FINITE DISTANCE TRANSITIVE GENERALIZED POLYGONS

F. Buekenhout
Université Libre de Bruxelles
Campus Plaine 216,
B – 1050 Bruxelles
e-mail: fbueken@ulb.ac.be

H. Van Maldeghem*
Universiteit Gent
Vakgroep Fundamentele en Computergerichte Wiskunde
Galglaan 2,
B – 9000 Gent
e-mail: hvm@cage.rug.ac.be

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Abstract

Using the classification of the finite simple groups, we classify all finite generalized polygons having an automorphism group acting distance transitively on the set of points. This proves an old conjecture of J.Tits saying that every group with an irreducible rank 2 BN-pair arises from a group of Lie type.

*Research Associate of the National Fund for Scientific Research (Belgium)

1 Introduction.

In 1974, Tits [38],p.221 conjectured that *all finite generalized polygons having a group acting transitively on the pairs* (Σ, C)*, where* Σ *is an apartment and* C *a chamber contained in* Σ*, arise from absolutely simple algebraic groups over a finite field or from the Ree groups* ${}^2F_4(2^e)$*,* e *an odd non-negative integer*, in other words, all generalized polygons associated with a finite BN-pair are *"classical"*, as some authors would call them. Stronger conjectures have been made since then, such as : *(*) all finite generalized polygons with a group acting distance transitively on the point set are known*, see Brouwer, Cohen & Neumaier [4],p.205 or : *all flag-transitive finite generalized polygons are known*, see Kantor [23], or even more restrictive : *all (thick) finite generalized hexagons and octagons are known*, see Kantor [22]. Using the classification of the finite simple groups, it has been known to several people that Tits' conjecture, and even conjecture *(*)* above, was provable, see Kantor [23] (from p.252 of [23], we quote: *"... it is also clear that all generalized quadrangles whose automorphism groups have rank 3 on the set of points can be determined."*) for the case of generalized quadrangles, or Brouwer, Cohen & Neumaier [4] for the general case (from p.205 of [4], we quote: *"It is very likely that every thick distance-transitive generalized* $2d$*-gon with* $d \geq 3$ *is known. A proof of this fact is expected to emerge from the prospective classication of all primitive distance-transitive graphs*). In the present paper, we present a proof of conjecture *(*)* above, motivated by the fact that this could stimulate the search for weaker hypotheses, as, for instance, point-transitivity see Buekenhout & Van Maldeghem [8]. A more general result, on rank 2 geometries which are (g, d_p, d_l)-gon with $2 \leq g \leq d_p \leq d_l \leq g + 1$, can be found in Buekenhout & Van Maldeghem [9].

The approach we take is the one outlined in [4], p.229 for the general case of distance-transitive graphs: get to the primitive case, use a result of Preager, Saxl & Yokoyama [31], restrict to the almost simple case and perform a check of all examples arising in this way.

As an application, we show at the end of the paper that all finite half Moufang generalized polygons are Moufang (except for some small exceptions), a result which was already proved for generalized quadrangles without using the classification of finite simple groups by Thas, Payne & Van Maldeghem [36].

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2 Definitions and Statement of the Main Result.

A *finite generalized n-gon, $n \geq 3$, of order (s,t)* is a point-line incidence geometry $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$, with point set $\mathcal{P}$, line set $\mathcal{B}$ and symmetric incidence relation I such that the following axioms are satisfied :

(Gn1) Every point is incident with $t+1$ lines and two distinct lines are incident with at most one point.

(Gn2) Every line is incident with $s+1$ points and two distinct points are incident with at most one line.

(Gn3) The incidence graph has girth $2n$ and diameter n, i.e. two elements of $\mathcal{P} \cup \mathcal{B}$ can always be joined by a path of length $l \leq n$ (in the incidence graph) and if $l < n$, then the path is unique.

A *generalized polygon* is a generalized n-gon for some n. A generalized 3-gon has necessarily $s = t$ and if $s \geq 2$, then it is a usual projective plane. An *apartment* of a generalized n-gon is a chain of elements forming a closed path of length $2n$ in the incidence graph. Every apartment has exactly $2n$ natural cyclic orders if we agree on starting with a point.

Let $\mathcal{S}$ be a generalized n-gon and G a group of type preserving (i.e. preserving the sets $\mathcal{P}$ and $\mathcal{B}$) automorphisms. Then we say that $(\mathcal{S}, \mathcal{G})$ *is distance transitive* provided G acts transitively on each set of pairs of points at a certain distance from each other. We say that $(\mathcal{S}, \mathcal{G})$ has the *Tits property* provided G acts transitively on the ordered apartments. This is equivalent to saying that G is equipped with the structure of a BN-pair such that $\mathcal{S}$ amounts to the associated rank 2 building, see Tits [38],3.2.6 and 3.11. It is easy to check that the pair $(\mathcal{S}, \mathcal{G})$ is always distance transitive whenever it has the Tits property.

Before we can state our main result, we need some more notation. We denote by $W(q)$ (resp. $Q(4,q)$, $Q(5,q)$, $H(3,q^2)$, $H(4,q^2)$) the "classical" generalized quadrangle arising naturally from the classical group $PSp_4(q)$ (resp. $O_5(q)$, $O_6^-(q)$, $U_4(q)$, $U_5(q)$). The classical projective plane of order q is denoted by $PG(2,q)$. The unique generalized hexagon of order (q,q) arising from Dickson's group $G_2(q)$ and having ideal lines (this fixes the choice for the names *point* and *line* if q is not a power of 3; otherwise the geometry is self-dual and hence the two possible choices are equivalent) (see Ronan [32]) is denoted by $H(q)$. The unique generalized hexagon arising from the triality group ${}^3D_4(q)$ and having order (q,q^3) will be denoted here by $T(\sigma,q)$, where σ denotes the field automorphism of order 3 involved in the triality defining ${}^3D_4(q)$, see Tits [37]. Note that $H(q)$ is a subhexagon of the dual of $T(\sigma,q)$. The unique finite generalized octagon of order (q,q^2) arising from the Ree group ${}^2F_4(q)$ is denoted here by $R(q)$. All these examples are called *thick* because in each case $s,t \geq 2$. For a given generalized polygon $\mathcal{S}$, the dual is denoted by $\mathcal{S}^{\mathcal{D}}$, while $\mathcal{S}^{\mathcal{I}}$ is defined as follows : the points of $\mathcal{S}^{\mathcal{I}}$ are the points and lines of $\mathcal{S}$ and the lines of $\mathcal{S}^{\mathcal{I}}$ are the *flags* (i.e. the incident point-line pairs) of $\mathcal{S}$. Incidence is the natural one. For a given generalized n-gon $\mathcal{S}$ of order (s,s), $\mathcal{S}^{\mathcal{I}}$ is a generalized $2n$-gon of order $(1,s)$, which is non-thick. In fact, $\mathcal{S}^{\mathcal{I}}$ is isomorphic to the incidence graph of $\mathcal{S}$. All these examples are due to Tits [37].

There is one more example that will be needed, namely the unique generalized quadrangle of order $(3, 5)$. This example is one out of a class of generalized quadrangles of order $(s, s+2)$ due to Ahrens & Szekeres [1] and independently to Hall [18]. It can be constructd as follows. Consider the projective plane $PG(2, 4)$ and a *complete oval* O in it, i.e. a conic together with its kernel. Embed $PG(2, 4)$ as a hyperplane in $PG(3, 4)$ and define the following geometry $\Gamma = (\mathcal{P}, \mathcal{L}, I)$: the elements of $\mathcal{P}$ are the points of $PG(3, 4)$ not in $PG(2, 4)$; the elements of $\mathcal{L}$ are the lines in $PG(3, 4)$ meeting O in exactly 1 point; incidence is the natural one. Then Γ is a generalized quadrangle of order $(3, 5)$ and it is usually denoted by $T_2^*(O)$. For more information on this interesting quadrangle we refer to a recent paper of Payne [30].

We are now in a position to state our main result :

MAIN RESULT. *Suppose* $(\mathcal{S}, \mathcal{G})$ *is a distance transitive finite generalized n-gon,* $n \geq 3$*, then* (Γ, G) *is one of the examples of table 1 below (where q denotes an arbitrary prime power). In the case of* (GP13)*, there is a simple group* S *with* $S \times S \trianglelefteq G \leq S_o$ *wr* 2*, where* $S \trianglelefteq S_o \leq Aut(S)$ *and* S_o *acts* 2*-transitively on a set of* $s+1$ *points. No attempt has been made to classify the groups corresponding to case* (GP14).

3 Proof of the Main Result.

3.1 The non-thick case.

In this subsection, we prove our main result for the non-thick generalized polygons, assuming the result for the thick ones.

Suppose $\mathcal{S}$ is a non-thick generalized $2n'$-gon of order (s, t), $st > 1$, and G acts as a distance transitive automorphism group. First we consider the case $n' \neq 2$. Assume $s = 1$. Then $\mathcal{S}$ is the incidence graph of a thick generalized n'-gon $\mathcal{S}'$. The group G acts on $\mathcal{S}'$ transitively on the union of the set of points and lines (since this is the set of points of $\mathcal{S}$) and hence G must contain a correlation (graph automorphism). It is clear that, G acts distance transitively on $\mathcal{S}$ if and only if G acts distance transitively on both $\mathcal{S}'$ and its dual $\mathcal{S}'^{\mathcal{D}}$. This explains examples (GP18), (GP22), (GP24) and (GP25).

Now assume $t = 1$. Then $\mathcal{S}$ is the dual of the incidence graph of a thick generalized n'-gon $\mathcal{S}'$. The points of $\mathcal{S}$ are the flags of $\mathcal{S}'$. A sufficient condition for the existence of G is that $(\mathcal{S}', \mathcal{G})$ has the Tits property; indeed, in that case the pointwise stabilizer in G of a flag F in $\mathcal{S}'$ acts transitively on the set of flags in a given position w.r.t. F. The correlation does the rest. Reversing this argument, we see that G acts point distance transitively on $\mathcal{S}$ if and only if $(\mathcal{S}', \mathcal{G})$ has the Tits property. This implies the examples (GP19), (GP23) and (GP26).

Now let $n' = 2$, then the example (GP13) follows directly from the classification of all primitive rank 3 groups (it is easily seen that in this case the group must act primitively).

	n	$\mathcal{S}$	G	Restrictions
(GP1)	3	$PG(2,q)$	$L_3(q) \trianglelefteq G \leq P\Gamma L_3(q)$	
(GP2)	4	$W(q)$	$PSp_4(q) \trianglelefteq G \leq P\Gamma Sp_4(q)$	
(GP3)	4	$Q(4,q)$	$O_5(q) \trianglelefteq G \leq P\Gamma O_5(q)$	
(GP4)	4	$Q(5,q)$	$O_6^-(q) \trianglelefteq G \leq P\Gamma O_6^-(q)$	
(GP5)	4	$H(3,q^2)$	$U_4(q) \trianglelefteq G \leq P\Gamma U_4(q)$	
(GP6)	4	$H(4,q^2)$	$U_5(q) \trianglelefteq G \leq P\Gamma U_5(q)$	
(GP7)	6	$H(q)$	$G_2(q) \trianglelefteq G \leq \mathrm{Aut}(G_2(q))$	G contains no graph automorphism
(GP8)	6	$T(\sigma,q)$	${}^3D_4(q) \trianglelefteq G \leq \mathrm{Aut}({}^3D_4(q))$	
(GP9)	8	$R(q)$	${}^2F_4(q) \trianglelefteq G \leq \mathrm{Aut}({}^2F_4(q)$	q odd power of 2
(GP10)	4	$H(4,q^2)^D$	$U_5(q) \trianglelefteq G \leq P\Gamma U_5(q)$	
(GP11)	4	$W(2)$	A_6	
(GP12)	4	$T_2^*(O)$	$2^6 : 3 : A_6 \leq G \leq 2^6 : 3 : S_6$	O a complete oval in $PG(2,4)$
(GP13)	4	$(s+1) \times (s+1)$-grid	GRID	
(GP14)	4	dual grid		
(GP15)	6	$H(q)^D$	$G_2(q) \trianglelefteq G \leq \mathrm{Aut}(G_2(q))$	G contains no graph automorphism
(GP16)	6	$T(\sigma,q)^D$	${}^3D_4(q) \trianglelefteq G \leq \mathrm{Aut}({}^3D_4(q))$	
(GP17)	6	$H(2)$	$U_3(3) \cong G_2(2)'$	
(GP18)	6	$PG(2,q)^I$	$L_3(q) : 2 \leq G \leq P\Gamma L_3(q) : 2$	G contains a graph automorphism
(GP19)	6	$(PG(2,q)^I)^D$	$L_3(q) : 2 \leq G \leq P\Gamma L_3(q) : 2$	G contains a graph automorphism
(GP20)	8	$R(2^e)^D$	${}^2F_4(q) \trianglelefteq G \leq \mathrm{Aut}({}^2F_4(q)$	e odd
(GP21)	8	$R(2)$	${}^2F_4(2)'$ (Tits' group)	
(GP22)	8	$W(q)^I$	$PSp_4(q).2 \leq G \leq P\Gamma Sp_4(q).2$	q even G contains a graph automorphism
(GP23)	8	$(W(q)^I)^D$	$PSp_4(q).2 \leq G \leq P\Gamma Sp_4(q).2$	q even G contains a graph automorphism
(GP24)	8	$W(2)^I$	$A_6 : 2$	G contains a graph automorphism
(GP25)	12	$H(q)^I$	$G_2(q).2 \leq G \leq \mathrm{Aut}(G_2(q))$	q is a power of 3 G contains a graph automorphism
(GP26)	12	$(H(q)^I)^D$	$G_2(q).2 \leq G \leq \mathrm{Aut}(G_2(q))$	q is a power of 3 G contains a graph automorphism

Table 1: Distance Transitive Generalized Polygons.

Finally, (GP14) is the remaining case and no attempt is made to classify the groups here.

3.2 The cases $n = 3, 6, 8$.

The case $n = 3$ immediately follows from a celebrated result of OSTROM & WAGNER [29]. So we can restrict to $n = 6$ and $n = 8$.

We may assume that $\mathcal{S}$ is thick. We first show some lemmas.

LEMMA 1. *A group G acting distance transitively on the points of a thick generalized n-gon $\mathcal{S}$, $n \geq 4$, acts primitively on the set of points of $\mathcal{S}$.*

PROOF. Fix a point x of $\mathcal{S}$. Suppose G acts imprimitively on points and let S be a non-trivial set of imprimitivity containing x. Then there is some further point y in S. By the transitivity assumption on G, we can fix x and map y to any point at distance $d(x,y)$ from x. Let L be any line containing y at distance $d(x,y)-1$ from x, then all points except one on L are at distance $d(x,y)$ from x. Since $\mathcal{S}$ is thick, this implies that S contains a point z at distance 2 from y. Now all points collinear to y are contained in S. As $\mathcal{S}$ is connected, $S = \mathcal{S}$. □

LEMMA 2. *If G is a group acting distance transitively on the points of a thick generalized hexagon or octagon, then G is almost simple.*

PROOF. The point graph Γ of a generalized polygon is distance regular. If we restrict to generalized hexagons and octagons, then clearly the number of points adjacent to two points at distance 2 is 1; with standard notation, this means $c_2 = 1$ (see e.g. BROUWER, COHEN & NEIMAIER [4],p.1). By a theorem of PRAEGER, SAXL & YOKAYAMA [31] (see also VAN BON [45]), there are three possibilities:

1. Γ is a Hamming graph, but then $c_2 \geq 2$, see e.g. BROUWER, COHEN & NEUMAIER [4],p.27.
2. G is of affine type. Again $c_2 \geq 2$.
3. G is almost simple.□

LEMMA 3. *If G_x is the stabilizer in G of a point x of a thick generalized hexagon or octagon and G acts distance transitively, then G_x is 2-transitive on the lines through x.*

PROOF. Let L_i, L'_i be two lines through x, $i = 1, 2$. Choose points x_i, x'_i resp. on L_i, L'_i, $i = 1, 2$, all distinct from x. All these points are at distance 4 from each other, so by distance transitivity, there is an element of G mapping (x_1, x'_1) to (x_2, x'_2). Since x is the unique point at distance 1 from both x_1 and x'_1, resp. x_2 and x'_2, x is fixed and the assertion follows.□

Now let $(\mathcal{S}, \mathcal{G})$ be a distance transitive generalized n-gon, $n = 6, 8$, of order (s,t). By lemma 2, G is almost simple and the point graph Γ of $\mathcal{S}$ is a distance transitive graph.

By Buekenhout & Van Maldeghem [8], G is not of sporadic type. By the thickness assumption, the number a_1 of vertices of Γ adjacent with two given adjacent vertices satisfies $a_1 > 0$. By Ivanov [21] and Liebeck, Praeger & Saxl [26], who classified all distance transitive graphs related to alternating groups, G cannot be of alternating type (because no such graph satisfies both $c_2 \geq 2$ and $a_1 = 0$).

Suppose G is of Lie type. Here, a complete classification of representations of rank ≤ 5 has been achieved by Cuypers [11]. By lemma 3, we can restrict to the cases where the point stabilizer G_x under consideration has a 2-transitive representation. And by Buekenhout & Van Maldeghem [8], we may assume that G is not displayed in the Atlas [10] (this gives us as sporadic examples (GP17) and (GP21)). Furthermore, if G_x is a parabolic subgroup, then by Brouwer, Cohen & Neumaier [4],pp.341–343, we obtain examples (GP7), (GP8), (GP9), (GP15), (GP16) and (GP20). Left are the following two cases :

1. $G_2(q)$ acting on the cosets of $SU_3(q) : 2$, $q = 8, 9, 16, 32$.
2. $G_2(8)$ acting on the cosets of $SL_3(8) : 2$, $q = 8$.

In the first case $t = q^3$, but the number of points of the generalized polygon is the index of G_x in G and that is equal to $q^3.\frac{q^3-1}{2}$, contradicting the value for t (since the number of points equals $(1 + s)(1 + st + s^2t^2) > t^2$).
In the second case, $t = 72$ and the number of points should be $2^8(2^9 + 1)$. No integer value for s matches these conditions.

This completes the cases $n = 6, 8$.

3.3 Distance transitive generalized quadrangles.

All primitive rank 3 groups are classified and they fall into three distinct classes: the almost simple case, the affine case and the "grid" case. The latter is case (GP13). Let us look at the other two classes.

The classification in the almost simple case (of rank 3 groups) has been achieved by various people for the respective classes of simple groups: Bannai [2] for the alternating groups, Kantor & Liebler [24] for the classical Chevalley groups, Liebeck & Saxl [27] for the exceptional Chevalley groups and the sporadic groups.

Each rank 3 group G acts on a strongly regular graph, which is the point graph of the generalized quadrangle $\mathcal{S}$ if $(\mathcal{S}, \mathcal{G})$ is a distance transitive quadrangle. Let (v, k, λ, μ) be the parameters of this graph (with standard notation), then $s = \lambda + 1$, $t = \mu - 1$ and $k = s(t + 1)$ (provided $k \leq v - k - 1$). Hence $k = \mu(\lambda + 1)$. This condition is very easy to check in all cases and it turns out that it is satisfied only in the well known cases, which give rise to examples (GP2), (GP3), (GP4), (GP5), (GP6), (GP10) and (GP11) (a reference for the parameters of the strongly regular graphs is Hubaut [19] or Buekenhout & Van Maldeghem [9]).

The same argument can be used for the affine case (of rank 3 groups). The classification in this case is due to Liebeck [25]. Here, G acts on an affine space $AG(n, q)$. One can see easily that the lines of $\mathcal{S}$ must be subspaces of the affine space $AG(d,q)$, provided $d > 1$. Indeed, G_x, $x \in AG(d,q)$ has two orbits "at infinity" (because in the unique case where it has only one orbit, $k = v - k - 1$, which is impossible for a generalized quadrangle). So if two points y and z are collinear, then all points on the line yz in $AG(d,q)$ are collinear to them both. This means that $s + 1 = \lambda + 2$ must be a power of the prime p, where $q = p^h$ for some positive integer h. The only example from [25] satisfying these conditions gives rise to example (GP4). The case $d = 1$ remains.

So let $d = 1$. Set $l = v - k - 1$. We have $|\mathcal{P}| = q = p^h$ and by Foulser & Kallaher [17], $k|l$. Hence $t + 1|st$, implying $t + 1|s$, so $t < s$. Now $|\mathcal{P}| = (1 + s)(1 + st) = p^h$, so $1+s = p^f$ for some integer $f < h$, and $1+st = p^e \geq p^f$, $e < h$. Hence $1+st \equiv 0 \pmod{p^f}$ and since $s \equiv -1 \pmod{p^f}$, we have $1 - t \equiv 0 \pmod{p^f}$. So $t \equiv 1 \pmod{p^f}$ and this implies $t = 1$ or $t \geq p^f + 1 > s$, both contradictions.

This completes the proof of our main result.

4 Some Corollaries.

4.1 The Tits Condition.

As immediate consequences of our main result, we have:

COROLLARY 1. *If $(\mathcal{S}, \mathcal{G})$ has the Tits property, then $\mathcal{S}$ is associated with a finite group L of Lie type in the standard manner, and G contains the derived group L', or $\mathcal{S}$ is non-thick and is one of the examples* (GP18), (GP19), (GP22), (GP23), (GP25) *or* (GP26) *of table* 1.

COROLLARY 2. *If G is a finite group with an irreducible (B, N)-pair of rank 2, then G arises from a group of Lie type.*

4.2 The Moufang and Half Moufang Condition.

Let $\mathcal{S}$ be a generalized polygon and G a group of automorphisms. A geodesic path γ of length n in $\mathcal{S}$ is called a **root**. Denote by $\dot{\gamma}$ the geodesic of length $n - 2$ obtained from γ by deleting its extremeties. The root γ is called **Moufang (with respect to G)** if the subgroup of G fixing every element incident with a variety of $\dot{\gamma}$ acts transitively (and hence regularly) on the set of apartments containing γ. Note that, if γ is Moufang, then so is every root δ for which $\dot{\delta} = \dot{\gamma}$. If every root of Γ is Moufang with respect to G, then (Γ, G) is called **Moufang**, see Tits [39]. All Moufang polygons (finite and infinite) are classified, see Tits [39, 43, 41, 44] and also Faulkner [12], in particular a thick Moufang generalized g-gon must satisfy $g \in \{2, 3, 4, 6, 8\}$, see Tits [40, 42] and Weiss [46]. In the finite case, the classification of Moufang polygons follows from a result of Fong & Seitz [14, 15] on split (B, N)-pairs.

An interesting thing happens when n is even, because in that case, there are 2 kinds of roots depending on the type of variety in the middle of the root. If we require all roots of only one type to be Moufang, then we obtain a **half Moufang** generalized polygon. An immediate question is whether the half Moufang property is enough to ensure the Moufang property. The question is only interesting for thick generalized polygons because of the following observations: (1) every generalized polygon of order $(1, t)$ is trivially half Moufang for the roots based at a line; (2) if a generalized $2n$-gon of order $(1, t)$ is Moufang, then the corresponding generalized n-gon is half Moufang. Note that for non-thick Moufang generalized polygons the corresponding group G need not act transitively on the points or the lines and then Moufang does not necessarily imply Tits.

We will answer the above question in this subsection for the finite case. Note that we can assume $n = 6$ or 8 since the case $n = 4$ is done by Thas, Payne & Van Maldeghem [36]. More exactly, we will show:

COROLLARY 3. *Let $\mathcal{S}$ be a thick finite generalized hexagon or octagon and G a collineation group of $\mathcal{S}$. The pair $(\mathcal{S}, \mathcal{G})$ is half Moufang if and only if it is distance transitive or its dual is distance transitive, depending on the type of Moufang roots in $\mathcal{S}$. In particular, half Moufang implies Moufang whenever $(s, t) \neq (2, 2)$ (for generalized hexagons) or $(s, t) \neq (2, 4), (4, 2)$ (for generalized octagons). Also, $\mathcal{S}$ is Moufang with respect to some collineation group if and only if it is half Moufang with respect to some (possibly other) collineation group. Finally, $(\mathcal{S}, \mathcal{G})$ is half Moufang if and only if G is flag-transitive on $\mathcal{S}$ (and $\mathcal{S}$ is classical).*

PROOF. Suppose $\mathcal{S}$ is half Moufang with respect to G for roots based at lines.

(1). Let x and y be two collinear points. Considering a root γ based at the line xy and not containing x nor y, we see that we can map x to y. An inductive argument shows that G is transitive on the set of points.

(2). Let $l_1 I x_1 I l_2 I x_2 I l_3$. Consider any root $\gamma = (l_2, x_2, l_3, \ldots, l_i)$, where $i = 5$ or 6 (resp. for $g = 6$ or 8). Let y be distinct from both x_1 and x_2 but incident with l_1. Suppose also that l_1' is a line through x_1 distinct from l_2. Let θ_1 be the collineation fixing every element incident with a variety of $\dot{\gamma}$ and mapping x_1 to y. Let l be the unique line concurrent with l_i and at distance $g-3$ from y. There is a unique geodesic of length $g-3$ based at a certain fixed point z on l_1' and ending in a line l' concurrent with l. There is also a unique root $\delta = (l_2, x_2, \ldots, l')$. Denote by θ_2 the unique collineation fixing all elements incident with some variety of $\dot{\delta}$ and mapping y to x_1. The mapping $\theta_1\theta_2$ is a collineation fixing x_1, l_2, x_2 and l_3 and mapping l_1 to l_1'. So G is flag-transitive.

(3). By choosing the root γ in (1) suitably, it follows from (1) and (2) that G acts transitively on the set of geodesics of length 6 based at points. So if $G = 6$, G acts distance transitively on the points of $\mathcal{S}$ and the result follows from proposition 1.

(4). Suppose now $g = 8$. It is easily seen that one can map any point y at maximal distance from some point x to any point u collinear with y and also at distance 8 from x while fixing the point x. Now, if $(s, t) \neq (2, 4)$, Brouwer [3] shows that the set of points at maximal distance from x endowed with the induced lines, is connected, hence we have a distance transitive group on the points of $\mathcal{S}$ and the result again follows from proposition 1.

(5). So suppose $(s,t) = (2,4)$ and $g = 8$. The number of points is $1755 = 3^3.5.13$, the number of points at distance 6 of a fixed point is $640 = 2^7.5$ and the number of points at maximal distance from a given point is $1024 = 2^{10}$. It follows that $|G|$ is divisible by $2^9.3^3.5^2.13$. So put $|G| = k.2^9.3^3.5^2.13$. Let x and y be two points at distance 6 and denote by l the unique line through y at distance 5 from x. The stabilizer $G_{x,y}$ has order $4k$ and does not act transitively on the set of lines through y distinct from l, otherwise G acts as a rank 5 group on the point set and the result follows from proposition 1. So $G_{x,y}$ acts transitively on 2, 4 or 6 points collinear to y and not incident with l. Let z be a point in the smallest orbit (an orbit of size 2 or 4). The "remaining" group $G_{x,y,z}$ has order k or $k/2$. No element of that group stabilizes at least 2 lines through z distinct from yz except the identity (otherwise a thick proper suboctagon is fixed, but there are no such). This implies that k or $k/2$ divides 12. Hence $|G|$ divides $2^{12}.3^4.5^2.13$. Now, it is readily seen that G acts primitively on the set of points of $\mathcal{S}$. Applying a theorem of O'Nan [28] and Scott [33] (in the version of Buekenhout [7]), we see that, since $|\mathcal{P}| = 1755$ is neither a prime nor a non-trivial power of an integer, G is almost simple. By the foregoing, the socle S of G must divide $2^{12}.3^4.5^2.13$. By inspection, no simple group of order less than $2^{12}.3^4.5^2.13$ has $\mathrm{Out}(G)$ divisible by 5 or 13, hence $5^2.13$ divides $|S|$. By inspection again, only $L_2(25)$, $U_3(4)$ and ${}^2F_4(2)'$ satisfy the given conditions. But the automorphism group of the first two groups has a size less than $2^9.3.5^2.13$. So only ${}^2F_4(2)'$ qualifies and the result follows from proposition 1 and the fact that the permutation representation of ${}^2F_4(2)'$ on 1755 points is unique up to conjugacy and has rank 5 (Atlas [10]).

The last assertion follows directly from the previous and a result by Seitz [34]. This completes the proof of corollary 3.□

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