PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 84, Number 2, February 1982

FINITE DIMENSIONAL RIGHT DUO ALGEBRAS ARE DUO

R. C. COURTER

ABSTRACT. Available examples of right (but not left) duo rings include rings without unity element which are two dimensional algebras over finite prime fields. We prove that a right duo ring with unity element which is a finite dimensional algebra over an arbitrary field K is a duo ring. This result is obtained as a corollary of a theorem on right duo, right artinian rings R with unity: left duo-ness is equivalent to each right ideal of R having equal right and left composition lengths, which is equivalent to the same property on R alone. Another result concerns algebras over a field which are semiprimary right duo rings: such an algebra is left duo provided (1) the algebra is finite dimensional modulo its radical and (2) the square of the radical is zero. These two provisions are shown to be essential by examples which are local algebras, duo on one side only.

1. Introduction. Unless the contrary is stated, rings and algebras will be assumed to have a unity element. We will denote the unity element by 1 and the Jacobson radical of a ring or algebra by P without further comment.

DEFINITIONS. A ring R with unity element is a right duo ring if it satisfies the following equivalent conditions:

(A) Each right ideal of R is an ideal.

(B) For each $x \in R$, $Rx \subseteq xR$.

Left duo-ness has the obvious definition. R is a duo ring if it is both left duo and right duo.

PROPOSITION 1.1. *If R is a right duo ring, so is each of its homomorphic images.*

REMARK 1.2. An invertible element x of a ring R satisfies $xR = Rx$: if $r \in R$, $rx = xx^{-1}rx \in xR$.

THEOREM 1.3. *Let e be an idempotent element of a right duo ring R. Then e belongs to the center of R. Consequently, right artinian, right duo simple rings are division rings.*

PROOF. Let $f = 1 - e$. If e is not a central element, there exists an element $x \in R$ satisfying either $exf \neq 0$ or $fxe \neq 0$. If $exf \neq 0$, the right duo-ness of R implies that $exf \in fR$. But $eR \cap fR = 0$, contradicting $exf \neq 0$. Thus each idempotent is central. The second conclusion is obvious now.

EXAMPLES. In [5, p. 149, Theorem 1] it is proved that a skew power series ring $D\langle t, \sigma\rangle$, defined over a commutative principal ideal domain D with the aid of a

Received by the editors September 23, 1980 and, in revised form, April 13, 1981.
1980 *Mathematics Subject Classification*. Primary 16A15; Secondary 16A46.

monomorphism σ from the quotient field of D into D, is a left duo, left principal ideal domain (with unity element), not right duo if σ is not onto D.

The following example has no unity element: Let K be the field with exactly p elements, p a prime. The subalgebra $R = KE_{11} + KE_{12}$ of the full matrix algebra K_n $(n > 1)$ is a right duo ring, since it is easy to prove that the ideals KE_{12}, R and 0 are the only right ideals. Since $R(E_{11} + E_{12}) = K(E_{11} + E_{12})$ is not a right ideal, R is not left duo. Up to isomorphism, these rings appear in [**1**, p. 90].

The preceding example motivated the author to look for one-sided duo finite dimensional algebras with unity element and to prove instead that such algebras do not exist. The proof of this result as it now appears in §2 is due to the referee. In particular his contribution of Theorem 2.2 on right artinian, right duo rings widens the scope of the paper.

2. Right artinian rings.

Notation. We denote by $c(M)$ the composition length of an R-module M. If I is a right (left) ideal of R, I^0 will denote the left (right) annihilator in R of I.

DEFINITION. A ring R is a quasi-Frobenius ring if and only if R is right artinian and each left (right) ideal is the left (right) annihilator of its right (left) annihilator.

A lattice anti-isomorphism exists, as a result of the definition of quasi-Frobenius ring, between the set of left ideals and the set of right ideals of such a ring. Thus the right composition length of a quasi-Frobenius ring equals its left composition length, proving half of the following lemma (it is well known that quasi-Frobenius rings satisfy the hypothesis below on simple modules [**3**, p. 204, Theorem 24.4(c)]):

LEMMA 2.1. *Let R be a right artinian ring such that the R-dual* $\mathrm{Hom}_R(S_R, R_R)$ *of a simple right R-module S is zero or a simple left R-module. Then R is a quasi-Frobenius ring if and only if* $c({}_RR) = c(R_R)$.

PROOF OF $\Leftarrow$. The condition on composition lengths implies that R is left artinian, so that results in [**2**] hold as follows: By [**2**, 3.4, p. 349] a perfect duality exists between finitely generated left and right modules, which is equivalent to R being quasi-Frobenius—please see [**2**, p. 346 and 3.1, p. 349].

We remark that for any ring R and simple module S the R-dual of S is isomorphic to M^0 for some maximal left [right] ideal M if S is a left [right] module.

THEOREM 2.2. *Let R be a right artinian, right duo ring. The following are equivalent*:

(1) *R is left duo.*

(2) $c(R_R) = c({}_RR)$.

(3) *If $I_1 \subset I_2$ are right ideals with $(I_2/I_1)_R$ simple, ${}_R(I_2/I_1)$ is also simple.*

(4) *For any right ideal I,* $c(I_R) = c({}_RI)$.

PROOF. (1) implies (2), since all one-sided ideals are two-sided. To prove that (2) implies (3), let $I_1 \subset I_2$ be extended to a composition series for R_R. This is also a normal series for ${}_RR$ with nonzero factors. By (2) it is a composition series for ${}_RR$; I_2/I_1 is a simple left R-module. If (3) is assumed, a composition series for I_R is a normal series for ${}_RI$ and the factors are simple by (3).

PROOF THAT (4) IMPLIES (1). Assume that this is false and let R be a counterexample with $n = c(R_R)$ minimal. In particular each proper quotient of R is a duo ring and any left ideal containing a nonzero right ideal is an ideal. Let M be any maximal right ideal; thus $D = R/M$ is a division ring. The left R-module $M^0 \cong \mathrm{Hom}_R(R/M, R_R)$ is a right vector space over D and a right R-module. To show that M^0 is a simple left ideal we will show that it is a simple right ideal and apply (4).

Let $y \in R$ be such that Ry is contained properly in yR: let $x \in M^0$. Since xD is a right ideal, xD and $Ry + xD$ are left ideals. Thus $Ry + xD$ is an ideal and contains yR. We have

$$Ry + xD = yR + xD.$$

As left modules, $c(yR) \leqslant c(Ry + xD) \leqslant c(Ry) + 1 \leqslant c(yR)$, so that

$$yR = Ry + xD, \qquad Ry \cap xD = 0.$$

Suppose $\dim(M^0) > 1$, say $xD \oplus x_1D \subseteq M^0$. Then $yR = Ry + xD = Ry + x_1D = Ry + (xD \oplus x_1D)$. As left modules, $c(Ry + (xD \oplus x_1D)) < c(Ry) + 2$, so that $Ry \cap (xD \oplus x_1D) \neq 0$. For some nonzero $x_2 \in M^0$ the two-sided ideal x_2D is contained in Ry making Ry an ideal. This contradiction proves that M^0 is one dimensional and is a simple right ideal. By (4), M^0 is a simple left ideal. Since the duals of simple right R-modules are simple, we may apply Lemma 2.1. By (4), $c(R_R) = c({}_RR)$; by Lemma 2.1, R is a quasi-Frobenius ring. Each left ideal is the left annihilator of a two-sided ideal, whence it is a right ideal. The ring R is left duo.

COROLLARY 2.3. *Let R be a right duo ring with unity element which is a finite dimensional algebra over a field K. Then R is left duo.*

PROOF. Let $I_1 \subset I_2$ be right ideals such that, for some maximal right ideal M, $(R/M)_R \cong (I_2/I_1)_R$. By Theorem 1.3 there is a central primitive idempotent e such that $M = (1 - e)R + eM$ and $eM = Me$ is the Jacobson radical of eRe. Since $(R/M)(1 - e) = 0$, the isomorphism of $(R/M)_R$ onto $(I_2/I_1)_R$ implies that $(1 - e)(I_2/I_1) = (I_2/I_1)(1 - e) = 0$. Evidently, I_2/I_1 is a left and right (eRe)-module. Clearly, as a left or right module, I_2/I_1 is simple for R if and only if it is simple for eRe; also a left or right (eRe)-module is simple if and only if its K-dimension equals that of $(eRe)/(eMe)$. Evidently, the simple right R-module I_2/I_1 is a simple left R-module; by Theorem 2.2, R is a left duo ring.

3. Semiprimary rings.

DEFINITIONS. If R/P is a division ring, the ring R with unity element is a local ring. If R/P is right artinian and P is nilpotent, R is a semiprimary ring.

THEOREM 3.1. *Let R be a right duo semiprimary ring. Then R is the ring direct sum of finitely many right duo local rings with nilpotent radicals.*

PROOF. We assume that the primitive idempotents of the decomposition of R/P to simple summands has been lifted [3, p. 41]. Thus, by Theorem 1.3, we have central primitive idempotents $e_1, \ldots, e_m$ such that R is the direct sum of the ideals e_iR. For each i the radical of e_iR is $P \cap e_iR$ [3, p. 33] and is nilpotent. $e_iR/(P \cap e_iR) \cong (P + e_iR)/P$ is simple by the opening sentence. As homomorphic images of R, e_iR and $e_iR/(P \cap e_iR)$ are right duo rings, so that the latter is a division ring by Theorem 1.3. Thus each summand e_iR is a right duo local ring with nilpotent radical.

LEMMA 3.2. *Let K be a field and let R be a right duo local K-algebra such that R/P has finite K-dimension. Let x belong to the left socle and the right socle of R. Then $xR = Rx$. Consequently, if $P^2 = 0$, R is a duo ring.*

PROOF. By hypothesis $Rx \subseteq xR$. Since, for nonzero x, Rx and xR are K-isomorphic with R/P which has finite K-dimension, $Rx = xR$.

Now let $P^2 = 0$. Then P is equal to both socles, so that $yR = Ry$ for $y \in P$. For $y \notin P$, y is invertible [3, p. 35]; $yR = Ry$ by Remark 1.2. R is a duo ring.

THEOREM 3.3. *Let K be a field and let R be a right duo semiprimary K-algebra such that R/P has finite K-dimension and $P^2 = 0$. Then R is a duo ring.*

PROOF. By Theorem 3.1, R is the direct sum of finitely many right duo local K-algebras R_i. The square of the radical of each ring R_i is zero, whence each R_i is a duo ring by Lemma 3.2. Thus each left ideal of R is a right ideal, as required.

The essentiality of the assumptions on R/P and P^2 in Theorem 3.3 is demonstrated by two examples:

EXAMPLE 3.4. *A right duo (not left duo) local K-algebra R with $R/P \cong K$, $P^2 \neq 0$, $P^3 = 0$.* Let K be an arbitrary field and let V be a K-space having countably infinite dimension. Let I be the identity function on V. $R = KI \oplus P$ consists of linear transformations on V in matrix form. The radical P of R consists of row-finite matrices; we use matrix unit notation to describe a basis $\{a_0, a_1, \ldots, a_j, \ldots\}$ of P:

$$\begin{aligned} a_0 &= E_{12}, \\ a_1 &= E_{13}, \\ a_2 &= E_{14} + E_{32}, \\ &\ldots, \\ a_j &= E_{1,j+2} + E_{j+1,2}, \\ &\ldots. \end{aligned}$$

Thus $Pa_0 = 0 = a_0P$ and, for $i \geqslant 1$, $a_ia_{i+1} = a_0$ and $a_iP = Ka_0$. Evidently, $P^2 = Ka_0$ and $P^3 = 0$. R is not a left duo ring, since $a_1a_2 = a_0 = E_{12}$ does not equal rE_{13} for any $r \in R$, so that $a_1a_2 \notin Ra_1$. If a and b are elements of R with b invertible, $ab \in bR$. Now let $b \in P$ and let $m > 0$ be such that the elements a and b belong to $KI + \sum_0^m Ka_i$. Let

$$a = kI + \sum_0^m c_ia_i, \qquad b = \sum_0^m d_ia_i$$

where $k, c_0, \ldots, c_m, d_0, \ldots, d_m \in K$. Then

$$ab = \sum_1^m (kd_i)a_i + (c_1d_2 + c_2d_3 + \cdots + c_{m-1}d_m + kd_0)a_0.$$

Let

$$v = kI + c_1a_3 + c_2a_4 + \dots + c_{m-1}a_{m+1}.$$

Then $ab = bv$, completing the proof that R is a right duo ring.

EXAMPLE 3.5. *A right duo (not left duo) local algebra, having countably infinite dimension modulo its radical P and satisfying* $P^2 = 0$. We begin with the ring of twisted polynomials; the existence on one side only of a classical field of quotients is one of the phenomena this ring has illustrated [**4**, p. 384]. Let K be a field and let F be an extension field of K which has a K-algebra isomorphism σ onto a proper subfield F_0 of F. Let R denote the set of polynomials in an indeterminate t with coefficients in F written on the right. R is provided with a noncommutative, associative multiplication by

$$ct = tc^\sigma, \qquad c \in F.$$

Provided with the usual addition of polynomials, R is a K-algebra with unity. Let $\bar{R}$ be the ring obtained from R by setting $t^2 = 0$. The radical P of $\bar{R}$ is $\{tc \mid c \in F\}$. Thus $P^2 = 0$. $\bar{R}/P \cong F$ has infinite K-dimension if we choose as F the field of rational functions $K(x)$ in one indeterminate. σ can be the extension to $K(x)$ of the map which takes $p(x) \in K[x]$ to $p(x^2)$.

Let a and b be the following polynomials in $\bar{R}$:

$$a = c + td, \qquad b = tg, \qquad c, d, g \in F.$$

If w is the polynomial c^σ, then $ab = c(tg) = t(c^\sigma g) = bw \in b\bar{R}$. The proof that $\bar{R}$ is right duo can be completed. Now consider two polynomials th and h^{-1} where h belongs to F but not to $F_0 = F^\sigma$. Suppose that the product $t = (th)(h^{-1})$ belongs to $\bar{R}(th)$. Thus for some m and $m' \in F$, $t = (m + tm')(th) = m(th) = tm^\sigma h$ and we have the contradiciton $h = (m^{-1})^\sigma \in F^\sigma$. $\bar{R}$ is not left duo.

REFERENCES

1. V. R. Chandran, *On duo rings*. VI, Pure Appl. Math. Sci. **3** (1976), 89–92.
2. J. Dieudonné, *Remarks on quasi-Frobenius rings*, Illinois J. Math. **2** (1958), 346–354.
3. C. Faith, *Algebra*. II. *Ring theory*, Springer-Verlag, Berlin and New York, 1976.
4. E. R. Gentile, *On rings with one-sided field of quotients*, Proc. Amer. Math. Soc. **11** (1960), 380–384.
5. A. V. Jategaonkar, *Left principal ideal domains*, J. Algebra **8** (1968), 148–155.

Current address: 300 West 107th Street, Apartment 5B, New York, New York 10025