

Finite dimensional subspaces of L_p

by

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Abstract. If $1 < p < \infty$ and $E \subset L_p(\mu)$ is an n -dimensional subspace, then $d(E, l_2^n) < n^{1/2-1/p}$; further, there is a projection u of $L_p(\mu)$ onto E with $\|u\| < n^{1/2-1/p}$.

For convenience only real normed spaces are considered. The notation and terminology is standard; we mention only that the p -absolutely summing, p -integral and L_p -factorization norms of operators are denoted by π_p , i_p and γ_p , respectively (cf. [13], [12], [7]). The main result of this paper is the following.

THEOREM 1. Let E be an n -dimensional subspace of $L_p(\mu)$, $1 \leq p < \infty$.

(1) There is a basis $(f_i)_{i \leq n}$ of E so that for all $x \in l_2^n$,

$$n^{-1} \|x\|_2^2 = \int \left| \sum_{i \leq n} x_i f_i \right|^2 |f|^{p-2} d\mu, \quad \text{where } f = \left(\sum_{i \leq n} |f_i|^2 \right)^{1/2}.$$

(2) If $(h_i)_{i \leq n}$ is another basis for E satisfying (1), there is an $n \times n$ orthogonal matrix (a_{ik}) such that

$$h_k = \sum_{i \leq n} a_{ik} f_i, \quad i \leq k \leq n.$$

The proof requires an easy lemma.

LEMMA 2. For $1 \leq p < \infty$ and $u: l_2^n \rightarrow L_p(\mu)$ any operator,

$$\pi_p(u) = \left\| \sup_{\|x\|=1} |u(x)| \right\|.$$

Proof of Lemma. The Hilbert space l_2^n is a quotient of an L_q -space, $1/p + 1/q = 1$, so $\pi_p(u') \leq \pi_p(u)$ by [7]. Since the domain of u' is $L_q(\mu)$, the Kwapien-Schwartz theorem [6] shows that $u = u''$ maps the unit ball of l_2^n into an order bounded set of the lattice $L_p(\mu)$, and that $\|f\| \leq \pi_p(u')$ for $f = \sup_{\|x\|=1} |u(x)|$. The other inequality is obvious.

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Proof of Theorem 1. By [8] there is an isomorphism $u: l_2^n \rightarrow E$ with $\pi_p(u) = 1$ and $i_q(u^{-1}) = n$, $1/p + 1/q = 1$. Write $(e_i)_{i \leq n}$ for the unit vector basis of l_2^n , $f_i = u(e_i)$ and $f = (\sum_{i \leq n} |f_i|^2)^{1/2}$. There is no harm in supposing $f > 0$ μ -a.e. since the f_i are a basis for E . The operator u^{-1} has an extension $w: L_p(\mu) \rightarrow l_2^n$ satisfying $i_q(w) = i_q(u^{-1}) = n$. Let $g_i = w'(e_i^*)$, where $e_i^* \in (l_2^n)'$ is inner product with e_i , and $g = (\sum_{i \leq n} |g_i|^2)^{1/2}$.

It is clear that $f = \sup_{\|x\|=1} |u(x)|$ and $g = \sup_{\|x\|=1} |w'(x)|$. By the lemma, $\|f\|_p = 1$ and $\|g\|_q \leq \pi_q(w) \leq n$, again by the Kwapien-Schwartz theorem. Since $\langle f_i, g_k \rangle = \delta_{ik}$,

$$n = \int \sum_{i \leq n} f_i g_i d\mu \leq \int f g d\mu \leq \|f\|_p \|g\|_q \leq n.$$

Thus $\langle f, g \rangle = n$ and $\|g\|_q = n$. For $1 < p < \infty$ this clearly implies $g = n|f|^{p-1}$ because $\|f\|_p = 1$. In case $p = 1$, we have $g = n$ μ -a.e. since $f > 0$ μ -a.e. Also $\sum_{i \leq n} f_i g_i = fg$ so that $f_i f^{-1} = g_i g^{-1}$ for $i = 1, \dots, n$.

Combining equalities, $g_k = n|f|^{p-2} f_k$ for each k and hence

$$\delta_{ik} = \langle f_i, g_k \rangle = n \int f_i f_k |f|^{p-2} d\mu,$$

which is enough to establish (1).

To prove (2), let $(f_i)_{i \leq n}$ be any basis for which (1) is true, and define $u: l_2^n \rightarrow E$, $w: L_p(\mu) \rightarrow l_2^n$ by $u(x) = \sum_{i \leq n} x_i f_i$ and $w(h) = n \langle h, f_i |f|^{p-2} \rangle_{i \leq n}$. Clearly, wu is the identity, $\pi_p(u) = 1$ by the lemma and $n = \text{tr}(uw^{-1}) \leq \pi_p(u) i_q(w) \leq i_q(w)$. Notice that $|w'(x)| \leq n|f|^{p-1}$ μ -a.e. whenever $\|x\|_2 \leq 1$. For $1 < p < \infty$ this implies $i_q(w') \leq n$ and hence $i_q(w) \leq i_q(w') \leq n$ since l_2^n is a quotient of an L_p -space [7]. In the case $p = 1$, $w: L_1(\mu) \rightarrow l_2^n$ has norm $\leq n$; since l_2^n is a quotient of a $C(K)$ and w has the lifting property, $i_\infty(w) = \gamma_\infty(w) \leq n$. In any event the isomorphism $u: l_2^n \rightarrow E$ which takes the i th unit vector to f_i satisfies $\pi_p(u) = 1$ and $i_q(u^{-1}) = n$ if the basis (f_i) satisfies (1). Now if $(h_i)_{i \leq n}$ also satisfies (1), then $\pi_p(v) = 1$ and $i_q(v^{-1}) = n$, where $v: l_2^n \rightarrow E$ maps e_i to h_i . By Theorem 1.1 of [8], $u^{-1}v$ is an isometry of l_2^n and representing $u^{-1}v$ as a matrix with respect to the unit vector basis proves (2).

THEOREM 3. Let $1 < p < \infty$, $E = L_p(\mu)$ and F be an n -dimensional space. Each operator $u: E \rightarrow F$ has an extension $w: L_p(\mu) \rightarrow F$ with $\|w\| \leq n^{1/2-1/p} \|u\|$; in case $2 \leq p < \infty$ the extension may be chosen to satisfy $\gamma_2(w) \leq n^{1/2-1/p} \|u\|$.

Proof. First suppose $2 < p < \infty$ and consider the special case in which $E = F$ and u is the identity. Let $f_1, f_2, \dots, f_n \in E$ and let f be as in Theorem 1; set $d\gamma = |f|^p d\mu$ and let $w = w_2 w_1 w_3$, where $w_1: L_p(\mu) \rightarrow L_2(\gamma)$

is multiplication by f^{-1} , $w_2: L_2(\gamma) \rightarrow w_1(E)$ is the orthogonal projection and $w_3: w_1(E) \rightarrow E$ is multiplication by f . Clearly, w is a projection onto E . By Hölder's inequality,

$$\|w_1(h)\| = \left[\int |h|^2 |f|^{p-2} d\mu \right]^{1/2} \leq \|h\|_p \|f\|_p^{(p-2)/2p}$$

for all $h \in L_p(\mu)$, and hence $\|w_1\| \leq 1$. Also for $x \in \mathbb{R}^n$

$$\begin{aligned} \left\| w_3 \left[\sum_{i \leq n} x_i (f_i f^{-1}) \right] \right\|^p &= \int \left| \sum_{i \leq n} x_i f_i \right|^p \left| \sum_{i \leq n} x_i f_i \right|^{p-2} d\mu \\ &\leq \int \left| \sum_{i \leq n} x_i f_i \right|^2 \|x\|_2^{p-2} |f|^{p-2} d\mu = n^{(p/2)-1} \left[\int \left| \sum_{i \leq n} x_i (f_i f^{-1}) \right|^2 |f|^p d\mu \right]^{p/2} \end{aligned}$$

by Theorem 1, so $\|w_3\| \leq n^{1/2-1/p}$. This establishes the special case. More generally, for $2 < p < \infty$, let φ be the norm one bilinear form

$$\varphi: L(L_p(\mu), l_2) \times L(l_2, F) \rightarrow L(E, F)$$

defined by $\varphi(a, b) = ba/E$, and also denote by φ the induced linear operator on the projective tensor product of $L(L_p(\mu), l_2)$ and $L(l_2, F)$. After making the natural identification $L(E, F)' = i_1(F, E)$ and $L(l_2, F)' = i_1(F, l_2)$ (possible since F is finite dimensional) the adjoint

$$\varphi': i_1(F, E) \rightarrow L(L(L_p(\mu), l_2), i_1(F, l_2))$$

is given by $\varphi'(u)(v) = vju$, where $j: E \rightarrow L_p(\mu)$ is the natural embedding. For $u \in i_1(F, E)$ the special case proven above shows that there are operators $\alpha: L_p(\mu) \rightarrow l_2$, $\beta: l_2 \rightarrow ju(F)$ with $\|\alpha\| \leq 1$, $\|\beta\| \leq n^{1/2-1/p}$ and $\beta\alpha|ju(F) = \text{identity}$. For clarity we temporarily write $i_1(\gamma: A \rightarrow B)$ to denote the i_1 -norm of an operator γ considered as a map from A to B . With this notation,

$$\begin{aligned} i_1(u: F \rightarrow E) &\leq i_1(u: F \rightarrow u(F)) \\ &= i_1(ju: F \rightarrow ju(F)) \\ &= i_1(\beta\alpha ju: F \rightarrow ju(F)) \\ &\leq \|\beta\| i_1(\alpha ju: F \rightarrow l_2) \\ &\leq n^{1/2-1/p} \|\varphi'(u)\|. \end{aligned}$$

Thus $\|\varphi'(u)\| \leq i_1(u) \leq n^{1/2-1/p} \|\varphi'(u)\|$ for all $u \in i_1(F, E)$ so that φ' is an $n^{1/2-1/p}$ -into isomorphism and hence φ , on the tensor product, is $(1 + \varepsilon)n^{1/2-1/p}$ -quotient for every $\varepsilon > 0$.

For $u: E \rightarrow F$ and $\varepsilon > 0$, let t_ε in the projective tensor product satisfy $\varphi(t_\varepsilon) = u$ and $\|t_\varepsilon\| \leq (1 + \varepsilon)n^{1/2-1/p} \|u\|$. Expand t_ε as an absolutely convergent series $t_\varepsilon = \sum_{k \geq 1} \lambda_k a_k \otimes b_k$, with $(a_k)_{k \geq 1}$ and $(b_k)_{k \geq 1}$ sequences in the closed unit balls of $L(L_p(\mu), l_2)$ and $L(l_2, F)$, respectively, and $\lambda = (\lambda_k) \in l_1$

a positive sequence with $\|\lambda_k\| \leq (1 + \varepsilon) |t_k|$. Define operators $a: L_p(\mu) \rightarrow (\otimes_{k \geq 1} L_2)_2$ and $b: (\otimes_{k \geq 1} L_2)_2 \rightarrow F$ by $a(h) = (\lambda_k^{1/2} a_k(h))_{k \geq 1}$ and $b((w_k)_{k \geq 1}) = (\lambda_k^{1/2} b_k(w_k))_{k \geq 1}$. Set $w_* = ba$. Since $\|a\|$ and $\|b\|$ are both at most $\|\lambda\|_1^{1/2}$, $\gamma_2(w) \leq \|\lambda\|_1$. Since $\varphi(t_k) = u$, $w_*|E = u$ and combining inequalities shows $\gamma_2(w_*) \leq (1 + \varepsilon)^2 n^{1/2-1/p} \|u\|$. For each $\varepsilon > 0$ choose such an extension w_* . Then $\gamma_2(L_p(\mu), F) = \gamma_2^*(F, L_p(\mu))'$ since F is finite dimensional, so as ε tends to zero the net $(w_*)_{\varepsilon > 0}$ clusters wk^* to some $w \in \gamma_2(L_p(\mu), F)$. Clearly, w is the desired extension of u .

The case $1 < p < 2$ follows by duality. By a theorem of Maurey ([10], or [11], Proposition 9.2) the conclusion of Theorem 8 holds for $1 < p < 2$ if the inequality $i_q(v) \leq n^{1/2-1/q} \pi_q(v)$ is true for every operator v defined on an n -dimensional space F , where $1/p + 1/q = 1$. To see this let $v: F \rightarrow G$ be any map, let C be an injective space containing G isometrically and factor v as

$$F \xrightarrow{a} L_\infty(\mu) \xrightarrow{\beta} L_q(\mu) \xrightarrow{\gamma} C,$$

for μ a probability measure, β inclusion and a and γ operators satisfying $\|a\| \|\gamma\| = \pi_q(v)$. Since $q > 2$, the previously proven part of the theorem gives a projection w of $L_q(\mu)$ onto $\beta\alpha(F)$ with $\|w\| \leq n^{1/2-1/q}$. Since γ maps $\beta\alpha(F)$ into G , $i_q(v) = i_q(\gamma w \beta\alpha) \leq \|w\| \|a\| \|\gamma\|$, which completes the proof.

COROLLARY 4. *If $E \subset L_p(\mu)$ is n -dimensional and $1 < p < \infty$, there is a projection $w: L_p(\mu) \rightarrow E$ with $\|w\| \leq n^{1/2-1/p}$.*

Proof. In Theorem 4 take $E = F$ and u the identity.

The *Banach-Mazur distance* between isomorphic spaces E and F is defined as $d(E, F) = \inf \|u\| \|u^{-1}\|$, with the infimum taken over all isomorphisms $u: E \rightarrow F$. The next corollary answers a question raised in [1].

COROLLARY 5. *If F is n -dimensional and isometric to a quotient of a subspace of $L_p(\mu)$, $1 < p < \infty$, then $d(F, l_2^n) \leq n^{1/2-1/p}$.*

Proof. A subspace of a quotient of $L_p(\mu)$ is isometric to a quotient of a subspace of $L_p(\mu)$, so it suffices to consider the case $p > 2$. If $E \subset L_p(\mu)$ and $u: E \rightarrow F$ is a quotient map, then choosing w as in Theorem 3 shows $\gamma_2(u)' = \gamma_2(u) \leq \gamma_2(w) \leq n^{1/2-1/p}$, so the isometric embedding $u': E' \rightarrow F'$ factors nicely through a Hilbert space.

COROLLARY 6. *Let $v: E \rightarrow F$ be any linear operator and suppose that one of E, F is n -dimensional.*

- (1) For $1 \leq p \leq \infty$, $i_p(v) \leq n^{1/2-1/p} \pi_p(v)$.
- (2) For $2 \leq p \leq \infty$, $\pi_2(v) \leq (\pi/2)^{1/2} n^{1/2-1/p} \pi_p(v)$.
- (3) For $1 \leq q \leq 2$, $i_q(v) \leq (\pi/2)^{1/2} n^{1/2-1/q} \pi_2(v)$.

Proof. The extreme cases $p = 1$ and $p = \infty$ follow from John's Theorem [5], phrased as in [2] to say that $i_\infty(w) \leq \pi_2(w) \leq n^{1/2} \|w\|$ for

each operator w on or into an n -dimensional space; further, in (2) and (3) the constant $(\pi/2)^{1/2}$ can clearly be replaced by 1. To prove (1) and (2) assume $\dim F = n$, let C be an injective space containing F , and write $v = \gamma\beta\alpha$, where μ is some probability measure, $\alpha: E \rightarrow L_\infty(\mu)$, $\beta: L_\infty(\mu) \rightarrow L_p(\mu)$ is inclusion and $\gamma: L_p(\mu) \rightarrow C$ satisfy $\|a\| \|\gamma\| = \pi_p(v)$. Choose an extension $w: L_p(\mu) \rightarrow F$ of $\gamma|\gamma^{-1}(F)$ as in Theorem 3, so that $i_p(v) = i_p(w\beta\alpha) \leq n^{1/2-1/p} \pi_p(v)$. For $p > 2$, Grothendieck's Theorem (cf. [9], or [2] for the constant $(\pi/2)^{1/2}$) shows that $\pi_p(v) = \pi_2(w\beta\alpha) \leq (\pi/2)^{1/2} \gamma_2(w) \gamma_\infty(\beta\alpha) \leq (\pi/2)^{1/2} n^{1/2-1/p} \|a\| \|\gamma\|$, which proves (2). The case $\dim E = n$ may be handled as in the proof of Theorem 3, and (2) and (3) are equivalent by a standard duality argument.

Remarks. (1) The distance estimate of Corollary 5 is best possible since $d(l_p^n, l_2^n) = n^{1/2-1/p}$ [4]. The corollary implies that given $p, q \in (1, +\infty)$, there is an $\alpha < 1$ such that $d(E, F) \leq n^\alpha$ whenever $E \subset L_p$ and $F \subset L_q$ are n -dimensional. It would be of interest to determine the smallest such α . In particular, is it true that $d(E, F) \leq \max d(l_r^n, l_s^n)$, $r, s \in \{p, q, 2\}$?

(2) Given $p, q \in (1, +\infty)$, there is a $\beta > 0$ such that if $E \subset L_p$ and $F \subset L_q$ are any n -dimensional subspaces, there is an operator $u: E \rightarrow F$ with $\|u\| \leq 1$ and $i_1(u) \geq n^\beta$. This follows by composing the isomorphisms given by Corollary 5, since the identity on l_2^n has 1-integral norm n .

(3) The norm estimates of Theorem 3, Corollary 4 and Corollary 6 (1) are asymptotically best possible. Sobczyk [14] has shown that for n a power of 2 and any p , there is an n -dimensional subspace S of L_p such that every projection onto S has norm at least $2^{-1}[(2n)^{1/2-1/p} - 1]$. By Maurey's Theorem [10], if Corollary 6 (1) is true with $n^{1/2-1/p}$ replaced by constant c , then Theorem 3 is also true with constant c . The estimates of Corollary 6 (2) and (3) are also asymptotically best possible. Garling and Gordon [2] show that $i_p(u) = \pi_p(u) = n^{1/p}$, $1 \leq p \leq \infty$, for u the identity on l_∞^n .

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