A comparison of (64) and (65) leads to

$$E_k(\zeta_j') = \frac{g_{k+1}}{g_k} \frac{\mathcal{E}_{k+1}(\zeta_j')}{k+1} \frac{1}{\zeta_j' - 1} h + o(h). \tag{66}$$

Finally, taking (10), (42), and $\zeta_j - \zeta_i' \to \zeta_j' - \zeta_i' \neq 0$ for $i \neq j$ into account yields (26) with (27) which proves item 3) of the Theorem 3. ∎

VI. Conclusion

In the paper, a theorem has been proved that, for small sampling periods, characterizes the accuracy of all limiting zeros of the pulse transfer function of a system composed of a zero-order hold followed by a continuous-time plant.

The main result has a form of a correction to the asymptotic result of Åström *et al.* [3] in the form of a power term of h, whose degree depends on the relative order of the continuous-time counterpart, and its contribution is expressed in terms of Bernoulli numbers and the poles and zeros of the continuous-time transfer function.

The discussion is based on two fundamental lemmas. The first one yields two terms of the Taylor series expansion of the pulse transfer function around $h = 0$ and the second characterizes the magnitude of the difference between the exact pulse transfer function and the principal term of its Poisson representation as a function of h.

Similar methods can be applied to study limiting zeros for pulse transfer functions of systems with a first-order hold.

One of possible applications of the result is investigation of the accuracy of approximate pulse-transfer functions [7].

References

[1] J. Ackermann, *Sampled-Data Control Systems*. Berlin, Germany: Springer-Verlag, 1985.

[2] ——, *Robust Control*. Berlin, Germany: Springer-Verlag, 1993.

[3] K. J. Åström, P. Hagander, and J. Sternby, "Zeros of sampled systems," *Automatica,* vol. 20, pp. 31–38, 1984.

[4] K. J. Åström and B. Wittenmark, *Computer Controlled Systems.* Englewood Cliffs, NJ: Prentice-Hall, 1997.

[5] M. J. Błachuta, "On zeros of sampled systems," in *Proc. 1997 American Control Conf.,* Albuquerque, NM, 1997, pp. 3205–3210.

[6] ——, "Continuous-time design of discrete-time control systems," in *Proc. 1997 European Control Conf.,* Brussels, Belgium, 1997.

[7] ——, "On approximate pulse transfer functions," in *Proc. 36th Conf. Decision and Control,* San Diego, CA, 1997, pp. 357–362.

[8] T. Chen and B. A. Francis, *Optimal Control of Sampled-Data Systems.* Berlin, Germany: Springer-Verlag, 1995.

[9] D. W. Clarke, "Self-tuning control of nonminimum-phase systems," *Automatica,* vol. 20, pp. 501–517, 1984.

[10] M. Edwards, *Riemann's Zeta Function.* New York: Academic, 1974.

[11] Y. Fu and G. A. Dumont, "Choice of sampling to ensure minimum-phase behavior," *IEEE Trans. Automat. Contr.,* vol. 34, pp. 560–563, 1989.

[12] G. Frobenius, "On Bernoulli numbers and Euler polynomials," *Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften zu Berlin,* pp. 809–847, 1910 (in German).

[13] R. Gessing, "About some properties of discrete-time transfer functions for small sampling periods," in *Proc. European Control Conf.,* Groningen, The Netherlands, 1993, pp. 1699–1702.

[14] P. Hagander, "Comments on 'Conditions for stable zeros of sampled systems,'" *IEEE Trans. Automat. Contr.,* vol. 38, pp. 830–831, 1993.

[15] T. Hagiwara, "Analytic study on the intrinsic zeros of sampled data systems," *IEEE Trans. Automat. Contr.,* vol. 41, pp. 261–263, 1996.

[16] T. Hagiwara, T. Yuasa, and M. Araki, "Stability of the limiting zeros of sampled-data systems with zero- and first-order holds," *Int. J. Contr.,* vol. 58, pp. 1325–1346, 1993.

[17] S. Hara, H. Katori, and R. Kondo, "The relation between real poles and real zeros in SISO sampled systems," *IEEE Trans. Automat. Contr.,* vol. 34, pp. 632–635, 1989.

[18] Y. Hayakawa, S. Hosoe, and M. Ito, "On the limiting zeros of sampled multivariable systems," *Syst. Contr. Lett.,* vol. 2, pp. 292–300, 1983.

[19] M. Ishitobi, "Conditions for stable zeros of sampled systems," *IEEE Trans. Automat. Contr.,* vol. 37, pp. 1558–1561, 1992.

[20] ——, "Stable zeros of sampled low-pass systems," *Int. J. Contr.,* vol. 57, pp. 1485–1498, 1993.

[21] E. I. Jury, *Theory and Application of the Z-Transform.* New York: Wiley, 1964.

[22] Z. Kowalczuk, "A normal triangle and normal methods of transformations in the discrete approximation of continuous systems," *Arch. Automat. Telemech.,* vol. 28, pp. 15–32, 1983.

[23] D. P. Lindorff, *Theory of Sampled-Data Control Systems.* New York: Wiley, 1965.

[24] S. L. Sobolev, "On the roots of Euler polynomials," *Dokl. Akad. Nauk SSSR,* vol. 235, p. 297, 1977; English trans. in *Soviet Math. Dokl.,* vol. 18, pp. 935–938, 1977.

[25] E. C. Titchmarsh, *The Theory of the Riemann Zeta Function.* Oxford, U.K.: Clarendon, 1986.

[26] C.-H. Wang, W.-Y. Wang, and C.-C. Hsu, "Minimum-phase criteria for sampled systems via symbolic approach," *Int. J. Syst. Sci.,* vol. 28, pp. 767–775, 1997.

[27] S. R. Weller, W. Moran, B. Ninness, and A. D. Pollington, "Sampling zeros and the Euler–Frobenius polynomials," in *Proc. 36th Conf. Decision and Control,* San Diego, CA, 1997, pp. 1471–1476.

Finite-Dimensional Risk-Sensitive Filters and Smoothers for Discrete-Time Nonlinear Systems

Subhrakanti Dey and John B. Moore

***Abstract*— Finite-dimensional optimal risk-sensitive filters and smoothers are obtained for discrete-time nonlinear systems by adjusting the standard exponential of a quadratic risk-sensitive cost index to one involving the plant nonlinearity. It is seen that these filters and smoothers are the same as those for a fictitious linear plant with the exponential of squared estimation error as the corresponding risk-sensitive cost index. Such finite-dimensional filters do not exist for nonlinear systems in the case of minimum variance filtering and control.**

***Index Terms*— Finite-dimensional, information state, minimum variance control, minimum variance estimation, risk-sensitive estimation, smoothing.**

I. Introduction

Risk-sensitive filtering for linear or nonlinear stochastic signal models involves minimization of the expectation of an exponential in quadratic cost criteria. The filters for linear signal models are finite-dimensional but for nonlinear models are infinite-dimensional in general. As opposed to L_2 filtering, (termed as *risk-neutral filtering* in [3]), which achieves the minimization of a quadratic error criteria, risk-sensitive filtering robustifies the filter against plant and noise uncertainties by penalizing all the higher order moments of the estimation error energy. It also allows a tradeoff between optimal

Manuscript received December 6, 1995; revised April 26, 1996. Recommended by Associate Editor, J. C. Spall. This work was supported in part by the Cooperative Research Center for Robust and Adaptive Systems by the Australian Commonwealth Government under the Cooperative Research Centers Program.

The authors are with the Department of Systems Engineering Research, School of Information Sciences and Engineering, Australian National University, Canberra ACT 0200, Australia (e-mail: subhra@syseng.anu.edu.au).

Publisher Item Identifier S 0018-9286(99)03951-3.

filtering for the nominal model case and the average noise situation and robustness to worst case noise and model uncertainty by weighting the index of the exponential by a risk-sensitive parameter. For example, it has been shown in [4] that risk-sensitive filters for hidden Markov models (HMM) with finite-discrete states perform better than standard HMM filters in situations involving uncertainties in the noise statistics. Also, in the small noise limit, risk-sensitive problems have been shown to be closely related to estimation/control problems in a deterministic worst case noise scenario given from a differential game (H_∞ estimation/control problems for linear discrete-time systems) [5], [7], [8].

Risk-sensitive control problems are relatively more abundant in the literature [10]–[12]. Recently, a solution to the output feedback risk-sensitive control problem for linear and nonlinear discrete-time stochastic systems has been proposed in [7] and [13] using a change of probability measure and information state techniques. The problem of risk-sensitive filtering has been studied in [2] for linear Gauss–Markov models. The techniques applied in [2] are not readily generalizable for nonlinear filtering. More general nonlinear problems have been studied in [3] which tackles the risk-sensitive estimation problem using the reference probability methods of [1]. The cost index considered in [3] consists of the sum of quadratic estimation errors to the present and so parallels closely risk-sensitive control/tracking problems considered in [7], [13], and [14].

Although optimal nonlinear filters are known to be infinite-dimensional in general, there are examples of finite-dimensional optimal filters in special cases [15], [16]. Optimal nonlinear risk-sensitive filters studied in [3] are no exceptions. However, recently in controller design, the risk-sensitive cost index has been exploited to cancel the nonlinearities for a class of nonlinear systems so that we can have a finite-dimensional information state and thus finite-dimensional controllers [17], [18]. In [18], finite-dimensional risk-sensitive controllers with finite-dimensional information states are obtained by adjusting the risk-sensitive cost index for a class of discrete-time nonlinear systems.

In this paper, we use similar techniques as in [18] to obtain finite-dimensional risk-sensitive optimal filters for a class of discrete-time nonlinear systems. It is of interest that in the nonlinear context there is no duality between filtering and control problems, yet similar techniques can be used for the solutions of filtering and control problems. In Section II, we present the nonlinear signal model, formulate the risk-sensitive filtering problem, and reformulate it in the new probability measure using reference probability methods. We obtain results for recursive information states and the optimal risk-sensitive filter. In Section III, we show how an appropriate choice of the risk-sensitive cost index can allow us to have a finite-dimensional information state and a finite-dimensional optimal risk-sensitive filter which happens to be the same as that for a linearized version of the nonlinear signal model with a standard exponential of a quadratic cost index. We also provide motivation behind using such a cost index and discuss the robustness issues. Discussions on small noise limits and risk-neutral results are also included.

II. Risk-Sensitive Estimation for Nonlinear Systems

In this section, we consider a class of discrete-time nonlinear state-space signal models. We introduce a risk-sensitive cost index, the justification of which will be clear when we derive the filtering equations. Next, we apply the change of measure technique and reformulate the cost in the new probability measure. Linear recursions in the information state are obtained and the risk-sensitive filter is obtained as the minimizing argument of an integral as a nonlinear (in general) function of the information state.

Signal Model: We consider the following discrete-time nonlinear state-space model defined on a probability space $(\Omega, \mathcal{F}, P)$:

$$\begin{aligned} x_{k+1} &= A_k x_k + a_k(x_k) + w_{k+1} \\ y_k &= C_k x_k + c_k(x_k) + v_k \end{aligned} \tag{1}$$

where $w_k \in \mathbb{R}^n$, $v_k \in \mathbb{R}^p$, $x_k \in \mathbb{R}^n$, and $y_k \in \mathbb{R}^p$. Here, x_k denotes the state of the system, y_k denotes the measurement, and w_k and v_k are the process noise and measurement noise, respectively. The vectors $a_k(x_k)$ and $c_k(x_k)$ have entries which are time-varying nonlinear functions of x_k and $k \in \{0, 1, \cdots, T\}$. We assume that w_k, $k \in \mathbb{N}$ is i.i.d. and has a density function ψ and v_k, and $k \in \mathbb{N}$ is i.i.d. and has a strictly positive density function ϕ. The initial state x_0 or its density is assumed to be known and w_k is independent of v_k.

A. Problem Definition

Define $X_k \triangleq \{x_0, x_1, \cdots, x_k\}$, $Y_k \triangleq \{y_0, y_1, \cdots, y_k\}$, the σ-field generated by Y_k as $\mathcal{Y}_k^0$ and the σ-field generated by X_k and Y_{k-1} by $\mathcal{G}_k^0$. The corresponding complete filtrations are denoted as $\mathcal{Y}_k$ and $\mathcal{G}_k$, respectively. We define $\hat{x}_{t|t}$ as the estimate of the state x_t given $\mathcal{Y}_t$ and work with recursive estimates which update $\hat{x}_{t|t}$ from knowledge of $\hat{x}_{k-1|k-1}$, $k = 1, 2, \cdots, t$.

In [3], we considered the following estimation task: Determine an estimate $\hat{x}_{t|t}$ of x_t such that

$$\hat{x}_{t|t} \in \operatorname*{argmin}_{\zeta} J_t(\zeta), \qquad \forall t = 0, 1, \cdots, T \tag{2}$$

where

$$J_t(\zeta) = E[\exp(\theta \Psi_{0,t}(\zeta)) | \mathcal{Y}_t] \tag{3}$$

is the risk-sensitive cost function. Here

$$\Psi_{0,t}(\zeta) = \hat{\Psi}_{0,t-1} + \tfrac{1}{2}(x_t - \zeta)' Q_t (x_t - \zeta) \tag{4}$$

where

$$\hat{\Psi}_{m,n} = \tfrac{1}{2} \sum_{k=m}^{n} (x_k - \hat{x}_{k|k})' Q_k (x_k - \hat{x}_{k|k}).$$

Assume $Q_k > 0$.

In this paper, we mildly generalize the above cost index so that the estimation task is to obtain $\hat{x}_{t|t}$ such that

$$\hat{x}_{t|t} \in \operatorname*{argmin}_{\zeta} E[\exp(\theta \overline{\Psi}_{0,t}(\zeta)) | \mathcal{Y}_t] \tag{5}$$

where, for some $\overline{L}_k(.,.,.)$ and $\mathcal{L}_t(.,.)$

$$\overline{\Psi}_{0,t}(\zeta) = \sum_{k=0}^{t-1} \overline{L}_k(x_{k+1}, x_k, \hat{x}_{k|k}) + \mathcal{L}_t(x_t, \zeta). \tag{6}$$

Here, $\overline{L}_k(.,.,.)$ and $\mathcal{L}_t(.,.)$, are assumed to be continuous and bounded by a quadratic in the norms of their arguments. This assumption is necessary for the small noise limit results.

B. Change of Measure and Reformulated Cost Index

Define

$$\overline{\lambda}_k = \frac{\phi(y_k - C_k x_k - c_k(x_k))}{\phi(y_k)}$$

$$\overline{\Lambda}_k = \prod_{l=0}^{k} \overline{\lambda}_l.$$

A new probability measure $\overline{P}$ can be defined where y_l, $l \in \mathbb{N}$ are independent with density functions ϕ and the dynamics of x are as under P.

By setting the restriction on the Radon–Nikodym derivative $dP/d\overline{P}|_{\mathcal{G}_k} = \overline{\Lambda}_k$, the measure P can be defined starting with $\overline{P}$. The existence of P follows from Kolmogorov's Extension theorem [1]. Also, under P, the $\{v_l\}$, $l \in \mathbb{N}$, are independent and identically distributed.

Now, we work under measure $\overline{P}$, where y_k, $k \in \mathbb{N}$ is a sequence of independent real random variables with densities ϕ and $x_{k+1} = A_k x_k + a_k(x_k) + w_k$ where w_k, $k \in \mathbb{N}$ are independent random variables with densities ψ. Note that y_k is also independent of x_k under $\overline{P}$.

From a version of Bayes' theorem, our cost index becomes

$$E[\exp(\theta \overline{\Psi}_{0,t}(\zeta))|\mathcal{Y}_t] = \frac{\overline{E}[\overline{\Lambda}_t \exp(\theta \overline{\Psi}_{0,t}(\zeta))|\mathcal{Y}_t]}{\overline{E}[\overline{\Lambda}_t|\mathcal{Y}_t]} \tag{7}$$

where $\overline{E}$ denotes expectation under $\overline{P}$. Hence, our problem objective becomes to determine an $\hat{x}_{t|t}$ such that

$$\hat{x}_{t|t} = \operatorname*{argmin}_{\zeta} \overline{E}[\overline{\Lambda}_t \exp(\theta \overline{\Psi}_{0,t}(\zeta))|\mathcal{Y}_t]. \tag{8}$$

C. Recursive Estimates

Definition II.1: Define $\alpha_{k|k-1}(x)$ as the unnormalized conditional measure such that

$$\alpha_{k|k-1}(x)\,dx = \overline{E}\left[\overline{\Lambda}_{k-1} \exp\left(\theta \sum_{l=0}^{k-1} \overline{L}_l(x_{l+1}, x_l, \hat{x}_{l|l})\right) \times I(x_k \in dx)|\mathcal{Y}_{k-1}\right]. \tag{9}$$

Remark II.1: It has been shown in [5] that $\alpha_{k|k-1}(x)$ can be interpreted as an information state of an augmented plant where the state includes the actual state of the system and part of the risk-sensitive cost. In fact, an alternative method for solving this risk-sensitive optimal filtering problem can be found in [5].

Lemma II.1: The unnormalized measure $\alpha_{k|k-1}(x)$ obeys the following recursion:

$$\begin{aligned}\alpha_{k+1|k}(x) = \frac{1}{\phi(y_k)} \int_{\mathbb{R}^n} &\phi(y_k - C_k z - c_k(z)) \\ &\times \exp(\theta \overline{L}_k(x, z, \hat{x}_{k|k}))\psi(x - A_k z - a_k(z)) \\ &\times \alpha_{k|k-1}(z)\,dz.\end{aligned} \tag{10}$$

Proof: Suppose $f\colon \mathbb{R}^n \to \mathbb{R}$ is any Borel test function. Then, using Definition II.1, we have

$$\begin{aligned}
&\overline{E}\left[f(x_{k+1})\overline{\Lambda}_k \exp\left(\theta \sum_{l=0}^{k} \overline{L}_l(x_{l+1}, x_l, \hat{x}_{l|l})\right)\Bigg|\mathcal{Y}_k\right] \\
&= \int_{\mathbb{R}^n} f(\xi)\alpha_{k+1|k}(\xi)\,d\xi \\
&= \overline{E}\Bigg[f(x_{k+1})\overline{\lambda}_k \exp(\theta \overline{L}_k(x_{k+1}, x_k, \hat{x}_{k|k}))\overline{\Lambda}_{k-1} \\
&\qquad \times \exp\left(\theta \sum_{l=0}^{k-1} \overline{L}_l(x_{l+1}, x_l, \hat{x}_{l|l})\right)\Bigg|\mathcal{Y}_k\Bigg] \\
&= \overline{E}\Bigg[f(A_k x_k + a_k(x_k) + w_{k+1})\,\frac{\phi(y_k - C_k x_k - c_k(x_k))}{\phi(y_k)} \\
&\qquad \times \exp(\theta \overline{L}_k(A_k x_k + a_k(x_k) + w_{k+1}, x_k, \hat{x}_{k|k})) \\
&\qquad \times \overline{\Lambda}_{k-1} \exp\left(\theta \sum_{l=0}^{k-1} \overline{L}_l(x_{l+1}, x_l, \hat{x}_{l|l})\right)\Bigg|\mathcal{Y}_k\Bigg] \\
&= \overline{E}\Bigg[\int_{\mathbb{R}^n} f(A_k x_k + a_k(x_k) + w)\,\frac{\phi(y_k - C_k x_k - c_k(x_k))}{\phi(y_k)} \\
&\qquad \times \exp(\theta \overline{L}_k(A_k x_k + a_k(x_k) + w, x_k, \hat{x}_{k|k})) \\
&\qquad \times \overline{\Lambda}_{k-1} \exp\left(\theta \sum_{l=0}^{k-1} \overline{L}_l(x_{l+1}, x_l, \hat{x}_{l|l})\right)\psi(w)\,dw\Bigg|\mathcal{Y}_{k-1}\Bigg] \\
&= \frac{1}{\phi(y_k)} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(A_k z + a_k(z) + w))\phi(y_k - C_k z - c_k z) \\
&\qquad \times \exp(\theta \overline{L}_k(A_k z + a_k(z) + w, z, \hat{x}_{k|k})) \\
&\qquad \times \psi(w)\alpha_{k|k-1}(z)\,dw\,dz \\
&= \frac{1}{\phi(y_k)} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(\xi)\phi(y_k - C_k z - c_k(z)) \\
&\qquad \times \exp(\theta \overline{L}_k(\xi, z, \hat{x}_{k|k})) \\
&\qquad \times \psi(D(\xi, z))\alpha_{k|k-1}(z)\,d\xi\,dz
\end{aligned} \tag{11}$$

where $\xi = A_k z + a_k(z) + w$, such that $w = D(\xi, z) = \xi - A_k z - a_k(z)$, $z = z$, and $dw\,dz = d\xi\,dz$.

Since this identity holds for every Borel test function f, we have

$$\begin{aligned}\alpha_{k+1|k}(x) = \frac{1}{\phi(y_k)} \int_{\mathbb{R}^n} &\phi(y_k - C_k z - c_k(z)) \\ &\times \exp(\theta \overline{L}_k(x, z, \hat{x}_{k|k}))\psi(x - A_k z - a_k(z)) \\ &\times \alpha_{k|k-1}(z)\,dz.\end{aligned} \tag{12}$$

■

Remark II.2: Supposing $\pi_0(z)$ is the density function of x_0, so for any Borel set $A \in \mathbb{R}^n$, we have $P(x_0 \in A) = \overline{P}(x_0 \in A) = \int_A \pi_0(z)\,dz$. Then $\alpha_{0|-1}(z) = \pi_0(z)$ and all the subsequent estimates follow from Lemma II.1.

Theorem II.1: The optimal $\hat{x}_{t|t}$ can be expressed as

$$\begin{aligned}\hat{x}_{t|t} \in \operatorname*{argmin}_{\zeta} \int_{\mathbb{R}^n} &\alpha_{t|t-1}(z)\,\frac{\phi(y_t - C_t z - c_t(z))}{\phi(y_t)} \\ &\times \exp(\theta \mathcal{L}_t(z, \zeta))\,dz.\end{aligned} \tag{13}$$

Proof: The proof follows easily from (8) and (9). ■

D. Smoothing

In this section we obtain the density function of the smoothed state estimates from a fixed set of observations $Y_T = (y_0, \cdots, y_T)'$. We assume knowledge of the optimal filtered estimates $\hat{X}_T = (\hat{x}_{0|0}, \cdots, \hat{x}_{T|T})'$. This smoothing is essentially an off-line processing and technically known as fixed-interval smoothing. We will also define $\hat{X}_m^n = (\hat{x}_{m|m}, \cdots, \hat{x}_{n|n})$ and $\overline{\Lambda}_{m,n} = \Pi_{k=m}^n \overline{\lambda}_k$. Now, we will define the unnormalized density of the smoothed estimate $\gamma_{k,T}(x)$ and the backward filtered unnormalized density (or backward information state) $\beta_{k,T}(x)$ as follows.

Definition II.2:

$$\begin{aligned}
\gamma_{k,T}(x)\,dx &= \overline{E}\left[\overline{\Lambda}_T \exp\left(\theta \sum_{l=0}^{T-1} \overline{L}(x_{l+1}, x_l, \hat{x}_{l|l}) + \mathcal{L}_T(x_T, \hat{x}_{T|T})\right) I(x_k \in dx)\Bigg|\mathcal{Y}_T\right] \\
\beta_{k,T}(x) &= \overline{E}\left[\overline{\Lambda}_{k,T} \exp\left(\theta \sum_{l=k}^{T-1} \overline{L}(x_{l+1}, x_l, \hat{x}_{l|l}) + \mathcal{L}_T(x_T, \hat{x}_{T|T})\right)\Bigg| x_k = x, \mathcal{Y}_T\right].
\end{aligned} \tag{14}$$

With these definitions, we present the following lemma and theorem. We do not provide the proofs here because they closely follow the proof of Lemma II.1. Also, similar proofs for risk-sensitive smoothers for HMM's can be found in [4].

Lemma II.2: The process $\beta_{k,T}(x)$ satisfies the following backward recursion:

$$\beta_{k,T}(x) = \frac{\phi(y_k - C_k x - c_k(x))}{\phi(y_k)} \int_{\mathbb{R}^n} \exp(\theta \overline{L}(\xi, x, \hat{x}_{k|k})) \times \psi(\xi - A_k x - a_k(x)) \beta_{k+1,T}(\xi)\, d\xi \tag{15}$$

where

$$\beta_{T,T}(x) = \frac{\phi(y_T - C_T x - c_T(x))}{\phi(y_T)} \exp(\theta \mathcal{L}_T(x_T, \hat{x}_{T|T})).$$

Theorem II.2: The unnormalized density function of the smoothed estimate $\gamma_{k,T}(x)$ can be expressed as

$$\gamma_{k,T}(x) = \alpha_{k|k-1}(x)\beta_{k,T}(x). \tag{16}$$

III. Finite-Dimensional Risk-Sensitive Filters and Smoothers

In this section, we show how a suitable choice of the cost kernel allows us to have finite-dimensional risk-sensitive filters. We consider the nonlinear signal model (1) where $w_k \sim N(0, W_k)$ and $v_k \sim N(0, V_k)$. We show that our particular choice of the cost kernel gives us the same finite-dimensional risk-sensitive filters as those for a fictitious linear signal (to be introduced later) model with an exponential of a quadratic cost index and Gaussian distributed noise. We also assume that the distribution of the initial state for (1) is Gaussian distributed. For the limiting case when the noise variances approach zero, we need an additional assumption on the nonlinearities in (1) such that $a_k(x)$ and $c_k(x)$, $k \in \mathbb{N}$ are uniformly continuous in x and bounded by an affine function of the norm of x. With these assumptions, the following theorem holds.

Theorem III.1: The unnormalized information state given by Definition II.1 and the optimal risk-sensitive filter given by (13) for the nonlinear signal model (1) with the cost index (5), (6) are finite-dimensional if in (6), $\overline{L}_k(x_{k+1}, x_k, \hat{x}_{k|k})$ and $\mathcal{L}_t(x_t, \zeta)$ are restricted as follows:

$$\begin{aligned}
&\overline{L}_k(x_{k+1}, x_k, \hat{x}_{k|k}) \\
&= \frac{1}{2\theta}\Bigg(\|y_k - C_k x_k - c_k(x_k)\|^2_{V_k^{-1}} \\
&\qquad + \|x_{k+1} - A_k x_k - a_k(x_k)\|^2_{W_{k+1}^{-1}} - \|y_k - C_k x_k\|^2_{\tilde{V}_k^{-1}} \\
&\qquad - \|x_{k+1} - A_k x_k\|^2_{\tilde{W}_{k+1}^{-1}} \Bigg) + L_k(x_k, \hat{x}_{k|k}) \\
&\mathcal{L}_t(x_t, \zeta) \\
&= \frac{1}{2\theta}\Big(\|y_k - C_k x_k - c_k(x_k)\|^2_{V_k^{-1}} - \|y_k - C_k x_k\|^2_{\tilde{V}_k^{-1}} \Big) \\
&\quad + L_t(x_t, \zeta)
\end{aligned} \tag{17}$$

where $\|x\|^2_A = x'Ax$ and $L_k(x, y) = \frac{1}{2}(x-y)'Q_k(x-y)$.

Proof: Consider Lemma II.1. Noting that ϕ, ψ are Gaussian we can see that the particular choice of our cost kernel as restricted in Theorem III.1 lets us rewrite (10) as

$$\begin{aligned}
\alpha_{k+1|k}(x) = N_k \int_{\mathbb{R}^n} \exp\Big[-\tfrac{1}{2}\Big\{ &(y_k - C_k z)'\tilde{V}_k^{-1}(y_k - C_k z) \\
&+ (x - A_k z)'\tilde{W}_{k+1}^{-1}(x - A_k z) \\
&- \theta(z - \hat{x}_{k|k})'Q_k(z - \hat{x}_{k|k})\Big\}\Big] \\
\times\, \alpha_{k|k-1}(z)\, dz&
\end{aligned} \tag{18}$$

(where N_k is a constant) and (13) as

$$\begin{aligned}
\hat{x}_{t|t} \in \operatorname*{argmin}_{\zeta} \int_{\mathbb{R}^n} &\alpha_{t|t-1}(z) \\
\times \exp\Big[-\tfrac{1}{2}\Big\{ &(y_t - C_t z)'\tilde{V}_t^{-1}(y_t - C_t z) \\
&- \theta(z - \zeta)'Q_t(z - \zeta)\Big\}\Big] dz.
\end{aligned} \tag{19}$$

Noting the quadratic nature of the index of the exponential in (18), it is obvious that if the initial state distribution $\alpha_{0|-1}(x)$ is Gaussian distributed, $\alpha_{k|k-1}(x)$, $\forall k \in \mathbb{N}$ will be also. Hence, $\alpha_{k|k-1}(x)$ is finite-dimensional. An explicit Gaussian density for $\alpha_{k|k-1}(x)$ can be obtained by completion of square and subsequent integration. We present this result in a subsequent corollary in this paper.

Again, it is easy to see from (19) by substituting the Gaussian density of $\alpha_{t|t-1}(x)$ that $\hat{x}_{t|t}$ can be expressed in terms of the parameters of $\alpha_{t|t-1}(x)$ and hence is finite-dimensional. The exact expression for $\hat{x}_{t|t}$ is given in a subsequent corollary. ■

Corollary III.1: The information state $\alpha_{k|k-1}(x)$ is an unnormalized Gaussian density given by

$$\begin{aligned}
\alpha_{k|k-1}(x) &= \alpha_{k|k-1}(x, \chi_k) \\
&= Z_k \exp\left(-\tfrac{1}{2}(x - \mu_k)'R_k^{-1}(x - \mu_k)\right)
\end{aligned} \tag{20}$$

where $\chi_k = (\mu_k, R_k, Z_k)$ and $R_k^{-1}\mu_k$, R_k, Z_k are given by the following algebraic recursions which do not involve integrations:

$$R_{k+1}^{-1}\mu_{k+1} = \tilde{W}_{k+1}^{-1} A_k \Sigma_k \left(R_k^{-1}\mu_k + C_k'\tilde{V}_k^{-1} y_k - \theta Q_k \hat{x}_{k|k}\right) \tag{21}$$

$$R_{k+1} = \tilde{W}_{k+1} + A_k\left(R_k^{-1} + C_k'\tilde{V}_k^{-1}C_k - \theta Q_k\right)^{-1} A_k' \tag{22}$$

$$Z_{k+1} = Z_k |\tilde{W}_{k+1}|^{-(1/2)} |\Sigma_k|^{1/2} M_k\left(R_k^{-1}\mu_k, y_k, \hat{x}_{k|k}\right) \tag{23}$$

where

$$\Sigma_k^{-1} = A_k'\tilde{W}_{k+1}^{-1}A_k + R_k^{-1} + C_k'\tilde{V}_k^{-1}C_k - \theta Q_k$$

and $M_k(R_k^{-1}\mu_k, y_k, \hat{x}_{k|k})$ is an exponential of a quadratic form involving its arguments.

Proof: Consider (18). As mentioned in the proof of Lemma II.1, the quadratic nature of the index of the exponential can be exploited by completion of square and subsequent integration over the Gaussian density if $\alpha_{k|k-1}(x)$ Gaussian. Hence we take up the inductive method. We assume $\alpha_{k|k-1}(x)$ to be Gaussian as described by (20). We evaluate $\alpha_{k+1|k}(x)$ from (18) using the Gaussian expression for $\alpha_{k|k-1}(x)$ from (20). Equating this with the Gaussian form suggested by (20), we obtain the algebraic recursions (21)–(23). Since, we assume that $\alpha_{k|k-1}(x)$ is Gaussian for $k = 0$, the proof is complete. ■

Remark III.1: It is assumed here that $(R_k^{-1} + C_k'\tilde{V}_k^{-1}C_k - \theta Q_k) > 0$, $\forall k$, $R_0 > 0$, which limits the range of acceptable θ. An equivalent condition is $\Sigma_k > 0$; see [2].

We state the following corollary without proof.

Corollary III.2: The optimal estimate $\hat{x}_{t|t}$ can be expressed as

$$\hat{x}_{t|t} = \mu_t + \left(R_t^{-1} + C_t'\tilde{V}_t^{-1}C_t\right)^{-1} C_t'\tilde{V}_t^{-1}(y_t - C_t\mu_t) \tag{24}$$

$$\mu_{t+1} = A_t \hat{x}_{t|t} \tag{25}$$

where $(R_t^{-1} + C_t'\tilde{V}_t^{-1}C_t - \theta Q_t) > 0 \quad \forall t$ and R_t satisfies the following Riccati equation:

$$\begin{aligned}
R_{t+1} &= \tilde{W}_{t+1} + A_t\left(R_t^{-1} + C_t'\tilde{V}_t^{-1}C_t - \theta Q_t\right)^{-1} A_t', \\
R_0 &> 0.
\end{aligned}$$

A. The Optimal Risk-Sensitive Filter in Relation to a Linear Signal Model

In [2] and [3], results have been obtained for the risk-sensitive information states and the optimizing risk-sensitive state estimate for linear Gauss–Markov models with an exponential of a quadratic cost criteria (3), (4). The information state given by Corollary III.1 and the risk-sensitive state estimate given by Corollary III.2 are optimal for the nonlinear signal model (1) and the cost criteria given by (6), (17). Compared with the results in [3], it is interesting to note that the finite-dimensional information state and the risk-sensitive state estimate (20), (24) are similar to those of a fictitious linear signal model given by

$$\begin{aligned}\tilde{x}_{k+1} &= A_k\tilde{x}_k + \tilde{w}_{k+1}\\ \tilde{y}_k &= C_k\tilde{x}_k + \tilde{v}_k\end{aligned} \tag{26}$$

with the cost criteria given by (3) and (4). Of course, we need to assume $\tilde{w}_k \sim N(0, \tilde{W}_k)$, $\tilde{v}_k \sim N(0, \tilde{V}_k)$. Also, $\tilde{w}_k$, $\tilde{v}_k$ are i.i.d. and mutually independent and the initial state $\tilde{x}_0$ is assumed to be identically distributed as x_0, i.e., the initial state of the nonlinear model (1). Now, obviously, $\tilde{y}_k$ cannot have the same statistics as y_k without relaxing the above assumptions on $\tilde{w}_k$, $\tilde{v}_k$. However, for the purpose of designing a finite-dimensional risk-sensitive filter driven from y_k, it is reasonable to work with a linear model as given by (26). Also, we treat $\tilde{W}_k$, $\tilde{V}_k$ as design parameters and are free to choose them so that the filter is realizable. Note that considering a linear signal model like (26) and designing the risk-sensitive filter for it is, heuristically speaking, neglecting the nonlinear terms in (1) and designing a filter for the linearized model. It is necessary therefore, to consider high values for the noise variances $\tilde{W}_k$, $\tilde{V}_k$ to allow for the nonlinearities. In the section where we are dealing with robustness issues, we will see how choosing high values for $\tilde{W}_k$, $\tilde{V}_k$ actually ties in with realizing a sensible cost function $\overline{L}(.,.,.)$ which is convex and acts as an upper bound for the quadratic in nature $L(.,.)$.

1) Example: Here we give a simple example with a scalar nonlinear system given by

$$\begin{aligned}x_{k+1} &= Ax_k - a\,\frac{x_k^2}{1+x_k^2} + w_{k+1}\\ y_k &= Cx_k + v_k\end{aligned} \tag{27}$$

with $|A| < 1$. We assume x_k, $y_k \in \mathbb{R}$, $w_k \sim N(0, \sigma_w^2)$, $v_k \sim N(0, \sigma_v^2)$, $\forall k \in \mathbb{N}$. Also, $L(x, \xi) = (x-\xi)^2$. Substituting these in (17) we obtain the expressions for $\overline{L}_k(x_{k+1}, x_k, \hat{x}_{k|k})$ and $\mathcal{L}_t(x_t, \zeta)$. Substituting these expressions in Lemma II.1 and Theorem II.1, we observe that

$$\begin{aligned}\alpha_{k+1|k}(x) &= \kappa_k \int_{\mathbf{R}^n} \exp\left[-\frac{1}{2}\left\{\frac{(y_k - Cz)^2}{\tilde{\sigma}_v^2} + \frac{(x-Az)^2}{\tilde{\sigma}_w^2}\right.\right.\\ &\qquad \left.\left. -\ \theta(z - \hat{x}_{k|k})^2\right\}\right]\alpha_{k|k-1}(z)\,dz\\ \hat{x}_{t|t} &\in \operatorname*{argmin}_{\zeta} \int_{\mathbf{R}^n} \alpha_{t|t-1}(z)\\ &\qquad \times \exp\left[-\frac{1}{2}\left\{\frac{(y_k - Cz)^2}{\tilde{\sigma}_v^2} - \theta(z-\zeta)^2\right\}\right]dz.\end{aligned} \tag{28}$$

We assume that $\alpha_{0|-1}$ is Gaussian, and it is easy to see from (28) that $\alpha_{k|k-1}(x)$ will be Gaussian $\forall k \in \mathbb{N}$ (by using the obvious completion of square technique and performing the integration) and it will be given by

$$\begin{aligned}\alpha_{k|k-1}(x) &= \alpha_{k|k-1}(x, \mu_k, \sigma_{\alpha_k}^2, Z_k)\\ &= Z_k \exp\left(-\frac{(x-\mu_k)^2}{2\sigma_{\alpha_k}^2}\right)\end{aligned} \tag{29}$$

where

$$\frac{\mu_{k+1}}{\sigma_{\alpha_{k+1}}^2} = \frac{A\left(\dfrac{\mu_k}{\sigma_{\alpha_k}^2} + \dfrac{Cy_k}{\tilde{\sigma}_v^2} - \theta\hat{x}_{k|k}\right)}{A^2 + \dfrac{\tilde{\sigma}_w^2}{\sigma_{\alpha_k}^2} + \dfrac{C^2\tilde{\sigma}_w^2}{\tilde{\sigma}_v^2} - \theta\tilde{\sigma}_w^2}$$

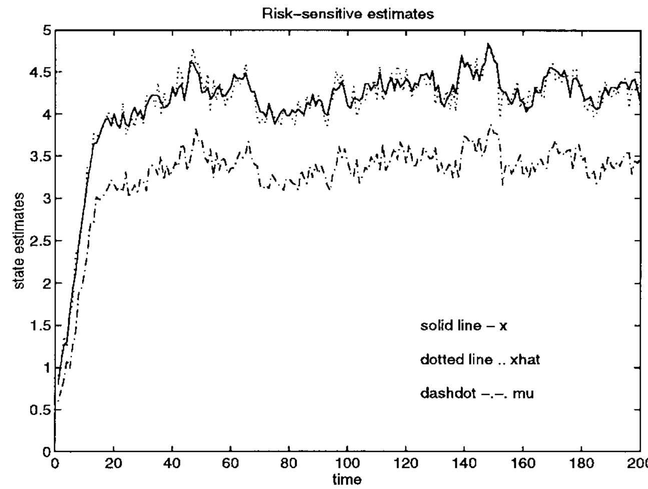

Fig. 1. Finite-dimensional risk-sensitive estimates for a nonlinear system.

$$\sigma_{\alpha_{k+1}}^2 = \tilde{\sigma}_w^2 + \frac{A^2}{\dfrac{1}{\sigma_{\alpha_k}^2} + \dfrac{C^2}{\tilde{\sigma}_v^2} - \theta}. \tag{30}$$

Also, using similar techniques, it is easy to see from Theorem II.1 that

$$\begin{aligned}\hat{x}_{t|t} &= \mu_t + \left(\sigma_{\alpha_t}^{-2} + \frac{C^2}{\tilde{\sigma}_v^2}\right)^{-1} C\tilde{\sigma}_v^{-2}(y_t - C\mu_t)\\ \mu_{t+1} &= A\hat{x}_{t|t}.\end{aligned} \tag{31}$$

We assume $(\sigma_{\alpha_k}^{-2} + (C^2/\tilde{\sigma}_v^2)) > \theta$, $\forall k \in \mathbb{N}$, $\sigma_{\alpha_0}^2 > 0$ in these derivations.

In a simulation study based on the above analysis, we chose $A = 0.8$, $a = -0.9$, $C = 1.0$, $\sigma_w = \sigma_v = 0.1$, and $x_0 = 0.8$. In designing the finite-dimensional risk-sensitive filter, we chose $\tilde{\sigma}_w = 0.5$, $\tilde{\sigma}_v = 0.1$, $\mu_0 = 0.6$, $\sigma_{\alpha_0}^2 = 10.0$, and $\theta = 1/\tilde{\sigma}_v^2$. Fig. 1 shows the time evolution of x_k (solid), $\hat{x}_{k|k}$ (dotted), and μ_k (dash dotted) for a set of 200 time points.

B. Smoothing

In this section, we do not provide the details of how we can obtain finite-dimensional smoothers with our special choice of the risk-sensitive cost index because the reasoning is similar to that for the finite-dimensional information state and the optimal risk-sensitive filter. We just like to make an observation that using Lemma II.2 and Theorem II.2 and the cost index restricted by (17) as in Theorem III.1, it is easy to see that the backward recursive information state and the density of the unnormalized smoothed estimate can be expressed as Gaussian densities which are the same (except for a scaling factor) as those for the linear model (26) with the cost index (3), (4), and $\tilde{y}_k$ replaced by y_k. More details about these densities can be found in [3].

C. Robustness Issues

It is quite obvious that in order to have a finite-dimensional risk-sensitive filter for a nonlinear plant, an appropriate cost index must be found. This result can, therefore, be termed as an inverse optimal estimation result. In this subsection, we discuss some robustness issues involved with selecting such a cost index.

Notice that the cost index given by (17) can be simplified and alternatively written as

$$\begin{aligned}&\overline{L}_k(x_{k+1}, x_k, \hat{x}_{k|k})\\&\quad= \frac{1}{2\theta}\Big(\|v_k\|^2_{V_k^{-1}} + \|w_{k+1}\|^2_{W_{k+1}^{-1}} - \|v_k + c_k(x_k)\|^2_{\tilde{V}_k^{-1}}\\&\qquad - \|w_{k+1} + a_k(x_k)\|^2_{\tilde{W}_{k+1}^{-1}}\Big) L_k(x_k, \hat{x}_{k|k})\\&\mathcal{L}_t(x_t, \zeta)\\&\quad= \frac{1}{2\theta}\Big(\|v_k\|^2_{V_k^{-1}} - \|v_k + c_k(x_k)\|^2_{\tilde{V}_k^{-1}}\Big) + L_t(x_t, \zeta). \end{aligned}\qquad (32)$$

It is easy to see that for plants which are nearly linear (i.e., the nonlinear terms $a(.)$ and $c(.)$ are small), we have

$$\overline{L}_k(.) \simeq L_k(.), \qquad \mathcal{L}_t(.) \simeq L_t(.).$$

Also, for plants where the nonlinear terms $a(.)$, $c(.)$ are exactly zero

$$\tilde{W}_k \geq W_k, \qquad \tilde{V}_k \geq V_k, \qquad \overline{L}_k(.) \geq L_k(.), \qquad \mathcal{L}_t(.) \geq L_t(.). \qquad (33)$$

Also, when the nonlinear terms are nonzero, choosing sufficiently high $\tilde{W}_k$, $\tilde{V}_k$ will ensure that

$$\overline{L}_k(.) \geq L_k(.), \qquad \mathcal{L}_t(.) \geq L_t(.).$$

Hence, the cost index for which the finite-dimensional risk-sensitive filter is optimal for the nonlinear stochastic plant (1) is actually an upper bound on the usual exponential of quadratic cost index for which the same filter is optimal for the artificial linear plant (26). Also, to incorporate any uncertainty in the plant or noise dynamics, the assumption (33) is a reasonable one. It is clear that under such circumstances, the risk-sensitive index given by (5) and (6) becomes more conservative than the one given by (3) and (4) and hence guarantees a more cautious and robust estimation policy which is the real motivation behind adopting a risk-sensitive estimation scheme.

Remark III.2: It should be mentioned here that assumption (33) justifies the choice of the risk-sensitive cost index given by (17), which in turn allows us to obtain a finite-dimensional filter for nonlinear systems. This also ties in with the robustness issues associated with the objective of risk-sensitive estimation. Note, however, that this robustness is with respect to uncertainties in the plant model or noise dynamics. Although suboptimal Kalman filtering techniques exist where noise covariance matrices are properly chosen to compensate for the nonlinearities, such filters will not be able to cope with uncertainties in the plant or noise dynamics. This is more evident in the fact that the finite-dimensional optimal filter [given by (20)–(25)] turns out to be the optimal risk-sensitive filter for a fictitious linear system, which is an H_∞ filter rather than a Kalman filter. This is rather expected because the optimal risk-sensitive filter in the linear Gaussian case is indeed an H_∞ filter. Evidences of robustness of risk-sensitive filters to uncertainties in the plant or noise dynamics have been given in [4] and [6], and applications of risk-sensitive filters to fault detection have been noted in [9]. While such applications are still in their adolescence, it is very important that we can obtain finite-dimensional risk-sensitive filters for nonlinear systems. Choosing sufficiently high $\tilde{W}_k$, $\tilde{V}_k$ merely allows us to justify the choice of the risk-sensitive cost kernel [given by (17)] which is crucial for obtaining finite-dimensional filters.

D. Small Noise Limit

It has been observed that stochastic risk-sensitive control/estimation can be interpreted in terms of a deterministic control/estimation problem in a worst case noise scenario given from a differential game as the risk-sensitive parameter θ approaches the small noise limit [5], [7], [8]. Also, for linear discrete-time systems, the optimal risk-sensitive controller/filter is an H_∞ controller/filter [2], [7]. Following the techniques of [5], we express θ as μ/ϵ and scale the noise variances by $\sqrt{\epsilon}$ so that the limit as $\epsilon \to 0$ can be regarded as the small noise limit. Obviously, using results from [5], the finite-dimensional risk-sensitive optimal filter for the nonlinear system (1) becomes an H_∞ filter which is also the optimal H_∞ filter for an artificial linear model (26). Hence, an appropriate choice of the cost kernel allows us to have a finite-dimensional optimal filter for a class of nonlinear discrete-time systems in a deterministic worst case noise scenario.

E. Risk-Neutral Results

Risk-neutral results are derived from risk-sensitive results by taking the limit as $\theta \to 0$. It is fairly obvious from Lemma II.1 and Theorem II.1 that as $\theta \to 0$, it is not possible to absorb the nonlinear terms anymore and hence, a finite-dimensional optimal filter cannot be obtained.

References

[1] R. J. Elliott, L. Aggoun, and J. B. Moore, *Hidden Markov Models: Estimation and Control* (Application of Mathematics Series). New York: Springer-Verlag, 1994.

[2] J. L. Speyer, C. Fan, and R. N. Banavar, "Optimal stochastic estimation with exponential cost criteria," in *Proc. 31st Conf. Decision and Control,* Dec. 1992, vol. 2, pp. 2293–2298.

[3] S. Dey and J. B. Moore, "Risk-sensitive filtering and smoothing via reference probability methods," *IEEE Trans. Automat. Contr.,* vol. 42, pp. 1587–1591, Nov. 1997.

[4] ——, "Risk-sensitive filtering and smoothing for hidden Markov models," *Syst. Contr. Lett.,* vol. 25, pp. 361–366, Aug. 1995.

[5] J. B. Moore, R. J. Elliott, and S. Dey, "Risk-sensitive generalizations of minimum variance estimation and control," *J. Math. Syst., Estimation and Contr.,* vol. 7, no. 1, pp. 123–126, 1997.

[6] R. J. Elliott, J. B. Moore, and S. Dey, "Risk-sensitive maximum likelihood sequence estimation," *IEEE Trans. Circuits and Syst., Part I,* vol. 43, pp. 805–810, Sept. 1996.

[7] M. R. James, J. S. Baras, and R. J. Elliott, "Risk-sensitive control and dynamic games for partially observed discrete-time systems," *IEEE Trans. Automat. Contr.,* vol. 39, pp. 780–792, Apr. 1994.

[8] R. S. Mangoubi, B. D. Appleby, and G. C. Verghese, "Stochastic Interpretation of H_∞ and robust estimation," in *Proc. 33rd Conf. Decision and Control,* Dec. 1993, vol. 4, pp. 3943–3948.

[9] R. S. Mangoubi *et al.,* "A robust failure detection and isolation algorithm," in *Proc. 35th IEEE Conf. Decision and Control,* New Orleans, LA, Dec. 1995, pp. 2377–2382.

[10] P. Whittle, "Risk-sensitive linear/quadratic/Gaussian control," *Adv. Appl. Probability 13,* pp. 746–777, 1981.

[11] A. Bensoussan and J. H. van Schuppen, "Optimal control of partially observable stochastic systems with an exponential-of-integral performance index," *SIAM J. Contr. Opt.,* vol. 23, pp. 599–613, 1985.

[12] D. H. Jacobson, "Optimal stochastic linear systems with exponential performance criteria and their relation to deterministic games," *IEEE Trans. Automat. Contr.,* vol. AC-18, pp. 124–131, Apr. 1973.

[13] I. B. Collings, M. R. James, and J. B. Moore, "An information-state approach to risk-sensitive tracking problems," *J. Math. Syst., Estimation, and Contr.,* vol. 6, no. 3, pp. 1–24, 1996.

[14] M. R. James, J. S. Baras, and R. J. Elliott, "Output feedback risk-sensitive control and differential games for continuous-time nonlinear systems," in *Proc. 32nd IEEE Conf. Decision and Control,* p. 1993.

[15] V. E. Benes, "Exact finite dimensional filters for certain diffusions with nonlinear drifts," *Stochastics,* vol. 5, pp. 65–92, 1981.

[16] J. Levine and G. Pignie, "Exact finite-dimensional filters for a class of nonlinear discrete-time systems," *Stochastics,* vol. 18, pp. 97–132, 1986.

[17] V. E. Benes and R. J. Elliott, "Finite-dimensional risk-sensitive information states," in *Proc. IFAC Symp. Nonlinear Control System Design (NOLCOS),* June 1995, vol. 2, pp. 471–476, June 1995.

[18] J. B. Moore and J. S. Baras, "Finite-dimensional optimal controllers for nonlinear plants," *Syst. Contr. Lett.,* vol. 26, pp. 223–230, 1995.