Finite-dimensional Sturm–Liouville vessels and their tau functions

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Abstract

We introduce a theory of a class of finite-dimensional vessels, a concept originating from the pioneering work of M. Livšic [Liv1]. This work may be considered as a first step toward analyzing and constructing Lax Phillips scattering theory for Sturm - Liouville differentiable equations on the half axis $(0, \infty)$ with singularity at 0. We also develop a rich and interesting theory of vessels with deep connections to the notion of τ function, arising in non linear differential equations and to the Galois differential theory for LDEs.

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1 Introduction

The Sturm–Liouville (SL) differential equation is a second order differential equation with real valued coefficients of the form

$$-\frac{d}{dx}\left(p(x)\frac{dy(x)}{dx}\right) + q(x)y(x) = \lambda w(x)y(x),$$

where $y(x)$ is a function of the free variable x. Here $p(x) > 0$, $q(x)$ and $w(x) > 0$ are specified and are integrable on the closed real interval $[a, b]$. It is usually considered with separated boundary conditions of the form

$$y(a)\cos(\alpha) - p(a)y'(a)\sin(\alpha) = 0,$$
$$y(b)\cos(\beta) - p(b)y'(b)\sin(\beta) = 0$$

where $\alpha, \beta \in [0, \pi)$.

It is well known that in the Hilbert space $L^2([a, b], w(x)dx)$ there is an orthonormal basis of solutions to this equation, see for example [Ha], and important examples for some choices of $p(x), q(x), w(x)$ are the Bessel and Legendre equations. A special case of this equation is obtained for $p(x) = w(x) = 1$, i.e.,

$$-\frac{d^2}{dx^2}y(x) + q(x)y(x) = \lambda y(x), \tag{1}$$

where the parameter $q(x)$ is usually called the potential. Investigating this class of equations is classical and extensive, dating back to C. Sturm [S] and R. Liouville [L] and over the years a wide spectrum of techniques was developed for solving this equation. For example, the monodromy preserving deformation problem of Linear Differential Equations (LDE) was extensively studied by Schlesinger [Shl], R. Fuchs [Fu] and Garnier [G]. They focused on the Sturm–Liouville equation primarily as the simplest non trivial LDE. (See bibliography for chapters 7,8,9 in [CoLe] for more information.) The Scattering theory of Lax Phillips [LxPh] focused on this equation as well, constructing the so called spectral function for a given potential and initial conditions $y'(0) - hy(0) = 0$ and *scattering data*. It is also worthwhile to mention the work of A. Povzner [Pov] who used Riemannian transformation to study the solutions of the PDE

$$\frac{\partial^2}{\partial x^2}u(x, y) - q(x)u(x, y) = \frac{\partial^2}{\partial y^2}u(x, y) - q(y)u(x, y),$$

which is closely connected to the study of solutions of the SL equation. The inverse scattering problem, which reconstructs the potential $q(x)$ from the scattering data was solved by a student of A. Povzner, V.A. Marchenko in [Mar] and by M.I. Gelfand and I.M. Levitan in [GL].

In the work of Moshe Livšic [Liv1], a theory of vessels was developed, connecting the theory of commuting non self-adjoint operators to the theory of systems intertwining solutions of PDEs. Using separation of variables in this theory, one finds that PDEs becomes LDEs with a spectral parameter [BV]. Some ideas of Moshe Livšic were further developed in [M], presenting a theory of overdetermined $2D$ systems, invariant in one direction. As we have just mentioned, the transfer functions of such a system map solutions of the *input* LDE with a spectral parameter λ to solutions of the *output* LDE with the same spectral parameter (this theory was further developed in this setting in [AMV]). In a special case of such systems, LDEs are constructed from solutions of the Sturm–Liouville differential equation (1). Using realization theory, developed in [M] one can construct more complicated differential equations at the output starting from a trivial SL equation ($q(t_2) = 0$) or more generally from SL equations with potentials, for which the solutions are obtainable.

This paper considers finite-dimensional vessels as a first step to understand the obtained potentials. As a result convergence problems do not arise and the tools are mostly (differentially) algebraic. In an analogy to [JMU] we consider this theory as a "deformation theory" of Sturm–Liouville differential equations. One of the reasons to consider it as a deformation theory is the appearance of an analogue of the $\tau(x)$-function, whose role is to generate a differential ring, to which all the involved objects belong. For example the formula for the potential at the output is $q_*(x) = -2\frac{\partial^2}{\partial x^2}\log\tau(x)$ which is identical to the classical case.

The main ideas in this paper are the following. First is the new Definition 2, which significantly simplifies the original definition of the vessel, appearing in [Liv1, M]. For example, it follows from this definition that the Lyapunov equation is redundant (Lemma 2.1). The second is a detailed study of a special example, appearing in Definition 4, which illustrates how one can apply this theory to the study of the differential equation (1). It is very well known that the role of the tau function is extremely important in the study of differential equations in general and in the study of (1) particularly. The τ-function arises as a determinant of a solution of a Lyapunov equation (16), associated to the vessel. A similar formula for the tau function appears in the work [KV] of V. Katsnelson and D. Volok, where it is used a Sylvester equation (which actually is an affine version of a Lyapunov equation, appearing in our case). One of the main theorems, Theorem 3.15, shows that $\dfrac{\tau'}{\tau}$ generates all the objects associated with the differential equation (1) and the corresponding vessel. In order to represent how the main result, Theorem 3.15 arises, we first show its form for the simple case when the input SL equation is trivial ($q(t_2)=0$). If one denotes by $\boldsymbol{\mathcal{R}}_*$ (Definition 6) the differential ring generated by $\{\frac{\tau'}{\tau}, 1\}$, then we prove in Theorem 3.10 that the entries of the transfer function and of the potential at the output $q_*(t_2)$ are in $\boldsymbol{\mathcal{R}}_*$, which probably explains the appearance of this function in many applications.

In the more general case (Definition 10), we distinguish between the input $\boldsymbol{\mathcal{R}}$ and the output $\boldsymbol{\mathcal{R}}_*$ differential rings. If the input SL equation (1) is defined by a potential $q_{in} = 2\dfrac{d^2}{dt_2}\eta$ and the output SL equation (1) by $q_{out} = 2\dfrac{d^2}{dt_2^2}\tau$, then one defines in Definition 10 the input differential ring $\boldsymbol{\mathcal{R}}$, generated by $\{\dfrac{\eta'}{\eta}, 1\}$, and the output differential ring $\boldsymbol{\mathcal{R}}_*$, generated by $\{\dfrac{\eta'}{\eta}, \dfrac{\tau'}{\tau}, 1\}$. In the first case when $q_{in}=0$, we obtain a special case, since then $\boldsymbol{\mathcal{R}} = \{1\}$ is a trivial differential ring.

Another innovation of this paper is the application of differential algebraic methods to linear differential equations [PS]. As a result of the main Theorem 3.15 the τ-function, together with the data at the input generate a differential Piccard-Vessiot ring for the output LDE. As a result the Galois differential group can be explicitly calculated [H] and turns out to be a finite group, as discussed in Conclusions. From the point of view of differential Galois theory, an interesting example arises of a finitely generated, filtered differential ring, whose properties may be axiomatized and studied in relationship to arbitrary rings (Corollary 3.16). In the general case, when the vessel is not finite-dimensional, one can study existence of Liouvillian solutions for the output LDE.

2 Overdetermined time invariant $2D$ systems

2.1 Conservative vessel [MVc]

The notion of a vessel as it appear in this article was defined by M.S. Livšic in [Liv2]. It is closely connected to the study of a pair of commuting non self-adjoint operators [LKMV] with compact imaginary parts and first appeared in [Liv1]. The origins of this theory are in the fundamental work of M. Livsic and B. Brodskii [BL] which study the connection between non self-adjoint operators and meromorphic functions in the upper half plane. For each non self-adjoint operator A_1 there corresponds a naturally defined *characteristic function* $S(\lambda)$ and conversely. Multiplicative structure of the function $S(\lambda)$ is in a correspondence with invariant subspaces of the operator A_1. A pair of commuting non self adjoint operators A_1, A_2 are studied via connection to their joint characteristic function of two variables $S(\lambda, w)$

[LKMV] and there are similar results concerning invariant subspaces of both A_1, A_2. The notion of a vessel arises as a collection of operators and spaces, which "encode" the properties of A_1, A_2. More precisely a (conservative) vessel is the collection

$$\mathfrak{V} = (A_1, A_2, B; \sigma_1, \sigma_2, \gamma, \gamma_*; \mathcal{H}, \mathcal{E}),$$

for which the following axioms hold:

$$\begin{cases} A_j + A_j^* + B\sigma_j B^* = 0, \quad j = 1, 2 \\ A_2 A_1 - A_1 A_2 = 0, \\ -A_2 B\sigma_1 + A_1 B\sigma_2 + B\gamma = 0, \\ A_2^* B\sigma_1 - A_1^* B\sigma_2 + B\gamma_* = 0, \\ \gamma = \gamma_* + \sigma_1 B^* B\sigma_2 - \sigma_2 B^* B\sigma_1. \end{cases}$$

Here the first axiom means that the operators are non self-adjoint, but their imaginary part may be decomposed through an auxiliary space $\mathcal{E}$. The second axiom is commutativity, the last three axioms determine additional connections between factorization operators B, σ_1, σ_2 and some operators γ, γ_*. These results were further explored in [BV] and applied to the theory of systems. The class of systems, arising in this manner is defined using the vessel operators

$$\Sigma : \begin{cases} \frac{\partial}{\partial t_1} x(t_1, t_2) = A_1 x(t_1, t_2) + B\ \sigma_1\ u(t_1, t_2) \\ \frac{\partial}{\partial t_2} x(t_1, t_2) = A_2 x(t_1, t_2) + B\ \sigma_2\ u(t_1, t_2) \\ y(t_1, t_2) = u(t_1, t_2) - B^* x(t_1, t_2) \end{cases}$$

and intertwines solutions of Partial Differential Equations PDEs. More precisely taking $u(t_1, t_2)$ as a solutions of the input PDE

$$[\sigma_2 \frac{\partial}{\partial t_1} - \sigma_1 \frac{\partial}{\partial t_2} + \gamma] u(t_1, t_2) = 0$$

one obtains that $y(t_1, t_2)$ is a solution of the output PDE

$$[\sigma_2 \frac{\partial}{\partial t_1} - \sigma_1 \frac{\partial}{\partial t_2} + \gamma_*] y(t_1, t_2) = 0$$

with constant coefficients $\sigma_1, \sigma_2, \gamma, \gamma_*$. It is shown [BV] how these axioms for a two operator vessel are derived from the system theory point of view. Independence of the system transition on the path and overdetermindness of the input/output signals is shown to be equivalent to the set of these axioms. These ideas have its origins in the work of M. Livšic [Liv1].

There are also some result, considering vessels on Riemann manifolds [Ga], whose vector bundles have fibers that are Hilbert spaces.

In a more general setting a t_1**-invariant conservative vessel** [M, MVc] is a collection of operators and spaces, defined for values of t_2 in an interval $\mathcal{I}$

$$\mathfrak{GV} = (A_1(t_2), A_2(t_2), B(t_2); \sigma_1(t_2), \sigma_2(t_2), \gamma(t_2), \gamma_*(t_2); \mathcal{H}, \mathcal{E}),$$

where $\mathcal{H}, \mathcal{E}$ are Hilbert spaces and

$$\begin{array}{ll} A_1(t_2), A_2(t_2) & : \mathcal{H} \to \mathcal{H}, \\ B(t_2) & : \mathcal{E} \to \mathcal{H}, \\ \sigma_1(t_2), \sigma_2(t_2), \gamma(t_2) & : \mathcal{E} \to \mathcal{E} \end{array}$$

are bounded operators, which satisfy the following axioms:

$$A_1(t_2) + A_1^*(t_2) + B(t_2)\sigma_1(t_2)B^*(t_2) = 0, \tag{2}$$

$$A_2(t_2) + A_2^*(t_2) + B(t_2)\sigma_2(t_2)B^*(t_2) = 0, \tag{3}$$

$$\frac{d}{dt_2}A_1(t_2) = A_2(t_2)A_1(t_2) - A_1(t_2)A_2(t_2) \tag{4}$$

$$\frac{d}{dt_2}\big(B(t_2)\sigma_1(t_2)\big) - A_2(t_2)B(t_2)\sigma_1(t_2) + A_1(t_2)B(t_2)\sigma_2(t_2) + B(t_2)\gamma(t_2) = 0 \tag{5}$$

$$\frac{d}{dt_2}\big(B(t_2)\big)\sigma_1(t_2) + A_2^*(t_2)B(t_2)\sigma_1(t_2) - A_1^*(t_2)B(t_2)\sigma_2(t_2) + B(t_2)\gamma_*(t_2) = 0 \tag{6}$$

$$\gamma(t_2) = \gamma_*(t_2) + \sigma_1(t_2)B^*(t_2)B(t_2)\sigma_2(t_2) - \sigma_2(t_2)B^*(t_2)B(t_2)\sigma_1(t_2) \tag{7}$$

$$\begin{array}{c} \sigma_1(t_2) = \sigma_1^*(t_2), \quad \sigma_2(t_2) = \sigma_2^*(t_2), \\ \gamma^*(t_2) + \gamma(t_2) = \gamma_*^*(t_2) + \gamma_*(t_2) = -\frac{d}{dt_2}\sigma_1(t_2). \end{array} \tag{8}$$

Using simple calculations one can show that the condition (6) is redundant, but it plays an important role in the theory of vessels and appears here for the completeness of the presentation. Since we are dealing with t_2 dependent operators, we have also to consider smoothness assumptions. In this article it is enough to make the following:

Assumption 1 *On an interval $\mathcal{I}$ the following conditions hold*

1. *Operators $A_1(t_2), A_2(t_2), B(t_2)$ are bounded operators for each $t_2 \in \mathcal{I}$,*
2. *Operators $A_1(t_2), \sigma_1(t_2), \sigma_1(t_2), \gamma(t_2), \gamma_*(t_2)$ are continuously differentiable,*
3. *Operator $\sigma_1(t_2)$ is invertible for each value of $t_2 \in \mathcal{I}$.*

The vessel is associated to the input/state/output (i/s/o) system

$$\Sigma : \begin{cases} \frac{\partial}{\partial t_1}x(t_1,t_2) = A_1(t_2)x(t_1,t_2) + B(t_2)\ \sigma_1(t_2)\ u(t_1,t_2) \\ \frac{\partial}{\partial t_2}x(t_1,t_2) = A_2(t_2)x(t_1,t_2) + B(t_2)\ \sigma_2(t_2)\ u(t_1,t_2) \\ y(t_1,t_2) = u(t_1,t_2) - B^*(t_2)x(t_1,t_2) \end{cases} \tag{9}$$

and compatibility conditions for the input/ output signals:

$$\sigma_2(t_2)\frac{\partial}{\partial t_1}u(t_1,t_2) - \sigma_1(t_2)\frac{\partial}{\partial t_2}u(t_1,t_2) + \gamma(t_2)u(t_1,t_2) = 0, \tag{10}$$

$$\sigma_2(t_2)\frac{\partial}{\partial t_1}y(t_1,t_2) - \sigma_1(t_2)\frac{\partial}{\partial t_2}y(t_1,t_2) + \gamma_*(t_2)y(t_1,t_2) = 0. \tag{11}$$

A natural notion of equivalence for vessels is called *gauge-equivalence* and is defined as follows. Two vessels

$$\begin{array}{c} \mathfrak{V} = (A_1(t_2), A_2(t_2), B(t_2); \sigma_1(t_2), \sigma_2(t_2), \gamma(t_2), \gamma_*(t_2); \mathcal{H}, \mathcal{E}), \\ \widetilde{\mathfrak{V}} = (\widetilde{A}_1(t_2), \widetilde{A}_2(t_2), \widetilde{B}(t_2); \sigma_1(t_2), \sigma_2(t_2), \gamma(t_2), \gamma_*(t_2); \widetilde{\mathcal{H}}, \mathcal{E}) \end{array} \tag{12}$$

are called *gauge-equivalent* if there exists an operator $T(t_2):\mathcal{H}\to\widetilde{\mathcal{H}}$ with densely defined (on at least the same dense subspace as $T(t_2)$) inverse and derivative such that:

$$\begin{cases} \widetilde{A}_1(t_2) & = T(t_2)A_1(t_2)T^{-1}(t_2) \\ \widetilde{A}_2(t_2) & = T(t_2)A_2(t_2)T^{-1}(t_2) + \dfrac{dT(t_2)}{dt_2}T^{-1}(t_2) \\ \widetilde{B}(t_2) & = T(t_2)B(t_2) \end{cases} \tag{13}$$

Moreover, the inner products $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ and $\langle\cdot,\cdot\rangle_{\widetilde{\mathcal{H}}}$ of the spaces $\mathcal{H}$ and $\widetilde{\mathcal{H}}$ respectively are related by the following formula

$$\langle T^{-1}(t_2)x, T^{-1}(t_2)x'\rangle_{\mathcal{H}} = \langle x,x'\rangle_{\widetilde{\mathcal{H}}}. \tag{14}$$

Gauge transformations have the same role as state space similarities in the realization theory of matrix-valued functions [KFA]. Since each transformation is realized by an operator, acting between Hilbert spaces one can compose such transformations, use the inverse $T^{-1}(t_2)$ operator for the inverse transformation and to use an identity operator as the trivial (identity) transformation. If we consider such transformations for the same (or identified) Hilbert spaces $\mathcal{H}$ and $\widetilde{\mathcal{H}}$, we will obtain a group. At the Theorem 2.4 we will see that gauge transformation does not change one of the most important notions, related to the system: its transfer function. Thus gauge transformations may be thought as a "change of coordinate" of a given system.

Using [MV1] Theorem 8.1, one can construct a gauge-equivalent vessel, such that A_1 is a constant operator and $A_2(t_2)=0$. The basic idea behind the existence of such an equivalence is the fact that from the Lax equation (4) it follows that $A_1(t_2)=T^{-1}(t_2)A_1T(t_2)$, where A_1 is a constant operator, and where $T(t_2)$ is an operator generated by $A_2(t_2)$ as follows: $T'(t_2)=-T(t_2)A_2(t_2)$. Using now the gauge transformation, defined by this $T(t_2)$, we shall obtain that

$$\widetilde{A}_1 = T(t_2)A_1(t_2)T^{-1}(t_2) = T(t_2)T^{-1}(t_2)A_1T(t_2)T^{-1}(t_2) = A_1,$$

is constant and

$$\widetilde{A}_2 = T(t_2)A_2(t_2)T^{-1}(t_2) + T'(t_2)T^{-1}(t_2) = T(t_2)A_2(t_2)T^{-1}(t_2) - T(t_2)A_2(t_2)T^{-1}(t_2) = 0.$$

Moreover defining $\mathbb{X}(t_2)=T(t_2)T^*(t_2)$, one can show that the condition (5) for $\widetilde{B}(t_2)=T(t_2)B(t_2)$ becomes [MV1, Lemma 8.2]

$$\frac{d}{dt_2}[\widetilde{B}(t_2)\sigma_1(t_2)] = -A_1\widetilde{B}(t_2)\sigma_2(t_2) - \widetilde{B}(t_2)\gamma(t_2), \quad A_1=\widetilde{A}_1 \text{ is constant},$$

and the condition (2) gives us

$$A_1\mathbb{X}(t_2) + \mathbb{X}(t_2)A_1^* + \widetilde{B}(t_2)\sigma_1(t_2)\widetilde{B}^*(t_2) = 0.$$

The condition (3) then results in

$$\frac{d}{dt_2}\mathbb{X}(t_2) = \widetilde{B}(t_2)\sigma_2(t_2)\widetilde{B}^*(t_2).$$

Using these ideas we define a notion of a vessel of a very special kind, in which we give up the positive definiteness of $\mathbb{X}(t_2)$ and which enables us to develop a theory of "perturbations" of the potential in the Sturm–Liouville differential equation. We can also motivate this as follows. In the case $\mathbb{X}(t_2)$ has a constant signature, it follows that any such $\mathbb{X}(t_2)$ factors as $\mathbb{X}(t_2) = T(t_2)JT^*(t_2)$, where J is a constant signature matrix. Then use the induced by this J, Krein space, instead of the Hilbert space $\mathcal{H}$ appearing in the vessel $\mathfrak{GV}$, with all the formulas remaining the same.

Definition 2 *A vessel is a collection*

$$\mathfrak{V} = (A_1, B(t_2), \mathbb{X}(t_2) = \mathbb{X}^*(t_2); \sigma_1(t_2), \sigma_2(t_2), \gamma(t_2), \gamma_*(t_2); \mathcal{H}, \mathcal{E}),$$

for which the following vessel conditions hold

$$\frac{d}{dt_2}(B(t_2)\sigma_1(t_2)) + A_1 B(t_2)\sigma_2(t_2) + B(t_2)\gamma = 0, \tag{15}$$

$$A_1\mathbb{X}(t_2) + \mathbb{X}(t_2)A_1^* + B(t_2)\sigma_1(t_2)B^*(t_2) = 0, \tag{16}$$

$$\frac{d}{dt_2}\mathbb{X}(t_2) = B(t_2)\sigma_2(t_2)B^*(t_2), \tag{17}$$

$$\gamma_*(t_2) = \gamma(t_2) + \sigma_1(t_2)B^*(t_2)\mathbb{X}^{-1}(t_2)B(t_2)\sigma_2(t_2) - \sigma_2(t_2)B^*(t_2)\mathbb{X}^{-1}(t_2)B(t_2)\sigma_1(t_2) \tag{18}$$

$$\begin{array}{c} \sigma_1(t_2) = \sigma_1^*(t_2), \quad \sigma_2(t_2) = \sigma_2^*(t_2), \\ \gamma^*(t_2) + \gamma(t_2) = \gamma_*^*(t_2) + \gamma_*(t_2) = -\frac{d}{dt_2}\sigma_1(t_2). \end{array} \tag{19}$$

The vessel exists on an interval $\mathcal{I} \subseteq \mathbb{R}$ on which $\mathbb{X}(t_2)$ is invertible and the regularity assumptions 1 hold. The vessel is called ***conservative*** *if it holds that $\mathbb{X}(t_2) > 0$ on the interval $\mathcal{I}$.*

It turns out that the equation (17) is redundant for appropriately chosen initial conditions:

Lemma 2.1 *Suppose that $B(t_2)$ satisfies* (15) *and $\mathbb{X}(t_2)$ satisfies* (17)*, then if the Lyapunov equation* (16)

$$A_1\mathbb{X}(t_2) + \mathbb{X}(t_2)A_1^* + B(t_2)\sigma_1 B^*(t_2) = 0$$

holds for a fixed t_2^0, then it holds for all t_2. If $\mathbb{X}(t_2^0) = \mathbb{X}^*(t_2^0)$ for a fixed value of $t_2 = t_2^0$, then $\mathbb{X}(t_2) = \mathbb{X}^*(t_2)$ for all t_2.

Proof: By differentiating and using equations (15), (17), and (19) it can be seen that left hand side is a constant. From the formula (17) it follows that

$$\mathbb{X}(t_2) = \mathbb{X}(t_2^0) + \int_{t_2^0}^{t_2} B^*(y)\sigma_2(y)B(y)dy,$$

and since $\sigma_2(y)$ is self-adjoint, the result on the self adjointness of $\mathbb{X}(t_2)$ follows. □

2.2 The transfer function of a vessel

Let us consider first the *conservative* vessel $\mathfrak{GV}$, satisfying condition (2)-(7) . Collecting all the trajectory data in the form

$$\begin{array}{c} u(t_1, t_2) = u_\lambda(t_2)e^{\lambda t_1}, \\ x(t_1, t_2) = x_\lambda(t_2)e^{\lambda t_1}, \\ y(t_1, t_2) = y_\lambda(t_2)e^{\lambda t_1}, \end{array}$$

we arrive at the notion of a transfer function. Note that $u(t_1,t_2), y(t_1,t_2)$ satisfy PDEs, but $u_\lambda(t_2), y_\lambda(t_2)$ are solutions of LDEs with a spectral parameter λ,

$$\begin{array}{c}\lambda\sigma_2(t_2)u_\lambda(t_2) - \sigma_1(t_2)\frac{\partial}{\partial t_2}u_\lambda(t_2) + \gamma(t_2)u_\lambda(t_2) = 0,\\ \lambda\sigma_2(t_2)y_\lambda(t_2) - \sigma_1(t_2)\frac{\partial}{\partial t_2}y_\lambda(t_2) + \gamma_*(t_2)y_\lambda(t_2) = 0.\end{array}$$

The corresponding i/s/o system becomes

$$\begin{cases} x_\lambda(t_2) = (\lambda I - A_1(t_2))^{-1}B(t_2)\sigma_1(t_2)u_\lambda(t_2) \\ \frac{d}{dt_2}x_\lambda(t_2) = A_2(t_2)x_\lambda(t_2) + B(t_2)\sigma_2(t_2)u_\lambda(t_2) \\ y_\lambda(t_2) = u_\lambda(t_2) - B^*(t_2)x_\lambda(t_2). \end{cases}$$

The output $y_\lambda(t_2) = u_\lambda(t_2) - B^*(t_2)x_\lambda(t_2)$ may be found from the first i/s/o equation:

$$y_\lambda(t_2) = S(\lambda, t_2)u_\lambda(t_2),$$

using the *transfer function*

$$S(\lambda, t_2) = I - B^*(t_2)(\lambda I - A_1(t_2))^{-1}B(t_2)\sigma_1(t_2). \tag{20}$$

Here λ is outside the spectrum of $A_1(t_2)$, which is independent of t_2 by (4). We emphasize here that $S(\lambda, t_2)$ is a function of t_2 for each λ (which is a frequency variable corresponding to t_1).

Proposition 2.2 ([MVc]) *$S(\lambda, t_2) = I - B^*(t_2)(\lambda I - A_1(t_2))^{-1}B(t_2)\sigma_1(t_2)$ has the following properties:*

1. *$S(\lambda, t_2)$ is an analytic function of λ in the neighborhood of ∞, where it satisfies:*

$$S(\infty, t_2) = I_{n\times n}$$

.

2. *For all λ, $S(\lambda, t_2)$ is a continuous function of t_2.*
3. *In the case $\mathbb{X}(t_2) > 0$ the following inequalities are satisfied:*

$$\begin{array}{ll} S(\lambda, t_2)^*\sigma_1(t_2)S(\lambda, t_2) = \sigma_1(t_2), & \Re\lambda = 0 \\ S(\lambda, t_2)^*\sigma_1(t_2)S(\lambda, t_2) \geq \sigma_1(t_2), & \Re\lambda \geq 0 \end{array}$$

for λ in the domain of analyticity of $S(\lambda, t_2)$.

4. *For each fixed λ, multiplication by $S(\lambda, t_2)$ maps solutions of the input LDE with a spectral parameter λ*

$$\lambda\sigma_2(t_2)u - \sigma_1(t_2)\frac{du}{dt_2} + \gamma(t_2)u = 0 \tag{21}$$

to solutions of the output LDE with the same spectral parameter λ

$$\lambda\sigma_2(t_2)y - \sigma_1(t_2)\frac{dy}{dt_2} + \gamma_*(t_2)y = 0 \tag{22}$$

The converse also holds (see [MVc] chapter 5 on the realization problem).

Theorem 2.3 ([MVc]) *For any functions of two variables $S(\lambda, t_2)$, satisfying conditions of the Proposition 2.2, there is a conservative t_1 invariant vessel whose transfer function is $S(\lambda, t_2)$.*

Recall [CoLe] that the fourth property actually means that

$$S(\lambda, t_2)\Phi(\lambda, t_2, t_2^0) = \Phi_*(\lambda, t_2, t_2^0)S(\lambda, t_2^0) \tag{23}$$

for fundamental matrices of the corresponding equations:

$$\begin{array}{c} \lambda\sigma_2(t_2)\Phi_*(\lambda, t_2, t_2^0) - \sigma_1(t_2)\frac{\partial}{\partial t_2}\Phi_*(\lambda, t_2, t_2^0) + \gamma_*(t_2)\Phi_*(\lambda, t_2, t_2^0) = 0, \\ \Phi_*(\lambda, t_2^0, t_2^0) = I \end{array} \tag{24}$$

and

$$\begin{array}{c} \lambda\sigma_2(t_2)\Phi(\lambda, t_2, t_2^0) - \sigma_1(t_2)\frac{\partial}{\partial t_2}\Phi(\lambda, t_2, t_2^0) + \gamma(t_2)\Phi(\lambda, t_2, t_2^0) = 0, \\ \Phi(\lambda, t_2^0, t_2^0) = I. \end{array} \tag{25}$$

From (23) we obtain that

$$S(\lambda, t_2) = \Phi_*(\lambda, t_2, t_2^0)S(\lambda, t_2^0)\Phi^{-1}(\lambda, t_2, t_2^0)$$

and as a result $S(\lambda, t_2)$ satisfies the following differential equation

$$\frac{\partial}{\partial t_2}S(\lambda, t_2) = \sigma_1^{-1}(\sigma_2\lambda + \gamma_*)S(\lambda, t_2) - S(\lambda, t_2)\sigma_1^{-1}(\sigma_2\lambda + \gamma). \tag{26}$$

For two gauge equivalent vessels $\mathfrak{V}, \widetilde{\mathfrak{V}}$, defined in (12) using the operator $T(t_2)$, recall (13) that

$$\widetilde{A}_1(t_2) = T(t_2)A_1(t_2)T^{-1}(t_2), \quad \widetilde{B}(t_2) = T(t_2)B(t_2)$$

and the adjoint $\widetilde{B}^{[*]}(t_2)$ is given by $\widetilde{B}^{[*]}(t_2) = \widetilde{B}^*(t_2)(T(t_2)T^*(t_2))^{-1}$ as a result of the inner product property (14). Then we shall obtain that

$$\begin{aligned} \widetilde{S}(\lambda, t_2) &= I - \widetilde{B}^{[*]}(t_2)(\lambda I - \widetilde{A}_1(t_2))^{-1}\widetilde{B}(t_2)\sigma_1(t_2) = \\ &= I - B^*(t_2)T^{-1*}(t_2)(T(t_2)T^*(t_2))^{-1}(\lambda I - T(t_2)A_1(t_2)T^{-1}(t_2))^{-1}T(t_2)B(t_2)\sigma_1(t_2) = \\ &= I - B^*(t_2)(\lambda I - A_1(t_2))^{-1}B(t_2)\sigma_1(t_2) = S(\lambda, t_2) \end{aligned}$$

But also the converse holds (see [MVc], Theorem 3.5). In order to prove it, we need a notion of minimal realization of a transfer function. From Theorem 2.3 it follows that for each transfer function there exists a vessel which realizes it, i.e. $S(\lambda, t_2)$ is the transfer function of the constructed vessel. There is a natural notion of minimal vessel. A vessel $\mathfrak{V}$ is called *minimal* if for each $t_2 \in \mathcal{I}$

$$\operatorname{span} A_1^n B(t_2)\sigma_1(t_2)\mathcal{E} = \mathcal{H}, \quad n = 0, 1, 2, \ldots$$

Realization of a transfer function is called minimal if a vessel, which realizes this function is minimal. For minimal realization the following Theorem holds

Theorem 2.4 ([MVc]) *Assume that we are given two minimal t_1-invariant vessels $\mathfrak{V}, \widetilde{\mathfrak{V}}$ with transfer functions $S(\lambda, t_2)$, $\widetilde{S}(\lambda, t_2)$ respectively. Then the vessels are gauge-equivalent iff $S(\lambda, t_2) = \widetilde{S}(\lambda, t_2)$ for all points of analyticity.*

Let us focus now on the generalization of the conservative vessel

$$\mathfrak{V} = (A_1, \widetilde{B}(t_2), \mathbb{X}(t_2) = \mathbb{X}^*(t_2); \sigma_1(t_2), \sigma_2(t_2), \gamma(t_2), \gamma_*(t_2); \mathcal{H}, \mathcal{E}),$$

appearing in Definition 2. Consider the following trajectories (only $x_\lambda(t_2)$ is changed comparatively to the conservative vessel $\mathfrak{G}\mathfrak{V}$)

$$\begin{array}{l} u(t_1,t_2) = u_\lambda(t_2)e^{\lambda t_1}, \\ \widetilde{x}(t_1,t_2) = T(t_2)x_\lambda(t_2)e^{\lambda t_1}, \\ y(t_1,t_2) = y_\lambda(t_2)e^{\lambda t_1}. \end{array}$$

Then one can check that the corresponding i/s/o system is rewritten as

$$\begin{cases} \widetilde{x}_\lambda(t_2) = (\lambda I - A_1)^{-1}\widetilde{B}(t_2)\sigma_1(t_2)u_\lambda(t_2) \\ \frac{d}{dt_2}\widetilde{x}_\lambda(t_2) = \widetilde{B}(t_2)\sigma_2(t_2)u_\lambda(t_2) \\ y_\lambda(t_2) = u_\lambda(t_2) - \widetilde{B}^{[*]}(t_2)\widetilde{x}_\lambda(t_2). \end{cases}$$

As a result, its transfer function becomes

$$\begin{array}{rl} S(\lambda,t_2) & = I - \widetilde{B}^{[*]}(t_2)(\lambda I - A_1)^{-1}\widetilde{B}(t_2)\sigma_1(t_2) = \\ & = I - \widetilde{B}^*(t_2)\mathbb{X}^{-1}(t_2)(\lambda I - A_1)^{-1}\widetilde{B}(t_2)\sigma_1(t_2) \end{array} \tag{27}$$

The analogues of Proposition 2.2 and Theorem 2.3 also exist in this case and will be considered in the future works [AMV, M1]. For the present work one do need to consider such generalizations, because everything is finite-dimensional and may be calculated.

Finally, notice that if we are interested only in the transfer function, then one can bring by gauge-equivalence the operator A_1 to the simplest (up to similarity) form. We can suppose that it is a Jordan block matrix.

3 Sturm–Liouville vessels

At the first stage elementary input Sturm–Liouville vessels are considered. This case is presented in order to prepare and discus main theorems and notions for the general case. Then arbitrary input vessel is considered, i.e. for arbitrary rapidly decreasing sufficiently differentiable $q(t_2)$. Notice that we use the general version of the vessel $\mathfrak{V}$ appearing in Definition 2.

3.1 Elementary input vessels

3.1.1 Definition of a vessel with elementary input

There exists a choice of parameters of the vessel $\mathfrak{V}$ such that the input LDE is constructed from solutions of Sturm–Liouville differential equation (1) with the trivial potential $q(t_2) = 0$. Notice that in this case the equation (1) is solved by exponents. In the Definition 2 we choose the space $\mathcal{E} = \mathbb{C}^2$, i.e., a Hilbert space of dimension 2.

Definition 3 *The Sturm–Liouville parameters are given by [Liv2]*

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \gamma = \begin{bmatrix} 0 & 0 \\ 0 & i \end{bmatrix}.$$

It easy to check that the equation (19) is satisfied. The input compatibility differential equation (21) then becomes

$$\begin{aligned}0 &= \lambda\sigma_2(t_2)u_\lambda(t_2) - \sigma_1(t_2)\tfrac{\partial}{\partial t_2}u_\lambda(t_2) + \gamma(t_2)u_\lambda(t_2) = \\ &= \lambda\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} u_\lambda(t_2) - \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\tfrac{\partial}{\partial t_2}u_\lambda(t_2) + \begin{bmatrix} 0 & 0 \\ 0 & i \end{bmatrix} u_\lambda(t_2) = \\ &= \begin{bmatrix} \lambda & -\frac{\partial}{\partial t_2} \\ -\frac{\partial}{\partial t_2} & i \end{bmatrix} u_\lambda(t_2)\end{aligned}$$

and if we denote $u_\lambda(t_2) = \begin{bmatrix} u_1(\lambda, t_2) \\ u_2(\lambda, t_2) \end{bmatrix}$, we shall obtain the system of equations

$$\begin{cases} \lambda u_1(\lambda, t_2) - \frac{\partial}{\partial t_2}u_2(\lambda, t_2) = 0 \\ -\frac{\partial}{\partial t_2}u_1(\lambda, t_2) + iu_2(\lambda, t_2) = 0 \end{cases} \tag{28}$$

From the second equation $u_2(\lambda, t_2) = -i\frac{\partial}{\partial t_2}u_1(\lambda, t_2)$ and substituting it back to the first equation, we shall obtain the trivial Sturm–Liouville differential equation with the spectral parameter $-i\lambda$ for $u_1(\lambda, t_2)$:

$$-\frac{\partial^2}{\partial t_2^2}u_1(\lambda, t_2) = -i\lambda u_1(\lambda, t_2)$$

For the output compatibility differential (22), we take $\gamma_*(t_2)$ to be of the following form

$$\gamma_*(t_2) = \begin{bmatrix} -i\pi_{11}(t_2) & -\beta(t_2) \\ \beta(t_2) & i \end{bmatrix}$$

for real valued functions $\pi_{11}(t_2), \beta(t_2)$. Consequently, for the output $y_\lambda(t_2) = \begin{bmatrix} y_1(\lambda, t_2) \\ y_2(\lambda, t_2) \end{bmatrix}$, we shall obtain that (22) is

$$\begin{aligned}0 &= \lambda\sigma_2(t_2)u_\lambda(t_2) - \sigma_1(t_2)\tfrac{\partial}{\partial t_2}y_\lambda(t_2) + \gamma(t_2)u_\lambda(t_2) \\ &= \lambda\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} u_\lambda(t_2) - \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\tfrac{\partial}{\partial t_2}y_\lambda(t_2) + \begin{bmatrix} -i\pi_{11}(t_2) & -\beta(t_2) \\ \beta(t_2) & i \end{bmatrix} u_\lambda(t_2) \\ &= \begin{bmatrix} \lambda - i\pi_{11}(t_2) & -\frac{\partial}{\partial t_2} - \beta(t_2) \\ \beta(t_2) - \frac{\partial}{\partial t_2} & i \end{bmatrix}\begin{bmatrix} y_1(\lambda, t_2) \\ y_2(\lambda, t_2) \end{bmatrix}\end{aligned}$$

and thus the system of equations must be satisfied

$$\begin{cases} (\lambda - i\pi_{11}(t_2))y_1(\lambda, t_2) - (\frac{\partial}{\partial t_2} + \beta(t_2))y_2(\lambda, t_2) = 0, \\ (\beta(t_2) - \frac{\partial}{\partial t_2})y_1(\lambda, t_2) + iy_2(\lambda, t_2) = 0. \end{cases}$$

From the second equation $y_2(\lambda, t_2) = i(\beta(t_2) - \frac{\partial}{\partial t_2})y_1(\lambda, t_2)$ and substituting it into the first equation

$$\begin{aligned}0 &= (\lambda - i\pi_{11}(t_2))y_1(\lambda, t_2) - i(\tfrac{\partial}{\partial t_2} + \beta(t_2))(\beta(t_2) - \tfrac{\partial}{\partial t_2})y_1(\lambda, t_2) \\ &= i\tfrac{\partial^2}{\partial t_2^2}y_1(\lambda, t_2) + \lambda y_1(\lambda, t_2) - i(\pi_{11}(t_2) + \beta'(t_2) + \beta^2(t_2))y_1(\lambda, t_2)\end{aligned}$$

and consequently,

$$-\frac{\partial^2}{\partial t_2^2}y_1(\lambda, t_2) + (\pi_{11}(t_2) + \beta'(t_2) + \beta^2(t_2))y_1(\lambda, t_2) = -i\lambda y_1(\lambda, t_2),$$

which means that $y_1(\lambda, t_2)$ satisfies the Sturm–Liouville differential equation (1) with the spectral parameter $-i\lambda$ and the potential is $q(t_2) = \pi_{11}(t_2) + \beta'(t_2) + \beta^2(t_2)$. It turns out that properties of the transfer function (27) of the vessel $\mathfrak{V}$ require the compatibility condition on π_{11}, β. In order to see this one needs to consider so called moments of $S(\lambda, t_2)$. Take the Taylor expansion of that function around infinity

$$S(\lambda, t_2) = I - B^*(t_2)\mathbb{X}^{-1}(t_2)(\lambda I - A_1)^{-1}B(t_2)\sigma_1 = I - \sum_{n=0}^{\infty} \frac{1}{\lambda^{n+1}} B^*(t_2)\mathbb{X}^{-1}(t_2)A_1^n B(t_2)\sigma_1. \quad (29)$$

We define coefficients of this Taylour series

$$H_n(t_2) = B^*(t_2)\mathbb{X}^{-1}(t_2)A_1^n B(t_2)\sigma_1 \quad (30)$$

as *moments* of $S(\lambda, x)$. Then the following lemma holds:

Lemma 3.1 *Let $S(\lambda, t_2)$ be a transfer function of a vessel $\mathfrak{V}$ defined with Sturm-Liouville vessel parameters. Then the following compatibility condition must hold*

$$\pi'_{11}(x) = \beta'(x) - \beta^2(x). \quad (31)$$

Proof: Notice that the zero moment is $H_0(t_2) = B^*(t_2)\mathbb{X}^{-1}(t_2)B(t_2)\sigma_1$ and using it, the linkage condition (18) becomes

$$\gamma_*(t_2) - \gamma = \sigma_2 H_0(t_2) - \sigma_1 H_0(t_2)\sigma_1^{-1}\sigma_2.$$

From here it follows, denoting the elements of $H_0(t_2) = [H_0^{ij}], i, j = 1, 2$:

$$\begin{bmatrix} -i\pi_{11}(t_2) & -\beta(t_2) \\ \beta(t_2) & 0 \end{bmatrix} = \begin{bmatrix} H_0^{11} - H_0^{22} & H_0^{12}(t_2) \\ -H_0^{12}(t_2) & 0 \end{bmatrix}$$

and consequently,

$$H_0^{12}(t_2) = -\beta(t_2), \quad H_0^{11} - H_0^{22} = -i\pi_{11}(t_2). \quad (32)$$

Using the differential equation (26) and inserting here the formula (29) we shall obtain that entries of the first moment $H_1(t_2) = [H_1^{ij}], i, j = 1, 2$ satisfy

$$\sigma_1^{-1}\sigma_2 H_1(t_2) - H_1(t_2)\sigma_1^{-1}\sigma_2 = \tfrac{d}{dt_2}H_0(t_2) - \sigma_1^{-1}\gamma_*(t_2)H_0(t_2) + H_0(t_2)\sigma_1^{-1}\gamma.$$

which means

$$\begin{cases} \frac{d}{dt_2}H_0^{11} - \beta H_0^{11} - iH_0^{21} & = -H_1^{12}, \\ \frac{d}{dt_2}H_0^{12} - \beta H_0^{12} + i(H_0^{11} - H_0^{22}) & = 0, \\ \frac{d}{dt_2}H_0^{21} + i\pi_{11}H_0^{11} + \beta H_0^{21} & = H_1^{11} - H_1^{22}, \\ \frac{d}{dt_2}H_0^{22} + i\pi_{11}H_0^{12} + \beta H_0^{22} + iH_0^{21} & = H_1^{12}. \end{cases}$$

Then H_1^{12} can be evaluated using the first and the last equations. When we equate these two equations, we shall obtain that the compatibility condition for this differential equation to hold is

$$i(H_0^{11} - H_0^{22}) = \frac{d}{dt_2}H_0^{12}(t_2) - (H_0^{12}(t_2))^2,$$

which is exactly (31) using formulas (32). □

So, without loss of generality we make the following

Definition 4 *In terms of the elementary input, Sturm–Liouville vessel $\mathfrak{El}_{SL}$ is the following collection*

$$\mathfrak{El}_{SL} = (A_1, B(t_2), \mathbb{X}(t_2) = \mathbb{X}^*(t_2); \sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$
$$\gamma = \begin{bmatrix} 0 & 0 \\ 0 & i \end{bmatrix}, \gamma_*(t_2) = \begin{bmatrix} -i(\beta'(t_2) - \beta^2(t_2)) & -\beta(t_2) \\ \beta(t_2) & i \end{bmatrix}; \mathcal{H}, \mathbb{C}^2).$$

satisfying the vessel conditions

$$\frac{d}{dt_2}(B(t_2)\sigma_1) + A_1 B(t_2)\sigma_2 + B(t_2)\gamma = 0, \qquad (15)$$
$$A_1\mathbb{X}(t_2) + \mathbb{X}(t_2)A_1^* + B(t_2)\sigma_1 B^*(t_2) = 0, \qquad (16)$$
$$\frac{d}{dt_2}\mathbb{X}(t_2) = B(t_2)\sigma_2 B^*(t_2), \qquad (17)$$
$$\gamma_*(t_2) = \gamma + \sigma_1 B^*(t_2)\mathbb{X}^{-1}(t_2)B(t_2)\sigma_2 - \sigma_2 B^*(t_2)\mathbb{X}^{-1}(t_2)B(t_2)\sigma_1. \qquad (18)$$

The vessel exists on an interval $\mathcal{I} \subseteq \mathbb{R}$ on which $\mathbb{X}(t_2)$ is invertible and the regularity assumptions 1 hold.

An interesting question, which is out of the scope of this present article is whether for each $\beta(t_2)$ there exists an elementary input vessel, which has the output parameters defined with that given $\beta(t_2)$. This question is considered at [AMV].

3.1.2 The τ function of an elementary input vessel

Following [MV1] (Theorem 8.1) there is developed a structure of the transfer functions of a vessel and it will be applied to $\mathfrak{C}_{SL}$. If $S_{\mathfrak{C}_{SL}}(\lambda, t_2^0)$ has a realization at t_2^0 [Br]

$$S_{\mathfrak{C}_{SL}}(\lambda, t_2^0) = I - B_0^*\mathbb{X}_0^{-1}(\lambda I - A_1)^{-1}B_0\sigma_1,$$
$$A_1\mathbb{X}_0 + \mathbb{X}_0 A_1^* + B_0^*\sigma_1 B_0 = 0, \quad \mathbb{X}_0 = \mathbb{X}_0^*,$$

then solving (15) with initial value $B(t_2^0) = B_0$ and (17) with $\mathbb{X}(t_2^0) = \mathbb{X}_0$ we obtain that formula (27)

$$S_{\mathfrak{C}_{SL}}(\lambda, t_2) = I - B^*(t_2)\mathbb{X}^{-1}(t_2)(\lambda I - A_1)^{-1}B(t_2)\sigma_1.$$

Notice that since $\sigma_2 \geq 0$ and (integration of (17))

$$\mathbb{X}(t_2^0) = \mathbb{X}_0 + \int_{t_2^0}^{t_2} B(y)\sigma_2 B^*(y)dy$$

we shall obtain that in the case $\mathbb{X}_0 > 0$ it holds that $\mathbb{X}(t_2) > 0$ for all $t_2 \geq t_2^0$, since $B(y)\sigma_2 B^*(y) > 0$. As a result in that case the vessel $\mathfrak{C}_{SL}$ exists at least on the interval $\mathcal{I} = [t_2^0, \infty)$. Of course it can be extended to the left by continuity considerations. In the case spec $A_1 \subseteq i\mathbb{R}$ it is possible to obtain a vessel, which exist on the whole axis. For this it is enough to take $\mathbb{X}_0$ big enough and chains of "companion solutions" of length one (which will create periodic, and hence bounded with bounded anti-derivative functions on the whole axis). More about this construction may be found in [AMV].

Let us now delineate this construction for the transfer function. First we have to consider the initial realization at t_2^0, which is very well known from the realization theory of rational matrix functions [KFA]. It can be brought, as we have already mentioned to a Jordan block form.

Write the function $B(t_2)$ as

$$B(t_2) = \begin{bmatrix} b_1^*(t_2) \\ b_2^*(t_2) \\ \vdots \\ b_N^*(t_2) \end{bmatrix}, \tag{33}$$

and suppose that $A_1 = \text{Jordan}(z_1, r_1, \ldots, z_n, r_n)$, where z_i is a spectral value and r_i is the size of Jordan block (notice that we can have the same eigenvalue appearing more than once). Then [MV1] $b_i(t_2)$ are defined as chains of companion solutions of the so called *adjoint output LDE*

$$[\sigma_1 \frac{d}{dt_2} - \mu\sigma_2 - \gamma]y_* = 0 \tag{34}$$

with the spectral parameter $\mu = -z_i^*$:

$$\sigma_1 \frac{d}{dt_2} b_{j+1}(z_i) + z_i^* \sigma_2 b_{j+1}(z_i) - \gamma b_{j+1}(z_i) = \sigma_2 b_j(z_i), \quad j = r_1 + \ldots + r_{i-1}, \ldots, r_1 + \ldots + r_{i-1} + r_i - 1 \tag{35}$$

and where the first vector function $b_{r_1+\ldots+r_{i-1}}(z_i)$ is a solution of (34) with the spectral parameter $-z_i^*$.

The operator $\mathbb{X}(t_2) = [x_{ij}]$ is a solution of (16) (or equivalently (17) due to Lemma 2.1). Thus the transfer function is

$$\begin{aligned} S_{\mathfrak{E}_{SL}}(\lambda, t_2) &= I - B^*(t_2)\mathbb{X}^{-1}(t_2)(\lambda I - A_1)^{-1} B(t_2)\sigma_1 = \\ &= I - \begin{bmatrix} b_1 & b_2 & \cdots & b_N \end{bmatrix} \left[\tfrac{(-1)^{i+j}}{\tau} M_{ji}\right] (\lambda I - A_1)^{-1} \begin{bmatrix} b_1^* \\ b_2^* \\ \vdots \\ b_N^* \end{bmatrix} \sigma_1, \end{aligned} \tag{36}$$

where $M_{ij}(t_2)$ denotes the minor i, j of the matrix $\mathbb{X}(t_2)$ and

Definition 5 *The tau function $\tau = \tau(t_2)$ for the Jordan block matrix A_1 and the initial condition $B_0, \mathbb{X}_0$ is defined as*

$$\tau = \det \mathbb{X}(t_2) = \det[x_{ij}]. \tag{37}$$

It turns out that the tau function $\tau(t_2)$ defines also $\gamma_*(t_2)$ as the following proposition states

Proposition 3.2 *For Sturm–Liouville elementary vessel the following formula for γ_* holds*

$$\gamma_* = \gamma + \begin{bmatrix} i\frac{\tau''}{\tau} & \frac{\tau'}{\tau} \\ -\frac{\tau'}{\tau} & 0 \end{bmatrix}$$

Proof: It follows from the formula for logarithmic derivative of a determinant using the vessel condition (17) and the definition of the zero moment:

$$\begin{aligned} \frac{\tau'(t_2)}{\tau(t_2)} &= \text{tr}(\mathbb{X}^{-1}(t_2)\mathbb{X}'(t_2)) = \text{tr}(\mathbb{X}^{-1}(t_2)B(t_2)\sigma_2 B^*(t_2)) \\ &= \text{tr}(B^*(t_2)\mathbb{X}^{-1}(t_2)B(t_2)\sigma_2) = H_0^{11}(t_2), \end{aligned}$$

which combined with (32) gives the desired result. □

3.1.3 Anti-adjoint spectral values

Suppose that there are two eigenvalues z_i, z_j which satisfy $z_j = -z_i^*$. Then the entry $b_i^* \sigma_1 b_j$, appearing at the matrix $B(t_2)\sigma_1 B^*(t_2)$ (which in turn appears at the Lyapunov equation (16)) must be zero, because the corresponding i,j entry for the expression $A_1\mathbb{X}(t_2) + \mathbb{X}(t_2)A_1^*$ is such. Notice that denoting $b_k(t_2) = \begin{bmatrix} b_{k1} \\ b_{k2} \end{bmatrix}$, we shall obtain that

$$b_i^* \sigma_1 b_j = b_{i1}^* b_{j2} + b_{i2}^* b_{j1} = 0.$$

Since b_k satisfies (34) or (22) it means (see (28)) that $b_{k2} = -\sqrt{-1} b_{k1}'$. Substituting this into the last equality we shall obtain:

$$0 = b_{i1}^*(-\sqrt{-1}b_{j1}') + (-\sqrt{-1}b_{i1}')^* b_{j1} = \sqrt{-1}(-b_{i1}^* b_{j1}' + (b_{i1}')^* b_{j1}), \tag{38}$$

from where it follows (by dividing on $b_{i1}b_{j1}^*$ and obtaining $\dfrac{d}{dt_2}\ln(b_{i1}^*/b_{j1}) = 0$) that $b_{i1}^* = b_{j1}c$ for a constant $c \in \mathbb{C}$.

Corollary 3.3 *If two chains of solutions correspond to the spectral values z_i, z_j and have the property $z_i = -z_j^*$, then these chains are of length one.*

Proof: From the previous calculation it follows that each element b_i at the chain corresponding to z_i must be equal to the adjoint of *each* element b_j at the second chain, which is possible only in the case the chains are of length one, because a companion solution b_{k+1} is obtained by multiplying the previous element b_k on t_2, which means that we can have the equality $b_i = b_j^*$ only for the first elements at these chains. □

Corollary 3.4 *Suppose that the spectral value z_i is purely imaginary: $z_i^* = -z_i$, then its chain must be of length one.*

Proof: Using the same idea as in the previous Corollary all elements in this chain will be equal to the adjoint of the first one, which is possible only if it is a chain of length one. □

Notice that the last result has a feature in common with the result on discrete spectrum in the inverse scattering problem [Fa], where it is proved that for the case $\int_0^\infty x|q(x)|dx < \infty$ the discrete spectrum is on the imaginary axis and is simple (i.e. each eigenvalue appears exactly once).

3.2 Vessels as Bäcklund transformations. Crum transformations

A Bäcklund transform is typically a system of first order partial differential equations relating two functions, and often depending on an additional parameter. It implies that the two functions separately satisfy partial differential equations, and each of the two functions is then said to be a Bäcklund transformation of the other. We can consider a vessel as an example of Bäcklund transformation, because if we consider inputs $u(t_1, t_2)$ which satisfy the input compatibility condition (10)

$$[\sigma_2 \frac{\partial}{\partial t_1} - \sigma_1 \frac{\partial}{\partial t_2} + \gamma] u(t_1, t_2) = 0$$

then the output $y(t_1,t_2)$ of the system (9) satisfies the output PDE (11)

$$[\sigma_2\frac{\partial}{\partial t_1}-\sigma_1\frac{\partial}{\partial t_2}+\gamma_*]y(t_1,t_2)=0.$$

The set of parameters of this Bäcklund transformation in our case is the initial realization for the transfer function of the vessel $\mathfrak{V}$

$$S_{\mathfrak{V}}(\lambda,t_2^0)=I-B_0^*\mathbb{X}_0^{-1}(\lambda I-A_1)^{-1}B_0\sigma_1.$$

It turns out that Crum transformations, which first appeared in [Crum] are particular case of our construction. The basic idea of a Crum transformation is to use a solution $b_1(t_2)$ of the equation (1) (with the potential $q(t_2)$) for a fixed spectral value $-i\lambda_0$, $\lambda_0\in\mathbb{R}$. Then take a solution $y(\lambda,t_2)$ of (1) and define

$$y_1(\lambda,t_2)=\frac{y'(\lambda,t_2)b_1(t_2)-y(\lambda,t_2)b_1'(t_2)}{(\lambda-\lambda_0)b_1(t_2)},$$

where $y'=\dfrac{\partial}{\partial t_2}y$. A simple calculation [Crum, Fa] shows that $y_1(\lambda,t_2)$ satisfies (1) with the potential

$$q_1(t_2)=q(t_2)+\Delta q(t_2),\quad \Delta q(t_2)=-2\frac{b_1'(t_2)}{b_1(t_2)}=-2\frac{d^2}{dt_2^2}\ln b_1(t_2).$$

This transformation is used in order to construct solutions for equation (1), having the property that the corresponding SL operator will have one more point (namely λ_0) at the discrete spectrum, comparatively to the previous one.

It turns out that we can obtain the same transformation. Consider now the one dimensional vessel

$$\mathfrak{El}_{Crum}=(\lambda_0,B(t_2),\mathbb{X}(t_2);\sigma_1,\sigma_2,\gamma,\gamma_*(t_2)=\begin{bmatrix}-i(\beta'(t_2)-\beta^2(t_2)) & -\beta(t_2)\\ \beta(t_2) & i\end{bmatrix};\mathbb{C},\mathbb{C}^2).$$

Denote $B(t_2)=\begin{bmatrix}b_1(t_2)\\ b_2(t_2)\end{bmatrix}$, where b_1^* is a solution of (1) with the spectral parameter λ_0. Then $\mathbb{X}(t_2)$ may be found from the differential equation

$$\mathbb{X}'(t_2)=B(t_2)\sigma_2B^*(t_2)=b_1b_1^*.$$

For example, in the case $\lambda_0\neq 0$ we can also solve the Lyapunov equation (16) in order to find $\mathbb{X}(t_2)$

$$\mathbb{X}(t_2)=\frac{B(t_2)\sigma_1B^*(t_2)}{\lambda_0+\lambda_0^*}=\frac{b_2^*(t_2)b_1(t_2)+b_1^*(t_2)b_1(t_2)}{\lambda_0+\lambda_0^*}$$

The tau function in this one dimensional case is $\tau=\mathbb{X}(t_2)$ and the potential is $q(t_2)+2\beta'(t_2)$, where $-\beta=\dfrac{\tau'}{\tau}=\dfrac{b_1(t_2)b_1^*(t_2)}{\mathbb{X}(t_2)}$.

Using the transfer function $S_{Crum}(\lambda,t_2)$ of the vessel $\mathfrak{El}_{Crum}$ we find that if the input

$$u(\lambda,t_2)=\begin{bmatrix}u_0(\lambda,t_2)\\ u_1(\lambda,t_2)\end{bmatrix}$$

satisfies (21) with the spectral parameter λ, then the output

$$y(\lambda,t_2)=\begin{bmatrix}y_1(\lambda,t_2)\\ y_2(\lambda,t_2)\end{bmatrix}$$

satisfies the output (22) with the same spectral parameter and the following formula holds

$$\begin{aligned} y(\lambda,t_2) &= S_{Crum}(\lambda,t_2)u(\lambda,t_2) = [I - B^*(t_2)\mathbb{X}^{-1}(t_2)(\lambda-\lambda_0)^{-1}B(t_2)\sigma_1]u(\lambda,t_2) \\ &= [I - \frac{1}{\tau(\lambda-\lambda_0)}\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}\begin{bmatrix} b_1^* & b_2^* \end{bmatrix}\sigma_1]u(\lambda,t_2) \\ &= [I - \frac{1}{\tau(\lambda-\lambda_0)}\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}\begin{bmatrix} b_2^* & b_1^* \end{bmatrix}]u(\lambda,t_2), \end{aligned}$$

where taking the upper part $y_1(\lambda,t_2)$ of the vector $y(\lambda,t_2) = \begin{bmatrix} y_1(t_2) \\ y_2(t_2) \end{bmatrix}$, we find that

$$\begin{aligned} y_1(\lambda,t_2) &= u_0(\lambda,t_2) - y_1(t_2)\frac{1}{\tau(\lambda-\lambda_0)}\begin{bmatrix} b_2^* & b_1^* \end{bmatrix}\begin{bmatrix} u_0 \\ u_1 \end{bmatrix} = \\ &= u_0(\lambda,t_2) - b_1(t_2)\frac{1}{\tau(\lambda-\lambda_0)}(b_2^*u_0 + b_1^*u_1) \end{aligned}$$

Using equation (47), which solves the input LDE (21), we may rewrite this as

$$y_1(\lambda,t_2) = u_0(\lambda,t_2) - b_1(t_2)\frac{1}{\tau(\lambda-\lambda_0)}i((b_1^*)'u_0 - b_1^*u_0')$$

which is identical to the Crum transformations [Fa, (15.33)]. The Crum transformations corresponds to the choice of the spectral values on the imaginary axis, each appearing once, which is consistent with Corollaries 3.3 and 3.4 in our case.

3.3 The differential ring $\boldsymbol{\mathcal{R}}_*$ associated to an elementary input SL vessel

We can see that the element $\frac{\tau'}{\tau}$ is used to construct γ_*, since $\frac{\tau''}{\tau} = \frac{d}{dt_2}\big(\frac{\tau'}{\tau}\big) + \big(\frac{\tau'}{\tau}\big)^2$. In the sequel we shall use the notion of a differential ring, which can be studied for example from [K]. A *differential ring* is a ring R with a linear operator, called *derivation* $\partial : R \to R$, satisfying the Leibnitz rule $\partial(ab) = (\partial a)b + a\partial b$ and such that $\partial R \subseteq R$. Notice [K] that intersection of two differential rings is again a differential ring, thus we make the following definition. The ring R is called *generated by the set* $\{a_1, \dots, a_n\}$ (n may be ∞) if R is the minimal (in inclusion) differential ring containing $\{a_1, \dots, a_n\}$.

Definition 6 *The differential ring $\boldsymbol{\mathcal{R}}_*$ is defined to be the ring generated by $\{\frac{\tau'}{\tau}, 1\}$.*

Notice that it follows from the definition that $\boldsymbol{\mathcal{R}}_*$ is the smallest algebra of functions, containing $\frac{\tau'}{\tau}$, and 1 which is invariant under the operation $\dfrac{d}{dt_2}$. We define a space $\mathcal{T}$, which plays an important role in analyzing $\boldsymbol{\mathcal{R}}_*$. This space is obtained by taking the linear span of all the derivatives of the tau function and its structure is reflected in Definition 7.

Without loss of generality (by using gauge equivalence) we may suppose the eigenvalues of A_1 are ordered so that first there appear purely imaginary ones with length one for its chain (due to Corollary 3.4), then there appear pairs $(p_i, -p_i^*)$ so that each one has a chain of length one (due to Corollary 3.3) and after that eigenvalues with an arbitrary length for its chain, which are different from the minus adjoint of all other eigenvalues:

$$A_1 = \begin{bmatrix} E & 0 & 0 \\ 0 & P & 0 \\ 0 & 0 & J \end{bmatrix} \tag{39}$$

$$E = \mathrm{diag}(e_1, e_2, \ldots, e_{r_e}), \tag{40}$$

$$P = \mathrm{diag}(p_{r_e+1}, -p^*_{r_e+1}, \ldots, p_{r_p}, -p^*_{r_p}), \tag{41}$$

$$J = \mathrm{Jordan}(z_{r_e+2r_p+1}, r_{r_e+2r_p+1}, \ldots, z_n, r_n) \tag{42}$$

so that $N = r_e + 2r_p + r_{r_e+2r_p+1} + \ldots + r_n$. Suppose also that the transpose of $B(t_2)$ is

$$B(t_2)^t = \begin{bmatrix} u^*_1 & \ldots & u^*_{r_e} & v^*_{r_e+1} & w^*_{r_e+2} \ldots & v^*_{2r_p-1} & w^*_{2r_p} & b^*_{r_e+2r_p+1} & \ldots & b^*_N \end{bmatrix}, \tag{43}$$

where v_i, w_i's are corresponding solutions of the adjoint input LDE (34) and b_i's are companion solutions of the same equation (34). The following Definition 7 consists of a basis of functions, defined on $\mathcal{I}$, which are created by successive differentiation of the tau function $\tau(t_2)$. Notice that from the Lyapunov equation (16) it follows that in the case $z_i \neq -z^*_j$ the i, j entry x_{ij} of the matrix $\mathbb{X}(t_2)$ is

$$x_{ij} = \frac{b^*_i \sigma_1 b_j}{z_i + z^*_j},$$

and satisfies $\frac{d}{dt_2} x_{ij} = b^*_i \sigma_2 b_j$. But if $z_i = -z^*_j$, then this element is found from the equation (17) and is equal to

$$x_{ij} = x_{ij}(t^0_2) + \int_{t^0_2}^{t_2} b^*_i \sigma_2 b_j dy,$$

and its derivative is still $\frac{d}{dt_2} x_{ij} = b^*_i \sigma_2 b_j$. Moreover, using the formula for the determinant of a function

$$\tau = \det \mathbb{X}(t_2) = \sum_p (-1)^{\mathrm{sign}(p)} x_{1,p(1)} x_{2,p(2)} \ldots x_{N,p(N)}, \quad \text{p - permutations of } \{1, 2, \ldots, N\},$$

we obtain that each index i appears exactly twice at *each* summand. This can be translated as appearance of each b^*_i and b_i exactly once via the formulas for x_{ij}.

Notice that the term $x_{1,p(1)} x_{2,p(2)} \ldots x_{N,p(N)}$ and all its derivatives are actually of finite degree, which means that differentiating it enough times we will obtain terms, which appeared before. A simpler reason for that is that it is a multiplication of exponential functions and polynomials (Lemma 3.6). Another reason, reflected in the combinatorics of the following Definition 7, where one learned the structure of such a term. Each index $1 \leq i \leq N$ appears twice as we have already mentioned and each x_{ij} and its derivatives are expressed through the companion solutions with smaller indexes, thus we obtain that differentiating it enough times, we will obtain expressions appearing in the lower derivatives. Moreover, one can "split" all the multiplications so that all terms corresponding to the same chain appear together. All these ideas are present at the proof of Lemma 3.5. Actually, all the difficulties in the proof of the Lemma 3.5 are inserted into the definition, so that one have to show only that the structure of $\mathcal{T}$ is preserved after derivation (see Lemma 3.5).

Definition 7 *The space $\mathcal{T}$ is defined to be the span of functions constructed from multiplication of the following terms*

$$\mathcal{T} = \text{span}\{y_1 y_2 \dots y_{r_e} y_{r_e+1} y_{r_e+3} \dots y_{r_p} y_{r_e+2r_p+1} y_{r_e+2r_p+r_1+1} \dots y_{N-r_n+1}\},$$

where y_i's corresponds to the chains in the structure of A_1. The first r_e variables y_i are one of the following two elements

$$y_i = \begin{cases} x_{i,i} \\ u_i^* E_i u_i \end{cases}, \quad 1 \le i \le r_e$$

In this definition, the y_i's corresponds to the pairs of eigenvalues $p_i, -p_i^*$ and are multiplications of one of the following two terms

$$y_i = \begin{cases} v_i^* E_i v_i w_{i+1}^* E_{i+1} w_{i+1}, \\ w_i^* E_i v_{i+1} x_{i,i+1}, & r_e + 1 \le i \le 2r_p, \ i - r_e \text{ is odd} \\ v_i^* E_i w_{i+1} x_{i+1,i}, \end{cases}$$

The last group of y_i's corresponds to the companion solutions and is a multiplication of r_i terms:

$$y_i = b_{i+k_1} E_{i,k_1,\ell_1} b_{i+\ell_1}^* \, b_{i+k_2} E_{i,k_2,\ell_2} b_{i+\ell_2}^* \dots b_{i+k_{r_i}} E_{i,k_{r_i},\ell_{r_i}} b_{i+k_{r_i}}^*, \quad i = r_e + 2r_p + \begin{cases} r_1 \\ r_2 \\ \vdots \\ r_{n-1} \end{cases}$$

where the r_i tuples $\langle k_1, k_2, \dots, k_{r_i}\rangle, \langle \ell_1, \ell_2, \dots, \ell_{r_i}\rangle$

1. *satisfy increasing property $k_1 \le k_2 \le \dots \le k_{r_i}$, $\ell_1 \le \ell_2 \le \dots \le \ell_{r_i}$, and*
2. *are less than or equal to $\langle 1, 2, \dots, r_i\rangle$ in the point-wise order of tuples (so, for example k_1 and ℓ_1 are actually 1).*

E_i's with different indexes are 2×2 matrices over $\mathbb{C}$. We shall call each function $y_1 y_2 \dots y_{N-r_n+1}$, satisfying these two conditions as a ***basic element****.*

Lemma 3.5 *$\mathcal{T}' \subseteq \mathcal{T}$.*

Proof: Using the Leibnitz rule for the derivative of multiplication of functions, it is enough to prove that the derivative of $b_i^* E_{ij_i} b_{j_i}$ and of x_{i_1,j_1} are linear combinations of elements of the same form. For x_{i_1,j_1} the derivative is $b_i^* \sigma_2 b_{j_i}$. If b_i is a companion solution corresponding to the spectral parameter z (which is usually of the form $-z_i^*$), then

$$b_i' = \sigma_1^{-1}(\sigma_2 z + \gamma) b_i + \sigma_2 b_{i-1}.$$

Suppose also that b_{j_i} is a companion solution corresponding to the spectral parameter w. Then

$$\begin{aligned} \tfrac{d}{dt_2} b_i^* E b_{j_i} &= b_i^*(\sigma_2 z^* - \gamma)\sigma_1^{-1} E b_{j_i} + b_{i-1}^* \sigma_2 \sigma_1^{-1} E b_{j_i} + \\ &\quad + b_i^* E \sigma_1^{-1}(\sigma_2 w + \gamma) b_{j_i} + b_i^* E \sigma_1^{-1} \sigma_2 b_{j_i - 1} \\ &= b_i^*[(\sigma_2 z^* - \gamma)\sigma_1^{-1} E + E\sigma_1^{-1}(\sigma_2 w + \gamma)] b_{j_i} + b_{i-1}^* \sigma_2 \sigma_1^{-1} E b_{j_i} + b_i^* E \sigma_1^{-1} \sigma_2 b_{j_i - 1}. \end{aligned} \tag{44}$$

and again we obtain elements of the same form. In order to see that we stay within the space $\mathcal{T}$, notice that differentiating $b_i^* E b_{j_i}$ we obtain an element of the same form

$$b_i^*[(\sigma_2 z^* - \gamma)\sigma_1^{-1}E + E\sigma_1^{-1}(\sigma_2 w + \gamma)]b_{j_i}$$

and two elements with smaller indexes

$$b_{i-1}^*\sigma_2\sigma_1^{-1}Eb_{j_i}, \quad b_i^*E\sigma_1^{-1}\sigma_2 b_{j_i-1}.$$

But if b_i or b_j are initial members at a companion chain of solutions, then these two elements does not exist.

In order to see that the condition of point-wise comparison holds, notice that if there appear $b_{j_1}, \ldots, b_{j_{r_i}}$ satisfying this condition, then the derivative of $b_{i_1}^* E_1 b_{j_1} \ldots b_{i_N}^* E_N b_{j_N}$ will only decrease indexes. On the other hand, taking E_k's as elementary matrices we are able to substitute elements at the chain. For example,

$$\begin{aligned} & b_i^* \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} b_j b_k^* \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} b_m = \\ & = i,k \text{ interchanged } = b_k^* \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} b_j b_i^* \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} b_m \\ & = i,k \text{ and } m,j \text{ interchanged } = b_k^* \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} b_m b_i^* \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} b_j \end{aligned}$$

and as a result we can always represent an element as a sum of basic elements, defined by elementary matrices and having the increasing and the pointwise inequalities. □

Since we are dealing with finite-dimensional vessels, with a trivial equation at the input, we can also find chains of solution of the input LDE (21) explicitly. Let us consider an arbitrary chain $b_1, b_2, \ldots, b_r$ corresponding to a spectral parameter z. Solving the input compatibility condition (21) we find that

$$b_1 = \begin{bmatrix} b_{11}e^{kt_2} + b_{12}e^{-kt_2} \\ -ik(b_{11}e^{kt_2} - b_{12}e^{-kt_2}) \end{bmatrix}, b_1(t_2^0) = b_1^0, \quad k = \sqrt{-i\bar{z}}.$$

Notice that in a generic case, if we consider real and imaginary parts of the numbers b_{11}, b_{12}, k, we shall obtain that b_1 is a sum of 4 different real exponents:

$$e^{\Re k t_2}, e^{-\Re k t_2}, e^{\Im k t_2}, e^{-\Im k t_2}.$$

The second element at the chain b_2, satisfies

$$b_2'(t_2) = \sigma_1^{-1}(-\sigma_2\bar{z} - \gamma)b_2(t_2) + \sigma_2 b_1(t_2), \quad b_2(t_2^0) = b_2^0$$

and as a result is of the form

$$b_2(t_2) = (t_2 b_{21} + c_{21})e^{kt_2} + (t_2 b_{22} + c_{22})e^{-kt_2},$$

which means that in a generic case it is a linear combination of the previous exponents and their multiple by t_2:

$$e^{\Re k t_2}, e^{-\Re k t_2}, e^{\Im k t_2}, e^{-\Im k t_2}, t_2 e^{\Re k t_2}, t_2 e^{-\Re k t_2}, t_2 e^{\Im k t_2}, t_2 e^{-\Im k t_2}$$

Obviously, in the generic case (i.e. where the coefficients of no exponent vanish) the ℓ-th element in this chain will be a linear combination of powers of t_2 multiplied by exponents:

$$t_2^i e^{\pm \Re k}, t_2^i e^{\pm \Im k}, i = 0, 1, \ldots, \ell.$$

The coefficients of $t_2^i e^{\pm \Re k}, t_2^i e^{\pm \Im k}$ are defined by the initial conditions and moreover are polynomial expressions in $\Re b_i^0, \Im b_i^0$. As a result of these discussions we have the following lemma

Lemma 3.6 *Let $b_1(t_2), b_2(t_2), \ldots, b_r(t_2)$ be a chain corresponding to the spectral value z with initial values $b_1^0, b_2^0, \ldots, b_r^0$. Then an element $b_\ell, 1 \leq \ell \leq r$ at that chain is a linear combination of*

$$t_2^i e^{\pm \Re k}, t_2^i e^{\pm \Im k}, i = 0, 1, \ldots, \ell$$

with coefficients, which are polynomial expressions in the initial values $\Re b_i, \Im b_i$, and $\Re k, \Im k$.

Proof: We will use induction on n, using the fact that the element b_{n+1} may be presented as

$$b_{n+1} = \sum_{i=1}^{n+1} [b_{n+1,i} t_2^i (e^{kt_2} + e^{-kt_2}) + c_{n+1,i}(e^{kt_2} + e^{-kt_2})].$$

The same conclusion can be derived using the Fundamental matrix of solutions $\Phi(z, t_2)$ of the equation (21):

$$\Phi(z, t_2) = \begin{bmatrix} \cos(k(t_2 - t_2^0)) & \dfrac{\sin(k(t_2 - t_2^0))}{ik} \\ ik \sin(k(t_2 - t_2^0)) & \cos(k(t_2 - t_2^0)) \end{bmatrix}.$$

Then

$$b_{n+1} = \Phi(z, t_2)[\int_{t_2^0}^{t_2} \Phi^{-1}(z, y)\sigma_1^{-1}\sigma_2 b_n(y) dy + b_{n+1}^0].$$

Further, if b_n satisfies the assumptions of the Lemma, then b_{n+1} will satisfy them too, by direct computations. □

Corollary 3.7 *Let $b_1(t_2), b_2(t_2), \ldots, b_r(t_2)$ be a chain corresponding to the spectral value z with initial values $b_1^0, b_2^0, \ldots, b_r^0$. Then*

1. *for fixed $i \neq j$, $\operatorname{span} b_i^* E b_j = \operatorname{span} b_j^* E b_i$,*
2. *for any indexes satisfying $i + j = k + m$, $\operatorname{span} b_i^* E b_j = \operatorname{span} b_k^* E b_m$.*

Proof: Using the general form of b_i, developed in Lemma 3.6, we understand that expression $b_i^* E b_j$ is a linear combination of exponents with real exponents, multiplied by powers of t_2, which is also real. Thus conjugation will result in a combination of the same real valued exponents, multiplied by powers of t_2. Similarly, in the case $i + j = k + m$ we obtain that $b_i^* E b_j$ and $b_k^* E b_m$ are linear combinations of the same exponents, multiplied by powers of t_2 from 0 to $2(i + j - 2) = 2(k + m - 2)$, so the result follows. □

Corollary 3.8 *The dimension of the space $\mathcal{T}$ is at most*

$$4^{r_e}4^{2(r_p-r_e)}\prod_{k=1}^{n}[1+r_n(r_n-1)]4^{r_n}.$$

Proof: First, the number of different exponents, corresponding to a purely imaginary element $z_i=-z_i^*$ is 4 (notice that $b^*\sigma_1 b = b^*E_{12}b + b^*E_{21}b = 0$) :

$$b^*E_{11}b, b^*E_{12}b, b^*E_{22}b, x_{ii}$$

For the pair, corresponding to the spectral value p_i and $-p_i^*$ there are 4^2 elements, which correspond to

$$v_i^*E_iv_iw_{i+1}^*E_{i+1}w_{i+1}, w_i^*E_iv_{i+1}x_{i,i+1}, v_i^*E_iw_{i+1}x_{i+1,i},$$

again elements $w_i^*\sigma_1 v_i = v_i^*\sigma_1 w_i = 0$ and we substitute them with corresponding x_{ij}.

Finally, for the general chain, we have to count the number of different pairs of r_n-tuples $\langle k_1,k_2,\ldots,k_{r_n}\rangle, \langle \ell_1,\ell_2,\ldots,\ell_{r_n}\rangle$, which additionally to being increasing and point-wize less or equal to $\langle 1,2,\ldots,r_n\rangle$ have the two properties, stated in Corollary 3.7. From the first property it follows that we can order the tuples so that $k_i\le \ell_i$. And using next the second property it follows that actually the total sum

$$\sum_{i}^{r_n}(k_i+\ell_i)$$

distinguishes between the pairs of tuples. The total sum of the indexes is between $2r_n$ and $r_n(r_n+1)$. Since each on of the tuples creates at most 4^{r_n} different terms, we obtain that their total number is at most

$$[r_n(r_n+1)-2r_n+1]4^{r_n}=[1+r_n(r_n-1)]4^{r_n}$$

which is exactly the term appearing in the Corollary. □

There is also an alternative proof for the calculation of maximal number of different elements for a general chain, which is presented in the next lemma. We choose to talk about one block for A_1, but it is easily generalized to any number of blocks, since there is a property of τ, which enables to "collect" all companion solutions together and then use the result of this Lemma for each block. Notice that we use the fact that the companion solutions are pure exponential functions, multiplied by powers of t_2:

Lemma 3.9 *Suppose that A_1 consist of one spectral value z and one chain of length r. Suppose also that $b_1,b_2,\ldots,b_n$ are companion solutions corresponding to this data. Then the tau function in this case is a linear combination of the following functions*

$$\underbrace{e^{\pm 2\Re k t_2 \pm 2\Im k t_2}\cdots e^{\pm 2\Re k t_2 \pm 2\Im k t_2}}_{r \text{ times}} t_2^i, \quad i=0,1,\ldots,1+r(r-1)$$

and their total number is

$$[1+r(r-1)]4^r$$

Proof: First, let us consider the maximal power of t_2 and proceed by a proof of induction on N. For $N=1$, we obtain that

$$\tau=\det\mathbb{X}=\det\frac{b_1^*\sigma_1 b_1}{\bar z_1+z_1},$$

which is a sum of 4 exponents

$$e^{2\Re k_1 t_2}, e^{-2\Re k_1 t_2}, e^{2\Im k_1 t_2}, e^{-2\Im k_1 t_2}, \quad k_1 = \sqrt{iz_1},$$

coefficients are rational functions of real and imaginary parts of b_1^0, k. Since increasing of the state space $\mathcal{H}$ is equivalent to adding one element b_N to the last chain or creating a new chain with different eigenvalue, we obtain that

$$\mathbb{X}_N = \begin{bmatrix} \mathbb{X}_{N-1} & C_N \\ C_N^* & \dfrac{b_N^* \sigma_1 b_N}{z_N + \bar{z}_N} \end{bmatrix},$$

where

$$C_N = \begin{bmatrix} \dfrac{b_1^* \sigma_1 b_N}{z_1 + z_N^*} \\ \dfrac{b_2^* \sigma_1 b_N}{z_2 + z_N^*} \\ \vdots \\ \dfrac{b_{N-1}^* \sigma_1 b_N}{z_{N-1} + z_N^*} \end{bmatrix}.$$

Using a formula for evaluating determinant of a block matrix, we shall obtain that

$$\det \mathbb{X}_N = \det \mathbb{X}_{N-1} \det(\frac{b_N^* \sigma_1 b_N}{z_N + z_N^*} - C_N^* \mathbb{X}_{N-1}^{-1} C_N) = \tau_{N-1} \frac{b_N^* \sigma_1 b_N}{z_N + z_N^*} - C_N^* \mathbb{X}_{N-1}^{-1} C_N \tau_{N-1}$$

The expression $\tau_{N-1} \dfrac{b_N^* \sigma_1 b_N}{z_N + z_N^*}$ can be easily understood, since we know by the induction hypothesis that:

$$\tau_{N-1} \in \operatorname{span}\{\underbrace{e^{\pm 2\Re k t_2 \pm 2\Im k t_2} \cdots e^{\pm 2\Re k t_2 \pm 2\Im k t_2}}_{r-1 \text{ times}} t_2^i \mid i = 0, 1, \ldots, 1 + (r-1)(r-2)\}$$

On the other hand, using Lemma 3.6 we obtain that the element $\dfrac{b_N^* \sigma_1 b_N}{z_N + z_N^*}$ is a linear combination of the following terms:

$$\underbrace{e^{\pm 2\Re k}, e^{\pm 2\Im k}}_{\text{no power of } t_2}, \underbrace{t_2 e^{\pm 2\Re k}, t_2 e^{\pm 2\Im k}}_{t_2 \text{ in power } 1}, \ldots, \underbrace{t_2^{2r} e^{\pm 2\Re k}, t_2^{2r} e^{\pm 2\Im k}}_{t_2 \text{ in power } 2r},$$

Notice that the highest power of t_2 is $2r$, which is multiplied on one of the four "basic" exponents:

$$t_2^{2r} e^{\pm 2\Re k}, t_2^{2r} e^{\pm 2\Im k}$$

On the other hand, the highest power for τ_{N-1}. If we collect the highest possible powers for each element b_i at the chain and multiply on the all possible exponents, we shall obtain that $\dfrac{b_N^* \sigma_1 b_N}{z_N + z_N^*}$ contains the element with the highest power of t_2:

$$1 + 2 + 4 + \cdots + 2(r-1) + 2r = 1 + r(r-1).$$

This element is multiplied by one of the following exponents

$$\underbrace{e^{\pm 2\Re k t_2 \pm 2\Im k t_2} \cdots e^{\pm 2\Re k t_2 \pm 2\Im k t_2}}_{\text{r times}},$$

because each $e^{\pm 2\Re k t_2}$ or $e^{\pm 2\Im k t_2}$ appears at each b_i.

The second term appearing in the expression for τ_N is actually of the same form (denoting by M_{ij} the minor (i,j) of $\mathbb{X}_{N-1}$):

$$\begin{aligned}
&C_N^* \mathbb{X}_{N-1}^{-1} C_N \tau_{N-1} = C_N^* [M_{ij}(-1)^{i+j}] C_N = \\
&= \begin{bmatrix} \frac{b_N^* \sigma_1 b_1}{z_1^* + z_N} & \frac{b_N^* \sigma_1 b_2}{z_2^* + z_N} & \cdots & \frac{b^* N \sigma_1 b_{N-1}}{z_N^* + z_{N-1}} \end{bmatrix} [M_{ij}(-1)^{i+j}] \begin{bmatrix} \frac{b_1^* \sigma_1 b_N}{z_1 + z_N^*} \\ \frac{b_2^* \sigma_1 b_N}{z_2 + z_N^*} \\ \vdots \\ \frac{b_{N-1}^* \sigma_1 b_N}{z_{N-1} + z_N^*} \end{bmatrix} \\
&= \sum_{ij} \frac{b_N^* \sigma_1 b_j}{z_j^* + z_N} M_{ij}(-1)^{i+j} \frac{b_i^* \sigma_1 b_N}{z_i + z_N^*} = b_N^* \big(\sum_{ij} M_{ij}(-1)^{i+j} \frac{\sigma_1 b_j}{z_j^* + z_N} \frac{b_i^* \sigma_1}{z_i + z_N^*} \big) b_N
\end{aligned}$$

and again it can be presented as a multiplication of exponents appearing at τ_{N-1} (since such are the expressions

$$\sum_{ij} M_{ij}(-1)^{i+j} \frac{\sigma_1 b_j}{z_j^* + z_N} \frac{b_i^* \sigma_1}{z_i + z_N^*}$$

and of exponents of $b_N^* b_N$. Thus we obtain the same exponents as for the term $\tau_{N-1} \frac{b_{N-1}^* \sigma_1 b_N}{z_{N-1} + z_N^*}$. The result follows. □

From this Lemma 3.6 it follows that the coefficients of the exponents may vanish on a variety, if we consider a "big" space $\mathbb{R}^K$, where K is the total number of real and imaginary parts of all initial conditions and all spectral values. Indeed, it is true for each block of A_1 separately, due to Lemma 3.6. As a result we make the following

Definition 8 *The choice of the initial spectral parameters $A_1, B(t_2^0), \mathbb{X}_0$ for which all the exponents does not vanish in the expression for $\tau(t_2)$ is called a **generic case**.*

Theorem 3.10 *The entries of γ_* and $\frac{1}{\tau}\mathcal{T}$ are in $\boldsymbol{\mathcal{R}}_*$. For each natural n, $\tau^{(n)} \in \mathcal{T}$ and as a result τ satisfies a linear differential equation of finite order with constant coefficients. In the generic case*

1. $\mathrm{span}_{n \in \mathbb{N}}(\tau^{(n)}) = \mathcal{T}$.
2. *The entires of the transfer function $S_{\mathfrak{E}_{SL}}(\lambda, t_2)$ of $\mathfrak{E}_{SL}$ are in $\frac{1}{\tau}\mathcal{T} \subseteq \boldsymbol{\mathcal{R}}_*$.*

Proof: Notice that $1 = \frac{\tau}{\tau}$ is in $\boldsymbol{\mathcal{R}}_*$ by definition. We will show that $\frac{\tau^{(n)}}{\tau} \in \boldsymbol{\mathcal{R}}_*$ by induction on $n \geq 1$. For $n = 1$ it is true by the definition, and generally,

$$\frac{\tau^{(n+1)}}{\tau} = \frac{d}{dt_2}\big(\frac{\tau^{(n)}}{\tau}\big) + \frac{\tau^{(n)}}{\tau}\frac{\tau'}{\tau},$$

The space $\mathcal{T}$ was constructed so that all the derivatives of τ are there. Thus the first part of the Lemma is proven.

Let us consider now the generic case. Comparing the result of corollaries 3.8 and 3.7 the maximal number of independent elements coincides with the minimal number of exponential functions, multiplied by polynomials:

$$4^{r_e}4^{2(r_p-r_e)}\prod_{k=1}^{n}[1+r_n(r_n-1)]4^{r_n}$$

and since in the generic case all the exponents, multiplied by powers of t_2 do not vanish, it is a well known fact (using the generalized Vandermonde determinant) that all their derivatives (up to their total order) are independent and we obtain that $\mathrm{span}_{n\in\mathbb{N}}(\tau^{(n)})=\mathcal{T}$.

Let us use the formula (36):

$$S_{\mathfrak{E}_{SL}}(\lambda,t_2)=I-\begin{bmatrix} b_1 & b_2 & \dots & b_{r_1+\cdots+r_n}\end{bmatrix}\big[\frac{(-1)^{i+j}}{\tau}M_{ji}\big](\lambda I-A_1)^{-1}\begin{bmatrix} b_1^* \\ b_2^* \\ \vdots \\ b_{r_1+\cdots+r_n}^*\end{bmatrix}\sigma_1.$$

Notice first that

$$\mathbb{X}^{-1}(t_2)(\lambda I-A_1)^{-1}=\big[\frac{(-1)^{i+j}}{\tau}M_{ji}\big](\lambda I-A_1)^{-1}=[K_{ij}]$$

where K_{ij} are linear combinations of $\frac{M_{ji}}{\tau}$ when we consider λ as a constant. Then

$$\begin{aligned} S(\lambda,t_2) &= I-\begin{bmatrix} b_1 & b_2 & \cdots & b_N\end{bmatrix}\mathbb{X}^{-1}(t_2)\big[\tfrac{(-1)^{i+j}}{\tau}M_{ji}\big](\lambda I-A_1)^{-1}\begin{bmatrix} b_1^* \\ b_2^* \\ \vdots \\ b_N^*\end{bmatrix}\sigma_1 \\ &= I-\textstyle\sum_{ij} b_i b_j^* K_{ij}\sigma_1. \end{aligned}$$

Since $b_i b_j^* K_{ij}$ has entries in $\frac{1}{\tau}\mathcal{T}$, we obtain the desired result. □

Let us finish this discussion with some properties of the differential ring $\boldsymbol{\mathcal{R}}_*$

Corollary 3.11 *$\boldsymbol{\mathcal{R}}_*$ is a finitely generated, filtered differential ring*

$$\boldsymbol{\mathcal{R}}_*=\mathbb{C}+\frac{\mathcal{T}}{\tau}+\frac{\mathcal{T}^2}{\tau^2}+\cdots.$$

for which the derivative respects the following rule

$$\frac{d}{dt_2}(\frac{\mathcal{T}^i}{\tau^i})\subseteq\frac{\mathcal{T}^i}{\tau^i}+\frac{\mathcal{T}^{i+1}}{\tau^{i+1}}.$$

Proof: By its definition $\boldsymbol{\mathcal{R}}_*$ is generated by $\frac{\tau'}{\tau}$. Linear combinations and multiplications obviously respect the filtering, since $\frac{\mathcal{T}^i}{\tau^i}\frac{\mathcal{T}^j}{\tau^j}\subseteq\frac{\mathcal{T}^{i+j}}{\tau^{i+j}}$. All the other elements of the ring are obtained by multiplication.

Let us evaluate the differentiation of an element of the filtering

$$\frac{d}{dt_2}(\frac{\mathcal{T}^i}{\tau^i})=\frac{\frac{d}{dt_2}\mathcal{T}^i}{\tau}-i\frac{\mathcal{T}^i}{\tau^{i+1}}\tau'\subseteq\frac{\mathcal{T}^i}{\tau^i}+\frac{\mathcal{T}^{i+1}}{\tau^{i+1}},$$

where $\frac{d}{dt_2}\mathcal{T}^i \subseteq \mathcal{T}^i$ holds by Lemma 3.5. $\square$

One can also calculate the Piccard Vessiot differential ring of the output LDE. This ring is by definition the minimal differential ring, containing the entries of the fundamental matrix. See the references [H, PS] for more material on this subject.

Corollary 3.12 *Let $\lambda \notin \{z_1, \dots, z_n\}$ be a parameter. The Piccard-Vessiot ring of the output LDE (22)*

$$Y' = \sigma_1^{-1}(\sigma_2\lambda + \gamma_*(t_2))Y(t_2)$$

is generated by $\{\frac{\tau'}{\tau}, e^{kt_2}, e^{-kt_2}\}$, where $k = \sqrt{-i\lambda}$.

Proof: The fundamental matrix of the input LDE (21) can be taken as follows

$$\Phi(\lambda, t_2) = \begin{bmatrix} e^{kt_2} & e^{-kt_2} \\ -ike^{kt_2} & ike^{-kt_2} \end{bmatrix}.$$

Moreover from (23) we obtain the fundamental matrix of the output LDE (22) to be given by

$$\Phi_*(t_2) = S(\lambda, t_2)\Phi(\lambda, t_2)S^{-1}(\lambda, t_2^0).$$

Since the entries of $S(\lambda, t_2)$ are generated by $\frac{\tau'}{\tau}$ and the entries of $\Phi(\lambda, t_2)$ are combinations of $\{e^{kt_2}, e^{-kt_2}\}$, we obtain the desired result. $\square$

3.4 General Sturm–Liouville vessels

In this section we want to consider an arbitrary input Sturm–Liouville vessel. In order to obtain a Sturm–Liouville equation at the input we will take

$$\gamma = \begin{bmatrix} i\frac{\eta''}{\eta} & \frac{\eta'}{\eta} \\ -\frac{\eta'}{\eta} & i \end{bmatrix}$$

for an analytic function $\eta = \eta(t_2)$ (actually it is enough for η to be differentiable a finite number of times, but then the notion of the differential ring $\boldsymbol{\mathcal{R}}$, appearing at the Definition 10 must be substituted by the ring, generated by the derivatives). Moreover, in order to use techniques similar to the trivial case, we shall suppose that

$$\lim_{t_2\to\infty} \eta^{(n)}(t_2) = 0, \tag{45}$$

for sufficiently large n, which will be clear from the proof of Lemma 3.13. Then entries of the input $u(t_2) = \begin{bmatrix} u_1(\lambda, t_2) \\ u_2(\lambda, t_2) \end{bmatrix}$ will satisfy

$$-\frac{\partial^2}{\partial t_2^2}u_1(\lambda, t_2) + 2\frac{d^2}{dt_2^2}\big(\log \eta\big)u_1(\lambda, t_2) = -i\lambda u_1(\lambda, t_2), \tag{46}$$

$$u_2(\lambda, t_2) = i(\eta - \frac{\partial}{\partial t_2})u_1(\lambda, t_2), \tag{47}$$

which means that $u_1(\lambda, t_2)$ satisfies the Sturm–Liouville differential equation (1) with potential $2\frac{d^2}{dt_2^2}\big(\log \eta\big)$ and the spectral parameter $-i\lambda$.

Definition 9 *A general Sturm–Liouville vessel is a collection*

$$\mathfrak{G}_{SL} = (A_1, B(t_2), \mathbb{X}(t_2); \sigma_1, \sigma_2, \gamma = \begin{bmatrix} i\frac{\eta''}{\eta} & \frac{\eta'}{\eta} \\ -\frac{\eta'}{\eta} & i \end{bmatrix}, \gamma_*(t_2) = \begin{bmatrix} -i(\beta' - \beta^2) & -\beta \\ \beta & i \end{bmatrix}; \mathcal{H}, \mathbb{C}^2)$$

satisfying the the vessel condition (15), (16), (17), (18) and existing on an interval $\mathcal{I}$ on which the regularity assumptions 1 hold.

If $A_1 = \text{Jordan}(z_1, r_1, \ldots, z_n, r_n)$ with z_i's spectral values and r_i's the corresponding sizes of Jordan blocks, then defining companion solutions b_i of (35) we obtain solving (15) (where $N = r_1 + \cdots + r_n$)

$$B(t_2) = \begin{bmatrix} b_1^* \\ b_2^* \\ \cdots \\ b_N^* \end{bmatrix}.$$

$\mathbb{X}(t_2) = [x_{ij}]$ is a solution of the Lyapunov equation (16) and satisfies (17). We will use the same definition as in the elementary input case $\tau = \det \mathbb{X}(t_2)$ and using Lemma 3.1 and Proposition 3.2 considered in this new setting, the same formula, appearing at the Proposition 3.2 is obtained for γ_*:

$$\gamma_* = \begin{bmatrix} i\frac{\eta''}{\eta} & \frac{\eta'}{\eta} \\ -\frac{\eta'}{\eta} & i \end{bmatrix} + \begin{bmatrix} i\frac{\tau''}{\tau} & \frac{\tau'}{\tau} \\ -\frac{\tau'}{\tau} & 0 \end{bmatrix}. \tag{48}$$

From here we can see that the differential ring $\boldsymbol{\mathcal{R}}_*$ must include the element $\frac{\eta'}{\eta}$. In the sequel, we will also use the ring, generated by $\frac{\eta'}{\eta}$ itself, so we make the following

Definition 10 *The input differential ring $\boldsymbol{\mathcal{R}}$ is the ring as defined to be generated by $\{\frac{\eta'}{\eta}, 1\}$. The output differential ring $\boldsymbol{\mathcal{R}}_*$ is the ring as defined to be generated by $\{\frac{\tau'}{\tau}, \frac{\eta'}{\eta}, 1\}$.*

This definition is a generalization of the Definition 6, because in the elementary input case we obtain that the input differential ring is generated by $\frac{\eta'}{\eta} = 0$ and 1, i.e it is trivial and as a result, the output differential ring is generated only by $\frac{\tau'}{\tau}$ and 1.

Notice also that the same restriction (corollaries 3.3 and 3.4) on the appearance of purely imaginary and pairs $(p_i, -p_i^*)$ hold and as a result we may consider A_1 of the same form as in the elementary input case

$$A_1 = \begin{bmatrix} E & 0 & 0 \\ 0 & P & 0 \\ 0 & 0 & J \end{bmatrix} \tag{39}$$

$$E = \text{diag}(e_1, e_2, \ldots, e_{r_e}), \tag{40}$$

$$P = \text{diag}(p_{r_e+1}, -p_{r_e+1}^*, \ldots, p_{r_p}, -p_{r_p}^*), \tag{41}$$

$$J = \text{Jordan}(z_{r_e+2r_p+1}, r_{r_e+2r_p+1}, \ldots, z_n, r_n) \tag{42}$$

so that $N = r_e + 2r_p + r_{r_e+2r_p+1} + \ldots + r_n$. Suppose also that as in (43) the transpose of $B(t_2)$ is

$$B(t_2)^t = \begin{bmatrix} u_1^* & \ldots & u_{r_e}^* & v_{r_e+1}^* & w_{r_e+2}^* \cdots & v_{2r_p-1}^* & w_{2r_p}^* & b_{r_e+2r_p+1}^* & \cdots & b_N^* \end{bmatrix},$$

where v_i, w_i's are corresponding solutions of the adjoint input LDE (34) (for which γ is not trivial) and b_i's are companion solutions of the same equation (34).

Let us define an analogue of the space $\mathcal{T}$ appearing in Definition 7 in the following way

Definition 11 *We define $\mathcal{T}_G$ to be the space defined by*

$$\mathcal{T}_G = \mathrm{span}\{y_1 y_2 \dots y_{r_e} y_{r_e+1} y_{r_e+3} \dots y_{r_p} y_{r_e+2r_p+1} y_{r_e+2r_p+r_1+1} \dots y_{N-r_n+1}\}\boldsymbol{\mathcal{R}},$$

where y_i's are defined as in Definition 7.

Remark: It follows from the definition that $\mathcal{T}_G = \mathcal{T}\boldsymbol{\mathcal{R}}$.

On the contrary to Theorem 3.10 this space will not usually be finite-dimensional, because the input differential ring $\boldsymbol{\mathcal{R}}$ is generally infinite-dimensional.

Lemma 3.13 *$\mathcal{T}_G' \subseteq \mathcal{T}_G$ and τ satisfies a linear differential equation of finite order over $\boldsymbol{\mathcal{R}}$.*

Proof: From the definition it follows that $\mathcal{T}_G = \mathcal{T}\boldsymbol{\mathcal{R}}$ and using Leibnitz rule,

$$\mathcal{T}_G' \subseteq \mathcal{T}'\boldsymbol{\mathcal{R}} + \mathcal{T}\boldsymbol{\mathcal{R}}'.$$

Consequently, it is enough to show that $\mathcal{T}' \subseteq \mathcal{T}\boldsymbol{\mathcal{R}}$, since then

$$\mathcal{T}'\boldsymbol{\mathcal{R}} + \mathcal{T}\boldsymbol{\mathcal{R}}' \subseteq \mathcal{T}\boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{R}} + \mathcal{T}\boldsymbol{\mathcal{R}} \subseteq \mathcal{T}\boldsymbol{\mathcal{R}} \subseteq \mathcal{T}_G$$

as desired. In order to see that $\mathcal{T}' \subseteq \mathcal{T}\boldsymbol{\mathcal{R}}$, we use the Leibnitz rule and evaluate the derivatives of $b_i^* E_{ij_1} b_{j_i}$ using their definition as companion solutions of the adjoint output LDE (34)

$$\begin{aligned}
&\tfrac{d}{dt_2} b_i^* E_{ij_i} b_{j_i} = \\
&= b_i^*[(\sigma_2 z^* - \gamma)\sigma_1^{-1} E_{ij_i} + E_{ij_i}\sigma_1^{-1}(\sigma_2 w + \gamma)] b_{j_i} + b_{i-1}^* \sigma_2 \sigma_1^{-1} E_{ij_i} b_{j_i} + b_i^* E_{ij_i} \sigma_1^{-1} \sigma_2 b_{j_i-1} \\
&= b_i^*[\begin{bmatrix} \frac{\eta'}{\eta} & z^* + i\frac{\eta''}{\eta} \\ -i & -\frac{\eta'}{\eta} \end{bmatrix} E_{ij_i} + E_{ij_i} \begin{bmatrix} \frac{\eta'}{\eta} & i \\ w - i\frac{\eta''}{\eta} & -\frac{\eta'}{\eta} \end{bmatrix}] b_{j_i} + \\
&\qquad\qquad + b_{i-1}^* \sigma_2 \sigma_1^{-1} E_{ij_i} b_{j_i} + b_i^* E_{ij_i} \sigma_1^{-1} \sigma_2 b_{j_i-1} \\
&= b_i^*[\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} E_{ij_i} + E_{ij_i} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}] b_{j_i} \frac{\eta'}{\eta} + b_i^*[\begin{bmatrix} 0 & z^* \\ -i & 0 \end{bmatrix} E_{ij_i} + E_{ij_i} \begin{bmatrix} 0 & i \\ w & 0 \end{bmatrix}] b_{j_i} + \\
&\qquad + b_i^*[\begin{bmatrix} 0 & i \\ 0 & 0 \end{bmatrix} E_{ij_i} + E_{ij_i} \begin{bmatrix} 0 & 0 \\ i & 0 \end{bmatrix}] b_{j_i} \frac{\eta''}{\eta} + b_{i-1}^* \sigma_2 \sigma_1^{-1} E_{ij_i} b_{j_i} + b_i^* E_{ij_i} \sigma_1^{-1} \sigma_2 b_{j_i-1},
\end{aligned}$$

which means that we obtain elements of the form $b_i^* E_{ij_1} b_{j_i} \boldsymbol{\mathcal{R}}$. In order to see that we stay at the space $\mathcal{T}_G$, notice that if b_i or b_{j_i} are initial members at a companion chain of solutions, then the two elements with smaller indexes

$$b_{i-1}^* \sigma_2 \sigma_1^{-1} E_{ij_i} b_{j_i}, \qquad b_i^* E_{ij_i} \sigma_1^{-1} \sigma_2 b_{j_i-1}$$

does not appear, which explains why the restriction on $\mathcal{T}_G$ in Definition 11 holds.

Since $\mathcal{T}_G = \mathcal{T}\boldsymbol{\mathcal{R}}$ and $\mathcal{T}$ is finite-dimensional, we obtain that $\tau \in \mathcal{T}_G$ satisfies a linear differential equation with coefficients in $\boldsymbol{\mathcal{R}}$. □

Although the solutions of the input LDE (21) are not exponents, multiplied by powers of t_2, they have the same properties appearing in Corollary 3.7:

Lemma 3.14 *Let $b_1(t_2), b_2(t_2), \dots, b_r(t_2)$ be a chain corresponding to the spectral value z with initial values $b_1^0, b_2^0, \dots, b_r^0$. Then*

1. *for fixed $i \neq j$,* $\mathrm{span}(b_i^* E b_j) = \mathrm{span}(b_j^* E b_i)$,

2. for any indexes satisfying $i+j=k+m$, $\mathrm{span}(b_i^* E b_j)=\mathrm{span}(b_k^* E b_m)$,

Proof: Denote by $y_{ij}=\dfrac{b_i^*\sigma_1 b_j}{z+z^*}$. As we saw at the previous Lemma 3.13 derivatives of y_{ij} involves terms with the same b_i, b_j and terms with lower indexes. Performing these calculations, we can find that y_{ij} satisfies the following differential equation

$$y_{ij}^{(4)}-[4(\frac{\eta''\eta-(\eta')^2}{\eta^2}+2\sqrt{-1}(z-z^*))y_{ij}^{(2)}-(z+\bar{z})^2 y_{ij}+K_{ij}=0, \tag{49}$$

where K_{ij} is a linear combination of elements y_{km} with lower indexes. Let $\Psi(t_2)$ be the fundamental matrix of solutions of the homogeneous part of the equation (49):

$$\Psi(t_2)^{(4)}-[4(\frac{\eta''\eta-(\eta')^2}{\eta^2}+2\sqrt{-1}(z-z^*))\Psi(t_2)^{(2)}-(z+\bar{z})^2\Psi(t_2)=0, \quad \Psi(t_2^0)=I, \tag{50}$$

then $\Psi(t_2)$ can be taken as a real valued function, since the coefficients are such. Then

$$y_{ij}=\Psi(t_2)[-\int_{t_2^0}^{t_2}\Psi(y)^{-1}K_{ij}(y)dy+y_{ij}(t_2^0)].$$

Since $\Psi(t_2)$ is real valued, we obtain that

$$y_{ji}=y_{ij}^*=\Psi(t_2)[-\int_{t_2^0}^{t_2}\Psi(y)^{-1}K_{ij}^*(y)dy+y_{ij}^*(t_2^0)].$$

If we suppose using induction that K_{ij} and K_{ij}^* are linear combinations of the same functions real-valued functions, then immediately we obtain that y_{ij} and y_{ji} are also linear combinations of the same functions. This finishes the first part of the Lemma.

Let us prove the second part using induction on $i+j$. We will use again the differential equation (50) for $\Psi(t_2)$, but now we will use the fact that the coefficients of this differential equation does not depend on i,j but on z,η only. We have also analyze more carefully the element K_{ij} appearing in (49). Differentiating twice and four times the function $y_{ij}=\dfrac{b_i^*\sigma_1 b_j}{z+z^*}$ we obtain elements of the form $b_{i_1}^* E_{i_1,j_1} b_{j_1}$ where $i-4\le i_1\le i, j-4\le j_1\le j$ and E_{i_1,j_1} is a constant matrix in $\mathbb{C}^{2\times 2}$. So, their sum satisfies

$$i+j-8\le i_1+j_1\le i+j-1$$

Consequently,

$$y_{ij}=\Psi(t_2)[-\int_{t_2^0}^{t_2}\Psi(s)^{-1}\sum_{i+j-8\le i_1+j_1\le i+j-1} b_{i_1}^* E_{i_1,j_1} b_{j_1} ds+y_{ij}(t_2^0)].$$

But the same formula holds for y_{mn}, for $m+n=i+j$

$$y_{mn}=\Psi(t_2)[-\int_{t_2^0}^{t_2}\Psi(s)^{-1}\sum_{m+n-8\le m_1+n_1\le m+n-1} b_{m_1}^* E_{m_1,n_1} b_{n_1} ds+y_{mn}(t_2^0)].$$

If we suppose, by the induction hypothesis, that

$$\sum_{i+j-8\leq i_1+j_1\leq i+j-1} b_{i_1}^* E_{i_1,j_1} b_{j_1}, \quad \text{and} \quad \sum_{m+n-8\leq m_1+n_1\leq m+n-1} b_{m_1}^* E_{m_1,n_1} b_{n_1}$$

are linear combinations of the same functions, we shall obtain the desired result. Notice that the basis for this induction is for $i+j=3$, which holds by the first property. □

From the assumption (45) on the function η it follows that in the neighborhood of infinity ($t_2 \to \infty$) the solutions and their derivatives of the input compatibility condition (21) are close to the exponential functions. Using this observation, we may define a generic case on the basis of the trivial input case:

Definition 12 *We define a notion of* ***generic case*** *as follows. For each choice of the initial spectral parameters $A_1, B(t_2^0), \mathbb{X}_0$ at the neighborhood of infinity, the solutions of the input LDE* (21) *are close to exponential solutions. A choice of these parameters, for which no exponent, appearing in τ, considered at the neighborhood of infinity, vanishes is called a generic case.*

The generalization of Theorem 3.10 is as follows

Theorem 3.15 *The entries of γ_* are in $\boldsymbol{\mathcal{R}}_*$ and in the generic case the entries of the transfer function $S_{\mathfrak{G}_{SL}}(\lambda, t_2)$ of the vessel $\mathfrak{G}_{SL}$ are in $\boldsymbol{\mathcal{R}}_*$. In the generic case, the dimension of the space $\mathcal{T}_G$ over $\boldsymbol{\mathcal{R}}$ is the dimension of $\mathcal{T}$, which is as in Corollary 3.8:*

$$4^{r_e} 4^{2(r_p - r_e)} \prod_{k=1}^{n} [1 + r_n(r_n - 1)] 4^{r_n}, \quad r_e + 2r_p + r_1 + r_2 + \cdots + r_n = N$$

Proof: From the formula (48) it follows that the entries of γ_* are in $\boldsymbol{\mathcal{R}}_*$.

Due to the assumption of the generic case, the maximal and the minimal dimension of the space $\mathcal{T}_G$ coincide and is given by the formula above. In order to prove the statement regarding the transfer function, let us use the formula (36):

$$S_{\mathfrak{G}_{SL}}(\lambda, t_2) = I - \begin{bmatrix} b_1 & b_2 & \ldots & b_{r_1+\cdots+r_n} \end{bmatrix} \frac{(-1)^{i+j}}{\tau} [M_{ji}] (\lambda I - A_1)^{-1} \begin{bmatrix} b_1^* \\ b_2^* \\ \vdots \\ b_{r_1+\cdots+r_n}^* \end{bmatrix} \sigma_1.$$

Notice first that

$$\mathbb{X}^{-1}(t_2)(\lambda I - A_1)^{-1} = \frac{(-1)^{i+j}}{\tau} [M_{ji}] (\lambda I - A_1)^{-1} = [K_{ij}]$$

where K_{ij} are linear combinations of $\frac{M_{ji}}{\tau}$ when we consider λ as a constant. Then

$$\begin{aligned} S(\lambda, t_2) &= I - \begin{bmatrix} b_1 & b_2 & \cdots & b_N \end{bmatrix} \mathbb{X}^{-1}(t_2) \tfrac{(-1)^{i+j}}{\tau} [M_{ji}] (\lambda I - A_1)^{-1} \begin{bmatrix} b_1^* \\ b_2^* \\ \vdots \\ b_N^* \end{bmatrix} \sigma_1 \\ &= I - \textstyle\sum_{ij} b_i b_j^* K_{ij} \sigma_1. \end{aligned}$$

Since $b_i b_j^* K_{ij}$ has entries in $\frac{1}{\tau}\mathcal{T}$, we obtain by the assumption of the generic case that these entries are in $\boldsymbol{\mathcal{R}}_*$. □

The following corollaries are proved identically to the elementary input case

Corollary 3.16 *$\boldsymbol{\mathcal{R}}_*$ is a finitely generated, filtered differential ring:*

$$\boldsymbol{\mathcal{R}}_* = \boldsymbol{\mathcal{R}} + \frac{\mathcal{T}_G}{\tau} + \frac{\mathcal{T}_G^2}{\tau^2} + \cdots$$

for which the derivative respects the following rule

$$\frac{d}{dt_2}(\frac{\mathcal{T}_G^i}{\tau^i}) \subseteq \frac{\mathcal{T}_G^i}{\tau^i} + \frac{\mathcal{T}_G^{i+1}}{\tau^{i+1}}.$$

Corollary 3.17 *The Piccard-Vessiot ring of the output LDE (22) (for $\lambda \notin \{z_1, \ldots, z_n\}$)*

$$\frac{d}{dt_2}Y = \sigma_1^{-1}(\sigma_2\lambda + \gamma_*(t_2))Y(t_2)$$

is generated by $\frac{\tau'}{\tau}$ and the entries of $\Phi(\lambda, t_2)$.

4 Conclusions and remarks

1. It is possible to generalize all the formulas appearing in this article to solutions of differential equations of greater order. For example, defining

$$\sigma_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \sigma_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \gamma = \begin{bmatrix} i\pi & \beta & \alpha \\ -\beta^* & 0 & 1 \\ -\alpha & -1 & 0 \end{bmatrix}$$

for real valued $\alpha(t_2), \pi(t_2)$, one obtains that for the input function $u_\lambda(t_2) = \begin{bmatrix} u_1(\lambda, t_2) \\ u_2(\lambda, t_2) \\ u_3(\lambda, t_2) \end{bmatrix}$ the first entry satisfies

$$-u_1''' + u_1'(\beta^* + \beta - 2c' + c^2) + u_1((\beta^*)' - c'' - i\alpha + cc' + c\beta - c\beta^*) = \lambda u_1$$

which is a general linear differential equation of order 3

$$u''' + q_1 u' + q_2 u = \lambda u.$$

For the output $y_\lambda(t_2) = \begin{bmatrix} y_1(\lambda, t_2) \\ y_2(\lambda, t_2) \\ y_3(\lambda, t_2) \end{bmatrix}$, the first entry satisfies

$$y_1''' + q_{1*}y_1' + q_{2*}y_1 = \lambda y_1$$

and there are relations between q_{1*}, q_{2*} and q_1, q_2.

More generally, defining σ_1 as anti diagonal, $\sigma_1 = E_{11}$ and γ of the same form as for n=3, we can study differential equations of order n.

We can also conclude that in the Sturm–Liouville case the role of the τ-function is a "generating element" of a universe (a generator of the output differential ring $\boldsymbol{\mathcal{R}}_*$ together with $\frac{\eta'}{\eta}$ corresponding to the input), where all the relevant objects (γ_*, transfer function, the potential $q_*(t_2)$) live.

2. The Galois Group [PS] of the differential equation (1) is defined as the group of automorphisms of $\Phi_*(\lambda, t_2)$ (see Definition 24), commuting with the derivative, which leave the ring, generated by the potential, invariant. In that case, it means that we are interested in automorphisms of $\boldsymbol{\mathcal{R}}_*$, which leave the ring $\boldsymbol{\mathcal{R}}$ invariant. Any automorphism of $\boldsymbol{\mathcal{R}}_*$ is uniquely determined by its action on the generating element τ. Since, τ satisfies a linear differential equation of finite order over $\boldsymbol{\mathcal{R}}$, it follows that τ can be mapped to a solution of the same differential equation only. So, the Galois group is finite in that case and has a maximal number of elements as the degree of $\mathcal{T}_G$ over $\boldsymbol{\mathcal{R}}$, i.e. as the degree of $\mathcal{T}$. This analysis can be carried out further, using the nature of the differential equation for τ.

3. Generalizing the ideas in this article, we may study a differential ring $\boldsymbol{\mathcal{R}}_*$ generated from another differential ring $\boldsymbol{\mathcal{R}}$ using solutions of the input Sturm–Liouville differential equation (1). The ring $\boldsymbol{\mathcal{R}}_*$ will have the filtering structure appearing in Corollary 3.16. Structure of the space $\mathcal{T}_G$ may be further analyzed.

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References

[AMV] D. Alpay, A. Melnikov, V. Vinnikov, *On the class $\boldsymbol{\mathcal{RSI}}$ of Schur functions intertwining solutions of linear differential equations*, arXiv:0912.2014.

[BV] J.A. Ball and V. Vinnikov, *Overdetermined Multidimensional Systems: State Space and Frequency Domain Methods*, Mathematical Systems Theory, F. Gilliam and J. Rosenthal, eds., Inst. Math. and its Appl. Volume Series, Vol. 134, Springer-Verlag, New York (2003), 63–120.

[BL] M. S. Brodskii, M. S. Livšic,*Spectral analysis of non-self-adjoint operators and intermediate systems*, (Russian) Uspehi Mat. Nauk (N.S.) 13 1958 no. 1(79), 3–85. (Reviewer: R. R. Kemp)

[Br] M.S. Brodskii, *Triangular and Jordan representation of linear operators*, Moscow, Nauka, 1969 (Russian); English trans.: Amer. Math. Soc., Providence, 1974.

[CoLe] E.A Coddington, N. Levinson, *Theory of ordinary differential equation*, Mc-Graw Hill, 1955.

[Crum] M.M. Crum, *Associated Sturm-Liouville systems*, Quart. J. Math. Oxford Ser. (2) 6 (1955), 121–127.

[Fa] L.D. Fadeev, *The inverse problem in the quantum theory of scattering*, J. Math. Phys., 4 (1), 1963, translated from Russian.

[Fu] R. Fuchs, *Über lineare homogene Differentialgleichungen zweiter Ordnung mit drei im Endlichen gelegenen wesentlich singulären Stellen.* (German) Math. Ann. 63 (1907), no. 3, 301–321.

[G] R. Garnier, *Sur des équations différentielles du troisième ordre dont l'intégrale générale est uniforme et sur une classe d'équations nouvelles d'ordre supérieur dont l'intégrale générale a ses points critiques fixes*, (French) Ann. Sci. École Norm. Sup. (3) 29 (1912), 1–126.

[Ga] H. Gauchman, *Connection colligations on Hilbert bundles*, Integral Equations Operator Theory 6 (1983), no. 1, 31–58.

[GL] I. M. Gelfand, B. M. Levitan, *On the determination of a differential equation from its spectral function*, (Russian) Izvestiya Akad. Nauk SSSR. Ser. Mat. 15, 1951.

[Ha] P. Hartman, *Ordinary differential equations*, John Wiley & Sons, New York, London, 1964. (see Section XI).

[H] E. Hrushovski, *Computing the Galois group of a linear differential equation*, Banach Center publications, 58, Warzawa 2002.

[JMU] M. Jimbo, T. Miwa, K. Ueno, *Monodromy preserving deformation of linear ordinary differential equations with rational coefficients, I. General theory and τ-function*, Physica **2D** (1981), pp. 306-352.

[KFA] R.E. Kalman, P.L. Falb, M.A. Arbib, *Topics in mathematical system theory*, McGraw-Hill,1969.

[KV] V. Katsnelson, D. Volok, *Rational solutions of the Schlesinger system and isoprincipal deformations of rational matrix functions*, part I, OT 149, part II, OT 159, Birkhäuser.

[K] E. R. Kolchin, *Differential algebra and algebraic groups*, Acad. Press, New York, 1973.

[Kov] Jerald J. Kovacic, *An algorithm for solving second order linear homogeneous differential equations*, J. Symbolic Comput. 2 (1986), no. 1, 3–43.

[LxPh] P.D. Lax and R.S. Philips, *Scattering theory*, Academic Press, New-York-London, 1967.

[L] R. Liouville, *Sur les équations de la dynamique*, (French) Acta Math. 19 (1895), no. 1, 251–283.

[Liv1] M.S. Livšic, *Commuting nonselfadjoint operators and solutions of systems of partial differential equations generated by them*, (Russian) Soobshch. Akad. Nauk Gruzin. SSR 91 (1978), no. 2, 281–284. (Reviewer: E. R. Tsekanovskii).

[Liv2] M.S. Livšic, *Vortices of 2D systems*, Operator Theory: Advances and Applications, Vol. 123, Birkhauser-Verlag, Basel 2001.

[LKMV] M.S. Livšic, N.Kravitsky, A.S. Markus and V. Vinnikov, *Theory of Commuting Nonselfadjoint Operators*, Kluwer Acad. Press, 1995.

[Mar] V.A. Marchenko, *Sturm–Liouville operators and their applications*, Kiev, Naokova Dumka, 1977.

[M] A. Melnikov, *Overdetermied 2D systems invariant in one direction and their transfer functions*, Phd Thesis, submitted to Ben Gurion University, November 2008.

[M1] A. Melnikov, *The inverse scattering and a theory of vessels*, in preparation.

[MV1] A. Melnikov, V. Vinnikov, *Overdetermined 2D Systems Invariant in One Direction and Their Transfer Functions*, http://arXiv.org/abs/0812.3779.

[MVc] A. Melnikov, V. Vinnikov, *Overdetermined conservative 2D Systems, Invariant in One Direction and a Generalization of Potapov's theorem*, http://arxiv.org/abs/0812.3970.

[N] A. C. Newell, *Solitons in mathematics and physics*, Society for Industr. and Appl. Math., 1985.

[P] V.P. Potapov, *On the multiplicative structure of J-nonexpanding matrix functions*, Trudy Moskov. Mat. Obschestva, 4 (1955), 125-236 (Russian); English transl.: Amer. Math. Soc. Transl. (2), 15 (1960), 131-243.

[Pov] A. Povzner, *On differential equations of Sturm-Liouville type on a half-axis*, Amer. Math. Soc. Translation 1950, (1950). no. 5, 79

[R] B. Riemann, *Bernhard Gesammelte mathematische Werke und wissenschaftlicher Nachlas*, (German) Dover Publications, Inc., New York, N. Y., 1953. xix+558+viii+116

[Shl] L. Schlesinger, *Sur la solution du probléme de Riemann*, (French) Acta Math. 31 (1908), no. 1, 65–69.

[S] C. Sturm, *Sur les équations différentielles linéaires du second ordre*, J. Math. Pures Appl., 1(1), 1836, 106-186.

[PS] M. Van der Put, M. F. Singer, *Galois theory of linear differential equations*, Springer, 2003.