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FINITE ELEMENT APPROXIMATION OF A PARABOLIC INTEGRO-DIFFERENTIAL EQUATION WITH A WEAKLY SINGULAR KERNEL

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ABSTRACT. We give error estimates for the numerical solution by means of the Galerkin finite element method of an integro-differential equation of parabolic type with a memory term containing a weakly singular kernel. Optimal-order estimates are shown for spatially semidiscrete and completely discrete methods. Special attention is paid to the regularity of the exact solution.

1. INTRODUCTION

We shall consider the initial value problem (with $u_t = \partial u/\partial t$)

$$
\begin{aligned}
u_t + Au &= \int_0^t K(t-s)Bu(s)ds + f(t) \quad \text{in } \Omega, \quad \text{for } t > 0, \\
u &= 0 \quad \text{on } \partial\Omega, \quad t > 0, \\
u(0) &= u_0 \quad \text{in } \Omega,
\end{aligned} \tag{1.1}
$$

where A is a linear positive selfadjoint elliptic and B a general partial differential operator of second order with smooth, time-independent coefficients, where K is a weakly singular kernel $K(t)$ such that

$$
|K(t)| \le Ct^{-\alpha} \quad \text{with } 0 \le \alpha < 1, \quad \text{for } t > 0, \tag{1.2}
$$

and where Ω is a sufficiently smooth domain in R^d, $d \ge 1$. Integro-differential equations of this nature appear in applications such as heat conduction in materials with memory, population dynamics, and visco-elasticity; cf., e.g., Friedman and Shinbrot [3], Heard [5], and Renardy, Hrusa, and Nohel [12]. For equations with nonsmooth kernels such as in (1.2), we refer to Grimmer and Pritchard [4], Lunardi and Sinestrari [10], and Lorenzi and Sinestrari [9] and references therein. Finite element methods for problems of the form (1.1) with a smooth kernel K have been discussed in, e.g., Sloan and Thomée [13], Yanik and Fairweather [15], Thomée and Zhang [14], LeRoux and Thomée [6], Cannon and Lin [1], and Lin, Thomée, and Wahlbin [7].

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For the numerical solution we assume that we are given a family $\{S_h\}$ of finite-dimensional subspaces of $H_0^1 = H_0^1(\Omega)$ such that

$$\inf_{\chi\in S_h}\{\|v-\chi\| + h\|v-\chi\|_1\} \le Ch^2\|v\|_2, \quad \forall v\in H^2\cap H_0^1, \tag{1.3}$$

where $\|\cdot\|$ is the norm in $L_2 = L_2(\Omega)$ and $\|\cdot\|_s$ that in $H^s = H^s(\Omega)$.

We consider first the semidiscrete problem of finding $u_h : [0, \infty) \to S_h$ such that

$$\begin{aligned}(u_{h,t}, \chi) + A(u_h, \chi) &= \int_0^t K(t-s)B(u_h(s), \chi)\,ds + (f(t), \chi),\\ &\qquad\qquad \forall\chi\in S_h,\ t>0,\\ u_h(0) &= u_{0h},\end{aligned} \tag{1.4}$$

where $(\cdot\,,\cdot)$ is the inner product in L_2 and $A(\cdot\,,\cdot)$ and $B(\cdot\,,\cdot)$ are the bilinear forms on H_0^1 associated with the differential operators A and B, and where u_{0h} is an appropriate approximation in S_h of the initial data in (1.1). We shall show that, for each $T>0$, we then have the error estimate

$$\|u_h(t) - u(t)\| \le C_T h^2\left\{\|u_0\|_2 + \int_0^t \|u_t\|_2 ds\right\} \quad \text{for } t\le T. \tag{1.5}$$

We shall also consider the discretization in time of (1.4). Thus, let k be a time step, and let $U^n \in S_h$ be the approximation of the exact solution of (1.1) at time $t_n = nk$. The time discretization considered will be based on the backward difference quotient $\bar\partial_t U^n = (U^n - U^{n-1})/k$. The integral term then has to be evaluated by numerical quadrature from the values of the U^n, but since the integrand is singular, even when the solution is smooth, we shall use product integration: We shall approximate ϕ in $J_n(\phi) = \int_0^{t_n} K(t_n - s)\phi(s)\,ds$ by the piecewise constant function taking the value $\phi(t_j)$ in (t_j, t_{j+1}), and thus use

$$J_n(\phi) \approx Q_n(\phi) = \sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}} K(t_n-s)\phi(t_j)ds = \sum_{j=0}^{n-1}\kappa_{n-j}\phi(t_j),$$

where

$$\kappa_j = \int_{t_{j-1}}^{t_j} K(s)\,ds. \tag{1.6}$$

Our completely discrete scheme is therefore

$$\begin{aligned}(\bar\partial_t U^n, \chi) &+ A(U^n, \chi)\\ &= \sum_{j=0}^{n-1}\kappa_{n-j}B(U^j, \chi) + (f(t_n), \chi), \quad \forall\chi\in S_h,\ n\ge 1,\\ U^0 &= u_{0h}.\end{aligned} \tag{1.7}$$

For this completely discrete method we shall show

$$\|U^n - u(t_n)\| \le C_T(h^2+k)\left\{\|u_0\|_2 + \int_0^{t_n}(\|u_{tt}\| + \|u_t\|_2)\,ds\right\} \quad \text{for } t\le T. \tag{1.8}$$

Before we analyze these discrete methods, we shall discuss the existence and regularity of the solution of (1.1) and show, in particular, that the regularity required for the estimates (1.5) and (1.8) are satisfied under appropriate assumptions on the data. In the case of a weakly singular kernel the regularity of the solution with respect to time is limited, which makes higher-order quadrature formulas less attractive, as well as quadratures based on the use of sparser sets of time levels, such as those treated in [13] and [6].

2. An existence and regularity result

In this section we shall study the existence and regularity of the solutions of (1.1) and show, in particular, that the regularity required for the error estimates (1.5) and (1.8) holds under appropriate assumptions on the data of (1.1).

We shall need the following version of Gronwall's lemma.

Lemma 1. *Assume that* y *is a nonnegative function in* $L_1(0, T)$ *which satisfies*

$$y(t) \le b(t) + \beta \int_0^t (t-s)^{-\alpha} y(s)\, ds \quad \text{for } 0 < t \le T, \tag{2.1}$$

where $b(t) \ge 0$, $\beta \ge 0$. *Then there is a constant* C_T *such that*

$$y(t) \le b(t) + C_T \int_0^t (t-s)^{-\alpha} b(s)\, ds \quad \text{for } t \le T.$$

Proof. Let $K_1(s) = \beta s^{-\alpha}$ for $0 < s < T$, and let $K_1 * f$ denote the convolution

$$(K_1 * f)(t) = \int_0^t K_1(t-s) f(s)\, ds.$$

Recall that this is a bounded operator on $L_1(0, T)$. With K_i the kernel of the i times iterated convolution, we have

$$K_i(s) \le C(i, \alpha) s^{i(1-\alpha)-1},$$

and we easily see that $K_i * b(t) \le C K_1 * b(t)$ for $i \ge 2$. Hence, applying $K_1 *$ to (2.1) i times in succession, we obtain

$$y(t) \le b(t) + C(K_1 * b)(t) + (K_i * y)(t).$$

For $i(1-\alpha) - 1 > 0$, we have

$$(K_i * y)(t) \le C \int_0^t y(s)\, ds$$

and we can use the ordinary Gronwall lemma. Since

$$\int_0^t b(s)\, ds \le C(K_1 * b)(t),$$

this concludes the proof. □

We shall also need the following lemma.

Lemma 2. *Let* $K \in L_1(0, T)$. *Then for each* $\varepsilon > 0$ *there is a constant* $C_\varepsilon = C_\varepsilon(\|K\|_{L_1(0,T)})$ *such that*

$$\left| \int_0^T \int_0^t K(t-s) f(s) f(t)\, ds\, dt \right| \le \varepsilon \int_0^T f(t)^2\, dt + C_\varepsilon \int_0^T |K(T-t)| \int_0^t f(s)^2\, ds\, dt. \tag{2.2}$$

Proof. In this proof, let $(\cdot, \cdot)$ and $\|\cdot\|$ denote the inner product and norm in $L_2(0, T)$. We have, using the Cauchy–Schwarz inequality,

$$
\begin{aligned}
|(K * f)(t)|^2 &\le \left(\int_0^t |K(s)|^{1/2}|K(s)|^{1/2}|f(t-s)|\, ds\right)^2 \\
&\le \|K\|_{L_1(0,T)} \int_0^t |K(s)| f^2(t-s)\, ds.
\end{aligned}
$$

Hence, integrating with respect to t and changing the order of integration, and then changing variables,

$$
\begin{aligned}
\|K * f\|^2 &\le \|K\|_{L_1(0,T)} \int_0^T |K(s)| \int_s^T f^2(t-s)\, dt\, ds \\
&= \|K\|_{L_1(0,T)} \int_0^T |K(T-\tau)| \int_0^\tau f^2(\sigma)\, d\sigma\, d\tau.
\end{aligned}
$$

Hence, for the left-hand side of (2.2),

$$
\begin{aligned}
|(K * f, f)| &\le \|K * f\|\, \|f\| \le \varepsilon \|f\|^2 + \frac{1}{4\varepsilon}\|K * f\|^2 \\
&\le \varepsilon \|f\|^2 + \frac{1}{4\varepsilon}\|K\|_{L_1(0,T)} \int_0^T |K(T-t)| \int_0^t f(s)^2\, ds\, dt,
\end{aligned}
$$

which is the desired inequality. □

The following is our main existence and regularity result.

Theorem 1. *Assume that* $u_0 \in H^\beta \cap H_0^1$, $f \in \mathscr{C}([0, T]; H^{\beta-2})$ *and* $t^\gamma f_t \in L_\infty(0, T; H^{\beta-2})$ *with* $\beta > 2$, $0 < \gamma < 1$. *Then there exists a unique solution of* (1.1) *in* $\mathscr{C}([0, T]; L_2)$. *Furthermore,* $u \in \mathscr{C}([0, T]; H^2 \cap H_0^1)$, $u_t \in \mathscr{C}([0, T]; L_2) \cap L_1(0, T; H^2 \cap H_0^1)$, *and* $u_{tt} \in L_1(0, T; L_2)$.

Proof. We shall use the procedure of Faedo-Galerkin. Let $\{\phi_j\}_1^\infty$ be the eigenfunctions of A. We first seek $u^n \in \mathscr{S}_n = \mathrm{span}[\phi_1, \dots, \phi_n]$ satisfying

$$
\begin{aligned}
u_t^n + Au^n &= \int_0^t K(t-s) P_n B u^n(s) ds + P_n f(t) \quad \text{in } \Omega, \ \text{ for } t \ge 0, \\
u^n &= 0 \quad \text{on } \partial\Omega, \ t \ge 0, \\
u^n(0) &= P_n u_0 \quad \text{in } \Omega.
\end{aligned}
\tag{2.3}
$$

Here, P_n denotes the L_2 projection into $\mathscr{S}_n$. By standard arguments, cf., e.g., Linz [8], this system of ordinary integro-differential equations has a solution $u^n \in \mathscr{C}^1([0, T]) \cap \mathscr{C}^2((0, T])$.

We shall next derive a priori estimates for u^n. We first show that, independently of n,

$$
\left(\int_0^T \|u_t^n\|_2^p\, dt\right)^{1/p} + \left(\int_0^T \|u_{tt}^n\|^p\, dt\right)^{1/p} \le C_T M_{\beta,\gamma} \quad \text{for some } p = p(\alpha) > 1, \tag{2.4}
$$

where

$$M_{\beta,\gamma} = M_{\beta,\gamma}(u_0, f) \equiv \|u_0\|_\beta + \|f(0)\|_{\beta-2} + \sup_{s\le t}(s^\gamma \|f_t(s)\|_{\beta-2}).$$

Differentiating (2.3), we find that $v^n = u^n_t$ satisfies

$$\begin{aligned} v^n_t + Av^n &= K(t)P_nBP_nu_0 \\ &\quad + \int_0^t K(t-s)P_nBv^n(s)\,ds + P_nf_t(t) \quad \text{in } \Omega,\ t>0, \\ v^n &= 0 \quad \text{on } \partial\Omega,\quad t>0, \\ v^n(0) &= -AP_nu_0 + P_nf(0) \quad \text{in } \Omega. \end{aligned} \tag{2.5}$$

We now define $w^j = w^{n,j}$, $j \ge 1$, inductively by

$$\begin{aligned} w^1_t + Aw^1 &= K(t)P_nBP_nu_0 + P_nf_t(t) \quad \text{in } \Omega,\ t>0, \\ w^1 &= 0, \quad \text{on } \partial\Omega,\ t>0, \\ w^1(0) &= -AP_nu_0 + P_nf(0), \end{aligned}$$

and then, for $j \ge 2$,

$$\begin{aligned} w^j_t + Aw^j &= \int_0^t K(t-s)P_nBw^{j-1}(s)ds \equiv W^{j-1}(t) \quad \text{in } \Omega,\ t>0, \\ w^j &= 0 \quad \text{on } \partial\Omega,\ t>0, \\ w^j(0) &= 0 \quad \text{in } \Omega. \end{aligned}$$

Setting $z^j = v^n - \sum_{l=1}^j w^l$, we find for $j \ge 1$

$$\begin{aligned} z^j_t + Az^j &= \int_0^t K(t-s)P_nB(z^j + w^j)(s)\,ds \\ &\equiv \int_0^t K(t-s)P_nBz^j(s)ds + g^j \quad \text{in } \Omega,\ t>0, \\ z^j &= 0 \quad \text{on } \partial\Omega,\ t>0, \\ z^j(0) &= 0 \quad \text{in } \Omega. \end{aligned} \tag{2.6}$$

We shall show below that, for any $j \ge 1$ and δ with $2 < \delta < \beta$, there is a constant $C_j = C_j(\alpha, \delta)$ such that

$$\|w^j(t)\|_\delta \le C_j t^{-1+(\beta-\delta)/2+(j-1)(1-\alpha)} M_{\beta,\gamma}. \tag{2.7}$$

Assuming this for a moment, we conclude first that

$$\left(\int_0^T \|w^j\|_2^p\,dt\right)^{1/p} \le C_{j,T}M_{\beta,\gamma} \quad \text{for some } p>1,\ j\ge 1. \tag{2.8}$$

In order to bound z^j, we first note that by (2.7)

$$\begin{aligned}\|g^j\| &= \left\|\int_0^t K(t-s)P_nBw^j(s)\,ds\right\|\\ &\le C\int_0^t (t-s)^{-\alpha}s^{-1+(j-1)(1-\alpha)}ds M_{\beta,\gamma} \le CM_{\beta,\gamma} \quad \text{if} \quad j(1-\alpha)\ge 1.\end{aligned}$$

We now multiply (2.6) by $2Az^j(t)$ and integrate to obtain

$$\begin{aligned}&\|A^{1/2}z^j(T)\|^2 + 2\int_0^T \|Az^j\|^2\,dt\\ &\quad \le C\int_0^T\int_0^t (t-s)^{-\alpha}\|z^j(s)\|_2\|z^j(t)\|_2\,ds\,dt + C\int_0^T \|z^j\|_2\,ds\,M_{\beta,\gamma}.\end{aligned}$$

Hence, using Lemma 2 with ε suitably chosen for the double integral, and the Cauchy-Schwarz inequality for the last term, we have

$$2\int_0^T \|z^j\|_2^2\,dt \le CM_{\beta,\gamma}^2 + \int_0^T \|z^j\|_2^2\,dt + C\int_0^T (T-t)^{-\alpha}\int_0^t \|z^j(s)\|_2^2\,ds\,dt.$$

Moving the second term on the right over to the left and using Lemma 1, we conclude that

$$\int_0^T \|z^j\|_2^2 dt \le C_T M_{\beta,\gamma}^2 \quad \text{for } j(1-\alpha)\ge 1.$$

In particular, the estimate for u_t^n in (2.4) follows from this and (2.8).

It remains to show (2.7). For this purpose we first recall that the semigroup $E(t)$ generated by $-A$ satisfies, for $\phi\in H^\mu$ with $\phi = 0$ on $\partial\Omega$ if $\mu\ge\frac{1}{2}$,

$$\|E(t)\phi\|_\nu \le Ct^{-(\nu-\mu)/2}\|\phi\|_\mu, \quad 0\le\mu\le\nu,\ \mu<2.5. \tag{2.9}$$

(For $\mu\ge 2.5$, further boundary conditions have to be imposed on ϕ.) We have the representation

$$\begin{aligned}w^1 = &-E(t)AP_nu_0 + E(t)P_nf(0) + \int_0^t E(t-s)K(s)P_nBu_0\,ds\\ &+\int_0^t E(t-s)P_nf_t(s)\,ds,\end{aligned}$$

so that for $\delta<\beta$ (which we may clearly assume less than 2.5),

$$\begin{aligned}\|w^1\|_\delta &\le Ct^{-1+(\beta-\delta)/2}(\|u_0\|_\beta + \|f(0)\|_{\beta-2})\\ &\quad + C\int_0^t (t-s)^{-1+(\beta-\delta)/2}s^{-\alpha}\,ds\|u_0\|_\beta\\ &\quad + C\int_0^t (t-s)^{-1+(\beta-\delta)/2}s^{-\gamma}(s^\gamma\|f_t(s)\|_{\beta-2})\,ds\\ &\le Ct^{-1+(\beta-\delta)/2}M_{\beta,\gamma}.\end{aligned}$$

We now proceed with a proof of (2.7) by induction for $j\ge 2$ and assume the result holds for $j-1$. We note that then, for $2<\delta<\beta$,

$$\begin{aligned}\|W^{j-1}(t)\|_{\delta-2} &\le C\int_0^t (t-s)^{-\alpha}\|w^{j-1}(s)\|_\delta\,ds\\ &\le C_{j-1}\int_0^t (t-s)^{-\alpha}s^{-1+(\beta-\delta)/2+(j-2)(1-\alpha)}\,ds\,M_{\beta,\gamma}\\ &\le C_{j-1}t^{-1+(\beta-\delta)/2+(j-1)(1-\alpha)}M_{\beta,\gamma}.\end{aligned}$$

Thus, by (2.9), if $\varepsilon<\delta<\beta$, we obtain

$$\begin{aligned}\|w^j(t)\|_\varepsilon &= \left\|\int_0^t E(t-s)W^{j-1}(s)\,ds\right\|_\varepsilon\\ &\le C_{j-1}\int_0^t (t-s)^{-1+(\delta-\varepsilon)/2}s^{-1+(\beta-\delta)/2+(j-1)(1-\alpha)}\,ds\,M_{\beta,\gamma}\\ &\le C_{j-1}t^{-1+(\beta-\varepsilon)/2+(j-1)(1-\alpha)}\,M_{\beta,\gamma},\end{aligned}$$

which completes the proof of (2.7), and thus of the estimate for the first term in (2.4). Clearly, we then also have $\|u^n(t)\|_2\le C$, and it follows easily from (2.5) that the bound for u^n_{tt} in (2.4) is satisfied, and hence also that

$$\|u^n(t)\|_2+\|u^n_t(t)\|\ \le C\quad \text{for } 0\le t\le T. \tag{2.10}$$

We next proceed with a limiting argument. Writing (1.1) in weak form, we have

$$\begin{gathered}(u^n_t,\phi_m)+A(u^n,\phi_m)=\int_0^t K(t-s)B(u^n,\phi_m)ds+(f(t),\phi_m)\quad\text{for } m\le n,\\ u^n(0)=P_nu_0.\end{gathered}$$

By (2.10), a subsequence u^n converges weak* in $L_\infty(0,T;H^2)$, and we refer to that limit as u. By (2.4) we may also assume that a (further) subsequence u^n_t converges weakly in $L_p(0,T;H^2)$. Since u^n_t converges to u_t in the distribution sense, the weak limit is u_t also in $L_p(0,T;H^2)$. In particular, $u_t\in L_p(0,T;H^2)$. Similarly, by (2.4) again, $u^n_{tt}\rightharpoonup u_{tt}$ in $L_p(0,T;L_2)$. By (2.10) we may further assume that (u^n_t,ϕ_m), $A(u^n,\phi_m)$, and $B(u^n,\phi_m)$ all converge weak * in $L_\infty(0,T)$, and the limits are (u_t,ϕ_m), $A(u,\phi_m)$, and $B(u,\phi_m)$, respectively. Hence, for any $\psi\in L_1(0,T)$ and $m>0$,

$$\int_0^T\Big[(u_t(t),\phi_m)+A(u(t),\phi_m)-\int_0^t K(t-s)B(u(s),\phi_m)\,ds-(f(t),\phi_m)\Big]\,\psi(t)\,dt=0.$$

Since (u_t,ϕ_m) and (u_{tt},ϕ_m) both belong to $L_1(0,T)$, we have that (u_t,ϕ_m) is actually continuous on $[0,T]$. One similarly sees that $A(u,\phi_m)$ and $B(u,\phi_m)$ are continuous. Hence, using the density of the ϕ_m, one obtains the weak form of (1.1). Since $u\in L_1(0,T;H^2\cap H^1_0)$ and $u_t\in L_1(0,T;H^2\cap H^1_0)$, we have actually $u\in\mathscr{C}([0,T];H^2\cap H^1_0)$. Similarly, $u_t\in\mathscr{C}([0,T];L_2)$, and one concludes that (1.1) holds as an equation in $\mathscr{C}([0,T];L_2)$.

This completes the proof of the theorem. □

To see that, in general, u_{tt} blows up as $t \to 0$, consider the problem

$$\begin{aligned} u_t + Au &= \int_0^t (t-s)^{-\alpha} Au(s)\,ds \quad \text{in } \Omega, \quad \text{for } t > 0, \\ u &= 0 \quad \text{on } \partial\Omega, \quad \text{for } t > 0, \\ u(0) &= \phi \quad \text{in } \Omega, \end{aligned}$$

where ϕ is an eigenfunction of A corresponding to the eigenvalue λ. Setting $u(x, t) = \phi(x)y(t)$, we have for the scalar function y

$$\begin{aligned} y' + \lambda y &= \lambda \int_0^t (t-s)^{-\alpha} y(s)\,ds \quad \text{for } t > 0, \\ y(0) &= 1, \end{aligned}$$

and hence

$$y''(t) = \lambda t^{-\alpha} - \lambda y'(t) + \lambda \int_0^t (t-s)^{-\alpha} y'(s)\,ds.$$

Since $y' \in \mathscr{C}([0, T])$, we conclude that, for this particular function, (cf. also Miller and Feldstein [11])

$$\|u_{tt}\| \sim \lambda t^{-\alpha} \quad \text{as } t \to 0.$$

3. Discretization in space

In this section we shall derive the error estimate (1.5) stated in the introduction for the semidiscrete method (1.4).

For the analysis we introduce, following [1], the Ritz-Volterra projection V_h defined for an appropriately smooth function u by

$$\text{(3.1)} \quad A((V_h u - u)(t), \chi) = \int_0^t K(t-s) B((V_h u - u)(s), \chi)\,ds, \quad \forall\, \chi \in S_h, \ t \geq 0.$$

We have the following error estimate:

Lemma 3. *We have for the Ritz-Volterra projection*

$$\begin{aligned} &\|(V_h u - u)(t)\| + h\|(V_h u - u)(t)\|_1 \\ &\qquad \leq Ch^2 \sup_{s \leq t} \|u(s)\|_2 \leq Ch^2 \left\{ \|u_0\|_2 + \int_0^t \|u_t\|_2\,ds \right\}. \end{aligned}$$

Proof. Let $W = V_h u$ and $\rho = W - u$. We begin with an H^1 estimate, and introduce also the standard Ritz projection R_h defined by

$$A(R_h u - u, \chi) = 0, \quad \forall\, \chi \in S_h.$$

We recall that (see Ciarlet [2, (18.3) and (19.13)]), under the assumption (1.3),

$$\|R_h u - u\| + h\|R_h u - u\|_1 \leq Ch^2 \|u\|_2.$$

We have, using the definition of W, that, with $c > 0$,

$$\begin{aligned} c\|(W - R_h u)(t)\|_1^2 &\leq A(W - R_h u, W - R_h u) = A(\rho, W - R_h u)(t) \\ &= \int_0^t K(t-s) B(\rho(s), (W - R_h u)(t))\,ds \\ &\leq C\|(W - R_h u)(t)\|_1 \int_0^t (t-s)^{-\alpha} \|\rho(s)\|_1\,ds \end{aligned}$$

and hence

$$\|\rho(t)\|_1 \le C \int_0^t (t-s)^{-\alpha} \|\rho(s)\|_1 \, ds + \|(R_h u - u)(t)\|_1 .$$

Lemma 1 now implies

$$\|\rho(t)\|_1 \le C_T \sup_{s \le t} \|(R_h u - u)(s)\|_1 \le C_T h \sup_{s \le t} \|u(s)\|_2 .$$

We next turn to the L_2 estimate, which will be derived by a duality argument, thus using

$$\|\rho(t)\| = \sup_{\|\phi\|=1} (\rho(t), \phi).$$

For each such ϕ, we let ψ be the solution of

$$A\psi = \phi \quad \text{in } \Omega, \qquad \psi = 0 \quad \text{on } \partial\Omega,$$

and recall that

$$\|\psi\|_2 \le C\|\phi\| = C. \tag{3.2}$$

Then, for $\chi \in S_h$,

$$(\rho(t), \phi) = A(\rho, \psi) = A(\rho, \psi - \chi) + A(\rho, \chi).$$

Here,

$$\begin{aligned} A(\rho, \chi) &= \int_0^t K(t-s) B(\rho(s), \chi) \, ds \\ &= \int_0^t K(t-s) B(\rho(s), \chi - \psi) \, ds + \int_0^t K(t-s)(\rho(s), B^* \psi) \, ds , \end{aligned}$$

and hence, with $\chi = R_h \psi$, using (3.2),

$$\begin{aligned} (\rho(t), \phi) &\le C \sup_{s \le t} \|\rho(s)\|_1 \|R_h \psi - \psi\|_1 + C \int_0^t (t-s)^{-\alpha} \|\rho(s)\| \, ds \; \|\psi\|_2 \\ &\le C \left\{ h^2 \sup_{s \le t} \|u(s)\|_2 + \int_0^t (t-s)^{-\alpha} \|\rho(s)\| \, ds \right\} . \end{aligned}$$

Thus,

$$\|\rho(t)\| \le C h^2 \sup_{s \le t} \|u(s)\|_2 + C \int_0^t (t-s)^{-\alpha} \|\rho(s)\| ds ,$$

which by Lemma 1 completes the proof of Lemma 3. □

We shall also need the following estimate for the time derivative of the error in the Ritz-Volterra projection.

Lemma 4. *Under the assumptions of Lemma* 3 *we have, for* $\rho = V_h u - u$,

$$\int_0^t (\|\rho_t\| + h\|\rho_t\|_1) \, ds \le C h^2 \left\{ \|u_0\|_2 + \int_0^t \|u_t\|_2 \, ds \right\} .$$

Proof. Writing (3.1) in the form

$$A(\rho(t), \chi) = \int_0^t K(s) B(\rho(t-s), \chi) \, ds , \quad \forall \chi \in S_h ,$$

we obtain by differentiation

$$A(\rho_t(t), \chi) = K(t)B(\rho(0), \chi) + \int_0^t K(s)B(\rho_t(t-s), \chi)\, ds. \tag{3.3}$$

We begin with the H^1 estimate. We have, for $W = V_h u$,

$$\begin{aligned} c\|W_t - R_h u_t\|_1^2 &\le A(W_t - R_h u_t, W_t - R_h u_t) \\ &= A(\rho_t, W_t - R_h u_t) = K(t)B(\rho(0), W_t - R_h u_t) \\ &\quad + \int_0^t K(t-s)B(\rho_t(s), W_t - R_h u_t)\, ds. \end{aligned}$$

Hence,

$$\|(W_t - R_h u_t)(t)\|_1 \le Ct^{-\alpha}\|\rho(0)\|_1 + C\int_0^t (t-s)^{-\alpha}\|\rho_t(s)\|_1\, ds$$

or

$$\begin{aligned} \|\rho_t(t)\|_1 &\le Ct^{-\alpha}\|\rho(0)\|_1 + \|(R_h u_t - u_t)(t)\|_1 + C\int_0^t (t-s)^{-\alpha}\|\rho_t(s)\|_1\, ds \\ &\le Ch\{t^{-\alpha}\|u_0\|_2 + \|u_t(t)\|_2\} + C\int_0^t (t-s)^{-\alpha}\|\rho_t(s)\|_1\, ds. \end{aligned}$$

Thus by Lemma 1,

$$\|\rho_t(t)\|_1 \le Ch\left\{t^{-\alpha}\|u_0\|_2 + \|u_t(t)\|_2 + \int_0^t (t-s)^{-\alpha}\|u_t(s)\|_2\, ds\right\},$$

and finally

$$\begin{aligned} \int_0^t \|\rho_t\|_1 ds &\le Ch\left\{\|u_0\|_2 + \int_0^t \|u_t\|_2\, ds + \int_0^t\int_0^s (s-\tau)^{-\alpha}\|u_t(\tau)\|_2\, d\tau\, ds\right\} \\ &\le Ch\left\{\|u_0\|_2 + \int_0^t \|u_t\|_2\, ds\right\}. \end{aligned}$$

We now turn to the L_2 bound and write, with the notation of Lemma 3 and using (3.3),

$$\begin{aligned} (\rho_t(t), \phi) &= A(\rho_t(t), \psi) \\ &= A(\rho_t(t), \psi - \chi) + \int_0^t K(t-s)[B(\rho_t(t), \psi - \chi) + (\rho_t(s), B^*\psi)]\, ds \\ &\quad + K(t)[B(\rho(0), \psi - \chi) + (\rho(0), B^*\psi)]. \end{aligned}$$

With an appropriate choice of χ we obtain that

$$\begin{aligned} \|\rho_t(t)\| &\le Ch\left\{\|\rho_t(t)\|_1 + \int_0^t (t-s)^{-\alpha}\|\rho_t(s)\|_1\, ds\right\} + Ch^2 t^{-\alpha}\|u_0\|_2 \\ &\quad + C\int_0^t (t-s)^{-\alpha}\|\rho_t(s)\|\, ds, \end{aligned}$$

from which we conclude by Lemma 1 that

$$\|\rho_t(t)\| \le Ch\left\{\|\rho_t(t)\|_1 + \int_0^t (t-s)^{-\alpha}\|\rho_t(s)\|_1\, ds\right\} + Ch^2 t^{-\alpha}\|u_0\|_2.$$

After integration and using the H^1 estimate already derived we have

$$\int_0^t \|\rho_t\|\,ds \le Ch\int_0^t \|\rho_t(s)\|_1\,ds + Ch^2\|u_0\|_2 \le Ch^2\left\{\|u_0\|_2 + \int_0^t \|u_t(s)\|_2\,ds\right\},$$

which thus completes the proof. □

Theorem 2. *Assume that* u_{0h} *is chosen so that*

$$\|u_{0h} - u_0\| \le Ch^2\|u_0\|_2.$$

Then for each $T > 0$ *there is a constant* C_T *such that for the solutions of* (1.1) *and* (1.4)

$$\|u_h(t) - u(t)\| \le C_T h^2\left\{\|u_0\|_2 + \int_0^t \|u_t\|_2\,ds\right\} \quad \textit{for } t \le T.$$

Proof. In a standard fashion we write

$$u_h - u = (u_h - V_h u) + (V_h u - u) = \theta + \rho.$$

Lemma 3 immediately gives the desired estimate for ρ, so it remains to bound θ.

We have directly from our definitions

$$(\theta_t, \chi) + A(\theta, \chi) = \int_0^t K(t-s)B(\theta(s), \chi)\,ds + (\rho_t, \chi), \quad \forall \chi \in S_h,$$

and hence, setting $\chi = \theta$,

$$\frac{1}{2}\frac{d}{dt}\|\theta\|^2 + A(\theta, \theta) \le C\int_0^t (t-s)^{-\alpha}\|\theta(s)\|_1\|\theta(t)\|_1 ds + \|\rho_t\|\,\|\theta\|.$$

By integration this yields

$$\|\theta(T)\|^2 + \int_0^T \|\theta\|_1^2\,dt$$
$$\le C\left\{\|\theta(0)\|^2 + \int_0^T\int_0^t (t-s)^{-\alpha}\|\theta(s)\|_1\|\theta(t)\|_1\,ds\,dt + C\int_0^T \|\rho_t\|\,\|\theta\|\,dt\right\}.$$

Using Lemma 2 with a suitable choice of ε for the double integral, we thus have

$$\|\theta(T)\|^2 + \int_0^T \|\theta\|_1^2\,dt$$
$$\le C\left\{\|\theta(0)\|^2 + \int_0^T \|\rho_t\|\,\|\theta\|\,dt + \int_0^T (T-t)^{-\alpha}\int_0^t \|\theta(s)\|_1^2\,ds\,dt\right\}.$$

By Lemma 1, therefore, we obtain the bound

$$\|\theta(T)\|^2 + \int_0^T \|\theta\|_1^2\,dt \le C_T\left\{\|\theta(0)\|^2 + \int_0^T \|\rho_t\|\,\|\theta\|\,dt\right\},$$

whence, using also Lemma 4, and noting that $V_h(0) = R_h$,

$$\|\theta(T)\| \le C_T\left\{\|\theta(0)\| + \int_0^T \|\rho_t\|\,dt\right\}$$
$$\le C_T\left\{\|u_{0h} - R_h u_0\| + h^2\left(\|u_0\|_2 + \int_0^T \|u_t\|_2\,ds\right)\right\}.$$

In view of our choice of u_{h0} this completes the proof of the desired estimate for θ, and thus of the theorem. □

4. The completely discrete scheme

In this section we shall consider the completely discrete method (1.7).

In the next lemma we estimate a time-discrete $L_1(0, T; L_2(\Omega))$ type norm of the quadrature error

$$\varepsilon_n(\phi) = \sum_{j=0}^{n-1} \kappa_{n-j}\phi(t_j) - \int_0^{t_n} K(t_n - s)\phi(s)\,ds,$$

where κ_j is defined by (1.6).

Lemma 5. *For each $T>0$ there is a constant C_T such that, if $\phi_t \in L_1(0, T; L_2)$, then*

$$k\sum_{n=1}^{N} \|\varepsilon_n(\phi)\| \le C_T k \int_0^{t_N} \|\phi_t(s)\|\,ds \quad \textit{for } Nk \le T.$$

Proof. By the definition of the κ_j we have

$$\varepsilon_n(\phi) = \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} K(t_n - s)(\phi(t_j) - \phi(s))\,ds,$$

so that by (1.2), for each $x \in \Omega$,

$$\begin{aligned}|\varepsilon_n(\phi)| &\le \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} |K(t_n - s)| \int_{t_j}^{t_{j+1}} |\phi_t(\sigma)|\,d\sigma\,ds \\ &\le C\sum_{j=0}^{n-1} \mu_{\alpha, n-j} \int_{t_j}^{t_{j+1}} |\phi_t(\sigma)|\,d\sigma,\end{aligned}$$

where

$$\mu_{\alpha, j} = \int_{t_{j-1}}^{t_j} s^{-\alpha}\,ds = (1-\alpha)^{-1}(t_j^{1-\alpha} - t_{j-1}^{1-\alpha}). \tag{4.1}$$

By integration in x and use of Minkowski's inequality this yields

$$\|\varepsilon_n(\phi)\| \le C\sum_{j=0}^{n-1} \mu_{\alpha, n-j} \int_{t_j}^{t_{j+1}} \|\phi_t\|\,ds.$$

Hence, by interchanging the orders of summation we find

$$\sum_{n=1}^{N} \|\varepsilon_n(\phi)\| \le C\sum_{j=0}^{N-1} \sum_{n=j+1}^{N} \mu_{\alpha, n-j} \int_{t_j}^{t_{j+1}} \|\phi_t\|\,ds \le C_T \int_0^{T_N} \|\phi_t\|\,ds,$$

since

$$\sum_{n=j+1}^{N} \mu_{\alpha, n-j} = \int_0^{t_{N-j}} s^{-\alpha}\,ds \le C_T = (1-\alpha)^{-1} T^{1-\alpha}.$$

This completes the proof. □

The following two lemmas are discrete analogues of Lemmas 1 and 2, and are proved similarly to these.

Lemma 6. *Let* $\mu_{\alpha,j}$ *be defined by* (4.1) *and assume that* $y_n \geq 0$ *and satisfies*

$$y_n \leq b_n + \beta \sum_{j=0}^{n-1} \mu_{\alpha,n-j} y_j \quad \text{for } n \geq 0,$$

where $b_n \geq 0$, $\beta \geq 0$. *Then for each* $T > 0$ *there is a constant* C_T *such that*

$$y_n \leq b_n + C_T \sum_{j=0}^{n-1} \mu_{\alpha,n-j} b_j \quad \text{for } nk \leq T.$$

Lemma 7. *Let* $K \in L_1(0, T)$, *and let* κ_j *be defined by* (1.6). *Then for each* $\varepsilon > 0$ *there is a constant* $C_\varepsilon = C_\varepsilon(\|K\|_{L_1(0,T)})$ *such that*

$$\left| \sum_{n=1}^{N} \sum_{j=0}^{n-1} \kappa_{n-j} f_j f_n \right| \leq \varepsilon \sum_{n=1}^{N} f_n^2 + C_\varepsilon \sum_{n=0}^{N-1} |\kappa_{N-n}| \sum_{j=0}^{n-1} f_j^2.$$

The following error estimate is our main result of this section. Its proof will require the inverse estimate

$$\|\chi\|_1 \leq Ch^{-1}\|\chi\|, \quad \forall \chi \in S_h. \tag{4.2}$$

Theorem 3. *Assume that* S_h *satisfies* (4.2) *and that* u_{0h} *is chosen so that*

$$\|u_{0h} - u_0\| \leq h^2 \|u_0\|_2. \tag{4.3}$$

Then for each $T > 0$ *there is a constant* C_T *such that for the solutions of* (1.7) *and* (1.1)

$$\|U^n - u(t_n)\| \leq C_T(h^2 + k) \left\{ \|u_0\|_2 + \int_0^{t_n} (\|u_{tt}\| + \|u_t\|_2)\, ds \right\} \quad \text{for } t \leq T.$$

Proof. With V_h the Ritz-Volterra projection introduced in (3.1), we write

$$U^n - u(t_n) = (U^n - V_h u(t_n)) + (V_h u(t_n) - u(t_n)) = \theta^n + \rho^n.$$

The term ρ^n is estimated as desired by Lemma 3. For θ^n we have by our definitions

$$(\bar{\partial}_t \theta^n, \chi) + A(\theta^n, \chi) = \sum_{j=0}^{n-1} \kappa_{n-j} B(\theta^j, \chi) + (\tau_n, \chi), \tag{4.4}$$

where

$$(\tau_n, \chi) = (u_t^n - \bar{\partial}_t V_h u^n, \chi) + \sum_{j=0}^{n-1} \kappa_{n-j} B(V_h u^j, \chi) - \int_0^{t_n} K(t_n - s) B(V_h u(s), \chi)\, ds.$$

Defining $B_h \colon H_0^1 \to S_h$ by

$$(B_h \phi, \chi) = B(\phi, \chi), \quad \forall \chi \in S_h,$$

we may write

$$\tau_n = u_t^n - \bar{\partial}_t V_h u^n + \varepsilon_n(B_h V_h u).$$

We shall show by an energy argument that

$$\|\theta^N\| \leq C_T \left(\|\theta^0\| + k \sum_{n=1}^{N} \|\tau_n\| \right) \quad \text{for } Nk \leq T. \tag{4.5}$$

Assuming this for a moment, we then write $\tau_n = \sum_{l=1}^4 \tau_n^l$, where

$$\begin{aligned}\tau_n^1 &= u_t^n - \overline{\partial}_t u^n,\\ \tau_n^2 &= \overline{\partial}_t(u^n - V_h u^n) = -\overline{\partial}_t \rho^n,\\ \tau_n^3 &= \varepsilon_n(B_h u),\\ \tau_n^4 &= \varepsilon_n(B_h \rho).\end{aligned}$$

We have at once

$$k\sum_{n=1}^N \|\tau_n^1\| \le Ck\sum_{n=1}^N \int_{t_{n-1}}^{t_n} \|u_{tt}\|\,ds = Ck\int_0^{t_N} \|u_{tt}\|\,ds,$$

and, by Lemma 4,

$$\begin{aligned}k\sum_{n=1}^N \|\tau_n^2\| &\le \sum_{n=1}^N \int_{t_{n-1}}^{t_n} \|\rho_t\|\,ds = \int_0^{t_N} \|\rho_t\|\,ds\\ &\le Ch^2\left\{\|u_0\|_2 + \int_0^{t_N} \|u_t\|_2\,ds\right\}.\end{aligned}$$

To estimate τ_n^3, we note that when u is smooth, $B_h u = P_h B u$ and hence, by Lemma 5,

$$k\sum_{n=1}^N \|\tau_n^3\| \le Ck\int_0^{t_N} \|P_h B u_t\|\,ds \le Ck\int_0^{t_N} \|u_t\|_2\,ds.$$

Using the inverse assumption (4.2), we have

$$(B_h\rho, \chi) = B(\rho, \chi) \le C\|\rho\|_1\|\chi\|_1 \le Ch^{-1}\|\rho\|_1\|\chi\|,$$

so that

$$\|B_h\rho\| \le Ch^{-1}\|\rho\|_1.$$

Hence, for τ_n^4, we have by Lemmas 5 and 4,

$$\begin{aligned}k\sum_{n=1}^N \|\tau_n^4\| &\le C_T k\int_0^{t_N} \|B_h\rho_t\|\,ds\\ &\le C_T k h^{-1}\int_0^{t_N} \|\rho_t\|_1\,ds \le C_T k\left\{\|u_0\|_2 + \int_0^{t_N} \|u_t\|_2\,ds\right\}.\end{aligned}$$

Inserted into (4.5), these estimates show

$$\|\theta^N\| \le C_T\|u_{0h} - R_h u_0\| + C_T(h^2 + k)\left\{\|u_0\|_2 + \int_0^{t_N} (\|u_{tt}\| + \|u_t\|_2)\,ds\right\}.$$

In view of (4.3) this completes the proof.

It remains to show (4.5). For this we choose $\chi = \theta^n$ in (4.4), which yields

$$\frac{1}{2}\overline{\partial}_t\|\theta^n\|^2 + \frac{1}{2}k\|\overline{\partial}_t\theta^n\|^2 + A(\theta^n, \theta^n) = \sum_{j=0}^{n-1} \kappa_{n-j} B(\theta^j, \theta^n) + (\tau_n, \theta^n),$$

whence

$$\overline{\partial}_t\|\theta^n\|^2 + \|\theta^n\|_1^2 \le C\sum_{j=0}^{n-1} \mu_{\alpha, n-j}\|\theta^j\|_1\|\theta^n\|_1 + C\|\tau_n\|\,\|\theta^n\|,$$

and, after summation,

$$\|\theta^N\|^2 + k\sum_{n=1}^{N}\|\theta^n\|_1^2 \le \|\theta^0\|^2 + Ck\sum_{n=1}^{N}\sum_{j=0}^{n-1}\mu_{\alpha,n-j}\|\theta^j\|_1\|\theta^n\|_1 + Ck\sum_{n=1}^{N}\|\tau_n\|\,\|\theta^n\|.$$

Using Lemma 7 with $K(t) = Ct^{-\alpha}$, we may conclude

$$\|\theta^N\|^2 + k\sum_{n=1}^{N}\|\theta^n\|_1^2 \le \|\theta^0\|^2 + Ck\sum_{n=1}^{N}\|\tau_n\|\,\|\theta^n\| + C\sum_{n=0}^{N-1}\mu_{\alpha,N-n}\left(k\sum_{j=0}^{n-1}\|\theta^j\|_1^2\right).$$

In combination with Lemma 6, applied to $y_N = k\sum_{n=1}^{N}\|\theta^j\|_1^2$, this shows

$$\|\theta^N\|^2 \le C_T\left(\|\theta^0\|^2 + k\sum_{n=1}^{N}\|\tau_n\|\,\|\theta^n\|\right) \quad \text{for } Nk \le T,$$

from which (4.5) follows.

This completes the proof. □

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