$. Fix $G \in \mathscr{G}_0(\boldsymbol{R}^{2n})$. By (*) we have

$$\tfrac{1}{4}\mathrm{Id} = \int_G T^{-1} P T \, dh_G(T).$$

Thus, for every $x \in F_1$,

$$\tfrac{1}{4}(x, x) = \int_G (T^{-1} PTx, x)\, dh_G(T)$$

and

$$\tfrac{1}{4} = \int_{S_{F_1}} \Big(\int_G (T^{-1} PTx, x)\, dh_G(T)\Big) d\mu(x)$$
$$= \int_{S_{F_1}} \Big(\int_G (PTx, PTx)\, dh_G(T)\Big) d\mu(x) = \int_G \Big(\int_{S_{F_1}} |PTx|^2 d\mu(x)\Big) dh_G(T).$$

Thus, there is a $T_0$ in $G$ such that

$$\int_{S_{F_1}} |PT_0 x|^2 d\mu(x) \geqslant \tfrac{1}{4}.$$

Let $U = PT_0$. There exist orthonormal bases $(u_i)_{i=1}^{5k}$, $(v_i)_{i=1}^{5k}$ in $F_1$ and $F_2$, respectively, and non-negative numbers $\lambda_i$ such that

$$Ux = \sum_{i=1}^{5k} \lambda_i (u_i, x) v_i$$

for every $x \in F_1$. We have

$$\tfrac{1}{4} \leqslant \int_{S_{F_1}} |Ux|^2 d\mu(x) = \int_{S_{F_1}} \Big|\sum_{i=1}^{5k} \lambda_i (u_i, x) v_i\Big|^2 d\mu(x)$$
$$= \sum_{i=1}^{5k} \lambda_i^2 \int_{S_{F_1}} (u_i, x)^2 d\mu(x) = \frac{2}{n} \sum_{i=1}^{5k} \lambda_i^2.$$

Hence $\sum_{i=1}^{5k} \lambda_i^2 \geqslant \frac{1}{8} n$; we conclude that the cardinality of the set $\Lambda = \{i\colon \lambda_i \geqslant \frac{1}{4}\}$ is at least $k$. Indeed, if this were not so, then, since $\|U\| \leqslant 1$ and therefore $\lambda_i \leqslant 1$ for $i = 1, 2, \ldots, 5k$, we would have

$$\sum_{i=1}^{5k} \lambda_i^2 = \sum_{i \in \Lambda} \lambda_i^2 + \sum_{i \notin \Lambda} \lambda_i^2 \leqslant (\operatorname{card} \Lambda)\, 1 + (5k - \operatorname{card} \Lambda)(\tfrac{1}{4})^2$$
$$< k + 4k\tfrac{1}{16} = \tfrac{1}{8} n,$$

which gives a contradiction. To conclude the proof note that for $x$ in $\bar{F}_1 = \operatorname{lin}\{u_i\colon i \in \Lambda\}$ we have $|P_{F_2} T_0 x| = |Ux| \geqslant \frac{1}{4}|x|$.

We now turn our attention back to the case of a general group $G$ in $\mathscr{G}(\boldsymbol{R}^{2n})$. Using Lemma 1, we shall derive

Lemma 2. *For every group $G \in \mathscr{G}(\boldsymbol{R}^{2n})$ there is a permutation $(i, j)$ of $\{1, 2\}$ and an operator $T_0 \in G$ such that the operator $P_j^n T_0$ restricted to $E_i^n$ is thick (considered as an operator from $\boldsymbol{R}^n$ into $\boldsymbol{R}^n$).*

Proof. Fix $G \in \mathscr{G}(\boldsymbol{R}^{2n})$ and let $\varepsilon_G$ be the ellipsoid with property (**). Let $F_1$ and $F_2$ be $5k$-dimensional subspaces of $E_1^n$ and $E_2^n$, respectively, such that $|x| = c_i \|x\|_G$ for $x \in F_i$, $i = 1, 2$ (it is easy to show that such subspaces exist, cf. [1]). Assume that $c_1 \leqslant c_2$ (the other case can be treated in a "symmetric" way). Let $E$ be the orthogonal complement of $F_2$ in $(\boldsymbol{R}^{2n}, (\,,))$ and let $P$ be the orthogonal projection with respect to the scalar product $\langle\,,\rangle_G$ with $\ker P = E$. Set $\tilde{E} = \operatorname{Im} P$. Obviously $\dim \tilde{E} = 5k$ and $P|F_2$ is 1-1 mapping of $F_2$ onto $\tilde{E}$. By Lemma 1 applied to $(\boldsymbol{R}^{2n}, \langle\,,\rangle_G)$ there is a $T_0$ in $G$ with the property that $\|PT_0 x\|_G \geqslant \frac{1}{4}\|x\|_G$ for every $x$ in some $k$-dimensional subspace $\bar{F}_1$ of $F_1$. Let $Q\colon \tilde{E} \to F_2$ be the inverse of $P|F_2$. Since $\|Px\|_G \leqslant \|x\|_G$ we infer that $\|Qy\|_G \geqslant \|y\|_G$ for every $y \in \tilde{E}$. Note that $P_{F_2} = QP$ is the orthogonal projection of $\boldsymbol{R}^{2n}$ onto $F_2$ with respect to the scalar product $(\,,)$. Indeed, $P_{F_2}$ annihilates the orthogonal complement of $F_2$ and $P_{F_2}|F_2 = \mathrm{Id}_{F_2}$. Thus, for every $x \in \bar{F}_1$, we have

$$|P_2^n T_0 x| \geqslant |P_{F_2} T_0 x| = c_2 \|P_{F_2} T_0 x\|_G$$
$$\geqslant c_2 \|PT_0 x\|_G \geqslant c_2 \tfrac{1}{4}\|x\|_G \geqslant c_1 \tfrac{1}{4}\|x\|_G = \tfrac{1}{4}|x|,$$

which concludes the proof.

Remark. Note that, if for some $G \in \mathscr{G}(\boldsymbol{R}^{2n})$ and some $B_1, B_2 \in \mathscr{A}_n$ $s(B_1 \times B_2, G) \leqslant c\sqrt{n}$ and if $T_0$ is an operator which satisfies the conclusion of Lemma 2, then the operator

$$T = P_j^n T_0 | E_i^n\colon E_i^n \to E_j^n$$

has the following properties:

(a) $T$ is thick,

(b) $\|T\|_{B_i \to B_j} \leqslant c\sqrt{n}$ (because $\|P_j^n\|_{B_1 \times B_2 \to B_1 \times B_2} = 1$).

**3. Volume estimates and Gluskin spaces.** We begin with

Lemma 3. *Let $T\colon \boldsymbol{R}^n \to \boldsymbol{R}^n$ be such that $T$ restricted to some $k$-dimensional subspace $F$ of $\boldsymbol{R}^n$ is 1-1 and let $A$ be a subset of $\boldsymbol{R}^n$. Then*

$$\operatorname{vol}_n\{x \in K_n\colon Tx \in A\} \leqslant |\det T|F|^{-1} \operatorname{vol}_k P(A) \operatorname{vol}_{9k}(K_{9k}),$$

*where $P$ is the orthogonal projection onto* $\operatorname{Im} T|F$.

Proof. Let $P_1$ be the projection onto $F$ with $\ker P_1 = \ker PT$. Then $PT = PT_1$. Since, by the Hadamard inequality, for every $C \subset F$ and $D \subset \boldsymbol{R}^n$ we have

$$\operatorname{vol}_n\{x \in D\colon P_1 x \in C\} \leqslant \int_C \operatorname{vol}_{9k}\{x \in D\colon P_1 x = y\}\, dy,$$

we infer that

$$\operatorname{vol}_n\{x \in K_n\colon Tx \in A\} \leqslant \operatorname{vol}_n\{x \in K_n\colon PTx \in P(A)\}$$
$$= \operatorname{vol}_n\{x \in K_n\colon TP_1 x \in P(A)\} = \operatorname{vol}_n\{x \in K_n\colon P_1 x \in T^{-1}(P(A))\}$$
$$\leqslant \int_{T^{-1}(P(A))} \operatorname{vol}_{9k}\{x \in K_n\colon P_1 x = y\}\, dy \leqslant \int_{T^{-1}(P(A))} \operatorname{vol}_{9k}(K_{9k})\, dy$$
$$= \operatorname{vol}_{9k}(K_{9k}) \int_{P(A)} |\det T|F|^{-1} dx = |\det T|F|^{-1} \operatorname{vol}_k P(A) \operatorname{vol}_{9k}(K_{9k}).$$

Lemma 3 yields the following (cf. Gluskin [3], Cor. 1, and Szarek [7], Claim 6.2).

LEMMA 4. *There is an absolute constant $d_1 > 0$ such that for every $c > 0$ and every thick $T \in L(\mathbf{R}^n)$ and every $B_1 \in \mathscr{A}_n$ the following inequality holds*

$$\mathbf{P}_n\{B \in \mathscr{A}_n:\ \|T\|_{B \to B_1} \leqslant 2c\sqrt{n}\} \leqslant (cd_1)^{2n^2}.$$

Proof. By Lemma 3, with $A = 2c\sqrt{n}B_1$, we have for every fixed thick $T \in L(\mathbf{R}^n)$, $B_1 \in \mathscr{A}_n$ and $c > 0$

$$\mathrm{vol}_n\{x \in K_n:\ Tx \in 2c\sqrt{n}B_1\} \leqslant 4^k \mathrm{vol}_k(\mathrm{P}(2c\sqrt{n}B_1))\mathrm{vol}_{9k}(K_{9k}).$$

Let $\mathscr{D}$ be the family of all $k$-elements subsets of the set of extreme points of $B_1$. Since, by the Hadamard inequality,

$$\begin{aligned}\mathrm{vol}_k(P(2c\sqrt{n}B_1)) &\leqslant \sum_{D \in \mathscr{D}} (2c\sqrt{n})^k \mathrm{vol}_k(\mathrm{abs\ conv}\ P(D))\\ &\leqslant (2c\sqrt{n})^k \mathrm{card}\ \mathscr{D}\ \mathrm{vol}_k(\mathrm{abs\ conv}\{e_1, e_2, \ldots, e_k\})\\ &\leqslant (c\sqrt{n})^k \left(\frac{d_2}{n}\right)^k \leqslant \left(\frac{cd_2}{\sqrt{n}}\right)^k\end{aligned}$$

for suitable $d_2 > 0$, we infer that

$$\mathrm{vol}_n\{x \in K_n:\ Tx \in 2c\sqrt{n}B_1\} \leqslant 4^k \left(\frac{cd_2}{\sqrt{n}}\right)^k \left(\frac{d_3}{\sqrt{n}}\right)^{9k} \leqslant c^k \left(\frac{d_4}{\sqrt{n}}\right)^n,$$

for some absolute constants $d_3, d_4 > 0$. Hence, for sufficiently large $d_5 > 0$

$$\mu_n\{x \in S_n:\ Tx \in 2c\sqrt{n}B_1\} \leqslant \frac{\mathrm{vol}_n\{x \in K_n:\ Tx \in 2c\sqrt{n}B_1\}}{\mathrm{vol}_n(K_n)} \leqslant (d_5)^n c^k.$$

Therefore

$$\begin{aligned}&\mathbf{P}_n\{B \in \mathscr{A}_n:\ \|T\|_{B \to B_1} \leqslant 2c\sqrt{n}\}\\ &\qquad \leqslant \prod_{i=1}^{20n} \mu_n\{x_i \in S_n:\ Tx_i \in 2c\sqrt{n}B_1\} \leqslant (d_5^{10}c)^{2n^2},\end{aligned}$$

which gives the required inequality with $d_1 = d_5^{10}$.

**4. Proof of Theorem 2.** The rest of the proof is, essentially, a repetition of Gluskin's argument [3], but we shall present it for the sake of completeness. The next lemma can be found in [3].

LEMMA 5. *There exists an absolute constant $d_6 > 0$ such that for every $\varepsilon > 0$, every $B_1 \in \mathscr{A}_n$, every subset of the set of all operators $T \in L(\mathbf{R}^n)$ such that $Te_i \in \sqrt{n}B_1$, for $i = 1, 2, \ldots, n$, admits an $\varepsilon$-net $\mathscr{M}_\varepsilon^{B_1}$ with respect to the operator norm in $(L(\mathbf{R}^n), |\ |)$ with*

$$\mathrm{card}\ \mathscr{M}_\varepsilon^{B_1} \leqslant \left(\frac{d_6}{\varepsilon}\right)^{n^2}.$$



**Proof of Theorem 2.** Let $c_1 = (2d_1 d_6^2)^{-1}$. By Lemma 2 and the Remark which follows it, we have

$$\mathbf{P}_n \times \mathbf{P}_n\{(A, B) \in \mathscr{A}_n \times \mathscr{A}_n:\ s(A \times B) < c_1\sqrt{n}\} \leqslant \mathbf{P}_n \times \mathbf{P}_n(\mathscr{U}_1) + \mathbf{P}_n \times \mathbf{P}_n(\mathscr{U}_2)$$

where, for $i = 1, 2$ and $j \in \{1, 2\}$, $j \neq i$,

$$\mathscr{U}_i = \{(B_1, B_2) \in \mathscr{A}_n \times \mathscr{A}_n:\ \text{there exists a thick } T \in L(\mathbf{R}^n) \text{ such that } \|T\|_{B_i \to B_j} \leqslant c_1\sqrt{n}\}.$$

We shall show that $\mathbf{P}_n \times \mathbf{P}_n(\mathscr{U}_1) \leqslant (\frac{1}{2})^{n^2}$. Indeed, for every $B \in \mathscr{A}_n$ set $\mathscr{U}_{1,B} = \{A \in \mathscr{A}_n:\ (A, B) \in \mathscr{U}_1\}$ and let $\mathscr{M}_{c_1}^B$ be the $c_1$-net from Lemma 5 for the set $\mathscr{F}_B$ of all thick operators $T \in L(\mathbf{R}^n)$ such that $Te_i \in \sqrt{n}B$. Note that the set $\mathscr{F}_B$ contains all operators which appeared in the definition of $\mathscr{U}_1$. Since, for every $A \in \mathscr{A}_n$ and $T \in \mathscr{F}_B$ such that $\|T\|_{A \to B} \leqslant c_1\sqrt{n}$, we have

$$\begin{aligned}\inf\{\|T - \tilde{T}\|_{A \to B}:\ \tilde{T} \in \mathscr{M}_{c_1}^B\} &\leqslant \inf\{\|T - \tilde{T}\|_{l_2^n \to l_1^n}:\ \tilde{T} \in \mathscr{M}_{c_1}^B\}\\ &\leqslant \sqrt{n}\inf\{\|T - \tilde{T}\|_{l_2^n \to l_2^n}:\ \tilde{T} \in \mathscr{M}_{c_1}^B\} \leqslant c_1\sqrt{n}.\end{aligned}$$

we infer from the triangle inequality that

$$\mathscr{U}_{1,B} \subset \bigcup_{T \in \mathscr{M}_{c_1}^B} \{A \in \mathscr{A}_n:\ \|T\|_{A \to B} \leqslant 2c_1\sqrt{n}\}.$$

Hence, by Lemma 4, Lemma 5 and by the choice of $c_1$, for every $B \in \mathscr{A}_n$

$$\begin{aligned}\mathbf{P}_n(\mathscr{U}_{1,B}) &\leqslant \sum_{T \in \mathscr{M}_{c_1}^B} \mathbf{P}_n\{A \in \mathscr{A}_n:\ \|T\|_{A \to B} \leqslant 2c_1\sqrt{n}\}\\ &\leqslant (\mathrm{card}\ \mathscr{M}_{c_1}^B)(c_1 d_1)^{2n^2} \leqslant \left(\frac{d_6}{c_1}\right)^{n^2}(c_1 d_1)^{2n^2} = (\tfrac{1}{2})^{n^2}\end{aligned}$$

and therefore, by Fubini Theorem

$$\mathbf{P}_n \times \mathbf{P}_n(\mathscr{U}_1) = \int_{\mathscr{A}_n} \mathbf{P}_n(\mathscr{U}_{1,B})\,d\mathbf{P}_n(B) \leqslant (\tfrac{1}{2})^{n^2}.$$

By the same token $\mathbf{P}_n \times \mathbf{P}_n(\mathscr{U}_2) \leqslant (\frac{1}{2})^{n^2}$. Thus

$$\mathbf{P}_n \times \mathbf{P}_n\{(A, B) \in \mathscr{A}_n \times \mathscr{A}_n:\ s(A \times B) < c_1\sqrt{n}\} \leqslant 2(\tfrac{1}{2})^{n^2};$$

this concludes the proof.

Remark. It follows from the proof (not surprisingly) that if $(A, B) \notin \mathscr{U}_1 \cup \mathscr{U}_2$, then $d(A, B) \geqslant c_1^2 n$.

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