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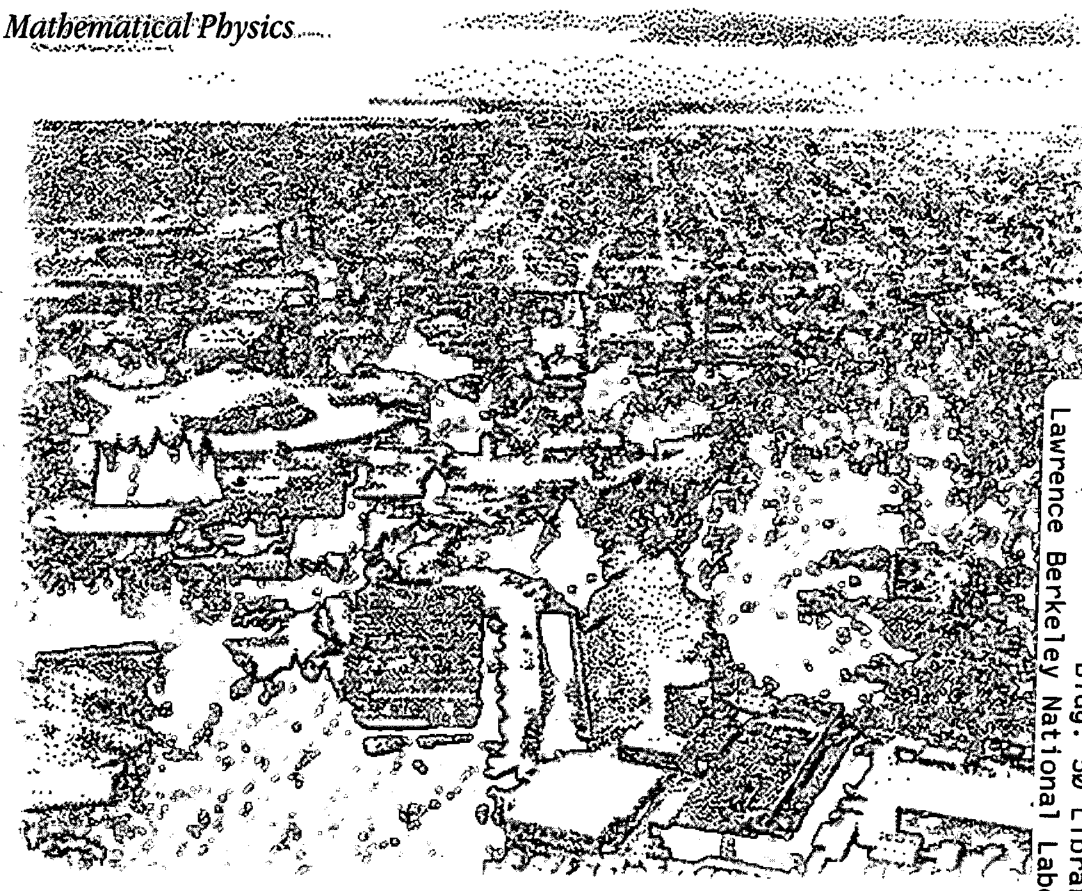

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Finite – dimensional unitary representations of quantum Anti – de Sitter groups at roots of unity [1]

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Abstract

We study unitary irreducible representations of $U_q(SO(2,1))$ and $U_q(SO(2,3))$ for q a root of unity, which are finite – dimensional. Among others, unitary representations corresponding to all classical one – particle representations with integral weights are found for $q = e^{i\pi/M}$ and M large enough. In the "massless" case with spin ≥ 1 in 4 dimensions, they are unitarizable only after factoring out a subspace of "pure gauges", as classically. A truncated associative tensor product describing unitary many – particle representations is defined for $q = e^{i\pi/M}$.

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1 Introduction

In recent years, the development of Noncommutative Geometry has sparked much interest in formulating physics and in particular quantum field theory on quantized, i.e. noncommutative spacetime. The idea is that if there are no more "points" in spacetime, such a theory should be well - behaved in the UV.

Quantum groups [5, 9, 4], although discovered in a different context, can be understood as generalized "symmetries" of certain quantum spaces. Thinking of elementary particles as irreducible unitary representations of the Poincare group, it is natural to try to formulate a quantum field theory based on some quantum Poincare group, i.e. on some quantized spacetime.

There have been many attempts (e.g. [21, 15]) in this direction. One of the difficulties with many versions of a quantum Poincare group comes from the fact that the classical Poincare group is not semisimple. This forbids using the well - developed theory of (semi)simple quantum groups, which is e.g. reviewed in [1, 18, 8]. In this paper, we consider instead the quantum Anti – de Sitter group $SO_q(2,3)$, resp. $SO_q(2,1)$ in 2 dimensions, thus taking advantage of much well- known mathematical machinery. In the classical case, these groups (as opposed to e.g. the de Sitter – group $SO(4,1)$) are known to have positive – energy representations for any spin [7], and e.g. allow supersymmetric extensions [25]. Furthermore, one could argue that the usual choice of flat spacetime is a singular choice, perhaps subject to some mathematical artefacts.

With this motivation, we study unitary representations of $SO_q(2,3) \equiv U_q(SO(2,3))$. Classically, all unitary representations are infinite- dimensional since the group is noncompact. It is well known that at roots of unity, the irreducible representations (irreps) of quantum groups are finite – dimensional. In this paper, we determine if they are unitarizable, and show in particular that for $q = e^{i\pi/M}$, all the irreps with positive energy and integral weights are unitarizable, as long as spin and rest energy are within some (q – dependent, large) limits. There is an intrinsic high – energy cutoff, and only finitely many such "physical" representations exist for given q. At low energies and for q close enough to 1, the structure is the same as in the classical case. Furthermore unitary representations exist only at roots of unity (if q is a phase). For generic roots of unity, their weights are non – integral. Analogous results are found for $SO_q(2,1)$. In general, there is a cell – like structure of unitary representations in weight space.

In the "massless" case, the naive representations with spin ≥ 1 are reducible and contain a null – subspace corresponding to "pure gauge" states. It is shown that they can be consistently factored out to obtain unitary representations with only the physical degrees of freedom ("helicities"), as in the classical case [7].

We also show that the class of "physical" (unitarizable) representations is closed under a new kind of associative truncated tensor product for $q = e^{i\pi/M}$, i.e. there exists a straightforward way to obtain many- particle representations.

Besides being very encouraging from the point of view of quantum field theory, this shows again the markedly different properties of quantum groups at roots of unity from the case of generic q and $q = 1$. The results are clearly not restricted to the groups considered here and should be of interest on purely mathematical grounds as well. We develop some methods to investigate the structure of representations of quantum groups at roots of unity and determine the structure of a large class of representations of $SO_q(2,3)$. Throughout this paper, $SO_q(2,3)$ will be equipped with a non – standard Hopf – algebra star structure.

The idea to find a quantum Poincare group from $SO_q(2,3)$ is not new: Already in [15], the so – called κ – Poincare group was based on $SO_q(3,2)$ by a contraction. This contraction however essentially takes $q \rightarrow 1$ (in a nontrivial way) and destroys the properties of the representations which we emphasize, in particular the finite - dimensionality.

Although it is not considered here, there exists a (space of functions on) quantum Anti – de Sitter space on which $SO_q(2,1)$ resp. $SO_q(2,3)$ operates, with an intrinsic mass parameter $m^2 = i(q - q^{-1})/R^2$ where R is the "radius" of Anti – de Sitter space (and the usual Minkowski signature for $q = 1$).

This paper is organized as follows: In section 2, we investigate the unitary representations of $SO_q(2,1)$, and define a truncated tensor product. In section 3, the most important facts about quantized universal enveloping algebras of higher rank are reviewed. In section 4, we consider $SO_q(2,3)$, determine the structure of the relevant irreducible representations (which are finite – dimensional) and investigate which ones are unitarizable with respect to $SO_q(5)$ or $SO_q(2,3)$. The truncated tensor product is generalized to the latter case. Finally we conclude and look at possible further developments.

2 Unitary representations of $SO_q(2,1)$

We first consider the simplest case of $SO_q(2,1) \equiv U_q(SO(2,1))$, which is a real form of $\mathcal{U} \equiv U_q(Sl(2,\mathbb{C}))$, the Hopf – algebra defined by [5, 9]

$$\begin{aligned} [H, X^{\pm}] &= \pm 2X^{\pm}, \quad [X^+, X^-] = [H] \\ \Delta(H) &= H \otimes 1 + 1 \otimes H, \\ \Delta(X^{\pm}) &= X^{\pm} \otimes q^{H/2} + q^{-H/2} \otimes X^{\pm}, \\ S(X^+) &= -qX^+, \quad S(X^-) = -q^{-1}X^-, \quad S(H) = -H \\ \varepsilon(X^{\pm}) &= \varepsilon(H) = 0 \end{aligned} \tag{1}$$

where $[n] \equiv [n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$. To talk about a real form of $SL_q(2, C)$, one has to impose a reality condition, i.e. a star – structure, and there may be several possibilities. Since we want the algebra to be implemented by a unitary representation on a Hilbert space, the star – operation should be an involution, an antilinear antihomomorphism of the algebra. Furthermore, we will see that to get finite – dimensional unitary representations, q must be a root of unity, so $|q| = 1$. Only at roots of unity the representation theory of quantum groups differs essentially from the classical case, and new features such as finite dimensional unitary representations of noncompact groups can appear. This suggests the following star – structure corresponding to $SO_q(2, 1)$:

$$\overline{H} = H, \quad \overline{X^+} = -X^- \tag{2}$$

whis is simply

$$\overline{x} = e^{-i\pi H/2}\theta(x)e^{i\pi H/2} \tag{3}$$

where θ is the usual Cartan – Weyl involution, for $x \in \mathcal{U}$. Since q is a phase, $\overline{q} = q^{-1}$, and

$$\overline{\Delta(x)} = \Delta(\overline{x}) \tag{4}$$

provided

$$\overline{a \otimes b} = \overline{b} \otimes \overline{a}. \tag{5}$$

Then $\overline{S(x)} = S(\overline{x})$, and $\overline{x}$ is a non – standard Hopf algebra star – structure. In particular (5) is different from the usual definition. It is however perfectly consistent as discussed in [19], and will be of no concern here; we plan to return to this issue in a future paper.

The irreps of $\mathcal{U}$ at roots of unity are well – known (see e.g. [10], whose notations we largely follow), and we list some facts. Let

$$q = e^{2\pi i n/m} \tag{6}$$

for positive relatively prime integers m, n and define $M = m$ if m is odd, and $M = m/2$ if m is even. Then it is consistent and appropriate in our context to set

$$(X^{\pm})^M = 0 \tag{7}$$

(if one uses q^H instead of H, then $(X^{\pm})^M$ is central). All finite - dimensional irreps are highest weight representations with dimension $d \leq M$. There are two types of irreps:

- $V_{d,z} = \{e^j_m; \quad j = (d-1) + \frac{m}{2n}z, \quad m = j, j-2, ..., -(d-1) + \frac{m}{2n}z\}$ with dimension d, for any $1 \leq d \leq M$ and $z \in \mathbb{Z}$, where $He^j_m = me^j_m$

- I_z^1 with dimension M and highest weight $(M-1)+\frac{m}{2n}z$, for $z \in \mathbb{C} \setminus \{\mathbb{Z} + \frac{2n}{m}r, 1 \leq r \leq M-1\}$.

Note that in the second type, $z \in \mathbb{Z}$ is allowed, in which case we will write $V_{M,z} \equiv I_z^1$ for convenience. We will concentrate on the $V_{d,z}$ – representations from now on. Furthermore, the fusion rules at roots of unity state that $V_{d,z} \otimes V_{d',z'}$ decomposes into $\oplus_{d''} V_{d'',z+z'} \bigoplus_p I_{z+z'}^p$ where I_z^p are the well - known reducible, but indecomposable representations of dimension $2M$, see figure 1 and [10]. If q is *not* a root of unity, then the universal $\mathcal{R} \in \mathcal{U} \otimes \mathcal{U}$ given by

$$\mathcal{R} = q^{\frac{1}{2}H \otimes H} \sum_{l=0}^{\infty} q^{-\frac{1}{2}l(l+1)} \frac{(q-q^{-1})^l}{[l]!} q^{lH/2}(X^+)^l \otimes q^{-lH/2}(X^-)^l \tag{8}$$

defines the quasitriangular structure of $\mathcal{U}$. It satisfies e.g.

$$\sigma(\Delta(u)) = \mathcal{R}\Delta(u)\mathcal{R}^{-1}, \quad u \in \mathcal{U} \tag{9}$$

where $\sigma(a \otimes b) = b \otimes a$. We will only consider representations with dimension $\leq M$; then $\mathcal{R}$ restricted to such representations is well defined for roots of unity as well, since the sum in (8) only goes up to $(M-1)$. Furthermore

$$\overline{\mathcal{R}} = (\mathcal{R})^{-1}. \tag{10}$$

To see this, (3) is useful.

Let us consider a Hermitian inner product (u, v) for $u, v \in V_{d,z}$. A hermitian inner product satisfies $(u, \lambda v) = \lambda(u, v) = (\overline{\lambda} u, v)$ for $\lambda \in \mathbb{C}$, $\overline{(u, v)} = (v, u)$ and

$$(u, x \cdot v) = (\overline{x} \cdot u, v), \tag{11}$$

i.e. $\overline{x}$ is the adjoint of x. If (,) is also positive – definite, we have a unitary representation.

Proposition 2.1 *The representations $V_{d,z}$ are unitarizable w.r.t $SO_q(2,1)$ if and only if*

$$(-1)^{z+1} \sin(2\pi nk/m) \sin(2\pi n(d-k)/m) > 0 \tag{12}$$

for all $k = 1, ..., (d-1)$.

For $d-1 < \frac{m}{2n}$, this holds precisely if z is odd. For $d-1 \geq \frac{m}{2n}$, it holds for isolated values of d only, i.e. if it holds for d, then it (generally) does not *hold for $d \pm 1, d \pm 2, \ldots$.*

The representations $V_{d,z}$ are unitarizable w.r.t $SU_q(2)$ if z is even and $d-1 < \frac{m}{2n}$.

Proof Let e^j_m be a basis of $V_{d,z}$ with highest weight j. After a straightforward calculation, invariance implies

$$\left((X^-)^k \cdot e^j_j, (X^-)^k \cdot e^j_j\right) = (-1)^k [k]![j][j-1]...[j-k+1] \left(e^j_j, e^j_j\right) \tag{13}$$

for $k = 1, ..., (d-1)$, where $[n]! = [1][2]...[n]$. Therefore we can have a positive definite inner product $(e^j_m, e^j_n) = \delta_{m,n}$ if and only if $a_k \equiv (-1)^k [k]![j][j-1]...[j-k+1]$ is a positive number for all $k = 1, ..., (d-1)$, in which case $e^j_{j-2k} = (a_k)^{-1/2} (X^-)^k \cdot e^j_j$.

Now $a_k = -[k][j-k+1] a_{k-1}$, and

$$\begin{aligned} -[k][j-k+1] &= -[k][d-k+\frac{m}{2n}z] = -[k][d-k]e^{i\pi z} & (14) \\ &= (-1)^{z+1} \sin(2\pi nk/m) \sin(2\pi n(d-k)/m) \frac{1}{sin(2\pi n/m)^2}, & (15) \end{aligned}$$

since z is an integer. Then the Proposition follows. The compact case is known [10]. □

In particular, all of them are finite - dimensional, and clearly if q is not a root of unity, none of the representations are unitarizable.

We will be particularly interested in the case of (half)integer representations of type $V_{d,z}$ and $n = 1, m$ even, for reasons to be discussed below. Then $d - 1 < \frac{m}{2n} = M$ always holds, and *the $V_{d,z}$ are unitarizable if and only if z is odd.* These representations are centered around Mz, with dimension $\leq M$.

Let us compare this with the classical case. For the Anti – de Sitter group $SO(2,1)$, H is nothing but the energy. At $q = 1$, the unitary irreps of $SO(2,1)$ are lowest - weight representations with lowest weight $j > 0$ resp. highest weight representations with highest weight $j < 0$. For any given such lowest resp. highest weight we can now find a *finite - dimensional* unitary representation with the same lowest resp. highest weight, provided M is large enough (we only consider (half)integer j here). These are unitary representations which for low energies look like the classical one - particle representations, but have an intrinsic high - energy cutoff if $q \neq 1$, which goes to infinity as $q \to 1$. The same will be true in the 4 – dimensional case.

So far we only considered what could be called one - particle representations. To talk about many – particle representations, there should be a tensor product of 2 or more such irreps, which gives a unitary representation as well and agrees with the classical case for low energies.

$\mathcal{U}$ being a Hopf – algebra, there is a natural notion of a tensor product of two representations, given by the coproduct Δ. However, it is not unitary a priori. As mentionned above,

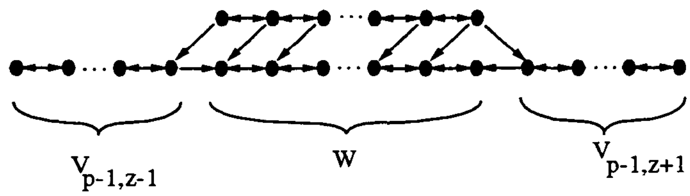

Figure 1: Indecomposable representation I_z^p

the tensor product of two irreps of type $V_{d,z}$ is

$$V_{d,z} \otimes V_{d',z'} = \oplus_{d''} V_{d'',z+z'} \bigoplus_{p=r,r+2,\ldots}^{d+d'-M} I^p_{z+z'} \tag{16}$$

where $r = 1$ if $d + d' - M$ is odd or else $r = 2$, and I_z^p is a indecomposable representation of dimenson $2M$ whose structure is shown in figure 1. The arrows indicate the rising resp. lowering operators.

In the case of $SU_q(2)$, one usually defines a truncated tensor product $\hat{\otimes}$ by omitting all I_z^p representations [19]. Then the remaining reps are unitary w.r.t. $SU_q(2)$; $\hat{\otimes}$ is associative only from the representation theory point of view [19].

This is not the right thing to do for $SO_q(2,1)$. Let $n = 1$ and m even, and consider e.g. $V_{M-1,1} \otimes V_{M-1,1}$. Both factors have lowest energy $H = 2$, and the tensor product of the two corresponding *classical* representations is the sum of representations with lowest weights 4,6,8,.... In our case, these weights are in the I_z^p representations, while the $V_{d'',z''}$ have $H \geq M \to \infty$! So we have to keep the I_z^p's and throw away the $V_{d'',z''}$'s in (16). The I_z^p's are not unitarizable, however. To get a unitary tensor product, note that as a vector space,

$$I_z^p = V_{p-1,z-1} \oplus W \oplus V_{p-1,z+1} \tag{17}$$

where

$$W = V_{M-p+1,z} \oplus V_{M-p+1,z} \tag{18}$$

as vector space, and $(X^-X^+) \cdot e_h = 0$ for e_h the highest weight vector of $V_{p-1,z-1}$ and similarly $(X^+X^-) \cdot e_l = 0$ for e_l the lowest - weight vector of $V_{p-1,z+1}$ (see figure 1). It is therefore consistent to put $W = 0$ (one cannot factor out W, since it is not a submodule; instead one has to take the subspace of the V's). The remaining two V - representations are unitarizable provided $n = 1$ and m is even, and one can keep both (notice the similarity with band structures in solid - state physics), or for simplicity keep the low - energy part only, in view of the physical application we have in mind. We therefore define a truncated tensor product as

Definition 2.2 *For $n = 1$ and even m,*

$$V_{d,z}\tilde{\otimes}V_{d',z'} \equiv \bigoplus_{\tilde{d}=r,r+2,...}^{d+d'-M} V_{\tilde{d},z+z'-1} \tag{19}$$

This can be stated as follows: Notice that any representation naturally decomposes as a vector space into sums of $V_{d,z}$'s, cp. (18); the definition of $\tilde{\otimes}$ simply means that only the smallest value of z in this decomposition is kept, which is the submodule of irreps with lowest weights $\leq \frac{m}{2n}(z + z' - 1)$. (Incidentally, z is the eigenvalue of D_3 in the *classical* $su(2)$ – algebra $\{D^{\pm} = \frac{(X^{\pm})^M}{[M]!}, 2D_3 = [D^+, D^-]\}$, where $\frac{(X^{\pm})^M}{[M]!}$ is understood by some limes procedure). With this in mind, it is obvious that $\tilde{\otimes}$ is associative: both in $(V_1\tilde{\otimes}V_2)\tilde{\otimes}V_3$ and in $V_1\tilde{\otimes}(V_2\tilde{\otimes}V_3)$, the result is simply the V's with minimal z, which is the *same* space, because the ordinary tensor product is associative and Δ is coassociative. This is in contrast with the "ordinary" truncated tensor product $\hat{\otimes}$ [19]. Of course, one could give a similar definition for negative – energy representations.

$V_{d,z}\tilde{\otimes}V_{d',z'}$ is unitarizable if all the V 's on the rhs of (19) are unitarizable. This is certainly true if $n = 1$ and m is even. In all other cases, there are no terms on the rhs of (19) if the factors on the lhs are unitarizable, since no I_z^p – type representations are generated (they are too large). This is the reason why we concentrate on this case, and furthermore on $z = z' = 1$ which corresponds to low – energy representations. Then $\tilde{\otimes}$ defines a two – particle Hilbert space with the correct classical limit. So

Proposition 2.3 *$\tilde{\otimes}$ is associative, and $V_{d,1}\tilde{\otimes}V_{d',1}$ is unitarizable.*

How the inner product can be induced from the single – particle Hilbert spaces will be explained in a future paper.

3 The quantum group $U_q(SO(2,3))$

In order to generalize the above results to the 4 – dimensional case, one has to use the general machinery of quantum groups, which is briefly reviewed (cp. e.g. [1]): Let $q \in C$ and $A_{ij} = 2\frac{(\alpha_i,\alpha_j)}{(\alpha_j,\alpha_j)}$ be the Cartan matrix of a classical simple Lie algebra g of rank r, where $(,)$ is the Killing metric and $\{\alpha_i, \ i = 1,...,r\}$ are the simple roots. Then the *quantized universal enveloping algebra* $U_q(g)$ is the Hopf - algebra generated by the elements $\{X_i^\pm, H_i; \quad i = 1, ..., r\}$ and relations [5, 9, 4]

$$\begin{aligned}
[H_i, H_j] &= 0 \\
\left[H_i, X_j^\pm\right] &= \pm A_{ji} X_j^\pm, \\
\left[X_i^+, X_j^-\right] &= \delta_{i,j} \frac{q^{d_i H_i} - q^{-d_i H_i}}{q^{d_i} - q^{-d_i}} = \delta_{i,j} [H_i]_{q_i} \\
\sum_{k=0}^{1-A_{ji}} & \left[\begin{array}{c} 1 - A_{ji} \\ k \end{array}\right]_{q_i} (X_i^\pm)^k X_j^\pm (X_i^\pm)^{1-A_{ji}-k} = 0, \quad i \neq j
\end{aligned} \tag{20}$$

where $d_i = (\alpha_i, \alpha_i)/2, \quad q_i = q^{d_i}, \quad [n]_{q_i} = \frac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}}$ and

$$\left[\begin{array}{c} n \\ m \end{array}\right]_{q_i} = \frac{[n]_{q_i}!}{[m]_{q_i}![n-m]_{q_i}!}. \tag{21}$$

The comultiplication is given by

$$\begin{aligned}
\Delta(H_i) &= H_i \otimes 1 + 1 \otimes H_i \\
\Delta(X_i^\pm) &= X_i^\pm \otimes q^{d_i H_i/2} + q^{-d_i H_i/2} \otimes X_i^\pm.
\end{aligned} \tag{22}$$

Antipode and counit are

$$\begin{aligned}
S(H_i) &= -H_i, \\
S(X_i^+) &= -q^{d_i} X_i^+, \quad S(X_i^-) = -q^{-d_i} X_i^-, \\
\varepsilon(H_i) &= \varepsilon(X_i^\pm) = 0.
\end{aligned} \tag{23}$$

(we use the conventions of [11], which differ slightly from e.g. [1]).

For $\mathcal{U} \equiv U_q(SO(5, C))$, $r = 2$ and

$$A_{ij} = \left(\begin{array}{cc} 2 & -2 \\ -1 & 2 \end{array}\right), \qquad (\alpha_i, \alpha_j) = \left(\begin{array}{cc} 2 & -1 \\ -1 & 1 \end{array}\right), \tag{24}$$

so $d_1 = 1$, $d_2 = 1/2$, to have the standard physics normalization (a rescaling of (,) can be absorbed by a redefinition of q). The weight diagrams of the vector and the spinor representations are given in figure 2 for illustration. The Weyl - vector is $\rho = \frac{1}{2}\sum_{\alpha>0} \alpha = \frac{3}{2}\alpha_1 + 2\alpha_2$.

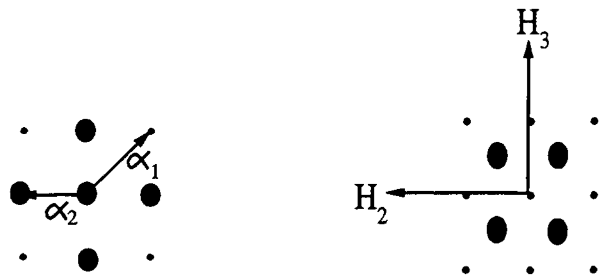

Figure 2: vector and spinor representations of $SO(2,3)$

The possible reality structures on $\mathcal{U}$ have been investigated in [14]. As in section 2, in order to obtain finite - dimensional unitary representations, q must be a root of unity. Furthermore, on physical grounds we insist upon having positive - energy representations; already in the classical case, that rules out e.g. $SO(4,1)$, cp. the discussion in [7]. It appears that then there is only one possibility, namely

$$\begin{aligned} \overline{H_i} &= H_i, \quad \overline{X_1^+} = -X_1^-, \quad \overline{X_2^+} = X_2^-, && (25) \\ \overline{a \otimes b} &= \overline{b} \otimes \overline{a}, \\ \overline{\Delta(u)} &= \Delta(\overline{u}), \quad \overline{S(u)} = S(\overline{u}),. && (26) \end{aligned}$$

which corresonds to the Anti - de Sitter group $SO_q(2,3) \equiv U_q(SO(2,3))$. Again with $E \equiv d_1 H_1 + d_2 H_2$, $(-1)^E \overline{x} (-1)^E = \theta(x)$ where θ is the usual Cartan - Weyl involution corresponding to $SO_q(5)$; note again that $\theta(q) = q^{-1}$.

Although we will not use it in the present paper, this algebra has the very important property of being *quasitriangular*, i.e. there exists a universal $\mathcal{R} \in \mathcal{U} \otimes \mathcal{U}$. It satisfies

$$\overline{\mathcal{R}} = (\mathcal{R})^{-1}, \tag{27}$$

which can be seen e.g. from uniqueness theorems, cp. [12, 1]. In the mathematical literature, usually a rational version of the above algebra, i.e. using $q^{d_i H_i}$ instead of H_i is considered. Since we are only interested in specific representations, we prefer to work with H_i.

Often the following generators are more useful:

$$h_i = d_i H_i, \quad e_{\pm i} = \sqrt{[d_i]} X_i^{\pm}, \tag{28}$$

so that

$$\begin{aligned} [h_i, e_{\pm j}] &= \pm(\alpha_i, \alpha_j) e_{\pm j}, \\ [e_i, e_{-j}] &= \delta_{i,j} [h_i]. && (29) \end{aligned}$$

In the present case, i.e. $h_1 = H_1, h_2 = \frac{1}{2}H_2$, $e_{\pm1} = X_1^\pm$ and $e_{\pm2} = \sqrt{[\frac{1}{2}]}X_2^\pm$.

So far we only have the generators corresponding to simple roots. A Cartan - Weyl basis corresponding to all roots can be obtained e.g. using the braid group action introduced by Lusztig [16], (see also [1, 8]) resp. the quantum Weyl group [11, 22, 13, 1]. If $\omega = \tau_{i_1}...\tau_{i_N}$ is a reduced expression for the longest element of the Weyl group where τ_i is the reflection along α_i, then $\{\alpha_{i_1}, \tau_{i_1}\alpha_{i_2}, ..., \tau_{i_1}...\tau_{i_{N-1}}\alpha_{i_N}\}$ is an ordered set of positive roots. We will use $\omega = \tau_1\tau_2\tau_1\tau_2$ and denote them $\beta_1 = \alpha_1, \beta_2 = \alpha_2, \beta_3 = \alpha_1 + \alpha_2, \beta_4 = \alpha_1 + 2\alpha_2$. A Cartan - Weyl basis of root vectors of $\mathcal{U}$ can then be defined as $\{e_{\pm1}, e_{\pm3}, e_{\pm4}, e_{\pm2}\} = \{e_{\pm1}, T_1e_{\pm2}, T_1T_2e_{\pm1}, T_1T_2T_1e_{\pm2}\}$ and similarly for the h_i 's, where the T_i represent the braid group on $\mathcal{U}$ [16]:

$$\begin{aligned} T_i(H_j) &= H_j - A_{ij}H_i, \quad T_iX_i^+ = -X_i^- q_i^{H_i}, \\ T_i(X_j^+) &= \sum_{r=0}^{-A_{ji}} (-1)^{r-A_{ji}} q_i^{-r} (X_i^+)^{-A_{ji}-r} X_j^+ (X_i^+)^r, \end{aligned} \tag{30}$$

and $T_i(\theta(x)) = \theta(T_i(x))$. We find

$$\begin{aligned} e_3 &= q^{-1}e_2e_1 - e_1e_2, \quad e_{-3} = qe_{-1}e_{-2} - e_{-2}e_{-1}, \quad h_3 = h_1 + h_2 \\ e_4 &= e_2e_3 - e_3e_2, \quad e_{-4} = e_{-3}e_{-2} - e_{-2}e_{-3}, \quad h_4 = h_1 + 2h_2. \end{aligned} \tag{31}$$

Similarly one defines the root vectors $X_{\beta_l}^\pm$. This can be used to obtain a Poincare - Birkhoff - Witt basis of $\mathcal{U} = \mathcal{U}^-\mathcal{U}^0U^+$ where $U^\pm$ is generated by the $X_i^\pm$ and $\mathcal{U}^0$ by the H_i: for $\underline{k} = (k_1, ..., k_N)$ where N is the number of positive roots, let $X_{\underline{k}}^+ = X_{\beta_1}^{+k_1}...X_{\beta_N}^{+k_N}$. Then the $X_{\underline{k}}^\pm$ form a P.B.W. basis of $\mathcal{U}^+$, and similarly for $\mathcal{U}^-$ [17] (assuming $q^4 \neq 1$).

Up to a trivial automorphism, (31) agrees with the basis used in [15]. The identification of the usual generators of the Poincare group has also been given there and will not be repeated here, except for pointing out that h_3 is the energy and h_2 is a component of angular momentum. All of the above form $SL_{\tilde{q}}(2, C)$ subalgebras with appropriate $\tilde{q}$ (but not as coalgebras), because the T_i's are algebra homomorphisms. The reality structure is

$$\overline{e_1} = -e_{-1}, \quad \overline{e_2} = e_{-2}, \quad \overline{e_3} = -e_{-3}, \quad \overline{e_4} = -e_{-4}. \tag{32}$$

So $\{e_{\pm2}, h_2\}$ is a $SU_{\tilde{q}}(2)$ algebra, and the other three $\{e_{\pm\alpha}, h_\alpha\}$ are noncompact $SO_{\tilde{q}}(2,1)$ algebras, as discussed in section 2.

Casimir elements of $\mathcal{U}$ are

$$c_l = Tr(q^{-2\rho}(L^+SL^-)^l) \tag{33}$$

for $l = 1,..,\mathrm{rank}(g)$, with $L^+ = (1 \otimes \pi)\mathcal{R}$ and $SL^- = (\pi \otimes 1)\mathcal{R}$ where π is the vector representation, and $\rho = \frac{3}{2}h_1 + 2h_2$ corresponds to the Weyl element. The c_k are central in $\mathcal{U}$, and $q^{2\rho}$ generates the square of the antipode: $q^{2\rho}xq^{-2\rho} = S^2(x)$ for $x \in \mathcal{U}$.

4 Unitary representations of $SO_q(2,3)$ and $SO_q(5)$

In this section, we consider representations of $SO_q(2,3)$ and show that for suitable roots of unity q, the irreducible positive resp. negative energy representations are again unitarizable, if the highest resp. lowest weight lies in some "bands" in weight space. Their structure for low energies is exactly as in the classical case including the appearance of "pure gauge" subspaces for spin ≥ 1 in the "massless" case, which have to be factored out to obtain the physical, unitary representations. At high energies, there is an intrinsic cutoff.

Most facts about representations of quantum groups we will use can be found e.g. in [2], see also [16]. It is useful to consider the Verma modules $M(\lambda)$ for a highest weight λ, which is the (unique) $\mathcal{U}$ - module having a highest – weight vector v_λ such that

$$\mathcal{U}^+ v_\lambda = 0, \quad H_i v_\lambda = \frac{(\lambda, \alpha_i)}{d_i} v_\lambda \tag{34}$$

where $\mathcal{U}^+$ is generated by the X^+, and the vectors $X^-_{\underline{k}} v_\lambda$ form a P.B.W. basis of $M(\lambda)$. On a Verma module, on can define a unique hermitian inner product (,) satisfying $(v_\lambda, v_\lambda) = 1$ and $(u, x \cdot v) = (\theta(x) \cdot u, v)$ for $x \in \mathcal{U}$ as in section 2 [2]; θ is the Cartan – Weyl involution corresponding to $SO_q(5)$.

The irreducible highest weight representations can be obtained from the corresponding Verma module by factoring out all submodules generated by the highest weight states in the Verma module. All these submodules are null spaces w.r.t. the above inner product, i.e. they are orthogonal to any state in $M(\lambda)$. Therefore one can consistently factor them out, and obtain a hermitian inner product on the quotient space. To see that they are null, let $w_\mu \in M(\lambda)$ be a highest weight vector, so $X^+ w_\mu = 0$ for every rising operator X^+. Now $M(\lambda)$ is generated by the vectors $v = X^-_{\underline{k}} v_\lambda$, $\underline{k} \in \mathbb{N}^N$ (cp. section 3). Therefore $(w_\mu, v) = (w_\mu, X^-_{\underline{k}} v_\lambda) = (\theta(X^-_{\underline{k}}) w_\mu, v_\lambda) = 0$. Again by hermiticity, it follows that all the descendants of w_μ are null too.

The following discussion until Theorem 4.5 is technical and may be skipped upon first reading. Let $Q = \sum \mathbb{Z}\alpha_i$ be the root lattice and $Q^+ = \sum \mathbb{Z}_+\alpha_i$ where $\mathbb{Z}_+ = \{0, 1, 2, ...\}$. We will write

$$\lambda \succ \mu \quad \text{if} \quad \lambda - \mu \in Q^+. \tag{35}$$

For $\eta \in Q$, denote [2]

$$\mathrm{Par}(\eta) = \{\underline{k} \in \mathbb{Z}_+^N; \quad \sum k_i \beta_i = \eta\}. \tag{36}$$

Let $M(\lambda)_\eta$ be the weight space with weight $\lambda - \eta$ in $M(\lambda)$. Then its dimension is given by $|\mathrm{Par}(\eta)|$. If $M(\lambda)$ contains a highest weight vector with weight σ, then the multiplicity of the weight space $(M(\lambda)/M(\sigma))_\eta$ is given by $|\mathrm{Par}(\eta)| - |\mathrm{Par}(\eta + \sigma - \lambda)|$, and so on. This way, the structure of the highest weight irreps can be determined once all the highest weight vectors in $M(\lambda)$ are known.

Our main tool to find them will be a remarkable formula by De Concini and Kac for $\det(M(\lambda)_\eta)$, the determinant of the inner product matrix of $M(\lambda)_\eta$. Before stating it, we point out its use for determining irreps:

Lemma 4.1 *Let v_λ be the highest weight vector in an irreducible highest weight representation $V(\lambda)$. If $(v_\lambda, v_\lambda) \neq 0$, then*

$$det(V(\lambda)_\eta) \neq 0 \tag{37}$$

for every weight space with weight $\lambda - \eta$ in $V(\lambda)$.

Proof Assume to the contrary that there is a vector v_μ which is orthogonal to all other vectors of the same weight (and therefore to all vectors of any weight). Because $V(\lambda)$ is assumed to be an irrep, there exists an i with $X_i^+ v_\mu \neq 0$. But then $(X_i^+ v_\mu, w) = (v_\mu, X_i^- w) = 0$ for any $w \in V(\lambda)$. This implies that $X_i^+ v_\mu \neq 0$ is orthogonal to *any* vectors in $V(\lambda)$. Repeating this argument, we come to the conclusion that the highest weight vector is null, which is a contradiction. □

Now we state the result of De Concini and Kac [2]:

$$\det(M(\lambda)_\eta) = \prod_{\beta \in R^+} \prod_{m \in \mathbb{N}} \left([m]_{d_\beta} \frac{q^{(\lambda+\rho-m\beta/2,\beta)} - q^{-(\lambda+\rho-m\beta/2,\beta)}}{q^{d_\beta} - q^{-d_\beta}} \right)^{|\mathrm{Par}(\eta - m\beta)|} \tag{38}$$

in a P.B.W. basis, where R^+ denotes the positive roots (cp. section 3), $d_\beta = (\beta, \beta)/2$, and $m = m_\beta$ really.

To get some insight, notice first of all that due to $|\mathrm{Par}(\eta - m\beta)|$ in the exponent, the product is finite. Now for some positive root β, let m_β be the smallest integer such that $D(\lambda)_{m_\beta,\beta} \equiv \left([m_\beta]_{d_\beta} \frac{q^{(\lambda+\rho-m_\beta\beta/2,\beta)} - q^{-(\lambda+\rho-m_\beta\beta/2,\beta)}}{q^{d_\beta} - q^{-d_\beta}} \right) = 0$ resp. $m_\beta = \infty$ if there is no such integer, and consider the weight space at weight $\lambda - m_\beta\beta$, i.e. $\eta_\beta = m_\beta\beta$. Then $|\mathrm{Par}(\eta_\beta - m_\beta\beta)| = 1$ and $\det(M(\lambda)_{\eta_\beta})$ is zero, so there is a highest weight vector w_β with weight $\lambda - \eta_\beta$. It generates a submodule which at weight $\lambda - \eta$ has dimension $|\mathrm{Par}(\eta - m_\beta\beta)|$. This is the origin of the exponent, and will be used to find the highest weight vectors in $M(\lambda)$. However these submodules may not be independent, i.e. they may contain common highest - weight vectors. To handle this complication, we introduce the following concept:

Definition 4.2 *A highest weight vector in $M(\lambda)$ is called* simple *if it is the only null vector at that particular weight.*

A weight space of weight μ in $M(\lambda)$ is called simple *if all highest weight vectors with weight $\succ \mu$ in the submodules generated by (and including) w_β are simple, for all $\beta \in R^+$.*

Notice that if two submodules are not linearly independent, they contain a common highest weight vector.

Thus we have identified null vectors w_β and their descendants. We want to know if there are others. To keep track of things, deform λ slightly to λ', so that $D(\lambda')_{m_\beta,\beta} \neq 0$ (there are no restrictions on λ). Then the following holds:

Proposition 4.3 *Assume that $M(\lambda)_\eta$ is simple. If it contains other null states besides the ones generated by the w_β's, then*

$$\left| \frac{\det(M(\lambda'))_\eta}{\prod_\beta (D(\lambda')_{m_\beta,\beta})^{|Par(\eta - m_\beta \beta)|}} \right| \rightarrow 0 \quad as \quad \lambda' \rightarrow \lambda. \tag{39}$$

Proof Define η_i, η_{ij} etc. such that the (simple) highest weight states are at weight $\lambda - \eta_i$ for w_{β_i}, $\lambda - \eta_{ij}$ if it is descendant of both w_{β_i} and w_{β_j}, etc. (there will be no η's with more than two indices; in particular, $\eta_i = m_{\beta_i}\beta_i$). At weight $\lambda' - \eta_i$ resp. $\lambda' - \eta_{ij}$ etc. choose an orthogonal basis $\{w_i, u_{i,l}\}$ resp. $\{w_{ij}, u_{ij,l}\}$ etc. with a unitary transformation matrix to the P.B.W. basis. From (38) and simplicity it follows that as $\lambda' \rightarrow \lambda$, the eigenvalues of the metric corresponding to the u's approach some nonzero limit, while its absolute value for w_i is $\leq c|D_i| \equiv c|D(\lambda')_{m_{\beta_i},\beta_i}|$, for w_{ij} it is $\leq c|D_i D_j|$ etc., with some (bounded) constant c.

For a positive root η, consider a basis $\{e_k\} = \{w_{ij,l}, w_{i,l}, v_l\}$ of $M(\lambda')_\eta$ of the form $w_{ij,l} = X_{\underline{k}}^- w_{ij}$, $w_{i,l} = X_{\underline{k}}^- w_i$ and $v_l = X_{\underline{k}}^- v_{\lambda'}$ with (suppressed) coefficients independent of λ', such that as $\lambda' \rightarrow \lambda$, the $w_{...,l}$'s become null, while the v_l's have finite norm *unless* they are the additional null states we want to find. Notice that the number of $w_{ij,l}$'s is $|\text{Par}(\eta - \eta_{ij})|$, while the number of $w_{i,l}$'s is $|\text{Par}(\eta - \eta_i)| - |\text{Par}(\eta - \eta_{ij})|$ if there is a w_{ij}, etc. The transition matrix T to a P.B.W. basis is bounded, because it is well - defined at $\lambda' = \lambda$. Then

$$\det(M(\lambda')_\eta) = \sum (e_1, e_{i_1})...(e_n, e_{i_n}) \varepsilon^{i_1 ... i_n} |\det T|^2. \tag{40}$$

We will show below that

$$|(w_{i,l}, e_k)| \leq c'|D_i|, \quad |(w_{ij,l}, e_k)| \leq c'|D_i D_j| \tag{41}$$

etc. for some constant c', if λ' is sufficiently close to λ. But then the limit in (39) can be nonzero only if there is a term in (40) where all the $w_{ij,l}$'s are paired, all the $w_{i,l}$'s are paired, etc., and there are no v_j's which are null.

It remains to show (41). Now $(w_{i,l}, e_k) = (X_{\underline{k}}^- w_i, e_k) = (w_i, X_{\underline{k}}^+ e_k)$, where again all coefficients are suppressed. However, $X_{\underline{k}}^+ e_k = a w_i + \sum_l b_l u_{i,l}$ with bounded coefficients a, b_l. Therefore $|(w_{i,l}, e_k)| = |a(w_i, w_i)| \leq c|a\bar{D}_i|$ as $\lambda' \to \lambda$; similarly for $w_{ij,l}$, and (41) follows. □

From now on, let $q = e^{2\pi i n/m}$ and M as is section 2. Then there exist remarkable non-trivial one – dimensional representations v_{λ_0} with weights $\lambda_0 = \sum \frac{m}{2n} k_i \alpha_i$ for integers k_i. By tensoring any representation with v_{λ_0}, one obtains another representation with identical structure, but all weights shifted by λ_0. We will see below that by such a shift, representations which are unitarizable w.r.t. $SO_q(2,3)$ are in one – to – one correspondence with representations which are unitarizable w.r.t. $SO_q(5)$. Furthermore, it is well – known (e.g. [2]) that all $(X_i^-)^m v_\lambda$ are highest – weight vectors, and are factored out in an irrep. It is therefore enough to consider highest weights in the following domain:

Definition 4.4 *A weight* $\lambda = E_0 \beta_3 + s\beta_2$ *is called* basic *if*

$$0 \leq (\lambda, \beta_3) = E_0 < \frac{m}{2n}, \quad 0 \leq (\lambda, \beta_4) = (E_0 + s) < \frac{m}{2n}. \tag{42}$$

In particular, $\lambda \succ 0$. *It is* compact *if in addition it is integral (i.e.* $(\lambda, \beta_i) \in \mathbb{Z} d_i$*),*

$$s \geq 0 \quad and \quad (\lambda, \beta_1) \geq 0. \tag{43}$$

An irrep with compact highest weight will be called compact.

The region of basic weights is drawn in figure 3, together with the lattice of v_{λ_0}'s. The compact representations are centered around 0 and invariant under the (quantum) Weyl group [11], as classically.

A representation with basic highest weight can be unitarizable w.r.t. $SO_q(5)$ only if λ is compact: all the $SU_q(2)$'s must be centered around 0, otherwise they cannot be unitary. This implies that λ is integral. Since $q_2 = e^{2\pi i n/2m}$, the relevant parameter for the $SU_q(2)$'s in the $\beta_{2,3}$ directions is $m_{(2)} = 2m$ and $M_{(2)} = m_{(2)}/2 = m$ according to section 2. Therefore the directions of $\beta_{2,3}$ are unitarizable in the compact case, and similarly for the directions

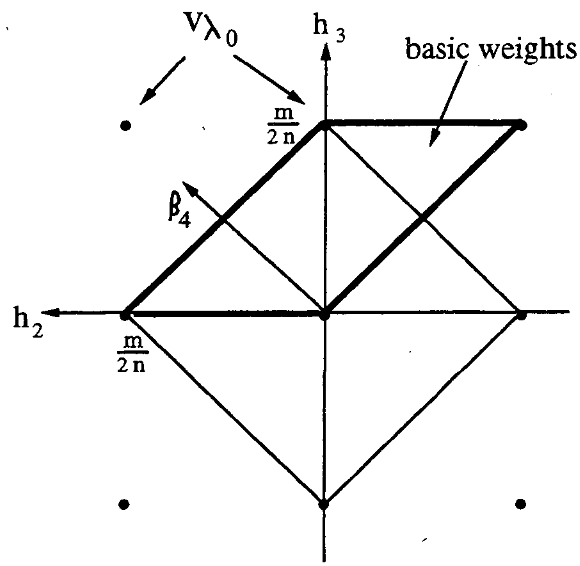

Figure 3: envelope of compact representations, basic weights and the lattice of v_{λ_0}

$\beta_{1,4}$. This alone however is not enough to show that they are unitarizable w.r.t. to the full group.

To show unitarizability, we first have to determine the structure of compact representations. For this it is enough to consider only basic integral weights (the structure for the remaining weights then follows e.g. by Weyl symmetry). Consider $M(\lambda)$ with highest weight vector v_λ for basic integral λ. To find the w_β's, notice that

$$D(\lambda)_{m_\beta,\beta} = 0 \Leftrightarrow [m_\beta]_\beta = 0 \quad \text{or} \quad (\lambda + \rho - \frac{m_\beta}{2}\beta, \beta) \in \frac{m}{2n}\mathbb{Z} \tag{44}$$

The first case is irrelevant, since w_β would not be basic. Using $\rho = \frac{3}{2}\beta_1 + 2\beta_2$, it follows that m_{β_1} is the smallest positive integer such that

$$(\lambda - \frac{m_{\beta_1} - 1}{2}\beta_1, \beta_1) \in \frac{m}{2n}\mathbb{Z}. \tag{45}$$

w_{β_1} has weight $\tau_1(\lambda) - \beta_1$ if λ is compact, or $\tau_1'(\lambda) - \beta_1$ if $(\lambda, \beta_1) \geq m/2$, $n = 1$ and m is even, where τ_1' is the reflection along β_1 with center $m/2\beta_3$. In all other cases w_{β_1} does not have basic weight. Similarly, w_{β_2} is determined by

$$(\lambda - \frac{m_{\beta_2} - 1}{2}\beta_2, \beta_2) \in \frac{m}{2n}\mathbb{Z}. \tag{46}$$

It has basic weight (namely $\tau_2(\lambda) - \beta_2$) only if $s \geq 0$. w_{β_3} is determined by

$$(\lambda - \frac{m_{\beta_3} - 3}{2}\beta_3, \beta_3) \in \frac{m}{2n}\mathbb{Z}. \tag{47}$$

Since $E_0 < m/(2n)$, the only case where it has basic weight is if $E_0 = m/2 - 1$ and $n = 1$, where $m_{\beta_3} = 1$. Finally, w_{β_4} is found from

$$(\lambda - \frac{m_{\beta_4} - 2}{2}\beta_4, \beta_4) \in \frac{m}{2n}\mathbb{Z}. \tag{48}$$

Since $(\lambda, \beta_4) < m/2$, the only cases where it has basic weight are $\lambda = (m/2 - 1 - s)\beta_3 + s\beta_2$ if $n = 1$ and m is even (otherwise λ is not integral), where $m_{\beta_4} = 1$.

From this, the only cases where basic weights might not be simple are if $n = 1$ and either $\lambda = (m/2 - 1 - s)\beta_3 + s\beta_2$ for $s > 0$ and m even, or $E_0 = m/2 - 1$. In the first case, there is indeed a highest weight vector with weight $(m/2 - s - 2)\beta_3 - s\beta_2$ which is descendant of both w_{β_2} and w_{β_4} (this can be seen from the multiplicities). It is however simple. The second case is only relevant for $s = 0$, where indeed there is a highest weight vector at $\lambda - \beta_2 - \beta_3$ which is not simple. However one can see directly using the casimir c_1 (33) that there are no basic highest weight vectors with $s \geq 0$ except w_{β_3} (this is all we really need to know), which is however descendant of w_{β_2}. Indeed, the value of c_1 on a highest weight module with $\lambda = E_0\beta_3 + s\beta_2$ is essentially $[E_0 + \frac{3}{2}]^2 + [s + \frac{1}{2}]^2$, and its value on any other such highest weight state besides w_{β_3} would be larger.

In all other cases, all basic weights are simple. (38) and Proposition 4.3 now imply that there are no other basic highest weight vectors.

Thus we have found all the relevant highest weight vectors in $M(\lambda)$ with compact λ and can prove the following:

Theorem 4.5 *The structure of the irreps $V(\lambda)$ with compact highest weight λ is the same as classically unless $\lambda = (m/2 - 1 - s)\beta_3 + s\beta_2$ for $s \geq 1$ and $\frac{m}{2n}$ is integer, where an additional highest weight state at weight $\lambda - \beta_4$ appears (and is factored out in the irrep). They are unitarizable w.r.t. $SO_q(5)$ (i.e. w.r.t. the involution θ).*

Proof For fixed λ, consider the representation before factoring out the additional highest - weight state at $\lambda - \beta_4$ in the case $\lambda = (m/2 - 1 - s)\beta_3 + s\beta_2$, so that the weight space is the same as classically. For $q = 1$, the representation is known to be unitarizable, so the inner product is positive definite. Consider the eigenvalues of the inner product matrix of $(\, , \,) \equiv (\, , \,)_q$ as q goes from 1 to $e^{2\pi i n/m}$ along the unit circle. The only way an eigenvalue could become negative is that it is zero for some q_0. This can only happen if q_0 is a root of unity, $q_0 = e^{2i\pi n'/m'}$ with $n'/m' \leq n/m$. According to the above analysis, this is only possible if $(\lambda, \beta_4) = m'/2 - 1$ and $\frac{m'}{2n'}$ is an integer; but $(\lambda, \beta_4) < m/(2n)$, so $n = n' = 1$ and $m' = m$. Therefore it happens precisely for $\lambda = (m/2 - 1 - s)\beta_3 + s\beta_2$ if $\frac{m}{2n}$ is integer and $s \geq 1$ (if $s = 1/2$, then w_{β_4} is descendant of w_{β_2}), in which case we have to factor out w_{β_4}, and all remaining eigenvalues are positive by continuity and the above analysis. □

So far all results were stated for highest - weight modules; of course the analogous statements for lowest - weight modules are true as well, which can be seen e.g. using the algebra automorphism $X_i^+ \to X_i^-$, $H \to -H$, $q \to q^{-1}$.

Now we want to find the "physical" representations which are unitarizable w.r.t. $SO_q(2,3)$. These (positive - energy) representations are most naturally considered as lowest - weight representations, and can be obtained from the compact case by a shift, as indicated above: if $V(\lambda)$ is a compact highest - weight representation, then

$$V(\lambda) \cdot \omega \equiv V(\lambda) \otimes \omega \tag{49}$$

with $\omega \equiv v_{\lambda_0}, \lambda_0 = \frac{m}{2n}\beta_3$ has lowest weight $\mu = -\lambda + \lambda_0 \equiv E_0\beta_3 - s\beta_2$ (short: $\mu = (E_0, s)$). It is a positive - energy representation, i.e. the eigenvalues of h_3 are positive.

For $\frac{m}{2n}$ integer, these representations correspond precisely to classical positive - energy representations with the same lowest weight [7]; in the "restframe", energy and spin are E_0 resp. s, and the structure for $h_3 \leq m/4$ is the same as classically, see figure 4. Otherwise, the weights are not integral.

The case $\mu = (s+1, s)$ for $s \geq 1$ and $\frac{m}{2n}$ integer will be called "massless" for two reasons. First, E_0 is the smallest possible energy for a unitarizable representation with given s (see below). The main reason however is the fact that as in the classical case [7], an additional lowest - weight state with $E_0' = E_0 + 1$ and $s' = s - 1$ appears, which generates a spin $s - 1$ null - subspace of what should be called "pure gauge" states. This corresponds precisely to the classical phenomenon in gauge theories, which ensures that the massless photon, graviton etc. have only their appropriate number of degrees of freedom (generally, the concept of mass in Anti - de Sitter space is not as clear as in flat space. Also notice that while "at rest" there are actually still $2s + 1$ states, the representation is nevertheless reduced by one spin $s - 1$ irrep). In the present case, all these representations are finite - dimensional!

Thus we are led to the following

Definition 4.6 *A lowest - weight irrep $V_{(\mu)}$ determined by its lowest weight $\mu = E_0\beta_3 - s\beta_2$ (resp. μ itself) is called* physical *if $-(\mu - \frac{m}{2n}\beta_3)$ is compact.*

It is called massless *if $E_0 = s + 1$, $s \geq 1$ and $\frac{m}{2n}$ is integer.*

Theorem 4.7 *The physical irreps are unitarizable w.r.t. $SO_q(2,3)$. For $h_3 \leq \frac{m}{4n}$, they are obtained by factoring out from a Verma module the subspace with lowest weight $(E_0, -(s+1))$ only, except in the massless case, where an additional lowest weight state with weight $(E_0+1, s-1)$ appears. This is the same as for $q=1$, see figure 4.*

Proof As mentionned before, we can write every vector in such a representation uniquely as $a \cdot \omega$, where a belongs to a compact irrep. Consider the inner product

$$\langle a \cdot \omega, b \cdot \omega \rangle \equiv (a, b), \tag{50}$$

where (a,b) is the hermitian inner product on the *compact* (shifted) representation. Then

$$\begin{aligned} \langle a \cdot \omega, e_1(b \cdot \omega) \rangle &= \langle a \cdot \omega, (e_1 \otimes q^{h_1/2} + q^{-h_1/2} \otimes e_1) b \otimes \omega \rangle \\ &= q^{h_1/2}|_\omega (a, e_1 b) = i(a, e_1 b) \end{aligned} \tag{51}$$

using $h_1|_\omega = \frac{m}{2n}$. Similarly,

$$\begin{aligned} \langle e_{-1}(a \cdot \omega), b \cdot \omega \rangle &= \langle (e_{-1} \otimes q^{h_1/2} + q^{-h_1/2} \otimes e_{-1}) a \otimes \omega, b \otimes \omega \rangle \\ &= q^{-h_1/2}|_\omega (e_{-1} a, b) = -i(e_{-1} a, b) \end{aligned} \tag{52}$$

because $\langle , \rangle$ is antilinar in the first argument and linear in the second. Therefore

$$\langle a \cdot \omega, e_1(b \cdot \omega) \rangle = -\langle e_{-1}(a \cdot \omega), b \cdot \omega \rangle. \tag{53}$$

Similarly $\langle a \cdot \omega, e_2(b \cdot \omega) \rangle = \langle e_{-2}(a \cdot \omega), b \cdot \omega \rangle$. This shows that $\langle \ , \ \rangle$ is hermitian w.r.t. $\overline{x}$, and positive definite because $(\ , \ )$ is positive definite according to Theorem 4.5.
□

Notice that Dirac's singleton representations [3] (which have non – integral weights) with $(E_0 = 1/2, s = 0)$ resp. $(E_0 = 1, s = 1/2)$ appear if m is odd and $n = 1$. We will see however that their tensor product is not unitarizable, and they cannot coexist with massless states.

As a consistency check, one can see again from section 2 that all the $SO_{\tilde{q}}(2,1)$ resp. $SU_{\tilde{q}}(2)$ subgroups are unitarizable in these representations, but this is not enough to show unitarizability for the full group. Note that for $n = 1$, one obtains the classical one – particle representations for given s, E_0 as $m \to \infty$. We have therefore also proved the unitarizability at $q = 1$ for (half)integer spin, which appears to be non – trivial in itself [7]. Furthermore, *all representations obtained from the above by shifting E_0 or s by a multiple of $\frac{m}{n}$ are unitarizable as well.* One obtains in weight space a cell - like structure of representations which are

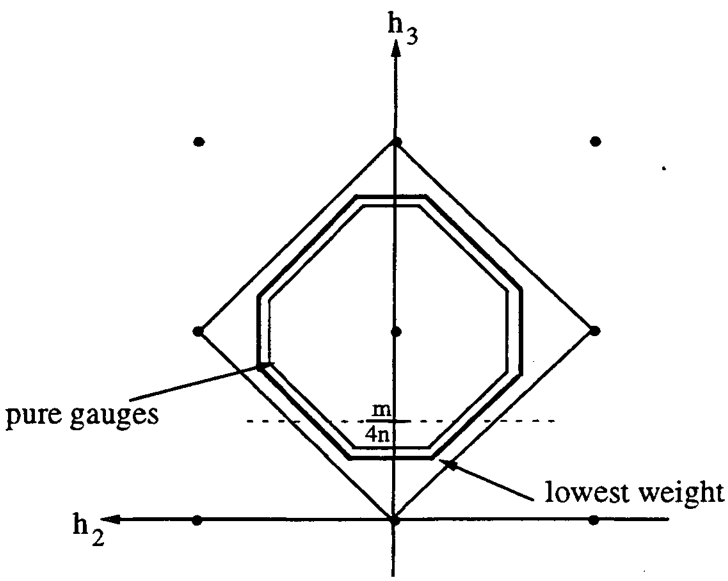

Figure 4: physical representation with subspace of pure gauges (only for $\frac{m}{2n}$ integer), schematically. For $h_3 \leq \frac{m}{4n}$, the structure is the same as for $q = 1$.

unitarizable w.r.t. $SO_q(2,3)$ resp. $SO_q(5)$. It is clear from the above that there are no other unitarizable representations.

Finally we want to consider many – particle representations, i.e. find a tensor product such that the tensor product of unitary representations is unitarizable, as in section 2. The idea is the same as there, the tensor product of 2 such representations will be a direct sum of lowest – weight representations, and we simply take the physical subspaces as determined by their lowest weights only. More precisely,

Definition 4.8 *Let $V_{(\mu)}$ and $V_{(\mu')}$ be two physical irreps as in Definition 4.6 resp. Theorem 4.7. For any physical lowest weight state $u_{\lambda'}$ in $V_{(\mu)} \otimes V_{(\mu')}$, let $W_{(\lambda')}$ be its lowest weight submodule, and $V_{(\lambda')}$ be the irreducible quotient of $W_{(\lambda')}$. Then define*

$$V_{(\mu)} \tilde{\otimes} V_{(\mu')} \equiv \bigoplus_{\lambda'} V_{(\lambda')} \tag{54}$$

where the sum goes over the physical λ', so the $V_{(\lambda')}$ are physical. It is nonzero only if $\frac{m}{2n}$ is integer.

Again as in section 2, one might also include a second "band" of high – energy states. Furthermore,

Theorem 4.9 *$\tilde{\otimes}$ is associative, and $V_{(\mu)} \tilde{\otimes} V_{(\mu')}$ is unitarizable w.r.t. $SO_q(2,3)$.*

Proof First, notice that the λ' in the above definition can be physical only if $\frac{m}{2n}$ is integer. Further, none of the λ' are massless, so none of the $W_{(\lambda')}$ contain physical lowest – weight states. Also, lowest weights for generic q cannot disappear at roots of unity. Therefore $\bigoplus_{\lambda'} V_{(\lambda')}$ contains all the physical states of the full tensor product, and the stucture is the same as classically for physical weights (since there are no massless representations, the classically inequivalent representations cannot recombine into indecomposable ones). Associativity now follows from associativity of the ordinary tensor product and coassociativity of the coproduct. □

Therefore $q = e^{2\pi i/m}$ with m even is the physically interesting case.

5 Conclusion

We have shown that in contrast to the classical case, there exist unitary finite – dimensional representations of noncompact quantum groups at roots on unity. In particular, the structure of such "physical" representations of $SO_q(2,3)$ for low energies is exactly the same as in the classical case, and thus they could be used to describe elementary particles for arbitrary spin. Representations for many non – identical particles are found.

Apart from purely mathematical interest, this is very encouraging for applications in QFT. In particular the appearance of pure gauge states should be a good guideline to construct gauge theories on quantum Anti – de Sitter space. If this is possible, one should expect it to be finite in light of these results. However to achieve that goal, more ingredients are needed, such as implementing a symmetrization axiom (cp. [6]), a dynamical principle (which would presumably involve integration over such a quantum space, cp. [23]), and efficient methods to do calculations in such a context. These are areas of current research.

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