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Finite Element Approximation of a Sixth Order Nonlinear Degenerate Parabolic Equation

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Summary We consider a finite element approximation of the sixth order nonlinear degenerate parabolic equation $u_t = \nabla.(\, b(u)\, \nabla \Delta^2 u)$, where generically $b(u) := |u|^\gamma$ for any given $\gamma \in (0,\infty)$. In addition to showing well-posedness of our approximation, we prove convergence in space dimensions $d \leq 3$. Furthermore an iterative scheme for solving the resulting nonlinear discrete system is analysed. Finally some numerical experiments in one and two space dimensions are presented.

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1 Introduction

Degenerate diffusion problems of the type $u_t = (-1)^k\, \nabla.(|u|^\gamma\, \nabla \Delta^k u)$, for given $\gamma \in (0,\infty)$ and nonnegative integer k, occur in mathematical models of many physical processes. The second order case, $k = 0$, leading to the porous medium equation has been widely studied by analysts and numerical analysts. Several mathematical models in fluid dynamics and material science have lead to the fourth order case ($k = 1$); e.g. lubrication approximation for thin viscous films ($\gamma = 3$), Hele Shaw flow and the Cahn-Hilliard equation with degenerate mobility ($\gamma = 1$). Over the last decade there has been a huge amount of work among analysts on this fourth order case, see the survey paper [7]. From the numerical analyst viewpoint, there has been

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very little work on this fourth order problem. A fully practical finite element approximation based on a variational inequality formulation was proposed and analysed in [4]. For extensions of this approach to degenerate fourth order systems arising in Cahn-Hilliard models of phase separation, see [5, 6, 3]. Schemes making use of entropy type estimates have also been proposed and analysed for the fourth order problem in [16] and [12]. The sixth order case, $k = 2$, with $\gamma = 3$ arises in a mathematical model of the oxidation of silicon in superconductor devices, see [13]. As stated there, with $\gamma = 3$ the case $k = 2$ is in the hierarchy of degenerate nonlinear parabolic equations describing the motion of thin viscous droplets under different driving forces: gravity ($k = 0$), surface tension ($k = 1$) and an elastic plate ($k = 2$). There are a few papers which include numerical experiments on the sixth order case, see for example [11]. This was restricted to one space dimension and moreover no attempt was made to analyse their finite difference approach. Our goal in this paper is to develop and analyse a fully practical scheme that works in all space dimensions. Our proposed scheme is the natural extension of the scheme for the corresponding fourth order problem in [4].

We consider the initial boundary value problem for the sixth order case, $k = 2$: **(P)** Find a function $u : \Omega \times [0, T] \to \mathbb{R}$ such that

$$\frac{\partial u}{\partial t} = \nabla.(b(u)\nabla\Delta^2 u) \qquad \text{in } \Omega_T := \Omega \times (0, T), \tag{1.1a}$$

$$u(x, 0) = u^0(x) \geq 0 \qquad \forall\, x \in \Omega, \tag{1.1b}$$

$$\frac{\partial u}{\partial \nu} = \frac{\partial \Delta u}{\partial \nu} = b(u)\frac{\partial \Delta^2 u}{\partial \nu} = 0 \qquad \text{on } \partial\Omega \times (0, T); \tag{1.1c}$$

where Ω is a bounded domain in $\mathbb{R}^d, d \leq 3$, with a Lipschitz boundary $\partial\Omega$, ν is normal to $\partial\Omega$ and $T > 0$ is a fixed positive time. To simplify our presentation we restrict ourselves to the case

$$b(u) := |u|^\gamma, \qquad \gamma \in (0, \infty), \tag{1.2}$$

but our results extend to more general mobilities of the form $b(u) := b_0(u)|u|^\gamma$ with a positive and sufficiently smooth b_0.

Degenerate parabolic equations of higher order ($k \geq 1$) exhibit some new characteristic features which are fundamentally different to those for second order degenerate parabolic equations. The key point is that there is no maximum or comparison principle for parabolic equations of higher order. This drastically complicates the analysis since a lot of results which are known for second order equations are proven with the help of comparison techniques. Related to this, is the fact that there is still no uniqueness result known for such problems. Although there is no comparison principle, one of the main features of these degenerate equations is the fact that one can show existence of nonnegative solutions if given nonnegative initial data.

This is in contrast to linear parabolic equations of higher order, where solutions which are initially positive may become negative in certain regions.

Let us review what is known for problem (P). Existence of Hölder continuous nonnegative solutions to (P) in one space dimension ($d = 1$) has been established in [8]. They used a very weak solution concept, which basically says that a function u solves (P) if for all $\eta \in L^2(0,T; H^1(\Omega))$

$$\int_0^T \langle \tfrac{\partial u}{\partial t}, \eta \rangle \,\mathrm{d}t + \int_{\{|u|>0\}} b(u) \nabla \Delta^2 u \nabla \eta \,\mathrm{d}x \,\mathrm{d}t = 0. \tag{1.3}$$

They showed that there exists a nonnegative solution

$$u \in L^\infty(0,T; H^2(\Omega)) \cap C^{1,\frac{1}{5}}_{x,t}(\overline{\Omega}_T) \quad \text{with} \quad u_x \in C^{\frac{1}{2},\frac{1}{12}}_{x,t}(\overline{\Omega}_T). \tag{1.4}$$

As stated above, there is no uniqueness result for (P) and as far as we are aware there are no other theoretical results on problem (P) in the literature.

In the case $\gamma = 1$, (P) has a source type similarity solution

$$u(x,t) = \frac{1}{5040(t+\vartheta)^{\frac{1}{7}}} \left[\omega^2 - \frac{x^2}{(t+\vartheta)^{\frac{2}{7}}} \right]_+^3, \tag{1.5}$$

where ϑ and ω are arbitrary positive constants, see [15]. Therefore, as to be expected with such degenerate diffusion problems, there exist “strong” solutions which have a finite speed of propagation property. This implies that the boundaries of where u is positive can be viewed as moving free boundaries. Hence, we require our numerical algorithm to be able to efficiently resolve such free boundaries.

In order to formulate a fully practical finite element approximation of problem (P), we extend the approach in [4] for the fourth order case by introducing potentials v and w. We then write the sixth order parabolic equation as the system of equations

$$\tfrac{\partial u}{\partial t} = \nabla.(b(u)\nabla w), \qquad w = -\Delta v, \qquad v = -\Delta u \qquad \text{in } \Omega_T.$$

On the discrete level, the nonnegativity of the approximation to u is not guaranteed when we discretise the above system in a naive way. We therefore impose the nonnegativity of the discrete solution as a constraint. Using a semi-implicit time discretisation we solve a discrete variational inequality at each time step.

The layout of this paper is as follows. In §2 we formulate our finite element approximation to (P) and prove its well-posedness, and derive stability bounds. The above results are direct analogues of those established for the corresponding fourth order problem in [4]. In §3 we establish convergence of our approximation. Unlike the numerical approximations of degenerate fourth order problems, see [4, 5, 12, 6, 3], where convergence is only established in one space dimension; we are able, by exploiting the fact that the operator is of higher order, to show convergence in all space dimensions ($1 \leq d \leq 3$) to a solution u satisfying the solution concept (1.3) of [8]. This, in particular, extends the existence and regularity results of [8] from one space dimension to higher space dimensions.

In §4 we introduce an algorithm, based on the general splitting algorithm of [14], to solve the discrete variational inequality at each time level. Moreover, we prove convergence of this algorithm. Finally in §5 we present some numerical computations in one and two space dimensions.

Notation and Auxiliary Results

We have adopted the standard notation for Sobolev spaces, denoting the norm of $W^{m,q}(G)$ ($m \in \mathbb{N}$, $q \in [1,\infty]$ and G a bounded domain in $\mathbb{R}^d$ with a Lipschitz boundary) by $\|\cdot\|_{m,q,G}$ and the semi-norm by $|\cdot|_{m,q,G}$. For $q=2$, $W^{m,2}(G)$ will be denoted by $H^m(G)$ with the associated norm and semi-norm written, as respectively, $\|\cdot\|_{m,G}$ and $|\cdot|_{m,G}$. For ease of notation, in the common case when $G \equiv \Omega$ the subscript "Ω" will be dropped on the above norms and semi-norms. Throughout $(\cdot,\cdot)$ denotes the standard L^2 inner product over Ω and $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $\left(H^1(\Omega)\right)'$ and $H^1(\Omega)$. In addition we define $⨍\,\eta := \frac{1}{\underline{m}(\Omega)}(\eta,1)$ for all $\eta \in L^1(\Omega)$, where $\underline{m}(\Omega)$ denotes the measure of Ω. We require also the standard Hölder space $C^{0,\alpha}(\overline{\Omega})$ and the Hölder space $C^{\alpha,\beta}_{x,t}(\overline{\Omega}_T)$ for $\alpha,\ \beta \in (0,1]$, which denotes those functions whose time(spatial) derivative(s) is(are) Hölder continuous over $\overline{\Omega}_T$ with exponent $\beta(\alpha)$.

For later purposes, we recall the following well-known Sobolev interpolation results, e.g. see [1]: Let $q \in [1,\infty]$ and $m \geq 1$, then for all $z \in W^{m,q}(\Omega)$ the inequality

$$|z|_{0,r} \leq C|z|_{0,q}^{1-\sigma}\|z\|_{m,q}^{\sigma} \quad \text{holds for } r \in \begin{cases} [q,\infty] & \text{if } m - \frac{d}{q} > 0, \\ [q,\infty) & \text{if } m - \frac{d}{q} = 0, \\ [q, -\frac{d}{m-(d/q)}] & \text{if } m - \frac{d}{q} < 0; \end{cases} \tag{1.6}$$

where $\sigma = \frac{d}{m}\left(\frac{1}{q} - \frac{1}{r}\right)$ and C is a constant depending only on Ω, q, r and m.

It is convenient to introduce the "inverse Laplacian" operator $\mathcal{G} : \mathcal{F} \to Z$ such that

$$(\nabla \mathcal{G} z, \nabla \eta) = \langle z, \eta\rangle \quad \forall\, \eta \in H^1(\Omega), \tag{1.7}$$

where $\mathcal{F} := \left\{z \in (H^1(\Omega))' : \langle z, 1\rangle = 0\right\}$ and $Z := \{z \in H^1(\Omega) : (z,1) = 0\}$. The well-posedness of $\mathcal{G}$ follows from the Lax-Milgram theorem and the Poincaré inequality

$$|\eta|_{0,q} \leq C(|\eta|_{1,q} + |(\eta,1)|) \quad \forall\, \eta \in W^{1,q}(\Omega) \quad \text{and} \quad q \in [1,\infty]. \tag{1.8}$$

Throughout C denotes a generic constant independent of h and τ, the mesh and temporal discretisation parameters. In addition $C(a_1, \cdots, a_I)$ denotes a constant depending on the arguments $\{a_i\}_{i=1}^{I}$.

2 Finite Element Approximation

We consider the finite element approximation of (P), firstly, under the following assumptions on the mesh:

(A1) Let Ω be a polyhedral domain. Let $\mathcal{T}^h$ be a regular partitioning of Ω into disjoint open simplices κ with $h_\kappa := \mathrm{diam}(\kappa)$ and $h := \max_{\kappa\in\mathcal{T}^h} h_\kappa$, so that $\overline{\Omega} = \cup_{\kappa\in\mathcal{T}^h}\overline{\kappa}$.

Associated with $\mathcal{T}^h$ is the finite element space

$$S^h := \{\chi \in C(\overline{\Omega}) : \chi\,|_\kappa \ \text{ is linear}\ \forall\,\kappa \in \mathcal{T}^h\} \subset H^1(\Omega).$$

We introduce also the closed convex sets

$$K^h := \{\chi \in S^h : \chi \geq 0 \text{ in } \Omega\} \subset K := \{\eta \in H^1(\Omega) : \eta \geq 0 \ a.e. \text{ in } \Omega\}.$$

Let J be the set of nodes of $\mathcal{T}^h$ and $\{p_j\}_{j\in J}$ the coordinates of these nodes. Let $\{\chi_j\}_{j\in J}$ be the standard basis functions for S^h; that is $\chi_j \in K^h$ and $\chi_j(p_i) = \delta_{ij}$ for all $i, j \in J$. We introduce $\pi^h : C(\overline{\Omega}) \to S^h$, the interpolation operator, such that $(\pi^h\eta)(p_j) = \eta(p_j)$ for all $j \in J$. A discrete semi-inner product on $C(\overline{\Omega})$ is then defined by

$$(\eta_1, \eta_2)^h := (\pi^h[\eta_1\,\eta_2], 1) \equiv \textstyle\sum_{j\in J} \omega_j\, \eta_1(p_j)\, \eta_2(p_j), \qquad (2.1)$$

where $\omega_j := (1, \chi_j)$. The induced semi-norm is then $|\eta|_h := [\,(\eta, \eta)^h\,]^{\frac{1}{2}}$, where $\eta \in C(\overline{\Omega})$.

Let $0 \equiv t_0 < t_1 < \ldots < t_{N-1} < t_N \equiv T$ be a partitioning of $[0, T]$ into variable time steps $\tau_n := t_n - t_{n-1}$, $n = 1 \to N$. Let $\tau := \max_{n=1\to N} \tau_n$. We then consider the following fully practical finite element approximation of (P):

(P$^{h,\tau}$) For $n \geq 1$, find $\{U^n, V^n, W^n\} \in K^h \times [S^h]^2$ such that for all $\chi \in S^h$ and for all $\eta^h \in K^h$

$$\left(\tfrac{U^n - U^{n-1}}{\tau_n}, \chi\right)^h + (\pi^h[b(U^{n-1})]\nabla W^n, \nabla\chi) = 0\,, \qquad (2.2a)$$

$$(\nabla U^n, \nabla\chi) = (V^n, \chi)^h\,, \qquad (2.2b)$$

$$(\nabla V^n, \nabla(\eta^h - U^n)) \geq (W^n, \eta^h - U^n)^h\,; \qquad (2.2c)$$

where $U^0 \in K^h$ is an approximation of u^0, e.g. $U^0 \equiv \pi^h u^0$.

(P$^{h,\tau}$) is the natural extension of the finite element approximation of the corresponding fourth order nonlinear degenerate parabolic equation, which was proposed and analysed in [4]. The only minor difference to this extension is that $\pi^h[b(U^{n-1})]$ is used instead of $b(U^{n-1})$ in (2.2a) to be more practical.

Introducing the "discrete Laplacian" operator $\Delta^h : S^h \to Z^h$ such that

$$(\Delta^h z^h, \chi)^h = -(\nabla z^h, \nabla \chi) \quad \forall\, \chi \in S^h, \tag{2.3}$$

where $Z^h := \{z^h \in S^h : (z^h, 1) = 0\} \subset Z$; (2.2b,c) can be rewritten as $V^n \equiv -\Delta^h U^n$ with

$$(\Delta^h U^n, \Delta^h(\chi - U^n))^h \geq (W^n, \chi - U^n)^h \quad \forall\, \chi \in K^h. \tag{2.4}$$

Below we recall some well-known results concerning S^h and the above operators. For any $\kappa \in \mathcal{T}^h$ and for $m = 0$ or 1, we have that

$$\left|\int_\kappa (I - \pi^h)(z^h\, \chi)\,\mathrm{d}x\right| \leq C h^{1+m} \|z^h\|_{m,\kappa} \|\chi\|_{1,\kappa} \quad \forall\, z^h, \chi \in S^h; \tag{2.5}$$

$$\int_\kappa \chi^2\,\mathrm{d}x \leq \int_\kappa \pi^h[\chi^2]\,\mathrm{d}x \leq (d+2) \int_\kappa \chi^2\,\mathrm{d}x \quad \forall\, \chi \in S^h; \tag{2.6}$$

$$\lim_{h \to 0} |(I - \pi^h)\eta|_{0,\infty} = 0 \quad \forall\, \eta \in C(\overline{\Omega}); \tag{2.7}$$

$$|(I - \pi^h)\eta|_{m,r,\kappa} \leq C h^\sigma |\eta|_{2,\kappa} \quad \forall\, \eta \in H^2(\kappa), \tag{2.8}$$

provided either $\sigma := 2 - m - \frac{d}{2} > 0$ if $r = \infty$ or $\sigma := 2 - m - d(\frac{1}{2} - \frac{1}{r}) \geq 0$ if $r \in [2, \infty)$.

Similarly to (1.7), we introduce the operator $\mathcal{G}^h : \mathcal{F} \to Z^h$ such that

$$(\nabla \mathcal{G}^h z, \nabla \chi) = \langle z, \chi \rangle \quad \forall\, \chi \in S^h. \tag{2.9}$$

In addition to (2.9) we introduce $\widehat{\mathcal{G}}^h : \mathcal{F}^h \to Z^h \subset \mathcal{F}^h$ such that

$$(\nabla \widehat{\mathcal{G}}^h z, \nabla \chi) = (z, \chi)^h \quad \forall\, \chi \in S^h, \tag{2.10}$$

where $\mathcal{F}^h := \{z \in C(\overline{\Omega}) : (z, 1)^h = 0\}$. A Young's inequality yields for all $z \in \mathcal{F}^h$, for all $\chi \in S^h$ and for all $\alpha > 0$

$$(z, \chi)^h \equiv (\nabla \widehat{\mathcal{G}}^h z, \nabla \chi) \leq |\widehat{\mathcal{G}}^h z|_1 \, |\chi|_1 \leq \tfrac{1}{2\alpha} |\widehat{\mathcal{G}}^h z|_1^2 + \tfrac{\alpha}{2} |\chi|_1^2 \,. \tag{2.11}$$

Finally, it follows from (2.3) and (2.11) that for all $z^h \in Z^h$

$$\begin{aligned} |z^h|_1^2 = -(z^h, \Delta^h z^h)^h &\leq |z^h|_h \, |\Delta^h z^h|_h \leq |\widehat{\mathcal{G}}^h z^h|_1^{\frac{1}{2}} \, |z^h|_1^{\frac{1}{2}} \, |\Delta^h z^h|_h \\ &\leq |\widehat{\mathcal{G}}^h z^h|_1^{\frac{2}{3}} \, |\Delta^h z^h|_h^{\frac{4}{3}}. \end{aligned} \tag{2.12}$$

We now adapt the approach taken in [4] to establish the existence of a solution $\{U^n, V^n, W^n\}_{n=1}^N$ to ($\mathrm{P}^{h,\tau}$). Firstly, we need to introduce some notation. In particular we define sets $Z^h(U^{n-1})$ in which we seek the update $U^n - U^{n-1}$. Given $q^h \in K^h$, we set $J_0(q^h) \subset J$ such that

$$j \in J_0(q^h) \iff (\pi^h[b(q^h)], \chi_j) = 0. \tag{2.13}$$

All other nodes we call active nodes and they can be uniquely partitioned so that $J_+(q^h) := J \setminus J_0(q^h) \equiv \bigcup_{m=1}^{M} I_m(q^h)$, $M \geq 1$; where $I_m(q^h)$, $m = 1 \to M$, are mutually disjoint and maximally connected in the following sense: $I_m(q^h)$ is said to be connected if for all $j,\ k \in I_m(q^h)$, there exist $\{\kappa_\ell\}_{\ell=1}^{L} \subseteq \mathcal{T}^h$, not necessarily distinct, such that

$$(a) \quad p_j \in \overline{\kappa}_1, \quad p_k \in \overline{\kappa}_L, \qquad (b) \quad \overline{\kappa}_\ell \cap \overline{\kappa}_{\ell+1} \neq \emptyset \quad \ell = 1 \to L-1,$$
$$(c) \quad q^h \not\equiv 0 \text{ on } \kappa_\ell \quad \ell = 1 \to L. \tag{2.14}$$

$I_m(q^h)$ is said to be maximally connected if there is no other connected subset of $J_+(q^h)$, which contains $I_m(q^h)$. We then set

$$Z^h(q^h) := \{\, z^h \in S^h : z^h(p_j) = 0 \quad \forall\, j \in J_0(q^h) \\ \text{and } (z^h, \Xi_m(q^h))^h = 0,\ m = 1 \to M \,\}, \tag{2.15}$$

where $\Xi_m(q^h) := \sum_{j \in I_m(q^h)} \chi_j$, $m = 1 \to M$.

For later reference we state that any $z^h \in S^h$ can be written as

$$z^h \equiv \overline{z}^h + \sum_{j \in J_0(q^h)} z^h(p_j)\,\chi_j + \sum_{m=1}^{M} \frac{(z^h, \Xi_m(q^h))^h}{(1, \Xi_m(q^h))}\, \Xi_m(q^h), \tag{2.16a}$$

where

$$\overline{z}^h := \sum_{m=1}^{M} \sum_{j \in I_m(q^h)} \left[z^h(p_j) - \frac{(z^h, \Xi_m(q^h))^h}{(1, \Xi_m(q^h))} \right] \chi_j \in Z^h(q^h) \tag{2.16b}$$

is the projection with respect to the $(\cdot,\cdot)^h$ scalar product of z^h onto $Z^h(q^h)$. In order to express V^n and W^n in terms of U^n and U^{n-1} we introduce for all $q^h \in K^h$ the discrete anisotropic Green's operator $\widehat{\mathcal{G}}^h_{q^h} : Z^h(q^h) \to Z^h(q^h)$ such that

$$(\pi^h[b(q^h)]\nabla \widehat{\mathcal{G}}^h_{q^h} z^h, \nabla \chi) = (z^h, \chi)^h \qquad \forall\, \chi \in S^h. \tag{2.17}$$

The well-posedness of $\widehat{\mathcal{G}}^h_{q^h}$ follows immediately from (2.13) and (2.15), see [4, §2] for details in the case when $\pi^h[b(q^h)]$ in (2.17) is replaced by $b(q^h)$. Finally, note that for all $q^h \in K^h$, $Z^h(q^h) \subseteq Z^h$ and in addition that $Z^h(q^h)$ defined in (2.15) is equal to Z^h if q^h is strictly positive.

Theorem 2.1 *Let the assumptions (A1) hold and $U^0 \in K^h$. Then for all h and all time partitions $\{\tau_n\}_{n=1}^{N}$, there exists a solution $\{U^n, V^n, W^n\}_{n=1}^{N}$ to (P$^{h,\tau}$). Moreover $\{U^n, V^n\}_{n=1}^{N}$ are unique and $U^n - U^0 \in Z^h$, $n =$*

$1 \to N$. *In addition* $W^n(p_j)$ *is unique if* $(\pi^h[b(U^{n-1})], \chi_j) > 0$*; for all* $j \in J$, $n = 1 \to N$. *Furthermore, the following stability bounds hold*

$$\max_{n=1\to N} \|U^n\|_1^2 + \max_{n=1\to N} |V^n|_0^2 + \sum_{n=1}^{N} (\tau_n)^2 \left[\|\tfrac{U^n - U^{n-1}}{\tau_n}\|_1^2 + |\tfrac{V^n - V^{n-1}}{\tau_n}|_0^2 \right]$$
$$+ \sum_{n=1}^{N} \tau_n \left[|(\pi^h[b(U^{n-1})])^{\frac{1}{2}} \nabla W^n|_0^2 + [b^{n-1}]^{-1} |\widehat{\mathcal{G}}^h[\tfrac{U^n - U^{n-1}}{\tau_n}]|_1^2 \right]$$
$$\leq C \left\{ |V^0|_0^2 + [\,(U^0, 1)\,]^2 \right\}, \qquad (2.18)$$

where $b^{n-1} := |b(U^{n-1})|_{0,\infty}$ *and* $V^0 := -\Delta^h U^0$*; and* C *is independent of* T*, as well as the mesh parameters.*

Proof It follows from (2.2a) and (2.17) that for $n \geq 1$, given $U^{n-1} \in K^h$, we seek $U^n \in \widetilde{K}^h(U^{n-1})$, where

$$\widetilde{K}^h(U^{n-1}) := K^h \cap \widetilde{Z}^h(U^{n-1})$$
$$\text{and} \quad \widetilde{Z}^h(U^{n-1}) := \{\, \chi \in S^h : \chi - U^{n-1} \in Z^h(U^{n-1}) \,\}. \qquad (2.19)$$

In addition a solution W^n to (2.2a) can be expressed in terms of U^n, on noting (2.17) and (2.16a,b), as

$$W^n \equiv -\widehat{\mathcal{G}}^h_{U^{n-1}}[\tfrac{U^n - U^{n-1}}{\tau_n}] + \sum_{j \in J_0(U^{n-1})} \mu_j^n \, \chi_j + \sum_{m=1}^{M} \lambda_m^n \, \Xi_m(U^{n-1}), \quad (2.20)$$

where $\{\mu_j^n\}_{j \in J_0(U^{n-1})}$ and $\{\lambda_m^n\}_{m=1}^M$ are arbitrary constants. Hence on noting (2.20) and (2.4), $(\mathrm{P}^{h,\tau})$ can be restated as:
For $n \geq 1$, find $U^n \in \widetilde{K}^h(U^{n-1})$ and constant Lagrange multipliers $\{\mu_j^n\}_{j \in J_0(U^{n-1})}$, $\{\lambda_m^n\}_{m=1}^M$ such that for all $\chi \in K^h$

$$a_{U^{n-1}}(U^n, \chi - U^n) \geq \Big(\sum_{j \in J_0(U^{n-1})} \mu_j^n \, \chi_j + \sum_{m=1}^{M} \lambda_m^n \, \Xi_m(U^{n-1}), \chi - U^n \Big)^h, \qquad (2.21)$$

where $a_{U^{n-1}}(\cdot,\cdot) : \widetilde{Z}^h(U^{n-1}) \times S^h \to \mathbb{R}$ is defined for all $z^h \in \widetilde{Z}^h(U^{n-1})$ and $\chi \in S^h$ by

$$a_{U^{n-1}}(z^h, \chi) := (\Delta^h z^h, \Delta^h \chi)^h + (\widehat{\mathcal{G}}^h_{U^{n-1}}[\tfrac{z^h - U^{n-1}}{\tau_n}], \chi)^h.$$

It follows from (2.21), (2.19) and (2.15) that $U^n \in \widetilde{K}^h(U^{n-1})$ is such that

$$a_{U^{n-1}}(U^n, \widetilde{z}^h - U^n) \geq 0 \qquad \forall\, \widetilde{z}^h \in \widetilde{K}^h(U^{n-1}). \qquad (2.22)$$

There exists $U^n \in \widetilde{K}^h(U^{n-1})$ satisfying (2.22), since this is the Euler-Lagrange variational inequality of the minimization problem

$$\min_{\widetilde{z}^h \in \widetilde{K}^h(U^{n-1})} \left\{ |\Delta^h \widetilde{z}^h|_h^2 + \tfrac{1}{\tau_n} |(\pi^h[b(U^{n-1})])^{\frac{1}{2}} \nabla \widehat{\mathcal{G}}^h_{U^{n-1}}(\widetilde{z}^h - U^{n-1})|_0^2 \right\}.$$

Following an identical argument to that in [4, §2], (2.22) yields existence of a solution to (2.21) with

$$\begin{aligned} \mu_j^n &= \tfrac{a_{U^{n-1}}(U^n,\chi_j)}{(1,\chi_j)} \equiv \tfrac{(\Delta^h U^n, \Delta^h \chi_j)^h}{(1,\chi_j)} \qquad && \forall\, j \in J_0(U^{n-1}) \\ \text{and} \quad \lambda_m^n &= \tfrac{a_{U^{n-1}}(U^n, \pi^h[U^n\, \Xi_m(U^{n-1})])}{(U^n, \Xi_m(U^{n-1}))^h} && m = 1 \to M. \end{aligned}$$

Hence, on noting (2.4), (2.21) and (2.20), we have existence of a solution $\{U^n, V^n, W^n\}_{n=1}^N$ to (P$^{h,\tau}$) with $U^n - U^0 \in Z^h$, $n = 1 \to N$.

For fixed $n \geq 1$, if (2.21) has two solutions $\{\, U^{n,i},\ \{\mu_j^{n,i}\}_{j\in J_0(U^{n-1})},$ $\{\lambda_m^{n,i}\}_{m=1}^M \,\}$, $i = 1,\, 2$; then it follows from (2.22) and (2.17) that $\overline{U}^n := U^{n,1} - U^{n,2} \in Z^h(U^{n-1}) \subset Z^h$ satisfies

$$|\Delta^h \overline{U}^n|_h^2 + \tfrac{1}{\tau_n} |\left(\pi^h[b(U^{n-1})]\right)^{\frac{1}{2}} \nabla(\widehat{\mathcal{G}}^h_{U^{n-1}} \overline{U}^n)|_0^2 \leq 0. \tag{2.23}$$

Therefore the uniqueness of $V^n \equiv -\Delta^h U^n$ follows directly from (2.23). Uniqueness of U^n then follows from (2.2b) and (1.8). For any $\delta \in (0,1)$, choosing $\chi \equiv U^n \pm \delta\, \pi^h[U^n\, \Xi_m(U^{n-1})] \equiv \pi^h[\,(1 \pm \delta\, \Xi_m(U^{n-1}))\, U^n\,]$ in (2.21), for $m = 1 \to M$, yields uniqueness of the Lagrange multipliers $\{\lambda_m^n\}_{m=1}^M$. Hence the desired uniqueness result on W^n follows from noting (2.20) and (2.13).

We now prove the stability bound (2.18). For fixed $n \geq 1$ choosing $\chi \equiv W^n$ in (2.2a), $\chi \equiv U^{n-1}$ in (2.2c) and combining yields that

$$(\nabla V^n, \nabla(U^n - U^{n-1})\,) + \tau_n\, (\pi^h[b(U^{n-1})] \nabla W^n, \nabla W^n) \leq 0.$$

Noting (2.2b), for $n = 0$ as well as for $n \geq 1$, and using the identity

$$2s\,(s-r) = s^2 - r^2 + (s-r)^2 \qquad \forall\, r, s \in \mathbb{R}; \tag{2.24}$$

we have that

$$\tfrac{1}{2}\,|V^n|_h^2 + \tfrac{1}{2}\,|V^n - V^{n-1}|_h^2 + \tau_n\,(\pi^h[b(U^{n-1})] \nabla W^n, \nabla W^n) \leq \tfrac{1}{2}\,|V^{n-1}|_h^2.$$

Summing this from $n = 1 \to m$, for $m = 1 \to N$, and noting (2.1) and (2.6) yields the bounds involving V^n and W^n in (2.18). The first two bounds involving U^n in (2.18) then follow from those involving V^n, (2.2b), (2.1), (2.6) and (1.8). Finally choosing $\chi \equiv \widehat{\mathcal{G}}^h(\frac{U^n-U^{n-1}}{\tau_n})$ in (2.2a) and noting (2.10) yields for $n \geq 1$ that

$$|\widehat{\mathcal{G}}^h[\tfrac{U^n-U^{n-1}}{\tau_n}]|_1^2 = -(\pi^h[b(U^{n-1})]\nabla W^n, \nabla\widehat{\mathcal{G}}^h[\tfrac{U^n-U^{n-1}}{\tau_n}])$$
$$\leq b^{n-1}\,|(\pi^h[b(U^{n-1})])^{\frac{1}{2}}\nabla W^n|_0^2. \qquad (2.25)$$

Summing (2.25) from $n = 1 \to N$ and noting the bound involving W^n in (2.18) yields the desired final bound in (2.18). □

3 Convergence

In this section we adapt and extend the techniques in [4] and [3] to prove convergence of our finite element approximation (P$^{h,\tau}$). The main difference is that for the above fourth order degenerate systems, we established convergence only in one space dimension $(d = 1)$. For the present sixth order problem one can establish convergence in one, two and three space dimensions $(d \leq 3)$. In order to achieve this, as in the references above, we need further restrictions on the mesh.

(A2) In addition to the assumptions (A1), we assume that Ω is convex and that $\mathcal{T}^h$ is a quasi-uniform partitioning of Ω into regular simplices.

As Ω is convex, we have the following well-known results for $m = 0$ or 1

$$\|\mathcal{G}z\|_2 \leq C|z|_0 \qquad \forall\, z \in L^2(\Omega)\cap\mathcal{F}, \qquad (3.1)$$
$$|(\mathcal{G}-\mathcal{G}^h)z|_m \leq Ch^{2-m}|z|_0 \qquad \forall\, z \in L^2(\Omega)\cap\mathcal{F}. \qquad (3.2)$$

The above quasi-uniformity condition on $\mathcal{T}^h$ yields, for any $\kappa \in \mathcal{T}^h$, the inverse inequality for $1 \leq r_1 \leq r_2 \leq \infty$ and for $m = 0$ or 1

$$|\chi|_{m,r_2,\kappa} \leq Ch^{\frac{d(r_1-r_2)}{r_1 r_2}}|\chi|_{m,r_1,\kappa} \qquad \forall\,\chi \in S^h\,. \qquad (3.3)$$

A simple consequence of (2.5), (2.8) and (3.3) is that for all $z \in C(\overline{\kappa})$ and for all $\eta \in H^2(\kappa)$

$$\left|\int_\kappa (I-\pi^h)(z\,\eta)\,\mathrm{d}x\right|$$
$$= \left|\int_\kappa [\,(I-\pi^h)(\,(\pi^h z)\,(\pi^h\eta)\,) + (\pi^h\eta)\,(I-\pi^h)z + z\,(I-\pi^h)\eta\,]\,\mathrm{d}x\right|$$
$$\leq C\left[\,|(I-\pi^h)z|_{0,\kappa} + h\,|z|_{0,\kappa}\right]\|\eta\|_{2,\kappa}\,. \qquad (3.4)$$

It follows from (2.1), (2.6), (2.3) and (3.3) that

$$|\Delta^h z^h|_0^2 \leq |\Delta^h z^h|_h^2 = -(\nabla z^h, \nabla(\Delta^h z^h)\,) \leq |z^h|_1|\Delta^h z^h|_1$$
$$\leq Ch^{-1}|z^h|_1|\Delta^h z^h|_0 \leq Ch^{-2}|z^h|_1^2 \leq Ch^{-4}|z^h|_0^2. \qquad (3.5)$$

Lemma 3.1 *Let the assumptions (A2) hold. Then we have for all $z^h \in S^h$ that*

$$|(I - \fint\,)z^h|_{0,\infty} \leq C|z^h|_0^{1-\frac{d}{4}}|\Delta^h z^h|_0^{\frac{d}{4}}; \tag{3.6}$$

$$|z^h|_{1,r} \leq C|\Delta^h z^h|_0, \qquad \textit{where} \quad \begin{cases} r = \infty & \textit{if } d = 1, \\ r < \infty & \textit{if } d = 2, \\ r = 6 & \textit{if } d = 3. \end{cases} \tag{3.7}$$

Proof It follows from (2.3), (2.10) and (2.9) that for all $z^h \in S^h$

$$(I - \fint\,)z^h = -\widehat{\mathcal{G}}^h(\Delta^h z^h) = -\mathcal{G}^h \xi^h, \tag{3.8}$$

where $\xi^h \in Z^h$ is such that

$$(\xi^h, \chi) = (\Delta^h z^h, \chi)^h \qquad \forall\, \chi \in S^h \tag{3.9}$$

From (3.9), (2.1) and (2.6) we have that

$$|\xi^h|_0 \leq C|\Delta^h z^h|_h \leq C|\Delta^h z^h|_0 \leq C|\xi^h|_0. \tag{3.10}$$

It follows from (3.8), (1.6), (2.8), (3.3), (3.2), (3.1), (3.10) and (3.5) that

$$\begin{aligned} |(I - \fint\,)z^h|_{0,\infty} &\leq |\mathcal{G}\xi^h|_{0,\infty} + |(I - \pi^h)\mathcal{G}\xi^h|_{0,\infty} + |(\pi^h\mathcal{G} - \mathcal{G}^h)\xi^h|_{0,\infty} \\ &\leq C|\mathcal{G}\xi^h|_0^{1-\frac{d}{4}}\|\mathcal{G}\xi^h\|_2^{\frac{d}{4}} + Ch^{2(1-\frac{d}{4})}|\mathcal{G}\xi^h|_2 + Ch^{-\frac{d}{2}}|(\pi^h\mathcal{G} - \mathcal{G}^h)\xi^h|_0 \\ &\leq C|z^h|_0^{1-\frac{d}{4}}\|\mathcal{G}\xi^h\|_2^{\frac{d}{4}} + Ch^{2(1-\frac{d}{4})}|\mathcal{G}\xi^h|_2 \\ &\leq C|z^h|_0^{1-\frac{d}{4}}|\Delta^h z^h|_0^{\frac{d}{4}} + Ch^{2(1-\frac{d}{4})}|\Delta^h z^h|_0 \leq C|z^h|_0^{1-\frac{d}{4}}|\Delta^h z^h|_0^{\frac{d}{4}}. \end{aligned}$$

Hence the desired result (3.6).

With $r \geq 2$ as defined in (3.7), we have from (3.8), (1.6), (2.8), (3.3), (3.2), (3.1) and (3.10) that

$$\begin{aligned} |z^h|_{1,r} &\leq |\mathcal{G}\xi^h|_{1,r} + |(I - \pi^h)\mathcal{G}\xi^h|_{1,r} + |(\pi^h\mathcal{G} - \mathcal{G}^h)\xi^h|_{1,r} \\ &\leq C\|\mathcal{G}\xi^h\|_2 + Ch^{\frac{d(2-r)}{2r}}|(\pi^h\mathcal{G} - \mathcal{G}^h)\xi^h|_1 \\ &\leq C(1 + h^{\frac{d(2-r)+2r}{2r}})\|\mathcal{G}\xi^h\|_2 \leq C|\Delta^h z^h|_0. \end{aligned}$$

Hence the desired result (3.7). □

Lemma 3.2 *Let $u^0 \in H^2(\Omega) \cap K$ with $\frac{\partial u^0}{\partial \nu} = 0$ on $\partial\Omega$. Let the assumptions (A2) hold. If $U^0 \equiv \pi^h u^0 \in K^h$, then it follows that*

$$|\Delta^h U^0|_0^2 + |\,(U^0, 1)\,|^2 \leq C \tag{3.11a}$$

$$\textit{and} \qquad \|u^0 - U^0\|_1, \quad |u^0 - U^0|_{0,\infty} \to 0 \qquad \textit{as } h \to 0. \tag{3.11b}$$

Proof It follows from (3.10), (2.3), (2.8) and (3.3) that

$$\begin{aligned} C\,|\Delta^h(\pi^h u^0)|_0^2 &\leq |\Delta^h(\pi^h u^0)|_h^2 = -(\nabla(\pi^h u^0), \nabla(\Delta^h(\pi^h u^0))) \\ &= -(\nabla u^0, \nabla(\Delta^h(\pi^h u^0))) + (\nabla(I-\pi^h)u^0, \nabla(\Delta^h(\pi^h u^0))) \\ &\leq |\Delta u^0|_0\,|\Delta^h(\pi^h u^0)|_0 + Ch|u^0|_2|\nabla(\Delta^h(\pi^h u^0))|_0 \leq C|u^0|_2^2 \leq C. \end{aligned}$$

Hence the first bound in (3.11a). The remaining results in (3.11a,b) follow directly from (2.8). □

Given $\{\chi^n\}_{n=0}^N$, $\chi^n \in S^h$, we introduce for $n \geq 1$

$$\chi(\cdot,t) := \tfrac{t-t_{n-1}}{\tau_n}\,\chi^n(\cdot) + \tfrac{t_n-t}{\tau_n}\,\chi^{n-1}(\cdot) \qquad t \in [t_{n-1}, t_n] \tag{3.12a}$$

$$\text{and} \quad \chi^+(\cdot,t) := \chi^n(\cdot), \quad \chi^-(\cdot,t) := \chi^{n-1}(\cdot) \quad t \in (t_{n-1}, t_n]\,. \tag{3.12b}$$

We note for future reference that

$$\chi - \chi^\pm = (t - t_n^\pm)\tfrac{\partial\chi}{\partial t} \qquad t \in (t_{n-1}, t_n) \quad n \geq 1, \tag{3.13}$$

where $t_n^+ := t_n$ and $t_n^- := t_{n-1}$. We introduce also

$$\overline{\tau}(t) := \tau_n \qquad t \in (t_{n-1}, t_n) \quad n \geq 1. \tag{3.14}$$

Using the above notation, (2.2a–c) can be restated as:
Find $\{U,\ V,\ W\} \in H^1(0,T;S^h) \times [L^2(0,T;S^h)]^2$ such that $U(\cdot,t) \in K^h$ and for all $\chi \in L^2(0,T;S^h)$, $\eta^h \in L^2(0,T;K^h)$

$$\int_0^T \left[\, \left(\tfrac{\partial U}{\partial t}, \chi\right)^h + \left(\pi^h[b(U^-)]\nabla W^+, \nabla\chi\right) \right] \mathrm{d}t = 0\,, \tag{3.15a}$$

$$\int_0^T \left[\, (\nabla U^+, \nabla\chi) - (V^+, \chi)^h \right] \mathrm{d}t = 0\,, \tag{3.15b}$$

$$\int_0^T \left[\, (\nabla V^+, \nabla(\eta^h - U^+)) - (W^+, \eta^h - U^+)^h \right] \mathrm{d}t \geq 0\,. \tag{3.15c}$$

Lemma 3.3 *Let u^0 satisfy the assumptions of Lemma 3.2. In addition to the assumptions (A2), we assume that $U^0 \equiv \pi^h u^0$ and $\tau \to 0$ as $h \to 0$. Adopting the notation (3.12a,b), there exists a subsequence of $\{U,V\}_h$ and a function*

$$u \in L^\infty(0,T;K) \cap C^{\alpha,\beta}_{x,t}(\overline{\Omega}_T), \quad \textit{where} \begin{cases} \alpha = 1,\ \beta = \frac14 & \textit{if } d = 1 \\ \alpha < 1,\ \beta = \frac16 & \textit{if } d = 2 \\ \alpha = \frac12,\ \beta = \frac{1}{12} & \textit{if } d = 3 \end{cases}, \tag{3.16}$$

with $⨍\, u(\cdot,t) = ⨍\, u^0$ for all $t \in [0,T]$ and a function

$$v \in L^\infty(0,T;L^2(\Omega)) \tag{3.17}$$

such that as $h \to 0$

$$U,\ U^{\pm} \to u \quad \textit{uniformly on } \overline{\Omega}_T, \quad \textit{weak-}* \textit{ in } L^\infty(0,T;H^1(\Omega)); \quad (3.18)$$
$$V,\ V^{\pm} \to v \quad \textit{weak-}* \textit{ in } L^\infty(0,T;L^2(\Omega)). \quad (3.19)$$

Proof Noting the definitions (3.12a,b), (3.14), (2.18), (3.11a) and (1.8) we have that

$$\|U\|^2_{L^\infty(0,T;H^1(\Omega))} + \|V\|^2_{L^\infty(0,T;L^2(\Omega))} + \|\overline{\tau}^{\frac{1}{2}} \tfrac{\partial U}{\partial t}\|^2_{L^2(0,T;H^1(\Omega))} + \|\overline{\tau}^{\frac{1}{2}} \tfrac{\partial V}{\partial t}\|^2_{L^2(\Omega_T)} + \|(\pi^h[b(U^-)])^{\frac{1}{2}} \nabla W^+\|^2_{L^2(\Omega_T)} + \|\widehat{\mathcal{G}}^h \tfrac{\partial U}{\partial t}\|^2_{L^2(0,T;H^1(\Omega))} \leq C. \quad (3.20)$$

Furthermore, we deduce from (3.13) and (3.20) that

$$\|U - U^{\pm}\|^2_{L^2(0,T;H^1(\Omega))} + \|V - V^{\pm}\|^2_{L^2(\Omega_T)} \leq \|\overline{\tau}\, \tfrac{\partial U}{\partial t}\|^2_{L^2(0,T;H^1(\Omega))} + \|\overline{\tau}\, \tfrac{\partial V}{\partial t}\|^2_{L^2(\Omega_T)} \leq C\,\tau. \quad (3.21)$$

The next step is to show that the discrete solutions U are uniformly Hölder continuous. Firstly we note from (3.20), (3.12a), $V \equiv -\Delta^h U$, (3.6), (3.7) and the imbedding result $W^{1,r}(\Omega) \subset C^{0,1-\frac{d}{r}}(\overline{\Omega})$, $r > d$, that

$$\|U\|_{C([0,T],C^{0,\alpha}(\overline{\Omega}))} \leq C\|U\|_{C([0,T],W^{1,r}(\Omega))} \leq C, \quad (3.22)$$

where r and α are as in (3.7) and (3.16), respectively, and C is independent of T. Secondly it follows from $\frac{\partial U}{\partial t} \in Z^h$, (3.6), (2.1), (2.6), (2.11), (2.12), $V \equiv -\Delta^h U$ and (3.20) that

$$\begin{aligned}
|U(x,t_b) - U(x,t_a)| &= \left| \int_{t_a}^{t_b} \tfrac{\partial U}{\partial t}(x,t)\,\mathrm{d}t \right| \leq \left| \int_{t_a}^{t_b} \tfrac{\partial U}{\partial t}(\cdot,t)\,\mathrm{d}t \right|_{0,\infty} \\
&\leq C \left| \int_{t_a}^{t_b} \tfrac{\partial U}{\partial t}(\cdot,t)\,\mathrm{d}t \right|_h^{1-\frac{d}{4}} \left| \Delta^h \left[\int_{t_a}^{t_b} \tfrac{\partial U}{\partial t}(\cdot,t)\,\mathrm{d}t \right] \right|_0^{\frac{d}{4}} \\
&\leq C \left| \widehat{\mathcal{G}}^h \left[\int_{t_a}^{t_b} \tfrac{\partial U}{\partial t}(\cdot,t)\,\mathrm{d}t \right] \right|_1^{\frac{2}{3}(1-\frac{d}{4})} \left| \Delta^h \left[\int_{t_a}^{t_b} \tfrac{\partial U}{\partial t}(\cdot,t)\,\mathrm{d}t \right] \right|_h^{\frac{2}{3}(1-\frac{d}{4})} \\
&\leq C \left(\int_{t_a}^{t_b} \left| \widehat{\mathcal{G}}^h \tfrac{\partial U}{\partial t} \right|_1 \mathrm{d}t \right)^{\frac{2}{3}(1-\frac{d}{4})} \left(2 \left\| \Delta^h U \right\|_{L^\infty(0,T;L^2(\Omega))} \right)^{\frac{1}{3}(1+\frac{d}{2})} \\
&\leq C(t_b - t_a)^{\frac{1}{3}(1-\frac{d}{4})} \left(\int_{t_a}^{t_b} \left| \widehat{\mathcal{G}}^h \tfrac{\partial U}{\partial t} \right|_1^2 \mathrm{d}t \right)^{\frac{1}{3}(1-\frac{d}{4})} \\
&\leq C(t_b - t_a)^{\beta} \qquad \forall\, t_b \geq t_a \geq 0, \quad \forall\, x \in \overline{\Omega}, \quad (3.23)
\end{aligned}$$

where β is defined as in (3.16) and C is independent of T.

An immediate consequence of (3.23) is that

$$\|U - U^{\pm}\|_{L^\infty(\Omega_T)} \leq C\,\tau^{\frac{2}{3}(1-\frac{d}{4})}. \tag{3.24}$$

(3.22) and (3.23) imply that the $C^{\alpha,\beta}_{x,t}(\overline{\Omega}_T)$ norm of U is bounded independently of h, τ and T. Hence, under the stated assumptions on τ, every sequence $\{U\}_h$ is uniformly bounded and equicontinuous on $\overline{\Omega}_T$, for any $T > 0$. Therefore by the Arzelà-Ascoli theorem there exists a subsequence and a $u \geq 0$, as $U(\cdot,t) \in K^h$, such that

$$U \to u \in C^{\alpha,\beta}_{x,t}(\overline{\Omega}_T) \qquad \text{uniformly on } \overline{\Omega}_T \text{ as } h \to 0. \tag{3.25}$$

Combining (3.25) and (3.24) yields the Hölder continuity result in (3.16) and the uniform convergence result in (3.18). As $(U(\cdot,t) - \pi^h u^0(\cdot), 1) = 0$ for all $t \in [0,T]$, it follows from (3.25) and (3.11b) that $(u(\cdot,t) - u^0(\cdot), 1) = 0$ for all $t \in [0,T]$. Finally, (3.20) and (3.21) imply that a further subsequence $\{U,V\}_h$ can be extracted such that the weak-$\star$ convergence result in (3.18), and (3.19) hold. Hence the first inclusion in (3.16), on recalling that $u \geq 0$, and (3.17) hold. $\Box$

From (3.20), (3.18) and (2.8) we see that we can only control ∇W^+ on those sets where $u > 0$. Therefore in order to construct the appropriate limits as $h \to 0$, we introduce the following open subsets of $\overline{\Omega}$ and $\overline{\Omega}_T$. For any $\delta > 0$, we set

$$B_\delta := \{(x,t) \in \overline{\Omega}_T : u(x,t) > \delta\} \;\text{ and }\; B_\delta(t) := \{x \in \overline{\Omega} : u(x,t) > \delta\}. \tag{3.26}$$

From (3.16), we have that there exist positive constants C_x and C_t such that for all $y_1, y_2, x \in \overline{\Omega}$

$$|u(y_2,t) - u(y_1,t)| \leq C_x\,|y_2 - y_1|^\alpha \qquad \forall\, t \in [0,T]; \tag{3.27a}$$

$$|u(x,t_b) - u(x,t_a)| \leq C_t\,|t_b - t_a|^\beta \qquad \forall\, t_a,\, t_b \in [0,T]. \tag{3.27b}$$

As $⨍\, u(\cdot,t) = ⨍\, u^0 > 0$ for all $t \in [0,T]$, it follows that there exists a $\delta_0 \in (0, ⨍\, u^0)$ such that $B_{\delta_0}(t) \not\equiv \emptyset$ for all $t \in [0,T]$. It immediately follows from (3.26) and (3.27a,b) for any $t_a,\, t_b \in [0,T]$ and for any $\delta_1,\, \delta_2 \in (0,\delta_0)$ with $\delta_1 > \delta_2$ that

$$y_1 \in B_{\delta_1}(t_a) \text{ and } y_2 \in \partial B_{\delta_2}(t_b) \text{ with } y_2 \notin \partial\Omega \quad \Longrightarrow$$
$$C_x\,|y_2 - y_1|^\alpha + C_t\,|t_b - t_a|^\beta \geq u(y_1,t_a) - u(y_2,t_b) > \delta_1 - \delta_2, \tag{3.28}$$

where $\partial B_\delta(t)$ is the boundary of $B_\delta(t)$. Therefore (3.28) implies that for any $\delta \in (0,\delta_0)$, there exists an $h_0(\delta)$ such that for all $h \leq h_0(\delta)$ there exist collections of simplices $\mathcal{T}^h_\delta(t) \subset \mathcal{T}^h$ such that

$$B_\delta(t) \subset B^h_\delta(t) := \cup_{\kappa \in \mathcal{T}^h_\delta(t) \subset \mathcal{T}^h} \overline{\kappa} \subset B_{\frac{\delta}{2}}(t) \qquad \forall\, t \in [0,T]. \tag{3.29}$$

Similarly it follows from (3.28) that for any $\delta \in (0, \delta_0)$, there exists a $\tau_0(\delta)$ such that for all $\tau \leq \tau_0(\delta)$

$$B_\delta(t) \subset B_{\frac{\delta}{2}}(t_n) \subset B_{\frac{\delta}{4}}(t) \qquad \forall\, t \in (t_{n-1}, t_n], \quad n = 1 \to N. \quad (3.30)$$

Clearly, we have from (3.29) and (3.30) that $\delta_2 < \delta_1 < \delta_0$ implies that $h_0(\delta_2) \leq h_0(\delta_1)$ and $\tau_0(\delta_2) \leq \tau_0(\delta_1)$. For a fixed $\delta \in (0, \delta_0)$, it follows from (3.26), (3.18) and our assumption on τ in Lemma 3.3 that there exists an $\widehat{h}_0(\delta) \leq h_0(\delta)$ such that for $h \leq \widehat{h}_0(\delta)$

$$\begin{array}{ll} 0 \leq U^{\pm}(x,t) \leq 2\delta & \forall\, (x,t) \notin B_\delta, \\ \frac{1}{2}\delta \leq U^{\pm}(x,t) & \forall\, (x,t) \in B_\delta \end{array} \qquad \text{and} \qquad \tau \leq \tau_0(\delta). \quad (3.31)$$

In order to prove convergence of our approximation ($\mathrm{P}^{h,\tau}$), we make a final restriction on the mesh.

(A3) In addition to the assumptions (A2), we assume that $\mathcal{T}^h$ is a quasi-uniform partitioning of Ω into generic right-angled simplices (for $d = 3$ this means that all tetrahedra have two vertices at which two edges intersect at right angles, see below for more details).

We note that a cube is easily partitioned into such tetrahedra.

Let $\{e_i\}_{i=1}^d$ be the orthonormal vectors in $\mathbb{R}^d$, such that the j^{th} component of e_i is δ_{ij}, $i,\, j = 1 \to d$. Given non-zero constants ρ_i, $i = 1 \to d$; let $\widehat{\kappa}(\{\rho_i\}_{i=1}^d)$ be a reference simplex in $\mathbb{R}^d$ with vertices $\{\widehat{p}_i\}_{i=0}^d$, where $\widehat{p}_0$ is the origin and $\widehat{p}_i = \widehat{p}_{i-1} + \rho_i e_i$, $i = 1 \to d$. Given a $\kappa \in \mathcal{T}^h$ with vertices $\{p_{j_i}\}_{i=0}^d$, such that p_{j_0} is not a right-angled vertex, then there exists a rotation/reflection matrix R_κ and non-zero constants $\{\rho_i\}_{i=1}^d$ such that the mapping $\mathcal{R}_\kappa : \widehat{x} \in \mathbb{R}^d \to p_{j_0} + R_\kappa \widehat{x} \in \mathbb{R}^d$ maps the vertex $\widehat{p}_i$ to p_{j_i}, $i = 0 \to d$, and hence $\widehat{\kappa} \equiv \widehat{\kappa}(\{\rho_i\}_{i=1}^d)$ to κ. For all $\kappa \in \mathcal{T}^h$ an d $\eta \in C(\overline{\kappa})$, we set

$$\widehat{\eta}(\widehat{x}) \equiv \eta(\mathcal{R}_\kappa \widehat{x}) \quad \text{and} \quad (\widehat{\pi}^h \widehat{\eta})(\widehat{x}) \equiv (\pi^h \eta)(\mathcal{R}_\kappa \widehat{x}) \qquad \forall\, \widehat{x} \in \overline{\widehat{\kappa}}. \quad (3.32)$$

As $R_\kappa^T \equiv R_\kappa^{-1}$, we have for any $z^h \in S^h$ and $\kappa \in \mathcal{T}^h$ that

$$\nabla z^h \equiv R_\kappa \widehat{\nabla} \widehat{z}^h, \quad (3.33)$$

where $x \equiv (x_1, \cdots, x_d)^T$, $\nabla \equiv (\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_d})^T$, $\widehat{x} \equiv (\widehat{x}_1, \cdots, \widehat{x}_d)^T$ and $\widehat{\nabla} \equiv (\frac{\partial}{\partial \widehat{x}_1}, \cdots, \frac{\partial}{\partial \widehat{x}_d})^T$. From (3.32) and (3.33), it follows for all $\kappa \in \mathcal{T}^h$, $\eta_j \in C(\overline{\kappa})$ and $i = 1 \to d$ that

$$\begin{aligned} \tfrac{\partial}{\partial \widehat{x}_i}(\widehat{\pi}^h[\widehat{\eta}_1\, \widehat{\eta}_2]) &= \rho_i^{-1}\,[\,\widehat{\eta}_1(\widehat{p}_i)\,\widehat{\eta}_2(\widehat{p}_i) - \widehat{\eta}_1(\widehat{p}_{i-1})\,\widehat{\eta}_2(\widehat{p}_{i-1})\,] \\ &= (2\rho_i)^{-1}\,(\widehat{\eta}_1(\widehat{p}_{i-1}) + \widehat{\eta}_1(\widehat{p}_i))\,[\widehat{\eta}_2(\widehat{p}_i) - \widehat{\eta}_2(\widehat{p}_{i-1})] \\ &\quad + (2\rho_i)^{-1}\,(\widehat{\eta}_2(\widehat{p}_{i-1}) + \widehat{\eta}_2(\widehat{p}_i))\,[\widehat{\eta}_1(\widehat{p}_i) - \widehat{\eta}_1(\widehat{p}_{i-1})]. \end{aligned} \quad (3.34)$$

Therefore (3.34) yields for all $\kappa \in \mathcal{T}^h$ and $\eta_j \in C(\overline{\kappa})$ that

$$\widehat{\nabla}(\widehat{\pi}^h[\widehat{\eta}_1\,\widehat{\eta}_2]) = \widehat{D}_s(\widehat{\pi}^h\widehat{\eta}_1)\widehat{\nabla}(\widehat{\pi}^h\widehat{\eta}_2) + \widehat{D}_s(\widehat{\pi}^h\widehat{\eta}_2)\widehat{\nabla}(\widehat{\pi}^h\widehat{\eta}_1); \qquad (3.35)$$

where for any $z^h \in S^h$ and $\kappa \in \mathcal{T}^h$, $\widehat{D}_s(\widehat{z}^h)$ is the $d \times d$ diagonal matrix with diagonal entries

$$[\widehat{D}_s(\widehat{z}^h)]_{ii} := \tfrac{1}{2}\left[\widehat{z}^h(\widehat{p}_{i-1}) + \widehat{z}^h(\widehat{p}_i)\right] \qquad i = 1 \to d. \qquad (3.36)$$

On combining (3.32), (3.33) and (3.35), we have for all $\eta_j \in C(\overline{\Omega})$ that

$$\nabla(\pi^h[\eta_1\,\eta_2]) = D_s(\pi^h\eta_1)\nabla(\pi^h\eta_2) + D_s(\pi^h\eta_2)\nabla(\pi^h\eta_1); \qquad (3.37)$$

where for any $z^h \in S^h$,

$$D_s(z^h)\mid_\kappa := R_\kappa\,\widehat{D}_s(\widehat{z}^h)R_\kappa^T \qquad \forall\,\kappa \in \mathcal{T}^h. \qquad (3.38)$$

Similarly to (3.35), we have for all $\eta_j \in C(\overline{\Omega})$ that

$$\nabla(\pi^h[\eta_1^2\,\eta_2]) = D_p(\pi^h[\eta_1^2])\nabla(\pi^h\eta_2) + 2\,D_s(\pi^h[\eta_1\,\eta_2])\nabla(\pi^h\eta_1); \quad (3.39)$$

where for any $z^h \in K^h$ and $\kappa \in \mathcal{T}^h$,

$$D_p(z^h)\mid_\kappa := R_\kappa\,\widehat{D}_p(\widehat{z}^h)R_\kappa^T \qquad (3.40)$$

and $\widehat{D}_p(\widehat{z}^h)$ is the $d \times d$ diagonal matrix with diagonal entries

$$[\widehat{D}_p(\widehat{z}^h)]_{ii} := [\widehat{z}^h(\widehat{p}_{i-1})\,\widehat{z}^h(\widehat{p}_i)]^{\frac{1}{2}} \leq [\widehat{D}_s(\widehat{z}^h)]_{ii} \qquad i = 1 \to d. \quad (3.41)$$

We note for later purposes that the symmetric matrices $D_s(z_i^h)$ and $D_p(z_i^h)$ are such that

$$D_s(z_1^h)D_s(z_2^h) = D_s(z_2^h)D_s(z_1^h) \qquad \forall\, z_i^h \in S^h, \quad (3.42a)$$
$$D_p(z_1^h)D_p(z_2^h) = D_p(\pi^h[z_1^h\,z_2^h]) = D_p(z_2^h)D_p(z_1^h) \quad \forall\, z_i^h \in K^h. \quad (3.42b)$$

It is the results (3.37) and (3.39) that require the right angle constraint on the partitioning $\mathcal{T}^h$ in (A3).

We now derive bounds for W^+ and V^+ locally on the set $\{u > 0\}$. For any $\delta \in (0, \delta_0)$, we introduce cut-off functions $\theta_\delta^n \in C^\infty(\overline{\Omega})$, $n = 1 \to N$, such that

$$\theta_\delta^n \equiv 1 \quad \text{on } B_\delta(t_n), \qquad 0 \leq \theta_\delta^n \leq 1 \quad \text{on } B_{\frac{\delta}{2}}(t_n) \setminus B_\delta(t_n),$$
$$\theta_\delta^n \equiv 0 \quad \text{on } \overline{\Omega} \setminus B_{\frac{\delta}{2}}(t_n) \qquad \text{and} \qquad |\nabla\theta_\delta^n| \leq C\,\delta^{-2}. \qquad (3.43)$$

It follows from (3.28) that this last property can be achieved. Then for any $\delta \in (0, \frac{1}{4}\delta_0)$, we have from (3.43), (2.1), (3.29), (2.6), (3.30), (3.31) and (3.12b) that for all $h \leq \widehat{h}_0(2\delta)$

$$\sum_{n=1}^{N} \tau_n \, |\theta_\delta^n \, \chi^n|_h^2 \geq \sum_{n=1}^{N} \tau_n \int_{B_{2\delta}^h(t_n)} \pi^h[(\theta_\delta^n \, \chi^n)^2] \, \mathrm{d}x$$
$$\equiv \sum_{n=1}^{N} \tau_n \int_{B_{2\delta}^h(t_n)} \pi^h[(\chi^n)^2] \, \mathrm{d}x \geq \sum_{n=1}^{N} \tau_n \int_{B_{2\delta}^h(t_n)} |\chi^n|^2 \, \mathrm{d}x$$
$$\geq \sum_{n=1}^{N} \tau_n \int_{B_{2\delta}(t_n)} |\chi^n|^2 \, \mathrm{d}x \geq \int_{B_{4\delta}} |\chi^+|^2 \, \mathrm{d}x \, \mathrm{d}t, \qquad (3.44)$$

and similarly

$$\sum_{n=1}^{N} \tau_n \, (D_s(\pi^h[(\theta_\delta^n)^2])\nabla\chi^n, \nabla\chi^n) \geq \int_{B_{4\delta}} |\nabla\chi^+|^2 \, \mathrm{d}x \, \mathrm{d}t. \qquad (3.45)$$

Lemma 3.4 *Let u^0, U^0 and τ satisfy the assumptions of Lemma 3.3. In addition let the assumptions (A3) hold. Then we have for any $\delta \leq 2\delta_0$ and for all $h \leq \widehat{h}_0(\frac{\delta}{32})$ that, for all $\chi \in L^2(0,T;S^h)$ with $\mathrm{supp}(\chi) \subset B_{\frac{\delta}{16}}$,*

$$\int_0^T \left[(\nabla V^+, \nabla\chi) - (W^+, \chi)^h \right] \, \mathrm{d}x \, \mathrm{d}t = 0; \qquad (3.46)$$

and

$$\int_{B_{\frac{\delta}{8}}} |\nabla W^+|^2 \, \mathrm{d}x \, \mathrm{d}t \leq C \, \delta^{-\gamma}, \qquad (3.47a)$$
$$\int_{B_{\frac{\delta}{2}}} \left[|\nabla V^+|^2 + |W^+|^2 \right] \, \mathrm{d}x \, \mathrm{d}t \leq C(\delta^{-1}). \qquad (3.47b)$$

Proof It follows from (3.20), (3.12b), (3.31), (3.29) and (3.30) that for all $h \leq \widehat{h}_0(\frac{\delta}{32})$

$$C \geq \int_{\Omega_T} \pi^h[b(U^-)] \, |\nabla W^+|^2 \, \mathrm{d}x \, \mathrm{d}t \equiv \sum_{n=1}^{N} \tau_n \int_{\Omega} \pi^h[b(U^{n-1})] \, |\nabla W^n|^2 \, \mathrm{d}x$$
$$\geq C_1 \delta^\gamma \sum_{n=1}^{N} \tau_n \int_{B^h_{\frac{\delta}{16}}(t_n)} |\nabla W^n|^2 \, \mathrm{d}x \geq C_1 \delta^\gamma \int_{B_{\frac{\delta}{8}}} |\nabla W^+|^2 \, \mathrm{d}x \, \mathrm{d}t. \qquad (3.48)$$

This yields the desired result (3.47a).

From (3.31) we have for all $h \leq \widehat{h}_0(\frac{\delta}{32})$ and for $n = 1 \to N$ that $\chi := U^n \pm \frac{\delta}{64}\,\eta^h/\|\eta^h\|_{L^\infty(\Omega)} \in K^h$ for any $\eta^h \in S^h$ with $\mathrm{supp}(\eta^h) \subset B_{\frac{\delta}{32}}(t_n)$. Choosing such χ in (2.2c) yields for all $h \leq \widehat{h}_0(\frac{\delta}{32})$ that

$$(\nabla V^n, \nabla \eta^h) = (W^n, \eta^h)^h \quad \forall\, \eta^h \in S^h \text{ with } \mathrm{supp}(\eta^h) \subset B_{\frac{\delta}{32}}(t_n). \tag{3.49}$$

The desired result (3.46) follows from (3.49), (3.30), (3.31) and (3.12b).

Noting (3.43) and (3.29) and as $h \leq \widehat{h}_0(\frac{\delta}{32})$, we can choose $\eta^h \equiv \pi^h[(\theta^n_{\frac{\delta}{8}})^2\, V^n]$ in (3.49) to obtain for all $\varepsilon_1 > 0$, on recalling (3.37) and (3.42a), that

$$\begin{aligned}
&(D_s(\pi^h[(\theta^n_{\frac{\delta}{8}})^2])\nabla V^n, \nabla V^n) \\
&\quad = ((\theta^n_{\frac{\delta}{8}})^2\, W^n, V^n)^h - 2\,(D_s(\pi^h\theta^n_{\frac{\delta}{8}})\,\nabla V^n, D_s(V^n)\,\nabla(\pi^h\theta^n_{\frac{\delta}{8}})\,) \\
&\quad \leq \varepsilon_1\,\delta^4\,|(\theta^n_{\frac{\delta}{8}})^2\,W^n|_h^2 + \varepsilon_1^{-1}\,\delta^{-4}\,|V^n|_h^2 + \tfrac{1}{2}\,|D_s(\pi^h\theta^n_{\frac{\delta}{8}})\,\nabla V^n|^2 \\
&\qquad + 2\,|D_s(V^n)\,\nabla(\pi^h\theta^n_{\frac{\delta}{8}})|^2.
\end{aligned} \tag{3.50}$$

It follows from (3.38), (3.36), (3.43), (2.1), (2.6) and (3.20) that

$$|D_s(\pi^h\theta^n_{\frac{\delta}{8}})\,\nabla V^n|^2 \leq (D_s(\pi^h[(\theta^n_{\frac{\delta}{8}})^2])\nabla V^n, \nabla V^n), \tag{3.51a}$$

$$|D_s(V^n)\,\nabla(\pi^h\theta^n_{\frac{\delta}{8}})|^2 \leq C_\star\,\delta^{-4}\,|V^n|_h^2 \leq C\,\delta^{-4}. \tag{3.51b}$$

Combining (3.50) and (3.51a,b) yields for $h \leq \widehat{h}_0(\frac{\delta}{32})$ and for all $\varepsilon_1 > 0$ that

$$(D_s(\pi^h[(\theta^n_{\frac{\delta}{8}})^2])\nabla V^n, \nabla V^n) \leq 2\,\varepsilon_1\,\delta^4\,|(\theta^n_{\frac{\delta}{8}})^2\,W^n|_h^2 + C(1+\varepsilon_1^{-1})\,\delta^{-4}. \tag{3.52}$$

Choosing $\eta^h \equiv \pi^h[(\theta^n_{\frac{\delta}{8}})^4\,W^+]$ in (3.49) yields on noting (3.39), (3.37), (3.42a,b) (3.43), (3.40), (3.41), (3.38) and (3.51a,b) that for $h \leq \widehat{h}_0(\frac{\delta}{32})$ and for all $\varepsilon_2 > 0$

$$\begin{aligned}
|(\theta^n_{\frac{\delta}{8}})^2\,W^n|_h^2 &= (\nabla V^n, D_p(\pi^h[(\theta^n_{\frac{\delta}{8}})^4])\,\nabla W^n) \\
&\qquad + 4\,(D_s(\pi^h\theta^n_{\frac{\delta}{8}})\nabla V^n, D_s(\pi^h[(\theta^n_{\frac{\delta}{8}})^2 W^n])\nabla(\pi^h\theta^n_{\frac{\delta}{8}})\,) \\
&\leq (D_p(\pi^h[(\theta^n_{\frac{\delta}{8}})^2])\nabla V^n, \nabla V^n) + (D_p(\pi^h[(\theta^n_{\frac{\delta}{8}})^6])\nabla W^n, \nabla W^n) \\
&\qquad + 4\,\varepsilon_2\,|D_s(\pi^h[(\theta^n_{\frac{\delta}{8}})^2 W^n])\nabla(\pi^h\theta^n_{\frac{\delta}{8}})|^2 + \varepsilon_2^{-1}\,|D_s(\pi^h\theta^n_{\frac{\delta}{8}})\nabla V^n|^2 \\
&\leq 4\,C_\star\,\varepsilon_2\,\delta^{-4}\,|(\theta^n_{\frac{\delta}{8}})^2\,W^n|_h^2 + (1+\varepsilon_2^{-1})\,(D_s(\pi^h[(\theta^n_{\frac{\delta}{8}})^2])\nabla V^n, \nabla V^n) \\
&\qquad + C\int_{B^h_{\frac{\delta}{16}}(t_n)} |\nabla W^n|^2\,\mathrm{d}x.
\end{aligned} \tag{3.53}$$

On choosing $\varepsilon_2 = \frac{1}{8} C_\star^{-1} \delta^4$ in (3.53), then multiplying by τ_n, summing from $n = 1 \to N$ and noting (3.48) we have that

$$\begin{aligned}&\sum_{n=1}^{N} \tau_n \, |(\theta^n_{\frac{\delta}{8}})^2 \, W^n|_h^2 \\ &\leq 2\,(1 + 8\,C_\star\,\delta^{-4}) \sum_{n=1}^{N} \tau_n \,(D_s(\pi^h[(\theta^n_{\frac{\delta}{8}})^2])\nabla V^n, \nabla V^n) + C\,\delta^{-\gamma}. \quad (3.54)\end{aligned}$$

Multiplying (3.52) by τ_n, summing from $n = 1 \to N$, noting (3.54) and choosing $\varepsilon_1 = (8\,\delta^4 + 64\,C_\star)^{-1}$, we obtain that

$$\sum_{n=1}^{N} \tau_n \,(D_s(\pi^h[(\theta^n_{\frac{\delta}{8}})^2])\nabla V^n, \nabla V^n) + \sum_{n=1}^{N} \tau_n \,|(\theta^n_{\frac{\delta}{8}})^2 \, W^n|_h^2 \leq C(\delta^{-1}). \quad (3.55)$$

The desired results (3.47a,b) then follow from (3.55) on noting (3.44) and (3.45). □

Theorem 3.1 *Let the assumptions of Lemma 3.4 hold. Then there exists a subsequence of* $\{U, V, W\}_h$ *and functions* $\{u, v, w\}$ *satisfying (3.16), (3.17) and*

$$u \in L^\infty(0,T; H^2(\Omega)) \cap H^1(0,T; (H^1(\Omega))'), \quad (3.56)$$

$$\nabla v,\, w,\, \nabla w \in L^2_{loc}(\{u > 0\}), \quad (3.57)$$

where $\{u > 0\} := \{(x,t) \in \overline{\Omega}_T \;:\; u(x,t) > 0\}$*; such that as* $h \to 0$ *(3.18), (3.19) hold and*

$$\nabla V^+ \to \nabla v,\; W^+ \to w,\; \nabla W^+ \to \nabla w \quad \textit{weakly in } L^2_{loc}(\{u > 0\}). \quad (3.58)$$

Furthermore u, v *and* w *fulfil* $u(\cdot,0) = u^0(\cdot)$ *and are such that for all* $\eta, z \in L^2(0,T; H^1(\Omega))$*, with* $\text{supp}(z) \subset \{u > 0\}$,

$$\int_0^T \langle \tfrac{\partial u}{\partial t}, \eta\rangle \,\mathrm{d}t + \int_{\{u>0\}} b(u)\nabla w \nabla \eta \,\mathrm{d}x\,\mathrm{d}t = 0, \quad (3.59a)$$

$$\int_{\Omega_T} [\,\nabla u \nabla \eta - v\,\eta\,]\;\mathrm{d}x\,\mathrm{d}t = 0, \quad (3.59b)$$

$$\int_{\{u>0\}} [\,\nabla v\,\nabla z - w\,z\,]\;\mathrm{d}x\,\mathrm{d}t = 0. \quad (3.59c)$$

Proof For any $\eta \in L^2(0,T; H^2(\Omega))$, we choose $\chi \equiv \pi^h \eta$ in (3.15b). From (2.8), (3.18), (2.1), (3.4) and (3.19) we have for all $\eta \in L^2(0,T; H^2(\Omega))$

that

$$\int_0^T (\nabla U^+, \nabla(\pi^h \eta))\,\mathrm{d}t \to \int_0^T (\nabla u, \nabla \eta)\,\mathrm{d}t, \tag{3.60a}$$

$$\int_0^T (V^+, \pi^h \eta)^h\,\mathrm{d}t \to \int_0^T (v, \eta)\,\mathrm{d}t \qquad \text{as } h \to 0. \tag{3.60b}$$

Combining (3.60a,b) yields (3.59b) by a density argument. As Ω is convex polyhedral, see (A2), (3.17), (3.59b) and elliptic regularity give rise to the first regularity result in (3.56).

For any $\eta \in H^1(0,T;H^2(\Omega))$ we choose $\chi \equiv \pi^h \eta$ in (3.15a) and now analyse the subsequent terms. Firstly, we have that

$$\int_0^T \left(\tfrac{\partial U}{\partial t}, \pi^h \eta\right)^h \,\mathrm{d}t = -\int_0^T \left(U, \tfrac{\partial(\pi^h \eta)}{\partial t}\right)^h \,\mathrm{d}t + (U(\cdot,T), \pi^h \eta(\cdot,T))^h - (U(\cdot,0), \pi^h \eta(\cdot,0))^h.$$

We conclude from (2.1), (3.4) and (3.18) for all $\eta \in H^1(0,T;H^2(\Omega))$ that as $h \to 0$

$$\int_0^T \left(\tfrac{\partial U}{\partial t}, \pi^h \eta\right)^h \mathrm{d}t \to -\int_0^T (u, \tfrac{\partial \eta}{\partial t})\,\mathrm{d}t + (u(\cdot,T), \eta(\cdot,T)) - (u(\cdot,0), \eta(\cdot,0)). \tag{3.61}$$

In view of (1.2), (3.12a,b), (3.20) and (3.6), and as $V \equiv -\Delta^h U$ we deduce that

$$\begin{aligned}
&\left| \int_{\Omega_T} \pi^h[b(U^-)] \nabla W^+ \nabla (I - \pi^h)\eta \,\mathrm{d}x\,\mathrm{d}t \right| \\
&\leq \|U^-\|^{\frac{\gamma}{2}}_{L^\infty(\Omega_T)} \, \|(\pi^h[b(U^-)])^{\frac{1}{2}} \nabla W^+\|_{L^2(\Omega_T)} \, \|(I-\pi^h)\eta\|_{L^2(0,T;H^1(\Omega))} \\
&\leq C \, \|(I-\pi^h)\eta\|_{L^2(0,T;H^1(\Omega))}.
\end{aligned} \tag{3.62}$$

We now consider a fixed $\delta \in (0, \frac{1}{2}\delta_0)$. On noting (3.29), (3.31) and (3.20) we have for all $h \leq \widehat{h}_0(2\delta)$ and $\eta \in L^2(0,T;H^1(\Omega))$ that

$$\begin{aligned}
&\left| \int_{\Omega_T \setminus B_\delta} \pi^h[b(U^-)] \nabla W^+ \nabla \eta \,\mathrm{d}x\,\mathrm{d}t \right| \\
&\leq \|\pi^h[b(U^-)]\|^{\frac{1}{2}}_{L^\infty(\Omega_T \setminus B^h_{2\delta})} \|(\pi^h[b(U^-)])^{\frac{1}{2}} \nabla W^+\|_{L^2(\Omega_T)} \|\eta\|_{L^2(0,T;H^1(\Omega))} \\
&\leq C \|b(U^-)\|^{\frac{1}{2}}_{L^\infty(\Omega_T \setminus B_{2\delta})} \, \|\eta\|_{L^2(0,T;H^1(\Omega))} \\
&\leq C \delta^{\frac{\gamma}{2}} \, \|\eta\|_{L^2(0,T;H^1(\Omega))},
\end{aligned} \tag{3.63}$$

where $B^h_\delta := \{(x,t) \in \overline{\Omega}_T : x \in B^h_\delta(t)\}$. On noting (3.48), (1.2), (2.7) and (3.18), we conclude that for all $\eta \in L^2(0,T;H^1(\Omega))$

$$
\begin{aligned}
&\left| \int_{B_\delta} (\pi^h[b(U^-)] - b(u)) \nabla W^+ \nabla \eta \,\mathrm{d}x\,\mathrm{d}t \right| \\
&\le \|b(u) - \pi^h[b(U^-)]\|_{L^\infty(\Omega_T)} \, \|\nabla W^+\|_{L^2(B_\delta)} \, \|\eta\|_{L^2(0,T;H^1(\Omega))} \\
&\le C(\delta^{-1}) \Big[\|(I-\pi^h) b(u) + \pi^h[b(u) - b(U^-)]\|_{L^\infty(\Omega_T)} \Big] \|\eta\|_{L^2(0,T;H^1(\Omega))}
\end{aligned} \tag{3.64}
$$

will converge to 0 as $h \to 0$. Combining (3.20) and (3.47a,b), and noting (3.12a,b) we have for all $h \le \widehat{h}_0(\frac{\delta}{32})$ that

$$
\|V^+\|_{L^2(0,T;H^1(B_{\frac{\delta}{2}}(t)))} + \|W^+\|_{L^2(0,T;H^1(B_{\frac{\delta}{2}}(t)))} \le C(\delta^{-1}). \tag{3.65}
$$

The bounds (3.65) imply the existence of a subsequence of $\{U,V,W\}_h$, of the subsequence $\{U,V\}_h$ satisfying (3.18) and (3.19), and a function $w \in L^2(0,T;H^1(B_{\frac{\delta}{2}}(t)))$ such that as $h \to 0$

$$
\nabla V^+ \to \nabla v, \; W^+ \to w, \; \nabla W^+ \to \nabla w \quad \text{weakly in } L^2(B_{\frac{\delta}{2}}) \,. \tag{3.66}
$$

It follows from (3.66), (1.2) and (3.16) that for all $\eta \in L^2(0,T;H^1(\Omega))$

$$
\int_{B_\delta} b(u) \nabla W^+ \nabla \eta \,\mathrm{d}x\,\mathrm{d}t \to \int_{B_\delta} b(u) \nabla w \nabla \eta \,\mathrm{d}x\,\mathrm{d}t \qquad \text{as } h \to 0. \tag{3.67}
$$

Combining (3.62), (3.64) and (3.67), and noting (1.2), (2.7) and (3.18) yields for all $\eta \in L^2(0,T;H^2(\Omega))$ that as $h \to 0$

$$
\int_{B_\delta} \pi^h[b(U^-)] \nabla W^+ \nabla (\pi^h \eta) \,\mathrm{d}x\,\mathrm{d}t \to \int_{B_\delta} b(u) \nabla w \nabla \eta \,\mathrm{d}x\,\mathrm{d}t. \tag{3.68}
$$

We now consider the inequality (3.15c) of (P$^{h,\tau}$). For any $\eta \in L^2(0,T;H^2(\Omega))$, with supp$(\eta) \subset D_\delta$, we choose $\eta^h \equiv \pi^h \eta$ in (3.15c). It follows immediately from (3.29) that for all $\eta \in L^2(0,T;H^2(\Omega))$ and for all $h \le h_0(\delta)$

$$
\mathrm{supp}(\eta) \subset B_\delta \;\Rightarrow\; \mathrm{supp}(\pi^h \eta) \subset B^h_\delta \subset B_{\frac{\delta}{2}}. \tag{3.69}
$$

We now analyse the subsequent terms in (3.15c). From (2.1), (3.4), (3.69) and (3.65) we deduce for all $h \le \widehat{h}_0(\frac{\delta}{32})$ and $\eta \in L^2(0,T;H^2(\Omega))$ with supp$(\eta) \subset B_\delta$ that

$$
\begin{aligned}
&\left| \int_0^T \Big[(W^+, \pi^h \eta)^h - (W^+, \eta) \Big] \,\mathrm{d}t \right| \\
&\qquad \le C h \, \|W^+\|_{L^2(0,T;H^1(B_{\frac{\delta}{2}}(t)))} \, \|\eta\|_{L^2(0,T;H^2(\Omega))} \\
&\qquad \le C(\delta^{-1}) \, h \, \|\eta\|_{L^2(0,T;H^2(\Omega))}.
\end{aligned} \tag{3.70}
$$

It follows from (3.70) and (3.66) that for all $\eta \in L^2(0,T;H^2(\Omega))$ with $\mathrm{supp}(\eta) \subset B_\delta$

$$\int_0^T (W^+, \pi^h \eta)^h \,\mathrm{d}t \to \int_0^T (w,\eta)\,\mathrm{d}t \equiv \int_{B_\delta} w\,\eta\,\mathrm{d}x\,\mathrm{d}t \quad \text{as } h \to 0. \tag{3.71}$$

Similarly to (3.70) and (3.71), we deduce from (3.69), (2.8), (3.65) and (3.66) that for all $\eta \in L^2(0,T;H^2(\Omega))$ with $\mathrm{supp}(\eta) \subset B_\delta$

$$\int_0^T (\nabla V^+, \nabla(\pi^h\eta))\,\mathrm{d}t \to \int_0^T (\nabla v, \nabla \eta)\,\mathrm{d}t \quad \text{as } h \to 0. \tag{3.72}$$

Combining (3.72) and (3.71), noting (3.46) and (3.69), and applying a density argument yields that for all $\eta \in L^2(0,T;H^1(\Omega))$ with $\mathrm{supp}(\eta) \subset B_\delta$

$$\int_{B_\delta} [\,\nabla v \nabla \eta - w\,\eta\,]\,\mathrm{d}x\,\mathrm{d}t = 0. \tag{3.73}$$

Repeating (3.63)–(3.68) for all $\delta > 0$, and noting (3.62) and (2.8) yields the desired results (3.57), (3.58) and for all $\eta \in L^2(0,T;H^2(\Omega))$ that as $h \to 0$

$$\int_{\Omega_T} \pi^h[b(U^-)]\nabla W^+ \nabla(\pi^h\eta)\,\mathrm{d}x\,\mathrm{d}t \to \int_{B_0} b(u)\nabla w \nabla \eta\,\mathrm{d}x\,\mathrm{d}t. \tag{3.74}$$

Combining (3.61), (3.74) and (3.15a) we have for all $\eta \in H^1(0,T;H^2(\Omega))$ that

$$\int_0^T (u, \tfrac{\partial \eta}{\partial t})\,\mathrm{d}t - \int_{B_0} b(u)\nabla w \nabla \eta\,\mathrm{d}x\,\mathrm{d}t = (u(\cdot,T),\eta(\cdot,T)) - (u(\cdot,0),\eta(\cdot,0))\,. \tag{3.75}$$

As $\{\pi^h[b(U^-)]\nabla W^+\}_h$ is uniformly bounded in $L^2(\Omega_T)$, see (3.20), (1.2) and (3.22), it follows that $b(u)\nabla w \in L^2(B_0)$. Therefore from (3.75) we conclude the second regularity result in (3.56) and, on noting a density argument, that

$$\int_0^T \langle \tfrac{\partial u}{\partial t}, \eta \rangle\,\mathrm{d}t + \int_{B_0} b(u)\nabla w \nabla \eta\,\mathrm{d}x\,\mathrm{d}t = 0 \quad \forall\, \eta \in L^2(0,T;H^1(\Omega))$$

and hence the desired result (3.59a). Finally repeating (3.69)–(3.73) for all $\delta > 0$ yields that $\int_{B_0}[\nabla v \nabla \eta - w\,\eta]\,\mathrm{d}x\,\mathrm{d}t = 0$ for all $\eta \in L^2(0,T;H^1(\Omega))$ with $\mathrm{supp}(\eta) \subset B_0$, and hence the desired result (3.59c). □

Remark 3.1 The identity (3.59c) and (3.57) imply that $w = -\Delta v$ in a weak sense locally on $\{u > 0\}$. As $v = -\Delta u$, see (3.59b), (3.56) and (3.17), we deduce that $w = \Delta^2 u$ in a weak sense locally on $\{u > 0\}$. Hence we conclude from (3.59a–c) that (1.3) holds with $\{|u| > 0\}$ replaced by $\{u >$

$0\}$. This is the weak formulation of (1.1a) introduced by [8] in one space dimension. A weak formulation of the boundary condition $b(u)\frac{\partial \Delta^2 u}{\partial \nu} = 0$ is also incorporated in (1.3). We note that (3.59b) implies that $\frac{\partial u}{\partial \nu}(x,t) = 0$ for $(x,t) \in \partial\Omega \times (0,T)$, whereas (3.59c) implies that $\frac{\partial \Delta u}{\partial \nu}(x,t) = 0$ for $(x,t) \in \partial\Omega \times (0,T)$ whenever $u(x,t) > 0$. In addition $u \in C^{1,\frac{1}{4}}_{x,t}(\overline{\Omega}_T)$, see (3.16) when $d = 1$, improves on $u \in C^{1,\frac{1}{5}}_{x,t}(\overline{\Omega}_T)$, see (1.4), as proved in §7 of [8]. Moreover, the above extends their existence and regularity results to higher space dimensions.

4 Solution of the Discrete System

We now consider an algorithm for solving the discrete system at each time level in (P$^{h,\tau}$). This is based on the general splitting algorithm of [14]; see also [10, 2, 5] where this algorithm has been adapted to solve similar variational inequality problems arising from Cahn-Hilliard systems. We remark that the alternative algorithm in §3 of [4] can also be adapted to the present problem.

For n fixed, multiplying (2.2c) by $\zeta > 0$, adding $(U^n, \chi - U^n)^h$ to both sides and rearranging on noting (2.2a) it follows that $\{U^n, V^n, W^n\} \in K^h \times [S^h]^2$ satisfy for all $\eta^h \in K^h$, $\chi \in S^h$

$$\Big(U^n, \eta^h - U^n\Big)^h \geq (Y^n, \eta^h - U^n)^h, \tag{4.1a}$$

$$\left(\frac{U^n - U^{n-1}}{\tau_n}, \chi\right)^h + b^{n-1}(\nabla W^n, \nabla \chi) = ([b^{n-1} - \pi^h[b(U^{n-1})]\,]\nabla W^n, \nabla \chi), \tag{4.1b}$$

where $Y^n \in S^h$ is such that

$$(Y^n, \chi)^h := (U^n, \chi)^h - \zeta\,[\,(\nabla V^n, \nabla \chi) - (W^n, \chi)^h\,] \quad \forall\, \chi \in S^h, \tag{4.1c}$$

$V^n = -\Delta^h U^n$ and $b^{n-1} := |b(U^{n-1})|_{0,\infty}$. We introduce also $X^n \in S^h$ such that

$$(X^n, \chi)^h := (U^n, \chi)^h + \zeta\,[\,(\nabla V^n, \nabla \chi) - (W^n, \chi)^h\,] \quad \forall\, \chi \in S^h \tag{4.1d}$$

and note that $X^n = 2U^n - Y^n$. We use this as a basis for constructing our iterative procedure:
For $n \geq 1$ set $\{U^{n,0}, V^{n,0}, W^{n,0}\} \equiv \{U^{n-1}, V^{n-1}, W^{n-1}\} \in K^h \times [S^h]^2$, where V^0, $W^0 \in S^h$ are arbitrary if $n = 1$. For $k \geq 0$ we define $Y^{n,k} \in S^h$ such that for all $\chi \in S^h$

$$(Y^{n,k}, \chi)^h = (U^{n,k}, \chi)^h - \zeta\,[\,(\nabla V^{n,k}, \nabla \chi) - (W^{n,k}, \chi)^h\,]. \tag{4.2a}$$

Then set $U^{n,k+\frac{1}{2}} = \pi^h[Y^{n,k}]_+ \in K^h$ and find $\{U^{n,k+1}, V^{n,k+1}, W^{n,k+1}\} \in [S^h]^3$ such that for all $\chi \in S^h$

$$(\tfrac{U^{n,k+1}-U^{n-1}}{\tau_n}, \chi)^h + b^{n-1}(\nabla W^{n,k+1}, \nabla \chi) = ([b^{n-1} - \pi^h[b(U^{n-1})]\,]\nabla W^{n,k}, \nabla \chi), \quad (4.2b)$$

$$(U^{n,k+1}, \chi)^h + \zeta\,[\,(\nabla V^{n,k+1}, \nabla \chi) - (W^{n,k+1}, \chi)^h\,] = (X^{n,k+1}, \chi)^h; \quad (4.2c)$$

where $V^{n,k+1} = -\Delta^h U^{n,k+1}$ and $X^{n,k+1} := 2U^{n,k+\frac{1}{2}} - Y^{n,k}$.

In order to establish the well-posedness of (4.2b,c), let $R^{n,k} \in Z^h$ be such that

$$(R^{n,k}, \chi)^h = (\pi^h[b(U^{n-1})]\nabla W^{n,k}, \nabla \chi) \quad \forall\, \chi \in S^h.$$

It then follows from (4.2b), (2.10) and (4.2c) with $\chi \equiv 1$ that

$$W^{n,k+1} = (I - ⨍\,)W^{n,k} - [b^{n-1}]^{-1}\widehat{\mathcal{G}}^h(\tfrac{U^{n,k+1}-U^{n-1}}{\tau_n} + R^{n,k}) + \tfrac{1}{\zeta} ⨍\,(U^{n,k+1} - X^{n,k+1}).$$

Therefore (4.2b,c) may be written equivalently as find $U^{n,k+1} \in S^h_m := \{\chi \in S^h : ⨍\,\chi = ⨍\,U^0\}$ such that for all $\chi \in S^h$

$$(U^{n,k+1}, (I - ⨍\,)\chi)^h + \zeta\,[\,(\Delta^h U^{n,k+1}, \Delta^h \chi)^h + ([b^{n-1}]^{-1}\widehat{\mathcal{G}}^h[\frac{U^{n,k+1} - U^{n-1}}{\tau_n}], \chi)^h\,] = (X^{n,k+1} + \zeta(W^{n,k} - [b^{n-1}]^{-1}\widehat{\mathcal{G}}^h R^{n,k}), (I - ⨍\,)\chi)^h. \quad (4.3)$$

Existence and uniqueness of $\{U^{n,k+1}, V^{n,k+1}, W^{n,k+1}\} \in S^h_m \times [S^h]^2$ then follows as (4.3) is the Euler-Lagrange equation of the strictly convex minimisation problem

$$\min_{\chi \in S^h_m} \Big\{ |\chi|_h^2 + \zeta\,\Big[|\Delta^h \chi|_h^2 + \tfrac{1}{b^{n-1}\tau_n}|\nabla \widehat{\mathcal{G}}^h(\chi - U^{n-1})|_0^2\Big] - 2\,(X^{n,k+1} + \zeta(W^{n,k} - [b^{n-1}]^{-1}\widehat{\mathcal{G}}^h R^{n,k}), \chi)^h \Big\}.$$

Hence the iterative procedure (4.2a–c) is well-defined.

Theorem 4.1 *For all $\zeta \in \mathbb{R}^+$ and $\{U^{n,0}, V^{n,0}, W^{n,0}\} \in [S^h]^3$ the sequence $\{U^{n,k}, V^{n,k}, W^{n,k}\}_{k\geq 0}$ generated by the algorithm (4.2a–c) satisfies as $k \to \infty$*

$$U^{n,k} \to U^n,\ V^{n,k} \to V^n \quad \textit{and} \int_\Omega b(U^{n-1})|\nabla(W^{n,k} - W^n)|^2 \,\mathrm{d}x \to 0. \quad (4.4)$$

Proof It follows from (4.1c,d), (4.2a,c) and the definition of $X^{n,k+1}$ that for $k \geq 0$

$$U^n = \tfrac{1}{2}(X^n + Y^n),\ U^{n,k} = \tfrac{1}{2}(X^{n,k} + Y^{n,k}),\ U^{n,k+\frac{1}{2}} = \tfrac{1}{2}(X^{n,k+1} + Y^{n,k}).$$

We now introduce for $k \geq 0$ the notation

$$E_U^k := U^{n,k} - U^n, \quad E_U^{k+\frac{1}{2}} := U^{n,k+\frac{1}{2}} - U^n, \quad E_V^k := V^{n,k} - V^n,$$
$$E_W^k := W^{n,k} - W^n,\ E_Y^k := Y^{n,k} - Y^n,\ E_X^{k+1} := X^{n,k+1} - X^n; \quad (4.5)$$

and hence we have for $k \geq 0$ that

$$E_U^{k+\frac{1}{2}} = \tfrac{1}{2}(E_X^{k+1} + E_Y^k), \qquad E_U^{k+1} = \tfrac{1}{2}(E_X^{k+1} + E_Y^{k+1}). \quad (4.6)$$

Adding (4.1d) to (4.2c), and noting (2.3) and that $E_V^{k+1} \equiv -\Delta^h E_U^{k+1}$ yields for all $\chi \in S^h$

$$(E_U^{k+1}, \chi)^h + \zeta\,[\,(\Delta^h E_U^{k+1}, \Delta^h \chi)^h - (E_W^{k+1}, \chi)^h\,] = (E_X^{k+1}, \chi)^h. \quad (4.7)$$

It follows from (4.7) and (4.6) that for $k \geq 0$

$$|\Delta^h E_U^{k+1}|_h^2 - (E_W^{k+1}, E_U^{k+1})^h = \tfrac{1}{\zeta}(E_X^{k+1} - E_U^{k+1}, E_U^{k+1})$$
$$= \tfrac{1}{4\zeta}(|E_X^{k+1}|_h^2 - |E_Y^{k+1}|_h^2). \quad (4.8)$$

It is easily established from (4.1a), $U^{n,k+\frac{1}{2}} = \pi^h[Y^{n,k}]_+$ and (4.6) that for $k \geq 0$

$$(E_U^{k+\frac{1}{2}} - E_Y^k, E_U^{k+\frac{1}{2}})^h \leq 0 \qquad \Longrightarrow \qquad |E_X^{k+1}|_h^2 \leq |E_Y^k|_h^2. \quad (4.9)$$

From (4.2b), (4.1b), (4.5) and (2.24) it follows for $k \geq 0$ that

$$\begin{aligned}
&-(E_W^{k+1}, E_U^{k+1})^h = -(E_W^{k+1}, (U^{n,k+1} - U^{n-1}) - (U^n - U^{n-1}))^h \\
&= \tau_n |(\pi^h[b(U^{n-1})])^{\frac{1}{2}} \nabla E_W^{k+1}|_0^2 \\
&\qquad + \tau_n([b^{n-1} - \pi^h[b(U^{n-1})]\,]\nabla(W^{n,k+1} - W^{n,k}), \nabla E_W^{k+1}) \\
&= \tau_n |(\pi^h[b(U^{n-1})])^{\frac{1}{2}} \nabla E_W^{k+1}|_0^2 + \tfrac{\tau_n}{2}\Big[|[b^{n-1} - \pi^h[b(U^{n-1})]\,]^{\frac{1}{2}} \nabla E_W^{k+1}|_0^2 \\
&\qquad + |[b^{n-1} - \pi^h[b(U^{n-1})]\,]^{\frac{1}{2}} \nabla(W^{n,k+1} - W^{n,k})|_0^2 \\
&\qquad\quad - |[b^{n-1} - \pi^h[b(U^{n-1})]\,]^{\frac{1}{2}} \nabla E_W^k|_0^2\Big]. \qquad (4.10)
\end{aligned}$$

Combining (4.8), (4.10) and (4.9) yields for $k \geq 0$ that

$$\begin{aligned}
&|\Delta^h E_U^{k+1}|_h^2 + \tau_n |(\pi^h[b(U^{n-1})])^{\frac{1}{2}} \nabla E_W^{k+1}|_0^2 \\
&\qquad + \tfrac{\tau_n}{2} |[b^{n-1} - \pi^h[b(U^{n-1})]\,]^{\frac{1}{2}} \nabla E_W^{k+1}|_0^2 + \tfrac{1}{4\zeta}|E_Y^{k+1}|_h^2 \\
&\leq \tfrac{1}{2}\tau_n |[b^{n-1} - \pi^h[b(U^{n-1})]\,]^{\frac{1}{2}} \nabla E_W^k|_0^2 + \tfrac{1}{4\zeta}|E_Y^k|_h^2. \qquad (4.11)
\end{aligned}$$

We conclude from (4.11) that

$$|\Delta^h E_U^{k+1}|_h^2 + \tau_n |(\pi^h[b(U^{n-1})])^{\frac{1}{2}} \nabla E_W^{k+1}|_0^2 \to 0 \qquad \text{as } k \to \infty. \quad (4.12)$$

As $E_V^{k+1} \equiv -\Delta^h E_U^{k+1}$ and $\fint E_U^{k+1} = 0$, the desired results (4.4) follow from (4.12), (4.5), (2.3) and (1.8). □

Remark 4.1 We see from (4.2a–c) and (4.3) that at each iteration one needs to solve only a fixed linear system with constant coefficients. On a uniform mesh this can be done efficiently using a discrete cosine transform; see [9, §5], where a similar problem is solved.

5 Numerical Experiments

Firstly, we present numerical experiments in one space dimension on a uniform partitioning of $\Omega = (0,1)$ with mesh points $p_j = (j-1)h$, $j = 1 \to \#J$, where $h = 1/(\#J-1)$. In addition, we chose a uniform time step $\tau_n \equiv \tau := T/N$, so that $t_n := n\tau$, $n = 0 \to N$. Similarly to [4], on recalling that $U^n \in \widetilde{K}^h(U^{n-1})$, $n = 1 \to N$, one characteristic feature of the discretisation (P$^{h,\tau}$) is that

$$U^{n-1}(p_{j-1}) = U^{n-1}(p_j) = U^{n-1}(p_{j+1}) = 0 \quad \Rightarrow \quad U^n(p_j) = 0, \quad (5.1)$$

so that the free boundary advances at most one mesh point locally from one time level to the next. To be able to track a free boundary which moves with a finite but a-priori unknown speed, one needs to choose τ and h such that $\tau^{-1}h \to \infty$. If we choose the time step too large, e.g. if $\tau^{-1}h \to 0$, the solution we obtain in the limit as $h,\, \tau \to 0$ would not spread at all. This gives the existence of non-spreading solutions for all $\gamma \in (0,\infty)$ and all initial data u^0 satisfying the assumptions of Lemma 3.2.

For the algorithm (4.2a–c) we chose $V^0 = -\Delta^h U^0$, $W^0 = -\Delta^h V^0$, $\zeta \propto h$ (from experimental evidence) in order to improve its convergence, and for each n adopted the stopping criterion $|U^{n,k} - U^{n,k-1}|_{0,\infty} < tol$ with $tol \leq 10^{-8}$. Similarly to [4], we imposed the additional requirement that the discrete free boundary had not moved more than one mesh point locally, recall (5.1). To ensure this we introduced approximate analogues of the sets $I_m(q^h)$ denoted by $\widetilde{I}_m(q^h)$, which were defined by replacing (c) in (2.14) by $(\widetilde{c})$ $q^h > tol_1 := 10^{-12}$ at some point in κ_l, $l = 1 \to L$. We then set $\{U^n, V^n, W^n\} \equiv \{\overline{U}^{n,k}, V^{n,k}, W^{n,k}\}$, where $\overline{U}^{n,k} \in K^h$ was defined by

$$\overline{U}^{n,k}(p_j) := \begin{cases} [U^{n,k}(p_j)]_+ & \text{if } j \in \widetilde{J}_+(U^{n-1}) := \bigcup_{m=1}^M \widetilde{I}_m(U^{n-1}), \\ 0 & \text{if } j \in \widetilde{J}_0(U^{n-1}) := J \setminus \widetilde{J}_+(U^{n-1}). \end{cases} \quad (5.2)$$

In the first set of numerical experiments we set $\gamma = 1$ and consider the source type similarity solution (1.5). The corresponding positive free boundary point is $x_F(t) = \omega(t+\vartheta)^{\frac{1}{7}}$. We chose $\omega = 2$, $\vartheta = 4^{-7}$ and noted the symmetry about $x = 0$. We set $U^0(x) \equiv (\pi^h u)(x,0) \equiv 16\pi^h(\left[1-4x^2\right]_+^3)/315$. We estimated the true free boundary, $x_F(t_n)$, at each time level t_n by x_C^n using inverse quadratic interpolation through the last three mesh points where $U^n(p_j) > tol_1$; that is, $Q^n(x_C^n) = 0$ where Q^n is the unique quadratic such that $Q^n(p_j) = U^n(p_j) > tol_1$, $j = j_n - 1 \to j_n + 1$, and $U^n(p_{j_n+2}) \leq tol_1$. For $n = 1 \to N$, we computed the quantities

$$|\pi^h u(\cdot,t_n) - U^n(\cdot)|_{0,\infty} \qquad \text{and} \qquad x_F(t_n) - x_C^n;$$

where $x_F(T_\star) = 1$, i.e. $T_\star = \omega^{-7} - \vartheta = 2^{-7}(1-2^{-7}) \approx 7.7515 \times 10^{-3}$, and $N := \max\{n : n\tau \leq T_\star\}$. In Figure 5.1 we show the graph of the

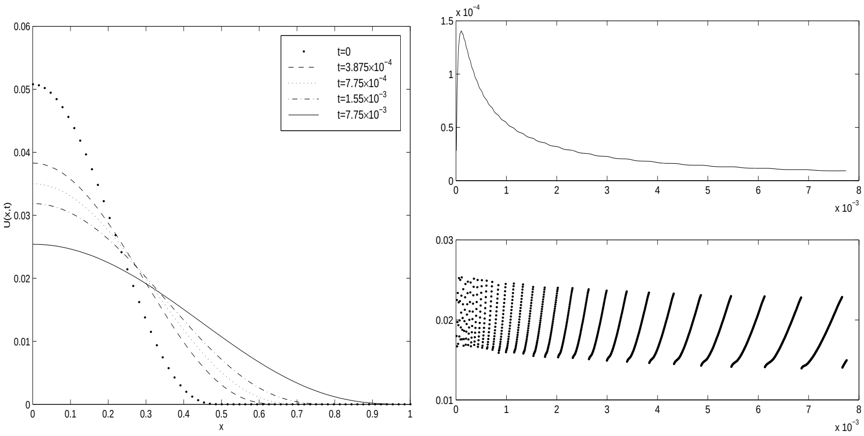

Fig. 5.1. $\gamma = 1$, $U(x,t)$ plotted against x for various t (left); $|\pi^h u(\cdot,t_n) - U^n(\cdot)|_{0,\infty}$ and $x_F(t_n) - x_C^n$ against t_n (right), where $\#J = 65$ and $\tau = 6.0546875 \times 10^{-6}$.

computed solution for different times and we plot $|\pi^h u(\cdot,t_n) - U^n(\cdot)|_{0,\infty}$ and $x_F(t_n) - x_C^n$ against t_n. The computations were performed for $h = 2^{-6}$, i.e. $\#J = 65$, and $T = 7.75 \times 10^{-3}$ with $\tau = T/1280 = 0.05\,T\,h = 6.0546875 \times 10^{-6}$.

We see there that the maximum $|\pi^h u(\cdot,t_n) - U^n(\cdot)|_{0,\infty}$ occurred for small $t_n \approx 0.0001$. This is not surprising, since the true free boundary, $x_F(t)$, moves very fast initially. In addition we see that the numerical free boundary, x_C^n, always underestimated $x_F(t_n)$. We repeated the above experiment with a final time $T = 7.75 \times 10^{-3}$ for various choices of h with $\tau = 4.0 \times 10^{-4} h$ and $\tau = 0.3968\,h^2$; see Table 5.1, where all values are cor-

Table 5.1. $\gamma = 1$, source type solution errors.

$\tau = 4.0 \times 10^{-4} h$	$\#J$	65	129	257	513	1025
$\max_{n=1\to N} \lvert\pi^h u(\cdot, t_n) - U^n(\cdot)\rvert_{0,\infty} / 10^{-5}$		14.05	7.105	3.408	1.653	0.7231

$\tau = 0.3968\, h^2$	$\#J$	65	129	257	513	1025
$\max_{n=1\to N} \lvert\pi^h u(\cdot, t_n) - U^n(\cdot)\rvert_{0,\infty} / 10^{-5}$		214.6	61.23	13.74	3.445	0.7992

rect to four significant figures. We note that the constant in the relationship $\tau = 4 \times 10^{-4} h$ was chosen to be sufficiently small so that the discrete free boundary could move faster than $x_F(t)$, i.e. $x'_F(t) \leq x'_F(0) = 8192/7 \leq (4.0 \times 10^{-4})^{-1}$ for all $t \geq 0$.

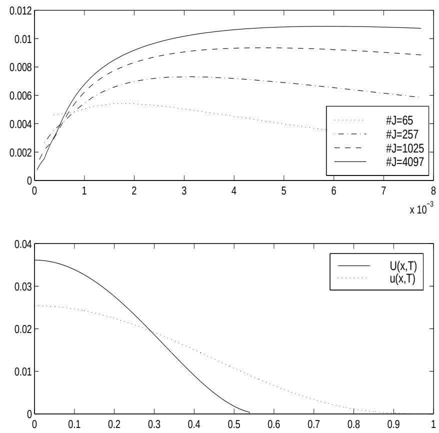

Fig. 5.2. $\gamma = 1$, $\lvert\pi^h u(\cdot, t_n) - U^n(\cdot)\rvert_{0,\infty}$ plotted against t_n, and $u(x, T)$ and $U(x, T)$ ($T = 7.75 \times 10^{-3}$) with $\#J = 4097$ plotted against x.

As noted earlier in this section there exist non-spreading solutions for all $\gamma \in (0, \infty)$ and all initial data u^0 satisfying the assumptions of Lemma 3.2. Clearly, the source type similarity solution, (1.5), at $t = 0$ satisfies these assumptions. Repeating the above numerical experiment with $\tau = 3.1 \times 10^{-3} h^{\frac{1}{2}}$, we found that the computed solution U did not converge to the source type similarity solution, see Figure 5.2; since it must converge to a non-spreading solution as $h \to 0$.

Remark 5.1 The obstacle formulation for ($\mathrm{P}^{h,\tau}$) is not crucial in proving the convergence of the resulting approximation U to a solution, u, of (P). Replacing the inequality by an equality, K^h by S^h, $b(s) := |s|^\gamma$ by $[s]_+^\gamma$

and adapting the proofs in §3 one can easily pass to a limit u which solves the equation in the sense of (1.3). Using the negative part of u, i.e. $[u]_-$, as a test function in the weak formulation (1.3) one recovers nonnegativity of the solution. The iterative method described in §4 can also be easily adapted to this approximation. Similarly to the fourth order problem in [4], although we found the resulting errors to be comparable with those in Table 5.1, there were a number of drawbacks. $U^n(\cdot)$ was negative (many orders of magnitude less than $-tol$) in many disconnected regions where $u(\cdot, t_n) \equiv 0$, which made the location of the approximate free boundary more difficult. In addition the CPU times were increased.

In the second set of experiments we took $u^0(x) \equiv [\frac{1}{4} - x^2]_+^2$, $U^0 \equiv \pi^h u^0$ and $\#J = 2l + 1$. A simple calculation yields that

$$V^0(jh) = -(\Delta^h U^0)(jh) = \begin{cases} 1 - 2h^2(1 + 6j^2) & j = 0 \to l - 1, \\ -(1 - h)^2 & j = l, \\ 0 & j = l + 1 \to 2l; \end{cases}$$

$$\text{and } W^0(jh) = -(\Delta^h V^0)(jh) = \begin{cases} 24 & j = 0 \to l - 2, \\ (23h^2 - 2h - 1)/h^2 & j = l - 1, \\ (12h - 8)/h & j = l, \\ (1 - h)^2/h^2 & j = l + 1, \\ 0 & j = l + 2 \to 2l. \end{cases}$$

It follows that

$$(\pi^h[b(U^0)]\nabla W^0, \nabla W^0) = \tfrac{h^{2\gamma-5}}{2} \{(1 + h)^4 [\,(2 - 4\,h)^{2\gamma} + (1 - h)^{2\gamma}\,] + (1 - 6h - 11h^2)^2 \,(1 - h)^{2\gamma}\,\}. \quad (5.3)$$

It follows from (2.2a) for $n = 1$ with $\chi \equiv W^1 - W^0$, (2.4) for $n = 1$ with $\chi \equiv U^0$, $W^0 \equiv (\Delta^h)^2 U^0$, (2.3) and (2.24) that

$$2\,|\Delta^h(U^1 - U^0)|_h^2 + \tau\,(\pi^h[b(U^0)]\nabla(W^1 - W^0), \nabla(W^1 - W^0)) \leq \tau\,(\pi^h[b(U^0)]\nabla W^0, \nabla W^0). \quad (5.4)$$

Combining (5.3) and (5.4) and noting (2.3), (1.8) and $\fint (U^1 - U^0) = 0$, we have for $\gamma > 2.5$ and any $\tau > 0$ that $U^1 \to U^0$ as $h \to 0$. Hence noting (2.8), we have that for any $\gamma > 2.5$ and $\tau > 0$ that $\|U^1(\cdot) - u^0(\cdot)\|_1 \to 0$ as $h \to 0$. Similarly to (5.4), it follows from (4.2a–c) with $n = 1$ and $k = 0$, $W^0 = (\Delta^h)^2 U^0$, (2.3) and (2.24) that

$$2\,|\Delta^h(U^{1,1} - U^0)|_h^2 + \tau\, b^0\, |W^{1,1} - W^0|_1^2 \leq \tau\,(\pi^h[b(U^0)]\nabla W^0, \nabla W^0).$$

Hence noting (5.3), (2.3), (1.8), $⨍\,(U^{1,1} - U^0) = 0$ and the stopping criterion we have for any $\gamma > 2.5$ and $\tau > 0$ that $\{U^1, V^1, W^1\} \equiv \{\overline{U}^{1,1}, V^{1,1}, W^{1,1}\}$ for $h \leq h_0(\gamma, tol)$, where $\overline{U}^{1,1}$ is defined by (5.2) above.

As in the fourth order case, see [4], it is possible that (P) may possess at least two solutions for certain values of γ. It is interesting to see how the numerical approximation (P$^{h,\tau}$) behaves in such circumstances. We performed experiments with $\gamma \in \{3.0, 2.5, 2.0\}$, $T = 7.75 \times 10^{-3}$ and $\tau = 50.7904\,h^3$. For the algorithm (4.2a–c) we chose $\zeta = 10^{-8}$ and $tol = 10^{-12}$. In Figure 5.3 we plot $\pi^h u^0(x) - U(x,T)$ for $\#J = 129,\ 257$ and 513. For $\gamma = 3.0 > 2.5$, we see that $U(x,T) \to u^0(x)$ as $h \to 0$. From Figure 5.3, we conclude that $U(x,T)$ converges to $u^0(x)$ also in the case $\gamma = 2.5$. The same experiment for $\gamma = 2.0$ shows that the computed solution spreads. Finally we remark that computations on these uniform

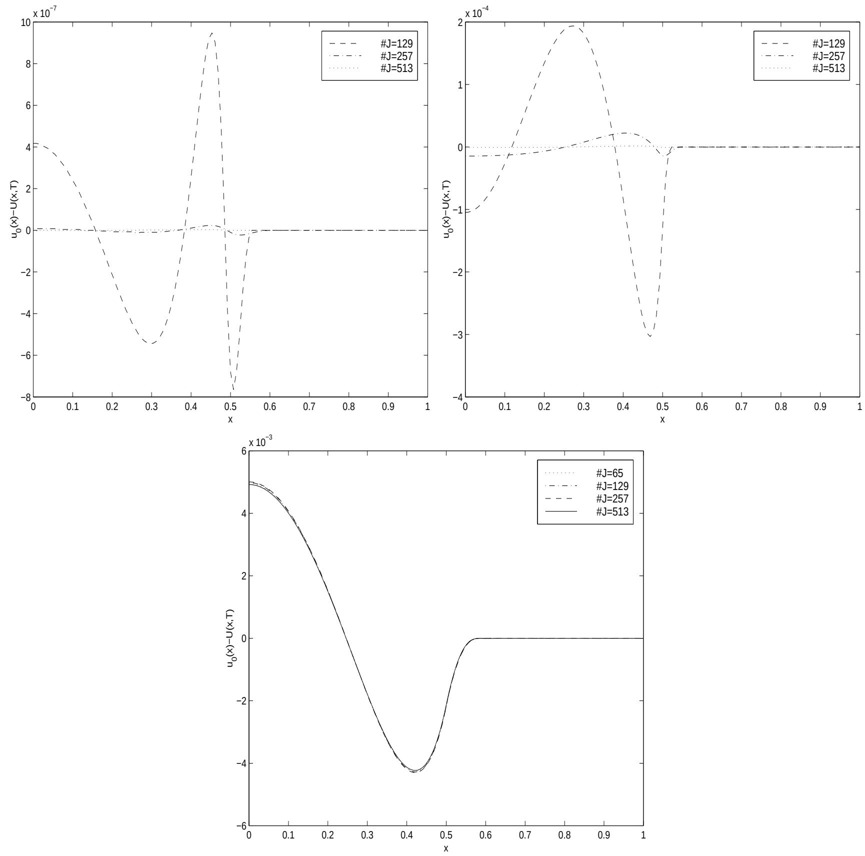

Fig. 5.3. $\pi^h u^0(x) - U(x,T)$ plotted against x for $\gamma = 3.0, 2.5$ and 2.0, respectively.

partitionings for $\gamma < 2.5$, but close to 2.5, were inconclusive, with it not being clear whether the computed solution spread or converged to $u^0(x)$. We remark that $\gamma = 3$ is the borderline value for spreading in the fourth order case, see [4].

Numerical Results for $d = 2$

Finally, we present numerical experiments in two space dimensions with $\Omega = (0,1)\times(0,1)$. We took a uniform mesh of squares ς of length $h = \frac{1}{256}$, each of which was divided into two triangles by its north east diagonal. We used the modified discrete semi-inner product on $C(\overline{\Omega})$

$$(\eta_1, \eta_2)^h_* := \int_\Omega \Pi^h(\eta_1(x)\,\eta_2(x))\,\mathrm{d}x\,. \tag{5.5}$$

Here Π^h is the piecewise continuous bilinear interpolant on $\overline{\Omega}$, which on each square ς is bilinear and interpolates at the vertices. Using (5.5) instead of (2.1) enables us to solve (4.2a–c) efficiently using a "discrete cosine transform" approach, see [2]. We note that, similarly to (2.6), the semi-inner product (5.5) is equivalent on S^h to the standard L^2 inner product. In place of (2.5), we have for $m = 0$ or 1 that

$$|(z^h,\chi)^h - (z^h,\chi)^h_*| \le Ch^{1+m}[\ln(\tfrac{1}{h})]^2\,\|z^h\|_m\|\chi\|_1 \qquad \forall\, z^h,\,\chi \in S^h\,.$$

Therefore it is easy to adapt the proofs to show that all the results in this paper remain unchanged with the choice (5.5).

We report on an experiment with similar initial data for (P) as in $d = 1$ for Figure 5.3. In particular, we took $u^0(x) = [(0.6)^2 - (x_1^2 + 4\,x_2^2)]_+^2$, $U^0 \equiv \pi^h u^0$ and set $\tau_n \equiv \tau = 5\times10^{-8}$, $T = 1.5\times10^{-3}$. For the algorithm (4.2a–c) we used $\zeta = 10^{-9}$ and for each n adopted the stopping criterion $|U^{n,k} - U^{n,k-1}|_{0,\infty} < 10^{-7}$. The results for different values of γ are shown in Figure 5.4, where we plot the $U(x,t_n) = 2\times10^{-3}$ contour lines at times $t_n = 0, 10^{-4}, T$. We note that the respective contour lines for much smaller values than 2×10^{-3} become very irregular as u approaches zero flatly; i.e. a "zero contact angle". For $\gamma = 1.0$, the elliptical support of U^0 spreads in all directions and the support of U^n becomes more circular. For $\gamma = 2.0$, there is no spreading in the x_1, major axis, direction; but once again the support of U^n becomes more circular. For $\gamma = 3.0$, there is virtually no spreading in any direction.

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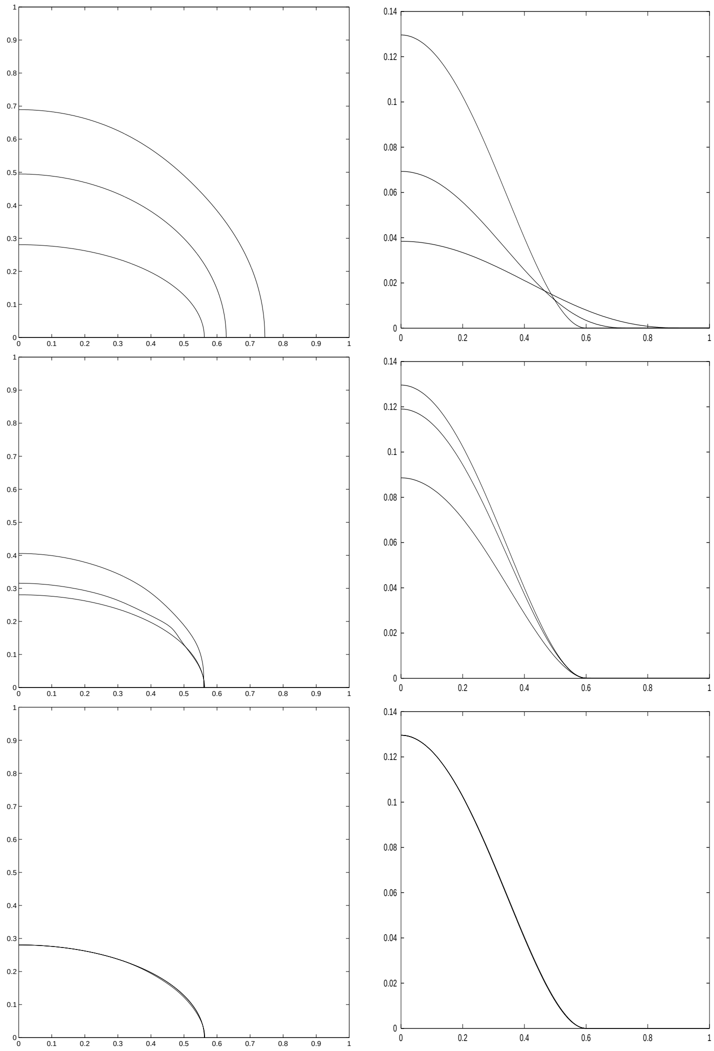

Fig. 5.4. The $U(x,t_n) = 2 \times 10^{-3}$ contour lines (left) and $U(x,t_n)\,|_{x_2=0}$ plotted against x_1 (right) for $\gamma = 1.0$, 2.0 and 3.0, respectively, and $t_n = 0, 10^{-4}, T = 1.5 \times 10^{-3}$.

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