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Finite element approximation of eigenvalue problems

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We discuss the finite element approximation of eigenvalue problems associated with compact operators. While the main emphasis is on symmetric problems, some comments are present for non-self-adjoint operators as well. The topics covered include standard Galerkin approximations, non-conforming approximations, and approximation of eigenvalue problems in mixed form. Some applications of the theory are presented and, in particular, the approximation of the Maxwell eigenvalue problem is discussed in detail. The final part tries to introduce the reader to the fascinating setting of differential forms and homological techniques with the description of the Hodge–Laplace eigenvalue problem and its mixed equivalent formulations. Several examples and numerical computations complete the paper, ranging from very basic exercises to more significant applications of the developed theory.

CONTENTS

1. Introduction

The aim of this paper is to provide the reader with an overview of the state of the art in the numerical analysis of the finite element approximation of eigenvalue problems arising from partial differential equations.

The work consists of four parts, which are ordered according to their increasing difficulty. The material is arranged in such a way that it should be possible to use it (or part of it) as a reference for a graduate course.

Part 1 presents several examples and reports on some academic numerical computations. The results presented range from a very basic level (such as the approximation of the one-dimensional Laplace operator), suited to those just starting work in this subject, to more involved examples. In particular, we give a comprehensive review of the Galerkin approximation of the Laplace eigenvalue problem (also in the presence of a singular domain and of non-conforming schemes), of the mixed approximation of the Laplace eigenvalue problem (with stable or unstable schemes), and of the Maxwell eigenvalue problem. Some of the presented material is new, in particular, the numerical results for the one-dimensional mixed Laplacian with the $\mathcal{P}_1 - \mathcal{P}_1$ and the $\mathcal{P}_2 - \mathcal{P}_0$ scheme.

Part 2 contains the main core of the theory concerning the Galerkin approximation of variationally posed eigenvalue problems. With a didactic purpose, we included a direct proof of convergence for the eigenvalues and eigenfunctions of the Laplace equation approximated with piecewise linear elements. By direct proof, we mean a proof which does not make use of the abstract spectral approximation theory, but is based on basic properties of the Rayleigh quotient. This proof is not new, but particular care has been paid to the analysis of the case of multiple eigenfunctions. In Section 9 we describe the so-called Babuška–Osborn theory. As an example of application we analyse the approximation of the eigensolutions of an elliptic operator. Then, we provide another application which involves the non-conforming Crouzeix–Raviart element for the approximation of the Laplace eigenvalue problem. The results of this section are probably not new, but we could not find a reference providing a complete analysis of this form.

Part 3 is devoted to the approximation theory of eigenvalue problems in mixed form. We recall that the natural conditions for the well-posedness and stability of source mixed problems (the classical inf-sup conditions) are not good hypotheses for convergence of the eigensolutions. It is standard to consider two different mixed formulations: problems of the first type (also known as $(f, 0)$ problems) and of the second type $(0, g)$. The first family is used, for instance, when the Stokes system is considered, and an example of an application for the second one is the mixed Laplace eigenvalue problem. The sufficient and necessary conditions for the convergence of eigenvalues and eigenfunctions of either type of mixed problem are discussed.

Finally, Part 4 deals with the homological techniques which lead to the finite element exterior calculus. We recall the Hodge–Laplace eigenvalue problem and show the links between this problem in the language of differential forms and standard eigenvalue problems for differential operators. In particular, we study the Maxwell eigenvalue problem and discuss the main tools for its analysis.

In a project like this one, it is responsibility of the author to make some choices about the material to be included. We acknowledge that we would have added some more subjects, but finally we had to trim our original plan. In particular, we completely ignored the topic of *a posteriori* and adaptive error analysis for eigenvalue problems. For this active and fundamental research field the reader is referred to the following papers and to the references therein: Hackbusch (1979), Larson (2000), Morin, Nochetto and Siebert (2000), Heuveline and Rannacher (2001), Neymeyr (2002), Durán, Padra and Rodríguez (2003), Gardini (2004), Carstensen (2008), Giani and Graham (2009), Grubišić and Ovall (2009) and Garau, Morin and Zuppa (2009). The p and hp version of finite elements is pretty much related to this topic: we give some references on this issue in Section 20 for the approximation of Maxwell's eigenvalue problem. Another area that deserves attention is the discontinuous Galerkin approximation of eigenvalue problems. We refer to the following papers and to the references therein: Hesthaven and Warburton (2004), Antonietti, Buffa and Perugia (2006), Buffa and Perugia (2006), Warburton and Embree (2006), Creusé and Nicaise (2006), Buffa, Houston and Perugia (2007) and Brenner, Li and Sung (2008). Nonstandard approximations, including mimetic schemes (Cangiani, Gardini and Manzini 2010), have not been discussed. Another important result we did not include deals with the approximation of non-compact operators (Descloux, Nassif and Rappaz 1978*a*, 1978*b*). It is interesting to note that such results have often been used for the analysis of the non-conforming approximation of compact operators and, in particular, of the approximation of Maxwell's eigenvalue problem.

Throughout this paper we quote in each section the references we need. We tried to include all significant references we were aware of, but obviously many others have not been included. We apologize for that in advance and encourage all readers to inform the author about results that would have deserved more discussion.

PART ONE
Some preliminary examples

In this section we discuss some numerical results concerning the finite element approximation of eigenvalue problems arising from partial differential equations. The presented examples provide motivation for the rest of this survey and will be used for the applications of the developed theory. We only consider *symmetric* eigenvalue problems, so we are looking for *real* eigenvalues.

2. The one-dimensional Laplace eigenvalue problem

We start with a very basic and well-known one-dimensional example. Let Ω be the open interval $]0,\pi[$ and consider the problem of finding eigenvalues λ and eigenfunctions u with $u \neq 0$ such that

$$-u''(x) = \lambda u(x) \qquad \text{in } \Omega, \tag{2.1a}$$

$$u(0) = u(\pi) = 0. \tag{2.1b}$$

It is well known that the eigenvalues are given by the squares of the integer numbers $\lambda = 1, 4, 9, 16, \dots$ and that the corresponding eigenspaces are spanned by the eigenfunctions $\sin(kx)$ for $k = 1, 2, 3, 4 \dots$. A standard finite element approximation of problem (2.1) is obtained by considering a suitable variational formulation. Given $V = H_0^1(\Omega)$, multiplying our equation by $v \in V$, and integrating by parts, yields the following: find $\lambda \in \mathbb{R}$ and a non-vanishing $u \in V$ such that

$$\int_0^\pi u'(x)v'(x)\,\mathrm{d}x = \lambda \int_0^\pi u(x)v(x)\,\mathrm{d}x \quad \forall v \in V. \tag{2.2}$$

A Galerkin approximation of this variational formulation is based on a finite-dimensional subspace $V_h = \operatorname{span}\{\varphi_1, \dots, \varphi_N\} \subset V$, and consists in looking for discrete eigenvalues $\lambda_h \in \mathbb{R}$ and non-vanishing eigenfunctions $u_h \in V_h$ such that

$$\int_0^\pi u_h'(x)v'(x)\,\mathrm{d}x = \lambda_h \int_0^\pi u_h(x)v(x)\,\mathrm{d}x \quad \forall v \in V_h.$$

It is well known that this gives an algebraic problem of the form

$$A\mathsf{x} = \lambda M\mathsf{x},$$

where the stiffness matrix $A = \{a_{ij}\}_{i,j=1}^N$ is defined as

$$a_{ij} = \int_0^\pi \varphi_j'(x)\varphi_i'(x)\,\mathrm{d}x$$

and the mass matrix $M = \{m_{ij}\}_{i,j=1}^N$ is

$$m_{ij} = \int_0^\pi \varphi_j(x)\varphi_i(x)\,\mathrm{d}x.$$

Given a uniform partition of $[0,\pi]$ of size h, let V_h be the space of continuous piecewise linear polynomials vanishing at the end-points (standard conforming $\mathcal{P}_1$ finite elements); then the associated matrices read

$$a_{ij} = \frac{1}{h}\cdot\begin{cases} 2 & \text{for } i=j,\\ -1 & \text{for } |i-j|=1,\\ 0 & \text{otherwise,}\end{cases} \qquad m_{ij} = h\cdot\begin{cases} 4/6 & \text{for } i=j,\\ 1/6 & \text{for } |i-j|=1,\\ 0 & \text{otherwise,}\end{cases}$$

with $i,j = 1,\dots,N$, where the dimension N is the number of internal nodes in the interval $[0,\pi]$. It is well known that in this case it is possible to compute the explicit eigenmodes: given $k \in \mathbb{N}$, the kth eigenspace is generated by the interpolant of the continuous solution

$$u_h^{(k)}(ih) = \sin(kih), \quad i = 1,\dots,N, \tag{2.3}$$

and the corresponding eigenvalue is

$$\lambda_h^{(k)} = (6/h^2)\frac{1-\cos kh}{2+\cos kh}. \tag{2.4}$$

It is then immediate to deduce the optimal estimates (as $h \to 0$)

$$\|u^{(k)} - u_h^{(k)}\|_V = O(h) \qquad |\lambda^{(k)} - \lambda_h^{(k)}| = O(h^2) \tag{2.5}$$

with $u^{(k)}(x) = \sin(kx)$ and $\lambda^{(k)} = k^2$.

We would like to make some comments about this first example. Although here the picture is very simple and widely known, some of the observations generalize to more complicated situations and will follow from the abstract theory, which is the main object of this survey.

First of all, it is worth noticing that, even if not explicitly stated, estimates (2.5) depend on k. In particular, the estimate on the eigenvalues can be made more precise by observing that

$$\lambda_h^{(k)} = k^2 + (k^4/12)h^2 + O(k^6h^4), \quad \text{as } h \to 0. \tag{2.6}$$

This property has a clear physical meaning: since the eigenfunctions present more and more oscillations when the frequency increases, an increasingly fine mesh is required to keep the approximation error within the same accuracy.

The second important consequence of (2.4) is that all eigenvalues are approximated from above. This behaviour, which is related to the so-called min-max property (see Proposition 7.2), can be stated as follows:

$$\lambda^{(k)} \le \lambda_h^{(k)} \le \lambda^{(k)} + C(k)h^2.$$

The first estimate in (2.5) on the convergence of the eigenfunctions requires some additional comments. It is clear that the solution of the algebraic system arising from (2.2) does not give, in general, the eigenfunctions described in (2.3). Since in this simple example all eigenspaces are one-dimensional, we might expect that the numerical solver will provide us with multiples of the functions in (2.3). It is evident that if we want to perform an automatic error estimation, a first step will be to normalize the computed eigenfunctions so that they have the same norm as the continuous ones. This, however, is not enough, since there can be a difference in sign, so we have to multiply them by ± 1 in order for the scalar product with the continuous eigenfunctions to be positive.

Remark 2.1. If the same eigenvalue computation is performed with V_h equal to the space of continuous piecewise polynomials of degree at most p and vanishing at the end-points (standard conforming $\mathcal{P}_p$ finite elements), then estimates (2.5) become

$$\|u^{(k)} - u_h^{(k)}\|_V = O(h^p) \qquad |\lambda^{(k)} - \lambda_h^{(k)}| = O(h^{2p}).$$

In any case, the order of approximation for the eigenvalues is double with respect to the approximation rate of the corresponding eigenfunctions. This is the typical behaviour of *symmetric* eigenvalue problems.

3. Some numerical results for the two-dimensional Laplace eigenvalue problem

In this section we present some numerical results for the Laplace eigenvalue problem in two dimensions. We use different domains and finite elements.

Given an open Lipschitz domain $\Omega \subset \mathbb{R}^2$, we are interested in the following problem: find eigenvalues λ and eigenfunctions u with $u \neq 0$ such that

$$-\Delta u(x,y) = \lambda u(x,y) \quad \text{in } \Omega, \tag{3.1a}$$
$$u = 0 \quad \text{on } \partial\Omega. \tag{3.1b}$$

Given $V = H_0^1(\Omega)$, a variational formulation of (3.1) can be obtained as follows: find $\lambda \in \mathbb{R}$ and $u \in V$, with $u \neq 0$, such that

$$\int_\Omega \mathbf{grad}\, u(x,y) \cdot \mathbf{grad}\, v(x,y)\, \mathrm{d}x\, \mathrm{d}y = \lambda \int_\Omega u(x,y) v(x,y)\, \mathrm{d}x\, \mathrm{d}y \quad \forall v \in V.$$

A Galerkin approximation based on a finite-dimensional subspace $V_h \subset V$ then reads: find $\lambda_h \in \mathbb{R}$ and $u_h \in V_h$, with $u_h \neq 0$, such that

$$\int_\Omega \mathbf{grad}\, u_h(x,y) \cdot \mathbf{grad}\, v(x,y)\, \mathrm{d}x\, \mathrm{d}y = \lambda_h \int_\Omega u_h(x,y) v(x,y)\, \mathrm{d}x\, \mathrm{d}y \quad \forall v \in V_h.$$

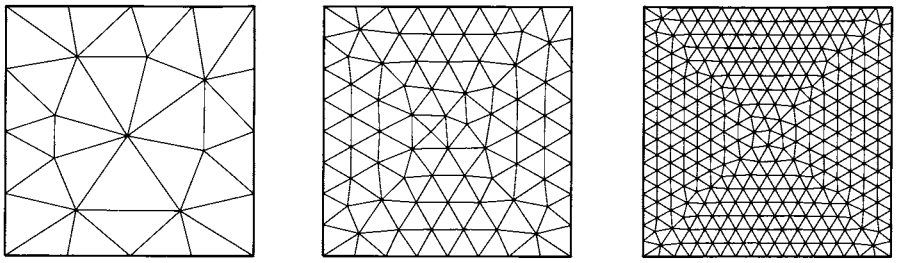

Figure 3.1. Sequence of unstructured meshes ($N = 4, 8, 16$).

3.1. *The Laplace eigenvalue problem on the square: continuous piecewise linears*

Let Ω be the square $]0,\pi[\times]0,\pi[$. It is well known that the eigenvalues of (3.1) are given by $\lambda_{m,n} = m^2 + n^2$ (with m and n strictly positive integers) and the corresponding eigenfunctions are $u_{m,n} = \sin(mx)\sin(ny)$. Throughout this subsection we are going to use continuous piecewise linear finite elements on triangles.

Our first computation involves a standard sequence of regular unstructured meshes, which is shown in Figure 3.1. Table 3.1 lists the first ten computed eigenvalues and their rate of convergence towards the exact values. It is evident that the scheme is convergent and that the convergence is quadratic. The abstract theory we are going to present will show that the eigenfunctions are first-order convergent in V.

Moreover, from Table 3.1 we can see behaviour similar to that observed in the one-dimensional example: all eigenvalues are approximated from above and the relative error increases with the rank of the eigenvalues in the spectrum (for instance, on the finest mesh, the relative error for the 10th eigenvalue is more than eight times the error for the first one).

This two-dimensional example allows us to make some important comments on multiple eigenvalues. If we look, for instance, at the double eigenvalue $\lambda = 5$, we see that there are two *distinct* discrete eigenvalues $\lambda_h^{(2)} < \lambda_h^{(3)}$ approximating it. Both eigenvalues are good approximations of the exact solution, and on the finest mesh their difference is smaller than 10^{-4}. A natural question concerns the behaviour of the corresponding eigenfunctions. The answer to this question is not trivial: indeed, the exact eigenspace has dimension equal to 2 and it is spanned by the functions $u_{1,2} = \sin x \sin(2y)$ and $u_{2,1} = \sin(2x)\sin y$. On the other hand, since the discrete eigenvalues are distinct, the approximating eigenspace consists of two separate one-dimensional eigenspaces. In particular, we cannot expect an estimate similar to the first one of (2.5) (even after normalization and choice of the sign for each discrete eigenfunction), since there is no rea-

Table 3.1. Eigenvalues computed on the unstructured mesh sequence.

Exact	Computed (rate)				
	$N = 4$	$N = 8$	$N = 16$	$N = 32$	$N = 64$
2	2.2468	2.0463 (2.4)	2.0106 (2.1)	2.0025 (2.1)	2.0006 (2.0)
5	6.5866	5.2732 (2.5)	5.0638 (2.1)	5.0154 (2.0)	5.0038 (2.0)
5	6.6230	5.2859 (2.5)	5.0643 (2.2)	5.0156 (2.0)	5.0038 (2.0)
8	10.2738	8.7064 (1.7)	8.1686 (2.1)	8.0402 (2.1)	8.0099 (2.0)
10	12.7165	11.0903 (1.3)	10.2550 (2.1)	10.0610 (2.1)	10.0152 (2.0)
10	14.3630	11.1308 (1.9)	10.2595 (2.1)	10.0622 (2.1)	10.0153 (2.0)
13	19.7789	14.8941 (1.8)	13.4370 (2.1)	13.1046 (2.1)	13.0258 (2.0)
13	24.2262	14.9689 (2.5)	13.4435 (2.2)	13.1053 (2.1)	13.0258 (2.0)
17	34.0569	20.1284 (2.4)	17.7468 (2.1)	17.1771 (2.1)	17.0440 (2.0)
17		20.2113	17.7528 (2.1)	17.1798 (2.1)	17.0443 (2.0)
DOF	9	56	257	1106	4573

son why, for instance, the eigenspace associated to $\lambda_h^{(2)}$ should provide a good approximation of $u_{1,2}$. The right approach to this problem makes use of the direct sum of the eigenspaces corresponding to $\lambda_h^{(2)}$ and $\lambda_h^{(3)}$, that is, $\operatorname{span}\{u_h^{(2)}, u_h^{(3)}\}$, which does in fact provide a good approximation to the two-dimensional eigenspace associated with $\lambda = 5$. The definition of such an approximation, which relies on the notion of a *gap* between Hilbert spaces, will be made more precise later on. For the moment, we make explicit the concept of convergence in this particular situation which can be stated as follows: there exist constants $\alpha_{1,2}(h)$, $\alpha_{2,1}(h)$, $\beta_{1,2}(h)$ and $\beta_{2,1}(h)$ such that

$$\begin{aligned} \|u_{1,2} - \alpha_{1,2}(h)u_h^{(2)} - \beta_{1,2}(h)u_h^{(3)}\|_V &= O(h), \\ \|u_{2,1} - \alpha_{2,1}(h)u_h^{(2)} - \beta_{2,1}(h)u_h^{(3)}\|_V &= O(h). \end{aligned} \tag{3.2}$$

It should be clear that the way $u_{1,2}$ and $u_{2,1}$ influence the behaviour of $u_h^{(2)}$ and $u_h^{(3)}$ is mesh-dependent: on the unstructured mesh sequences used for our computations, we cannot expect the α's and the β's to stabilize towards fixed numbers. In order to demonstrate this phenomenon, we display in Figure 3.2 the computed eigenfunctions associated with $\lambda_h^{(2)}$ for $N = 8$, 16, and 32. The corresponding plot for the computed eigenfunctions associated with $\lambda_h^{(3)}$ is shown in Figure 3.3. For the sake of comparison, the exact eigenfunctions $u_{1,2}$ and $u_{2,1}$ are plotted in Figure 3.4.

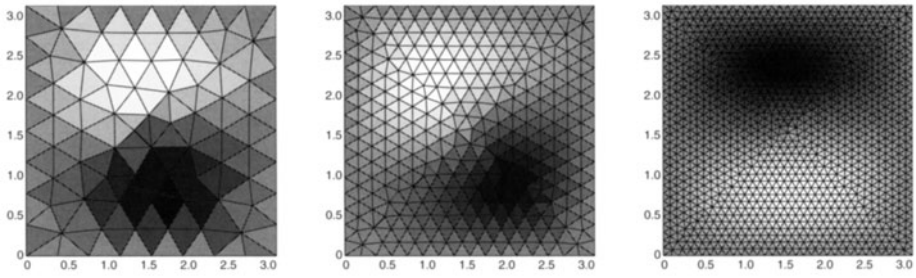

Figure 3.2. Eigenfunction associated with $\lambda_h^{(2)}$ on the unstructured mesh sequence.

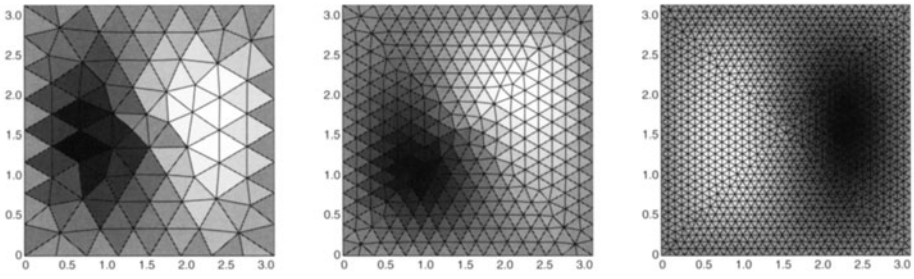

Figure 3.3. Eigenfunction associated with $\lambda_h^{(3)}$ on the unstructured mesh sequence.

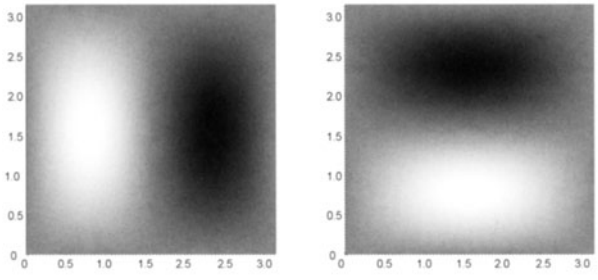

Figure 3.4. Eigenfunctions $u_{1,2}$ and $u_{2,1}$.