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FINITE ELEMENT APPROXIMATIONS FOR STOKES–DARCY FLOW WITH BEAVERS–JOSEPH INTERFACE CONDITIONS*

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Abstract. Numerical solutions using finite element methods are considered for transient flow in a porous medium coupled to free flow in embedded conduits. Such situations arise, for example, for groundwater flows in karst aquifers. The coupled flow is modeled by the Darcy equation in a porous medium and the Stokes equations in the conduit domain. On the interface between the matrix and conduit, Beavers–Joseph interface conditions, instead of the simplified Beavers–Joseph–Saffman conditions, are imposed. Convergence and error estimates for finite element approximations are obtained. Numerical experiments illustrate the validity of the theoretical results.

Key words. Stokes and Darcy equations, finite element approximation, error bound, initial-boundary value problem, fluid and porous media flow, Beavers–Joseph interface boundary condition

AMS subject classifications. 35M20, 35Q35, 65M12, 65M15, 65M60, 76D07, 76S05, 86A05

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1. Introduction.

The purpose of this paper is to investigate finite element approximations of the time-dependent Stokes–Darcy system with the Beavers–Joseph interface condition. Our motivation is the modeling and numerical simulation of groundwater flows in karst aquifers.

Karst aquifers represent a very significant source of water for public and private use. For instance, aquifers supply 90% of the water used for domestic and public purposes in the state of Florida. However, in comparison with large amounts of studies of groundwater in porous and fractured media, studies about karst aquifers are still very limited and inaccurate. One of the difficulties in the modeling of karst aquifers is that, in addition to a porous limestone matrix, a typical karst aquifer also has large cavernous conduits that are known to largely control groundwater flows within the aquifer. We refer to such a system as a conduit-matrix system. In this study, we develop a new modeling approach for water flows in conduit-matrix systems.

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Specifically, we conceptualize karst aquifers as consisting of two separated yet abutting domains: a conduit domain and a matrix domain. In the conduit domain, the flow is described by the Stokes equations, whereas in the matrix domain, it is described by the Darcy equation.

Although the use of the combination of the Darcy equation with the Stokes equations to simulate water flows in conduit-matrix systems is relatively new, there have been a few studies of the numerical solutions of the coupled Stokes–Darcy equations; see [3, 4, 10, 13, 14]. In [3], the Stokes–Darcy system is considered but the interaction in the normal direction is ignored; numerical simulation results using a domain decomposition method are provided. In [10], a formulation based on the Beavers–Joseph–Saffman–Jones interface conditions is considered. There, the existence and uniqueness of the weak solution as well as error analyses for mixed finite element approximations (see also [14]) are proved. All the works cited, however, consider only the steady state case and utilize the simplified interface conditions such as the Beavers–Joseph–Saffman–Jones condition.

In this paper, we study the time-dependent Stokes–Darcy coupled system. Instead of using the simplified Beavers–Joseph–Saffman–Jones [9, 15] interface condition, we use the Beavers–Joseph interface condition that was first observed through experiments [1]. Although the Beavers–Joseph–Saffman–Jones conditions have been mathematically proved to be valid under certain restrictive conditions [8, 9, 15], it is not clear if they are applicable to complicated matrix-conduit systems with curved interfaces together with inhomogeneous and anisotropic media.

One of the challenges of using the Beavers–Joseph interface conditions is that the bilinear form in the weak formulation is not coercive, which makes it difficult to carry out analysis for our work [2]. The remedy is a proper rescaling of the Darcy equation [2]. It turns out that the bilinear form for the new system satisfies a Gårding-type inequality for a sufficiently large scaling factor; this enables us to complete convergence and error analyses.

The paper is organized as follows. In section 2, we formulate the problem, including a specification of the interface conditions. In section 3, we study the convergence and error estimates for spatially semidiscrete finite element approximations for the time-dependent Stokes–Darcy problem. Then, in section 4, we consider the fully discrete finite element approximation based on the backward-Euler scheme. Finally, in section 5, we present a variety of computational results that illustrate our theoretical analyses.

2. Stokes–Darcy system with Beavers–Joseph interface condition.

2.1. Formulation of the problem. We assume that a conduit-matrix system consists of two domains, the conduit domain $\Omega_c \subset \mathbb{R}^d$ and the matrix domain $\Omega_m \subset \mathbb{R}^d$, $d = 2, 3$. See the sketch in Figure 2.1 for $d = 2$. Let $\Gamma_{cm} = \partial\Omega_c \cap \partial\Omega_m$, $\Gamma_c = \partial\Omega_c \setminus \Gamma_{cm}$, and $\Gamma_m = \partial\Omega_m \setminus \Gamma_{cm}$.

In the matrix, Ω_m, the flow is governed by the Darcy system

$$\left.\begin{array}{rcl} S\partial_t \phi_m + \nabla \cdot \mathbf{v}_m &=& f_2 \\ \mathbf{v}_m &=& -\mathbb{K}\nabla\phi_m \\ \phi_m(0) &=& \phi_0 \end{array}\right\} \quad \text{in } \Omega_m. \tag{2.1}$$

Here, $\partial_t := \frac{\partial}{\partial t}$, $\mathbf{v}_m$ denotes the specific discharge, ϕ_m the hydraulic (piezometric) head, S the mass storativity coefficient, $\mathbb{K}$ the hydraulic conductivity tensor of the porous media which is assumed to be symmetric and positive definite, and f_2 a sink/source

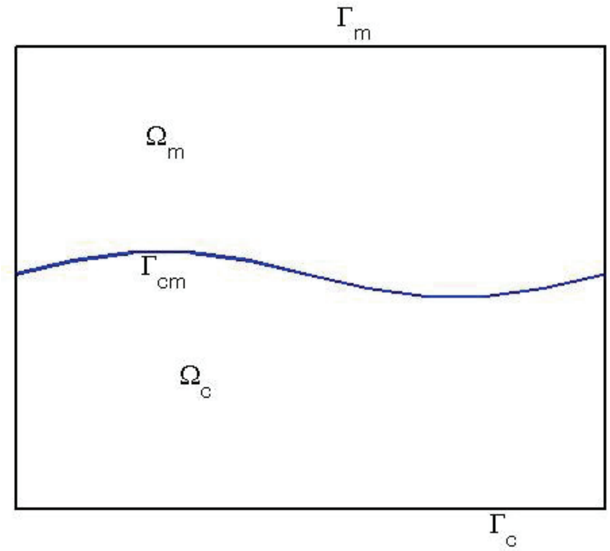

FIG. 2.1. *The conceptual domain of a conduit-matrix system.*

term. The hydraulic head ϕ_m is linearly related to the dynamic pressure of the fluid, p_m, via $\phi_m := z + \frac{p_m}{\rho g}$, where ρ denotes the density, g the gravitational acceleration, and z the relative depth from an arbitrary fixed reference height. By substituting the second equation in (2.1) into the first one, we obtain the parabolic equation that governs the hydraulic head:

$$S\partial_t\phi_m + \nabla \cdot (-\mathbb{K}\nabla\phi_m) = f_2 \quad \text{in } \Omega_m.$$

We impose the homogeneous Dirichlet condition along the boundary of the matrix:

$$\phi_m| = 0 \quad \text{on } \Gamma_m.$$

In the conduit domain of the problem, Ω_c, the flow is governed by the Stokes equations:

$$\left.\begin{array}{rcl} \partial_t\mathbf{v} &=& \nabla \cdot \big(-p\mathbb{I} + 2\nu\mathbb{D}(\mathbf{v}_c)\big) + \mathbf{f}_1 \\ \nabla \cdot \mathbf{v}_c &=& 0 \\ \mathbf{v}_c(0) &=& \mathbf{v}_0 \end{array}\right\} \quad \text{in } \Omega_c,$$

where $\mathbf{v}_c$ denotes the fluid velocity, $\mathbb{D}(\mathbf{v}_c) := \frac{1}{2}(\nabla\mathbf{v}_c + (\nabla\mathbf{v}_c)^T)$ the deformation tensor, ν the kinematic viscosity of the fluid, p the kinematic pressure, and $\mathbf{f}_1$ a general body forcing term that includes gravitational acceleration. For the sake of simplicity, the homogeneous Dirichlet condition is imposed on the boundary of the conduit:

$$\mathbf{v}_c = \mathbf{0} \quad \text{on } \Gamma_c.$$

We use the subscripts m and c to indicate where the variables belong. We omit these subscripts in what follows whenever there is no possibility for confusion.

On the interface Γ_{cm}, the Beavers–Joseph conditions are imposed:

$$\left.\begin{array}{rcl} \mathbf{v}_c \cdot \mathbf{n}_{cm} & = & \mathbf{v}_m \cdot \mathbf{n}_{cm} \\ -\mathbf{n}_{cm}^T \mathbb{T}(\mathbf{v}_c, p)\mathbf{n}_{cm} & = & g(\phi_m - z) \\ -P_\tau(\mathbb{T}(\mathbf{v}_c, p)\mathbf{n}_{cm}) & = & \dfrac{\alpha\nu\sqrt{3}}{\sqrt{\mathrm{trace}(\mathbf{\Pi})}} P_\tau(\mathbf{v}_c - \mathbf{v}_m) \end{array}\right\} \text{ on } \Gamma_{cm}, \tag{2.2}$$

where $\mathbf{n}_{cm}$ denotes the unit normal vector on Γ_{cm} pointing from Ω_c to Ω_m, $P_\tau(\cdot)$ the projection onto the local tangent plane on Γ_{cm}, g the gravitational acceleration, α a constant parameter, $\mathbf{\Pi}$ the intrinsic permeability that satisfies the relation $\mathbb{K} = \frac{\mathbf{\Pi} g}{\nu}$, and $\mathbb{T}(\mathbf{v}_c, p)$ the stress tensor defined as

$$\mathbb{T}(\mathbf{v}, p) = -p\mathbb{I} + 2\nu\mathbb{D}(\mathbf{v}).$$

The last equation in (2.2) is the Beavers–Joseph condition [1]. If the term $\mathbf{v}_c - \mathbf{v}_m$ is replaced by $\mathbf{v}_c$, then the Beavers–Joseph condition reduces to the Beavers–Joseph–Saffman–Jones condition.

2.2. Weak formulation of the time-dependent Stokes–Darcy model. For $s > \frac{1}{2}$, define the Hilbert spaces

$$\mathbf{H}_{c,0}^s := \{\mathbf{w} \in \left(H^s(\Omega_c)\right)^d \mid \mathbf{w} = \mathbf{0} \text{ on } \Gamma_c\},$$

$$H_{m,0}^s := \{\varphi \in H^s(\Omega_m) \mid \varphi = 0 \text{ on } \Gamma_m\},$$

$$Q := L^2(\Omega_c)$$

and the product Hilbert spaces

$$\underline{\mathbf{L}}^2 := \left(L^2(\Omega_c)\right)^d \times L^2(\Omega_m),$$

$$\underline{\mathbf{H}}^s := \mathbf{H}_{c,0}^s \times H_{m,0}^s.$$

A norm on Q is given by

$$\|q\|_0 := \|q\|_{L^2(\Omega_c)}$$

for $q \in Q$ and a norm in $\underline{\mathbf{H}}^s$ is given by

$$\|\underline{\mathbf{w}}\|_s := \left(\|\mathbf{w}\|_{(H^s(\Omega_c))^d}^2 + \|\varphi\|_{H^s(\Omega_m)}^2\right)^{1/2}$$

for $\underline{\mathbf{w}} = (\mathbf{w}, \varphi) \in \underline{\mathbf{H}}^s$. In what follows, we use $\underline{\mathbf{W}}$ to denote $\underline{\mathbf{H}}^1$ and $\underline{\mathbf{V}}$ the divergence free subspace of $\underline{\mathbf{W}}$, i.e.,

$$\underline{\mathbf{V}} := \mathbf{H}_{c,\mathrm{div}}^1 \times H_{m,0}^1,$$

where $\mathbf{H}_{c,\mathrm{div}}^1 = \left\{\mathbf{w} \in \mathbf{H}_{c,0}^1 \mid \mathrm{div}\mathbf{w} = 0\right\}$. For convenience in carrying out the convergence and error analyses of sections 3 and 4, we introduce an equivalent norm on $\underline{\mathbf{L}}^2$: for $\eta > 0$ and $\underline{\mathbf{v}} = (\mathbf{v}, \psi) \in \underline{\mathbf{L}}^2$,

$$\|\underline{\mathbf{v}}\|_{0,\eta} := \left(\|\mathbf{v}\|_{(L^2(\Omega_c))^d}^2 + \|\eta^{\frac{1}{2}}\psi\|_{L^2(\Omega_m)}^2\right)^{1/2},$$

whereas $\|\cdot\|_{\underline{\mathbf{L}}^2}$ denotes the standard norm. We also need the trace space $\mathbf{H}^{1/2}_{00}(\Gamma_{cm})$ defined as $\mathbf{H}^{1/2}_{00}(\Gamma_{cm}) := \mathbf{H}^1_{c,0}|_{\Gamma_{cm}}$.

We define the bilinear forms $a : \underline{\mathbf{W}} \times \underline{\mathbf{W}} \to \mathbb{R}$ and $b : \underline{\mathbf{W}} \times Q \to \mathbb{R}$ as follows. For $\underline{\mathbf{u}} = (\mathbf{u}, \phi)$ and $\underline{\mathbf{v}} = (\mathbf{v}, \psi)$ in $\underline{\mathbf{W}}$ and q in Q,

$$\begin{aligned} a(\underline{\mathbf{u}}, \underline{\mathbf{v}}) \;:=\; & 2\nu \int_{\Omega c} \mathbb{D}\mathbf{u} : \mathbb{D}\mathbf{v} \mathrm{d}\Omega_c + \frac{1}{S}\int_{\Omega_m} (\mathbb{K}\nabla\phi)\cdot\nabla\psi \mathrm{d}\Omega_m \\ & + g\int_{\Gamma_{cm}} \phi\mathbf{v}\cdot\mathbf{n}_{cm}\mathrm{d}\Gamma_{cm} - \frac{1}{S}\int_{\Gamma_{cm}} \mathbf{u}\cdot\mathbf{n}_{cm}\psi\mathrm{d}\Gamma_{cm} \\ & + \int_{\Gamma_{cm}} \frac{\nu\alpha\sqrt{d}}{\sqrt{\mathrm{trace}(\mathbf{\Pi})}} P_\tau(\mathbf{u} + \mathbb{K}\nabla\phi)\cdot\mathbf{v}\mathrm{d}\Gamma_{cm} \end{aligned}$$

and

$$b(\underline{\mathbf{u}}, q) := -\int_{\Omega_c} q\nabla\cdot\mathbf{u}\mathrm{d}\Omega_c.$$

Here, the integral of $P_\tau(\mathbb{K}\nabla\phi)\cdot\mathbf{v}$ on Γ_{cm} is understood to be the value of the functional $P_\tau(\mathbb{K}\nabla\phi)|_{\Gamma_{cm}} \in \big(H^{\frac{1}{2}}_{00}(\Gamma_{cm})\big)'$ applied to $\mathbf{v}|_{\Gamma_{cm}} \in H^{\frac{1}{2}}_{00}(\Gamma_{cm})$, which is well defined when $\mathbb{K}$ is isotropic (see [2, 11] for more details). Thus, from now on, we assume that $\mathbb{K}$ is isotropic. Finally, we define the linear functional $\underline{\mathbf{F}} : \underline{\mathbf{W}} \to \mathbb{R}$ by

$$\langle\underline{\mathbf{F}}, \underline{\mathbf{w}}\rangle := \langle\mathbf{f}_1, \mathbf{w}\rangle_c + \langle f_2, \varphi\rangle_m + g\int_{\Gamma_{cm}} z\mathbf{w}\cdot\mathbf{n}_{cm}\mathrm{d}\Gamma,$$

where $\mathbf{f}_1$ and f_2 are functionals on $\mathbf{H}^1_{c,0}$, respectively, and $H^1_{m,0}$, and $\langle\cdot,\cdot\rangle_c$ and $\langle\cdot,\cdot\rangle_m$ are the dualities induced by the L^2 inner product on Ω_c and Ω_m, respectively. The last integral results from the second equation in (2.2). The effect of the integral is to add the hydrostatic pressure profile to the Stokes equations. It does not affect the well-posedness or regularity of the problem. For convenience of discussion, it is omitted hereafter, although it is taken into account in the numerical experiments.

We can now define a weak formulation for the Stokes–Darcy problem: seek $\underline{\mathbf{u}} = (\mathbf{u}, \phi) \in \underline{\mathbf{W}}$ and $p \in Q$ such that

$$\text{(2.3)} \qquad \begin{cases} \langle\partial_t\underline{\mathbf{u}}, \underline{\mathbf{v}}\rangle + a(\underline{\mathbf{u}}, \underline{\mathbf{v}}) + b(\underline{\mathbf{v}}, p) = \langle\underline{\mathbf{F}}, \underline{\mathbf{v}}\rangle & \forall\, \underline{\mathbf{v}} \in \underline{\mathbf{W}}, \\ b(\underline{\mathbf{u}}, q) = 0 & \forall\, q \in Q, \\ \underline{\mathbf{u}}(0) = \underline{\mathbf{u}}_0 & \end{cases}$$

for almost all t, $0 < t \le T$, where $\underline{\mathbf{u}}_0 = (\mathbf{u}_0, \phi_0)$. We say that $(\underline{\mathbf{u}}, p) = (\underline{\mathbf{u}}(t), p(t))$ is a weak solution of the Stokes–Darcy problem if $(\underline{\mathbf{u}}, p) \in L^2(0, T; \underline{\mathbf{W}}) \times L^2(0, T; Q)$ with $\partial_t\underline{\mathbf{u}}(t) \in L^2(0, T; \underline{\mathbf{W}}')$ and satisfies (2.3). Here, $\underline{\mathbf{W}}'$ is the dual space of $\underline{\mathbf{W}}$.

The difficulty with the weak formulation (2.3) is that the bilinear form a is not coercive, which hinders the convergence and error analyses for finite element approximations. To overcome this difficulty, we multiply (2.1) by a scaling factor η. Obviously, the scaling factor does not change the Darcy equation itself. However, we need to modify the interface conditions accordingly in order to preserve the solution of the Stokes–Darcy problem. To this end, we modify the variational formulation as follows:

$$\text{(2.4)} \qquad \begin{cases} \langle\partial_t\underline{\mathbf{u}}, \underline{\mathbf{v}}\rangle_\eta + a_\eta(\underline{\mathbf{u}}, \underline{\mathbf{v}}) + b(\underline{\mathbf{v}}, p) = \langle\underline{\mathbf{F}}, \underline{\mathbf{v}}\rangle_\eta & \forall\, \underline{\mathbf{v}} \in \underline{\mathbf{W}}, \\ b(\underline{\mathbf{u}}, q) = 0 & \forall\, q \in Q, \\ \underline{\mathbf{u}}(0) = \underline{\mathbf{u}}_0, & \end{cases}$$

where

$$\langle \partial_t \underline{\mathbf{u}}, \underline{\mathbf{v}} \rangle_\eta := \left\langle \begin{pmatrix} \partial_t \mathbf{u} \\ \eta \partial_t \phi \end{pmatrix}, \underline{\mathbf{v}} \right\rangle$$

and

$$\begin{aligned} a_\eta(\underline{\mathbf{u}}, \underline{\mathbf{v}}) \quad := \quad & 2\nu \int_{\Omega_c} \mathbb{D}(\mathbf{u}) : \mathbb{D}(\mathbf{v}) \mathrm{d}\Omega_c + \frac{\eta}{S} \int_{\Omega_m} (\mathbb{K}\nabla\phi) \cdot \nabla\psi \mathrm{d}\Omega_m \\ & + g \int_{\Gamma_{cm}} \phi \mathbf{v} \cdot \mathbf{n}_{cm} \mathrm{d}\Gamma_{cm} - \frac{\eta}{S} \int_{\Gamma_{cm}} \mathbf{u} \cdot \mathbf{n}_{cm} \psi \mathrm{d}\Gamma_{cm} \\ & + \nu\alpha\sqrt{d} \int_{\Gamma_{cm}} \frac{1}{\sqrt{\mathrm{trace}(\mathbf{\Pi})}} P_\tau(\mathbf{u} + \mathbb{K}\nabla\phi) \cdot \mathbf{v} \mathrm{d}\Gamma_{cm}. \end{aligned}$$

A slightly simpler approach is to take the Leray–Hopf projection, and we work on the divergence free subspace only, i.e., seek $\underline{\mathbf{u}} = (\mathbf{u}, \phi) \in \underline{\mathbf{V}}$ and $p \in Q$ such that

$$\langle \partial_t \underline{\mathbf{u}}, \underline{\mathbf{v}} \rangle_\eta + a_\eta(\underline{\mathbf{u}}, \underline{\mathbf{v}}) = \langle \underline{\mathbf{F}}, \underline{\mathbf{v}} \rangle_\eta \quad \forall \underline{\mathbf{v}} \in \underline{\mathbf{V}} \tag{2.5}$$

for almost all t, $0 < t \leq T$.

From [2], we have the following result concerning the existence of a Gårding-type inequality for a_η and the existence and uniqueness of weak solutions for the Stokes–Darcy problem.

Proposition 2.1. *The bilinear form a_η satisfies the following Gårding-type inequality: for sufficiently large $\eta > 0$, there exist constants $C_{1,\eta} > 0$ and $C_0 > 0$ such that*

$$a_\eta(\underline{\mathbf{u}}, \underline{\mathbf{u}}) \geq C_{1,\eta} \|\underline{\mathbf{u}}\|_1^2 - C_0 \|\underline{\mathbf{u}}\|_{0,\eta}^2 . \tag{2.6}$$

Furthermore, (2.5)*, the Stokes–Darcy problem with the Beavers–Joseph interface condition, has a unique weak solution and* (2.4) *(or* (2.3)*) is equivalent to* (2.5)*.*

It is easy to verify that a_η is continuous on $\underline{\mathbf{W}}$; i.e., there exists a constant $C_{2,\eta}$ such that

$$a_\eta(\underline{\mathbf{u}}, \underline{\mathbf{v}}) \leq C_{2,\eta} \|\underline{\mathbf{u}}\|_1 \|\underline{\mathbf{v}}\|_1 \tag{2.7}$$

$\forall \ \underline{\mathbf{u}}, \underline{\mathbf{v}} \in \underline{\mathbf{W}}$.

In what follows, we assume that η is a sufficiently large fixed parameter so that (2.6) and (2.7) hold. Without confusion, we may drop the subscript η for these constants in the inequalities. We use C to denote a generic constant whose value may vary with context.

3. Spatial semidiscretization via finite element methods. For $i = 1, 2$, we partition Ω_j into the mesh $\{\mathcal{T}_j^h\}$ $(j = m, c)$ with $\overline{\Omega}_j = \cup_{K \in \{\mathcal{T}_j^h\}} \overline{K}$. We assume that the cells $K \in \{\mathcal{T}_j^h\}$ are affine equivalent and the grids of $\{\mathcal{T}_c^h\}$ and $\{\mathcal{T}_m^h\}$ match along Γ_{cm}.

Next, we introduce the finite element spaces $\underline{\mathbf{W}}^h$ and Q^h which are div-stable:[1] there exists a constant $\beta > 0$, independent of h, such that

$$\underline{\mathbf{W}}^h = \mathbf{H}_c^h \times H_m^h \subset \underline{\mathbf{W}}, \ Q^h \subset Q, \tag{3.1}$$

$$\inf_{0 \neq q^h \in Q^h} \sup_{0 \neq \mathbf{v}^h \in \underline{\mathbf{W}}^h} \frac{b(\mathbf{v}^h, q^h)}{\|\mathbf{v}^h\|_1 \|q^h\|_0} > \beta. \tag{3.2}$$

[1]This is also referred to as the inf-sup or LBB condition. Interested readers may refer to, e.g., [5, 6] for such element pairs.

We also assume Korn's inequality (see [2])

$$(\mathbb{D}(\mathbf{v}^h), \mathbb{D}(\mathbf{v}^h)) \geq C\|\mathbf{v}^h\|_1^2 \quad \forall\ \mathbf{v}^h \in \underline{\mathbf{W}}^h.$$

Furthermore, we assume that $\underline{\mathbf{W}}^h$ and Q^h include continuous piecewise polynomials of degree at least k and piecewise polynomials of degree at least $k-1$, respectively ($k \geq 1$), and satisfy the following approximation properties:

$$\inf_{\mathbf{v}^h \in \underline{\mathbf{W}}^h} \|\underline{\mathbf{v}} - \underline{\mathbf{v}}^h\|_1 \leq Ch^s \|\underline{\mathbf{v}}\|_{s+1} \quad \forall \underline{\mathbf{v}} \in \underline{\mathbf{H}}^{s+1},\ 0 < s \leq k, \tag{3.3}$$

$$\inf_{q \in Q^h} \|q - q^h\|_0 \leq Ch^s \|q\|_s \quad \forall q \in H^s(\Omega_c),\ 0 < s \leq k. \tag{3.4}$$

We also assume that for the chosen finite element spaces $\underline{\mathbf{W}}^h$ and Q^h there exists a projection operator $\Pi^h : \mathbf{H}^1_{c,0} \to \mathbf{H}^h_c$ such that

$$\Pi^h \mathbf{w} \in \mathbf{H}^h_c,\ (\nabla \cdot (\mathbf{w} - \Pi^h \mathbf{w}), q^h) = 0 \quad \forall q^h \in Q^h,\ \forall \mathbf{w} \in \mathbf{H}^1_{c,0}(\Omega_c), \tag{3.5}$$

$$\|\mathbf{w} - \Pi^h \mathbf{w}\|_1 \leq Ch^s \|\mathbf{w}\|_{s+1} \quad \forall \mathbf{w} \in \mathbf{H}^{s+1}_{c,0}(\Omega_c), \tag{3.6}$$

where $0 \leq s \leq k$ and C is a positive constant independent of h and $\mathbf{w}$. Details about the definitions of the finite element spaces $\underline{\mathbf{W}}^h$ and Q^h and of the existence of the projection operator Π^h can be found in [5, Chapter II, pages 136 and 146] for $k = 1$ and in [12] for $k = 1, 2, 3$.

Now we introduce the discretely divergence free space

$$\underline{\mathbf{V}}^h := \left\{ \underline{\mathbf{v}}^h \in \underline{\mathbf{W}}^h |\ b(\underline{\mathbf{v}}^h, q^h) = 0\ \forall q^h \in Q^h \right\}.$$

We define $P^h : \underline{\mathbf{L}}^2 \to \underline{\mathbf{V}}^h$ to be the projection operator with respect to the $\underline{\mathbf{L}}^2$ inner product, i.e.,

$$P^h \underline{\mathbf{w}} \in \underline{\mathbf{V}}^h,\ (P^h \underline{\mathbf{w}}, \underline{\mathbf{v}}^h) = (\underline{\mathbf{w}}, \underline{\mathbf{v}}^h) \quad \forall \underline{\mathbf{v}}^h \in \underline{\mathbf{V}}^h,\ \forall \underline{\mathbf{w}} \in \underline{\mathbf{L}}^2.$$

The following proposition is about the approximation properties of P^h [7].

PROPOSITION 3.1. *The operator P^h satisfies the following approximation properties:*

(1) $\|\underline{\mathbf{w}} - P^h \underline{\mathbf{w}}\|_1 \to 0$ *as* $h \to 0$ $\forall \underline{\mathbf{w}} \in \underline{\mathbf{V}}$;
(2) $\|\underline{\mathbf{w}} - P^h \underline{\mathbf{w}}\|_1 \leq Ch^r \|\underline{\mathbf{w}}\|_{r+1}$ $\forall \underline{\mathbf{w}} \in \underline{\mathbf{H}}^{r+1} \cap \underline{\mathbf{V}}$;
(3) $\|\underline{\mathbf{w}} - P^h \underline{\mathbf{w}}\|_0 \leq Ch^{r+1} \|\underline{\mathbf{w}}\|_{r+1}$ $\forall \underline{\mathbf{w}} \in \underline{\mathbf{H}}^{r+1} \cap \underline{\mathbf{V}}$;
(4) $\|\underline{\mathbf{w}} - P^h \underline{\mathbf{w}}\|_{L^2(0,T;\underline{\mathbf{W}})} \to 0$ *as* $h \to 0$ $\forall \underline{\mathbf{w}} \in L^2(0, T; \underline{\mathbf{V}})$;
(5) $\|\underline{\mathbf{w}} - P^h \underline{\mathbf{w}}\|_{L^2(0,T;\underline{\mathbf{W}})} \leq Ch^r \|\underline{\mathbf{w}}\|_{L^2(0,T;\underline{\mathbf{H}}^{r+1})}$ $\forall \underline{\mathbf{w}} \in L^2(0, T; \underline{\mathbf{H}}^{r+1} \cap \underline{\mathbf{V}})$.

Here $r \in [0, k]$.

We also need the following Gronwall's inequality in the error estimate.

LEMMA 3.2. *Assume that $K : [0, T] \to [0, \infty)$ is a continuous function. If the continuous function u satisfies*

$$u(t) \leq K(t) + \alpha \int_0^t u(s) ds \quad \forall t \in [0, T],$$

where $\alpha \geq 0$, *then*

$$u(t) \leq K(t) + \alpha \int_0^t K(s) e^{\alpha(t-s)} ds. \tag{3.7}$$

The finite element method for problem (2.4) is defined as finding $(\underline{\mathbf{u}}^h, p^h) \in H^1(0,T;\underline{\mathbf{W}}^h) \times L^2(0,T;Q^h)$ such that

$$\begin{cases} \langle \partial_t \underline{\mathbf{u}}^h(t), \underline{\mathbf{v}}^h \rangle_\eta + \, a_\eta(\underline{\mathbf{u}}^h(t), \underline{\mathbf{v}}^h) + b(\underline{\mathbf{v}}^h, p^h(t)) = \langle \underline{\mathbf{F}}(t), \underline{\mathbf{v}}^h \rangle_\eta, \\ b(\underline{\mathbf{u}}^h(t), q^h) = 0, \\ \underline{\mathbf{u}}^h(0) = P^h \underline{\mathbf{u}}_0 \end{cases} \tag{3.8}$$

$\forall \underline{\mathbf{v}}^h \in \underline{\mathbf{W}}^h$ and $\forall q^h \in Q^h$. Using the Gårding inequality (2.6), we can easily prove that if $\underline{\mathbf{F}} \in L^2(0,T;\underline{\mathbf{L}}^2)$, then (3.8) has a unique solution $(\underline{\mathbf{u}}^h, p^h) \in H^1(0,T;\underline{\mathbf{V}}^h) \times L^2(0,T;Q^h)$ (see the appendix for details).

THEOREM 3.3. *Assume that* $\underline{\mathbf{u}}_0 \in \underline{\mathbf{V}}$, $\underline{\mathbf{F}} \in L^2(0,T;\underline{\mathbf{L}}^2)$ *and* $(\underline{\mathbf{u}}, p) \in [L^2(0,T;\underline{\mathbf{V}}) \cap H^1(0,T;\underline{\mathbf{V}}^{'})] \times [L^2(0,T;L^2(\Omega_c))]$ *is the solution of* (2.4). *Then*

$$\|\underline{\mathbf{u}} - \underline{\mathbf{u}}^h\|_{L^2(0,T;\underline{\mathbf{H}}^1)} \to 0.$$

Furthermore, if $(\underline{\mathbf{u}}, p) \in L^2(0,T;\underline{\mathbf{H}}^{r+1}) \times L^2(0,T;H^r(\Omega_c))$, $0 < r \leq k$, *then*

$$\|\underline{\mathbf{u}} - \underline{\mathbf{u}}^h\|_{L^2(0,T;\underline{\mathbf{H}}^1)} \leq Ch^r(\|\underline{\mathbf{u}}\|_{L^2(0,T;\underline{\mathbf{H}}^{r+1})} + \|p\|_{L^2(0,T;H^r(\Omega_c))}).$$

Proof. Subtracting (3.8) from (2.4) we have that

$$\begin{cases} \langle \partial_t \underline{\mathbf{u}}(t) - \partial_t \underline{\mathbf{u}}^h(t), \underline{\mathbf{v}}^h \rangle_\eta + \, a_\eta(\underline{\mathbf{u}}(t) - \underline{\mathbf{u}}^h(t), \underline{\mathbf{v}}^h) \\ \qquad + \, b(\underline{\mathbf{v}}^h, p(t) - p^h(t)) = 0 \quad \forall \, \underline{\mathbf{v}}^h \in \underline{\mathbf{W}}^h, \\ \qquad b(\underline{\mathbf{u}}(t) - \underline{\mathbf{u}}^h(t), q^h) = 0 \quad \forall \, q^h \in Q^h. \end{cases}$$

From the above equation we deduce that for almost every $t \in [0,T]$

$$\begin{aligned} &\langle \partial_t \underline{\mathbf{u}}(t) - \partial_t \underline{\mathbf{u}}^h(t), \underline{\mathbf{u}}(t) - \underline{\mathbf{u}}^h(t) \rangle_\eta + \, a_\eta(\underline{\mathbf{u}}(t) - \underline{\mathbf{u}}^h(t), \underline{\mathbf{u}}(t) - \underline{\mathbf{u}}^h(t)) \\ &\qquad = \langle \partial_t \underline{\mathbf{u}}(t) - \partial_t \underline{\mathbf{u}}^h(t), \underline{\mathbf{u}}(t) - P^h \underline{\mathbf{u}}(t) \rangle_\eta + \, a_\eta(\underline{\mathbf{u}}(t) - \underline{\mathbf{u}}^h(t), \underline{\mathbf{u}}(t) - P^h \underline{\mathbf{u}}(t)) \\ &\qquad\quad - b(P^h \underline{\mathbf{u}}(t) - \underline{\mathbf{u}}^h(t), p(t) - p^h(t)). \end{aligned}$$

Noting that $P^h \underline{\mathbf{u}} - \underline{\mathbf{u}}^h \in \underline{\mathbf{V}}^h$ implies $\partial_t P^h \underline{\mathbf{u}} - \partial_t \underline{\mathbf{u}}^h \in \underline{\mathbf{V}}^h$, we also have

$$\langle \partial_t \underline{\mathbf{u}}(t) - \partial_t \underline{\mathbf{u}}^h(t), \underline{\mathbf{u}}(t) - P^h \underline{\mathbf{u}}(t) \rangle_\eta = \langle \partial_t \underline{\mathbf{u}}(t) - \partial_t P^h \underline{\mathbf{u}}(t), \underline{\mathbf{u}}(t) - P^h \underline{\mathbf{u}}(t) \rangle_\eta$$

and

$$b(P^h \underline{\mathbf{u}}(t), q^h) = 0 \quad \forall q^h \in Q^h.$$

Using the above equations and the Gårding inequality (2.6) we deduce that $\forall q^h \in L^2(0,T;Q^h)$ and for almost every t

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \|\underline{\mathbf{u}}(t) - \underline{\mathbf{u}}^h(t)\|_{0,\eta}^2 + C_1 \|\underline{\mathbf{u}}(t) - \underline{\mathbf{u}}^h(t)\|_1^2 \\ &\qquad \leq \frac{1}{2} \frac{d}{dt} \|\underline{\mathbf{u}}(t) - P^h \underline{\mathbf{u}}(t)\|_{0,\eta}^2 + C_0 \|\underline{\mathbf{u}}(t) - \underline{\mathbf{u}}^h(t)\|_{0,\eta}^2 \\ &\qquad\quad + a_\eta(\underline{\mathbf{u}}(t) - \underline{\mathbf{u}}^h(t), \underline{\mathbf{u}}(t) - P^h \underline{\mathbf{u}}(t)) \\ &\qquad\quad + b(\underline{\mathbf{u}}(t) - P^h \underline{\mathbf{u}}(t), p(t) - q^h(t)) - b(\underline{\mathbf{u}}(t) - \underline{\mathbf{u}}^h(t), p(t) - q^h(t)). \end{aligned}$$

The continuity of a_η and b and Young's inequality give

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\underline{\mathbf{u}}(t)-\underline{\mathbf{u}}^h(t)\|_{0,\eta}^2 + C_1\|\underline{\mathbf{u}}(t)-\underline{\mathbf{u}}^h(t)\|_1^2 \\
&\qquad\le \frac{1}{2}\frac{d}{dt}\|\underline{\mathbf{u}}(t)-P^h\underline{\mathbf{u}}(t)\|_{0,\eta}^2 + C_0\|\underline{\mathbf{u}}(t)-\underline{\mathbf{u}}^h(t)\|_{0,\eta}^2 \\
&\qquad\quad + \frac{1}{4}C_1\|\underline{\mathbf{u}}(t)-\underline{\mathbf{u}}^h(t)\|_1^2 + \frac{C_2^2}{C_1}\|\underline{\mathbf{u}}(t)-P^h\underline{\mathbf{u}}(t)\|_1^2 \\
&\qquad\quad + \frac{1}{2}\|\underline{\mathbf{u}}(t)-P^h\underline{\mathbf{u}}(t)\|_1^2 + \frac{1}{2}\|p(t)-q^h(t)\|_0^2 \\
&\qquad\quad + \frac{1}{4}C_1\|\underline{\mathbf{u}}(t)-\underline{\mathbf{u}}^h(t)\|_1^2 + \frac{1}{C_1}\|p(t)-q^h(t)\|_0^2,
\end{aligned}$$

where C_0, C_1, and C_2 are the constants that appeared in (2.6) and (2.7) with the subscript η dropped out.

From this estimate we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\underline{\mathbf{u}}(t)-\underline{\mathbf{u}}^h(t)\|_{0,\eta}^2 + \frac{1}{2}C_1\|\underline{\mathbf{u}}(t)-\underline{\mathbf{u}}^h(t)\|_1^2 \\
&\qquad\le \frac{1}{2}\frac{d}{dt}\|\underline{\mathbf{u}}(t)-P^h\underline{\mathbf{u}}(t)\|_{0,\eta}^2 + C_0\|\underline{\mathbf{u}}(t)-\underline{\mathbf{u}}^h(t)\|_{0,\eta}^2 \\
&\qquad\quad + C\|\underline{\mathbf{u}}(t)-P^h\underline{\mathbf{u}}(t)\|_1^2 + C\|p(t)-q^h(t)\|_0^2.
\end{aligned}$$

Integrating the above inequality from 0 to t, we have that

$$\begin{aligned}
&\|\underline{\mathbf{u}}(t)-\underline{\mathbf{u}}^h(t)\|_{0,\eta}^2 + C_1\int_0^t \|\underline{\mathbf{u}}(s)-\underline{\mathbf{u}}^h(s)\|_1^2 \mathrm{d}s \\
&\quad\le \|\underline{\mathbf{u}}(t)-P^h\underline{\mathbf{u}}(t)\|_{0,\eta}^2 + 2C_0\int_0^t \|\underline{\mathbf{u}}(s)-\underline{\mathbf{u}}^h(s)\|_{0,\eta}^2 \mathrm{d}s \\
&\qquad + 2C\int_0^t \|\underline{\mathbf{u}}(s)-P^h\underline{\mathbf{u}}(s)\|_1^2 \mathrm{d}s + 2C\int_0^t \|p(s)-q^h(s)\|_0^2 \mathrm{d}s.
\end{aligned}$$

By Gronwall's inequality (3.7), we have that

$$\begin{aligned}
\|\underline{\mathbf{u}}(t)-\underline{\mathbf{u}}^h(t)\|_{0,\eta}^2 &\le C\Bigg(\|\underline{\mathbf{u}}(t)-P^h\underline{\mathbf{u}}(t)\|_{0,\eta}^2 \\
&\quad + \int_0^t (\|\underline{\mathbf{u}}(s)-P^h\underline{\mathbf{u}}(s)\|_1^2 + \|p(s)-q^h(s)\|_0^2)\mathrm{d}s + e^{2C_0t}\int_0^t \big(\|\underline{\mathbf{u}}(s)-P^h\underline{\mathbf{u}}(s)\|_{0,\eta}^2 \\
&\quad + \int_0^s (\|\underline{\mathbf{u}}(\tau)-P^h\underline{\mathbf{u}}(\tau)\|_1^2 + \|p(\tau)-q^h(\tau)\|_0^2)\mathrm{d}\tau\big)\mathrm{d}s\Bigg) \\
&\le C(T)\big(\|\underline{\mathbf{u}}(t)-P^h\underline{\mathbf{u}}(t)\|_{0,\eta}^2 \\
&\quad + \|\underline{\mathbf{u}}-P^h\underline{\mathbf{u}}\|_{L^2(0,T;\underline{\mathbf{H}}^1)}^2 + \|p-q^h\|_{L^2(0,T;L^2(\Omega_c))}^2\big) \quad \forall\, q^h \in Q^h.
\end{aligned}$$

The result of the theorem is then the consequence of approximation properties of $P^h\underline{\mathbf{u}}$ and q^h. The proof is complete. ◻

Using an argument similar to the one in [7], we can obtain the following error estimates for the time derivative $\partial_t\underline{\mathbf{u}}$ and the pressure p.

THEOREM 3.4. *Assume that*

$$(\underline{\mathbf{u}}, p) \in [L^2(0,T;\underline{\mathbf{H}}^{r+1}) \cap H^1(0,T;\underline{\mathbf{H}}^{r-1})] \times L^2(0,T;H^r(\Omega_c))$$

for some $r \in [1,k]$ *is the solution of problem* (2.4)*, and let* $(\underline{\mathbf{u}}^h, p^h) \in H^1(0,T;\underline{\mathbf{W}}^h) \times L^2(0,T;Q^h)$ *be the solution of* (3.8)*. Then*

$$\|\partial_t \underline{\mathbf{u}} - \partial_t \underline{\mathbf{u}}^h\|_{L^2(0,T;\underline{\mathbf{L}}^2)} + \|p - p^h\|_{L^2(0,T;L^2(\Omega_c))} \leq Ch^{r-1}(\|\partial_t \underline{\mathbf{u}}\|_{H^1(0,T;\underline{\mathbf{H}}^{r-1})} + \|\underline{\mathbf{u}}\|_{L^2(0,T;\underline{\mathbf{H}}^{r+1})} + \|p\|_{L^2(0,T;H^r(\Omega_c))}).$$

4. Fully discrete finite element/backward-Euler discretization. Divide the time interval $[0,T]$ into N subintervals $[t_n, t_{n+1}]$ $(n = 0, 1, \ldots, N-1)$, satisfying

$$0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T.$$

Let $\Delta t_n = t_n - t_{n-1}$ be the time step with the biggest time step $\Delta t = \max_{1 \leq n \leq N} \Delta t_n$. Define the time backward difference operator d_{t_n} by $d_{t_n} \underline{\mathbf{u}}_h^n = \frac{\underline{\mathbf{u}}_h^n - \underline{\mathbf{u}}_h^{n-1}}{\Delta t_n}$ or $d_{t_n} \underline{\mathbf{u}} = \frac{\underline{\mathbf{u}}(t_n) - \underline{\mathbf{u}}(t_{n-1})}{\Delta t_n}$.

Based on the weak form (2.4), we propose the space-time fully discretized implicit Euler scheme as follows: given $(\underline{\mathbf{u}}_h^0, p_h^0) \in \underline{\mathbf{W}}^h \times Q^h$, find $(\underline{\mathbf{u}}_h^n, p_h^n) \in \underline{\mathbf{W}}^h \times Q^h$ such that

$$\left\{ \begin{array}{rcll} \langle d_{t_n} \underline{\mathbf{u}}_h^n, \underline{\mathbf{v}}^h \rangle_\eta + a_\eta(\underline{\mathbf{u}}_h^n, \underline{\mathbf{v}}^h) + b(\underline{\mathbf{v}}^h, p_h^n) & = & \langle \underline{\mathbf{F}}^n, \underline{\mathbf{v}}^h \rangle_\eta & \forall\ \underline{\mathbf{v}}^h \in \underline{\mathbf{W}}^h, \\ b(\underline{\mathbf{u}}_h^n, q^h) & = & 0 & \forall\ q^h \in Q^h \end{array} \right. \tag{4.1}$$

for $n = 1, 2, \ldots, N$, where $\underline{\mathbf{F}}^n := \underline{\mathbf{F}}(t_n)$.

A special version of this scheme restricted to the divergence free subspace $\underline{\mathbf{V}}$ and with uniform time step was discussed in our work [2].

To obtain the error estimates for the fully discretized implicit Euler scheme (4.1), we introduce the projection operator $\mathbb{P} = (\mathbb{P}_s, \mathbb{P}_p) : \underline{\mathbf{W}} \times Q \to \underline{\mathbf{W}}^h \times Q^h$ such that for $(\underline{\mathbf{u}}, p) \in \underline{\mathbf{W}} \times Q$ the projection $(\mathbb{P}_s \underline{\mathbf{u}}, \mathbb{P}_p p)$ satisfies

$$\left\{ \begin{array}{rcl} a_\eta(\underline{\mathbf{u}} - \mathbb{P}_s \underline{\mathbf{u}}, \underline{\mathbf{v}}^h) + C_0 \langle \underline{\mathbf{u}} - \mathbb{P}_s \underline{\mathbf{u}}, \underline{\mathbf{v}}^h \rangle_\eta + b(\underline{\mathbf{v}}^h, p - \mathbb{P}_p p) & = & 0, \\ b(\underline{\mathbf{u}} - \mathbb{P}_s \underline{\mathbf{u}}, q^h) & = & 0 \end{array} \right. \tag{4.2}$$

$\forall\ \underline{\mathbf{v}}^h \in \underline{\mathbf{W}}^h$ and $\forall\ q^h \in Q^h$. The Gårding-type inequality (2.6) in Proposition 2.1 implies that there is a unique solution $(\mathbb{P}_s \underline{\mathbf{u}}, \mathbb{P}_p p) \in \underline{\mathbf{W}}^h \times Q^h$ for a given $(\underline{\mathbf{u}}, p) \in \underline{\mathbf{W}} \times Q$.

Thanks to the Gårding-type inequality (2.6) in Proposition 2.1, we have the following approximate properties for $(\mathbb{P}_s \underline{\mathbf{u}}, \mathbb{P}_p p)$.

PROPOSITION 4.1.

Let $0 < r \leq k$*.*

1. *Assume that*

$$(\underline{\mathbf{u}}, p) = (\mathbf{u}, \phi, p) \in L^q(0,T;\underline{\mathbf{H}}^{r+1}) \times L^q(0,T;H^r(\Omega_c))$$

for some $q \in [1, \infty]$*. Let* $(\mathbb{P}_s \underline{\mathbf{u}}, \mathbb{P}_p p)$ *be the projection solution of* (4.2)*. Then we have*

$$\|\underline{\mathbf{u}} - \mathbb{P}_s \underline{\mathbf{u}}\|_{L^q(0,T;\underline{\mathbf{H}}^1)} + \|p - \mathbb{P}_p p\|_{L^q(0,T;L^2(\Omega_c))} \leq Ch^r(\|\underline{\mathbf{u}}\|_{L^q(0,T;\underline{\mathbf{H}}^{r+1})} + \|p\|_{L^q(0,T;H^r(\Omega_c))}). \tag{4.3}$$

2. *Assume that*

$$(\underline{\mathbf{u}}, p) \in H^l(0,T;\underline{\mathbf{H}}^{r+1}) \times H^l(0,T;H^r(\Omega_c)).$$

Then we have

$$\|\partial_t^l \underline{\mathbf{u}} - \partial_t^l \mathbb{P}_s \underline{\mathbf{u}}\|_{L^2(0,T;\underline{\mathbf{H}}^1)} + \|\partial_t^l p - \partial_t^l \mathbb{P}_p p\|_{L^2(0,T;L^2(\Omega_c))} \leq Ch^r(\|\underline{\mathbf{u}}\|_{H^l(0,T;\underline{\mathbf{H}}^{r+1})} + \|p\|_{H^l(0,T;H^r(\Omega_c))}). \tag{4.4}$$

Proof. Let $e_h = \underline{\mathbf{u}} - \mathbb{P}_s\underline{\mathbf{u}}$. Combining (4.2) and the equality

$$\begin{aligned} &a_\eta(e_h, e_h) + C_0\langle e_h, e_h\rangle_\eta \\ &\quad = a_\eta(e_h, \underline{\mathbf{u}} - \underline{\mathbf{v}}^h) + C_0\langle e_h, \underline{\mathbf{u}} - \underline{\mathbf{v}}^h\rangle_\eta \\ &\qquad + a_\eta(e_h, \underline{\mathbf{v}}^h - \mathbb{P}_s\underline{\mathbf{u}}) + C_0\langle e_h, \underline{\mathbf{v}}^h - \mathbb{P}_s\underline{\mathbf{u}}\rangle_\eta \end{aligned}$$

for $\underline{\mathbf{v}}^h = (\mathbf{v}^h, \psi^h) \in \underline{\mathbf{W}}^h$ and $p^h \in Q^h$, we obtain

$$\begin{aligned} &a_\eta(e_h, e_h) + C_0\langle e_h, e_h\rangle_\eta \\ &\quad = a_\eta(e_h, \underline{\mathbf{u}} - \underline{\mathbf{v}}^h) + C_0\langle e_h, \underline{\mathbf{u}} - \underline{\mathbf{v}}^h\rangle_\eta - b(\underline{\mathbf{v}}^h - \mathbb{P}_s\underline{\mathbf{u}}, p - \mathbb{P}_p p). \end{aligned}$$

From the definitions of Π^h in (3.5), it is easy to verify that

$$\begin{aligned} b(\underline{\mathbf{v}}^h - \mathbb{P}_s\underline{\mathbf{u}}, p - \mathbb{P}_p p) &= b(\underline{\mathbf{v}}^h - \underline{\mathbf{u}}, p - \mathbb{P}_p p) + b(\underline{\mathbf{u}} - \mathbb{P}_s\underline{\mathbf{u}}, p - \mathbb{P}_p p) \\ &= b(\underline{\mathbf{v}}^h - \Pi^h\underline{\mathbf{u}}, p - \mathbb{P}_p p) + b(\Pi^h\underline{\mathbf{u}} - \underline{\mathbf{u}}, p - p^h) \\ &\quad + b(\underline{\mathbf{u}} - \mathbb{P}_s\underline{\mathbf{u}}, p - p^h), \end{aligned} \tag{4.5}$$

where $\Pi^h\underline{\mathbf{u}} = (\Pi^h\mathbf{u}, \phi)$. Now letting $\underline{\mathbf{v}}^h = (\Pi^h\mathbf{u}, P^h\phi)$ in (4.5), using the Gårding inequality (2.6) and the continuity property of the bilinear operators a and b, we obtain the following estimate:

$$\begin{aligned} C_1\|e_h\|_1^2 &\le a_\eta(e_h, \underline{\mathbf{u}} - \underline{\mathbf{v}}^h) + C_0\langle e_h, \underline{\mathbf{u}} - \underline{\mathbf{v}}^h\rangle_\eta - b(\underline{\mathbf{v}}^h - \mathbb{P}_s\underline{\mathbf{u}}, p - \mathbb{P}_p p) \\ &\le C\|e_h\|_1\|\underline{\mathbf{u}} - \underline{\mathbf{v}}^h\|_1 + C_0\eta\|e_h\|_0\|\underline{\mathbf{u}} - \underline{\mathbf{v}}^h\|_{\underline{\mathbf{L}}^2} \\ &\quad + C\|e_h\|_1\|p - p^h\|_0 + C\|\underline{\mathbf{u}} - \Pi^h\underline{\mathbf{u}}\|_1\|p - p^h\|_0, \end{aligned}$$

which implies that

$$\|e_h\|_1 \le C\left(\|\underline{\mathbf{u}} - \underline{\mathbf{v}}^h\|_1 + \|\underline{\mathbf{u}} - \Pi^h\underline{\mathbf{u}}\|_1 + \inf_{p^h\in Q^h}\|p - p^h\|_0\right). \tag{4.6}$$

The div-stability condition implies that

$$\|p - \mathbb{P}_p p\| \le C\left(\|\underline{\mathbf{u}} - \underline{\mathbf{v}}^h\|_1 + \inf_{p^h\in Q^h}\|p - p^h\|_0\right). \tag{4.7}$$

Combining the estimates (4.6) and (4.7), the approximate properties (3.3) and (3.4) of the finite element spaces $\underline{\mathbf{W}}^h$ and Q^h, and the approximate property (3.6) of Π^h, we obtain the estimate (4.3). If the solution $(\underline{\mathbf{u}}, p)$ has the regularities needed, the estimate (4.4) holds true just by differentiating the equations in (4.2) with respect to time t and using the same methods of obtaining the estimate (4.3). □

Remark 4.2. In the proof of Proposition 4.1, we used the projection operator Π^h. If we do not use the operator Π^h, we can also prove that the estimates in Proposition 4.1 are true by using the same techniques used in the proof of Theorem 1.1 in Chapter II in [5].

To obtain the error estimates $\|\underline{\mathbf{u}} - \mathbb{P}_s\underline{\mathbf{u}}\|_{\underline{\mathbf{L}}^2}$, we assume that the solution $\underline{\mathbf{u}}$ of

$$\begin{cases} a_\eta(\underline{\mathbf{u}}, \underline{\mathbf{v}}) + C_0\langle\underline{\mathbf{u}}, \underline{\mathbf{v}}\rangle_\eta + b(\underline{\mathbf{v}}, p) = \langle\underline{\mathbf{F}}, \underline{\mathbf{v}}\rangle_\eta & \forall\ \underline{\mathbf{v}} \in \underline{\mathbf{W}}, \\ b(\underline{\mathbf{u}}, q) = 0 & \forall\ q \in Q \end{cases} \tag{4.8}$$

has the following regularity:

$$\|\underline{\mathbf{u}}\|_2 + \|p\|_{H^1(\Omega_c)} \le C\|\underline{\mathbf{F}}\|_{\underline{\mathbf{L}}^2}. \tag{4.9}$$

We also assume that the dual problem of (4.8) is regular in the sense of (4.9). Then we have the following Aubin–Nitsche-type estimate on $\|\underline{\mathbf{u}} - \mathbb{P}_s\underline{\mathbf{u}}\|_{\underline{\mathbf{L}}^2}$.

PROPOSITION 4.3. *Under the assumptions of Proposition* 4.1*, we have the estimates*

$$\begin{aligned}&\|\underline{\mathbf{u}} - \mathbb{P}_s\underline{\mathbf{u}}\|_{L^q(0,T;\underline{\mathbf{L}}^2)} + h\|p - \mathbb{P}_p p\|_{L^q(0,T;L^2(\Omega_c))}\\ &\quad \le Ch^{r+1}(\|\underline{\mathbf{u}}\|_{L^q(0,T;\underline{\mathbf{H}}^{r+1})} + \|p\|_{L^q(0,T;H^r(\Omega_c))})\end{aligned} \tag{4.10}$$

and

$$\begin{aligned}&\|\partial_t^l\underline{\mathbf{u}} - \partial_t^l\mathbb{P}_s\underline{\mathbf{u}}\|_{L^2(0,T;\underline{\mathbf{L}}^2)} + h\|\partial_t^l p - \partial_t^l\mathbb{P}_p p\|_{L^2(0,T;L^2(\Omega_c))}\\ &\quad \le Ch^{r+1}(\|\underline{\mathbf{u}}\|_{H^l(0,T;\underline{\mathbf{H}}^{r+1})} + \|p\|_{H^l(0,T;H^r(\Omega_c))}).\end{aligned} \tag{4.11}$$

Proof. Let $e_h = \underline{\mathbf{u}} - \mathbb{P}_s\underline{\mathbf{u}}$ and $(\underline{\mathbf{u}}^d, p^d)$ satisfy

$$\begin{cases} a_\eta(\underline{\mathbf{v}}, \underline{\mathbf{u}}^d) + C_0\langle\underline{\mathbf{v}}, \underline{\mathbf{u}}^d\rangle_\eta + b(\underline{\mathbf{v}}, p^d) = \langle e_h, \underline{\mathbf{v}}\rangle_\eta & \forall\ \underline{\mathbf{v}} \in \underline{\mathbf{W}},\\ b(\underline{\mathbf{u}}^d, q) = 0 & \forall\ q \in Q.\end{cases} \tag{4.12}$$

Choosing $\underline{\mathbf{v}} = e_h$ in (4.12), we have that

$$\begin{aligned}\|e_h\|_{0,\eta}^2 &= a_\eta(e_h, \underline{\mathbf{u}}^d) + C_0\langle e_h, \underline{\mathbf{u}}^d\rangle_\eta + b(e_h, p^d)\\ &= a_\eta(e_h, \underline{\mathbf{u}}^d - \underline{\mathbf{v}}^h) + C_0\langle e_h, \underline{\mathbf{u}}^d - \underline{\mathbf{v}}^h\rangle_\eta\\ &\quad + b(e_h, p^d - p^h) - b(\underline{\mathbf{u}}^d - \underline{\mathbf{v}}^h, p - \mathbb{P}_p p)\end{aligned} \tag{4.13}$$

$\forall\ \underline{\mathbf{v}}^h \in \underline{\mathbf{W}}^h$ and $p^h \in Q^h$, which implies that

$$\|e_h\|_{0,\eta}^2 \le C(\|e_h\|_1 + \|p - \mathbb{P}_p p\|_0)\left(\inf_{\underline{\mathbf{v}}^h\in\underline{\mathbf{W}}^h}\|\underline{\mathbf{u}}^d - \underline{\mathbf{v}}^h\|_1 + \inf_{p^h\in Q^h}\|p^d - p^h\|_0\right). \tag{4.14}$$

The approximation properties of the finite element spaces $\underline{\mathbf{v}}^h \in \underline{\mathbf{W}}^h$ and $p^h \in Q^h$ and the regularity inequality (4.9) give us

$$\begin{aligned}&\inf_{\underline{\mathbf{v}}^h\in\underline{\mathbf{W}}^h}\|\underline{\mathbf{u}}^d - \underline{\mathbf{v}}^h\|_1 + \inf_{p^h\in Q^h}\|p^d - p^h\|_0\\ &\le Ch(\|\underline{\mathbf{u}}^d\|_2 + \|p^d\|_{H^1(\Omega_c)}) \le Ch\|e_h\|_{\underline{\mathbf{L}}^2}.\end{aligned} \tag{4.15}$$

Now the estimates (4.3), (4.14), and (4.15) lead to the estimate (4.10). Under the conditions in the proposition, by differentiating the equations in (4.2) with respect to time t, we can similarly obtain the estimate (4.11). ∎

Now we turn to estimating the error $e_h^n = \underline{\mathbf{u}}(t_n) - \underline{\mathbf{u}}_h^n$, where $\underline{\mathbf{u}}(t_n)$ is the value of the solution $\underline{\mathbf{u}}$ of problem (2.4) at $t = t_n$, and $\underline{\mathbf{u}}_h^n$ is the solution of the fully discretized implicit Euler scheme (4.1).

THEOREM 4.4. *Assume that* $0 < r \le k$ *and the solution* $(\underline{\mathbf{u}}, p)$ *of problem* (2.4) *satisfies*

$$\begin{aligned}\partial_{tt}\underline{\mathbf{u}} &\in L^2(0,T;\underline{\mathbf{L}}^2),\\ (\underline{\mathbf{u}}, p) &\in H^1(0,T;\underline{\mathbf{H}}^{r+1}) \times H^1(0,T;H^r(\Omega_c)),\\ (\underline{\mathbf{u}}, p) &\in L^\infty(0,T;\underline{\mathbf{H}}^{r+1}) \times L^\infty(0,T;H^r(\Omega_c)).\end{aligned}$$

Also assume that the regularity property (4.9) *of* (4.8) *holds. Let* $(\underline{\mathbf{u}}_h^n, p_h^n)$*,* $n = 1, \dots, N$*, be the solution of the implicit Euler scheme* (4.1)*, and assume that the initial approximation* $\underline{\mathbf{u}}_h^0$ *of* $\underline{\mathbf{u}}(0)$ *satisfies*

$$\|\underline{\mathbf{u}}_h^0 - \mathbb{P}_s\underline{\mathbf{u}}(0)\|_{\underline{\mathbf{L}}^2} \le C^*h^{r+1}.$$

Then we have the following estimate:

$$\begin{aligned}\|\underline{\mathbf{u}}(t_n)-\underline{\mathbf{u}}_h^n\|_{0,\eta} \leq\ & C\big(h^{r+1}+\Delta t\,\|\partial_{tt}\underline{\mathbf{u}}\|_{L^2(0,T;\underline{\mathbf{L}}^2)} \\ &+h^{r+1}(\|\underline{\mathbf{u}}\|_{H^1(0,T;\underline{\mathbf{H}}^{r+1})}+\|p\|_{H^1(0,T;H^r(\Omega_c))}) \\ &+h^{r+1}(\|\underline{\mathbf{u}}\|_{L^\infty(0,T;\underline{\mathbf{H}}^{r+1})}+\|p\|_{L^\infty(0,T;H^r(\Omega_c))})\big).\end{aligned} \tag{4.16}$$

Proof. Denote by $(\underline{\mathbf{u}}^n,p^n)$ the value of $(\underline{\mathbf{u}}(t),p(t))$ at the time $t=t_n$. Let $e_h^n=\eta_h^n+\xi_h^n$ and $\zeta_h^n=\zeta_1^n+\zeta_2^n$, where $\eta_h^n=\underline{\mathbf{u}}^n-\mathbb{P}_s\underline{\mathbf{u}}^n$, $\xi_h^n=\mathbb{P}_s\underline{\mathbf{u}}^n-\underline{\mathbf{u}}_h^n=(\xi_{\mathbf{u}}^n,\xi_\phi^n)^T$, $\zeta_1^n=p^n-\mathbb{P}_pp^n$, and $\zeta_2^n=\mathbb{P}_pp^n-p_h^n$.

Let $(\underline{\mathbf{u}}(t),p(t))$ be the solution of (2.4), and let $(\mathbb{P}_s\underline{\mathbf{u}}(t),\mathbb{P}_pp(t))\in\underline{\mathbf{W}}^h\times Q^h$ be its projection solution defined by (4.2). Then $(\mathbb{P}_s\underline{\mathbf{u}}^n,\mathbb{P}_pp^n)$ satisfies

$$\begin{cases} a_\eta(\mathbb{P}_s\underline{\mathbf{u}}^n,\underline{\mathbf{v}}^h)+b(\underline{\mathbf{v}}^h,\mathbb{P}_pp^n) &= a_\eta(\underline{\mathbf{u}}^n,\underline{\mathbf{v}}^h)+b(\underline{\mathbf{v}}^h,p^n)+C_0\langle\eta_h^n,\underline{\mathbf{v}}^h\rangle_\eta \\ &= -\langle\partial_t\underline{\mathbf{u}}^n,\underline{\mathbf{v}}^h\rangle_\eta+\langle\underline{\mathbf{F}}^n,\underline{\mathbf{v}}^h\rangle_\eta \\ &\quad +C_0\langle\eta_h^n,\underline{\mathbf{v}}^h\rangle_\eta, \\ b(\mathbb{P}_s\underline{\mathbf{u}}^n,q^h) &= 0\end{cases} \tag{4.17}$$

$\forall\ \underline{\mathbf{v}}^h\in\underline{\mathbf{W}}^h$ and $q^h\in Q^h$. Subtracting (4.1) from (4.17), we deduce

$$\begin{cases} & \langle d_{t_n}\xi_h^n,\underline{\mathbf{v}}^h\rangle_\eta+a_\eta(\xi_h^n,\underline{\mathbf{v}}^h)+b(\underline{\mathbf{v}}^h,\zeta_2^n) \\ = & \langle d_{t_n}\mathbb{P}_s\underline{\mathbf{u}}-\partial_t\underline{\mathbf{u}}(t_n),\underline{\mathbf{v}}^h\rangle_\eta+C_0\langle\eta_h^n,\underline{\mathbf{v}}^h\rangle_\eta \\ = & \langle d_{t_n}\mathbb{P}_s\underline{\mathbf{u}}-d_{t_n}\underline{\mathbf{u}},\underline{\mathbf{v}}^h\rangle_\eta+\langle d_{t_n}\underline{\mathbf{u}}-\partial_t\underline{\mathbf{u}}(t_n),\underline{\mathbf{v}}^h\rangle_\eta \\ & +C_0\langle\eta_h^n,\underline{\mathbf{v}}^h\rangle_\eta, \\ b(\xi_h^n,q^h) = & 0.\end{cases} \tag{4.18}$$

Letting $\underline{\mathbf{v}}^h=\xi_h^n$ in (4.18), we have that

$$\begin{aligned} & \langle d_{t_n}\xi_h^n,\xi_h^n\rangle_\eta+a_\eta(\xi_h^n,\xi_h^n) \\ = & \langle d_{t_n}\mathbb{P}_s\underline{\mathbf{u}}-d_{t_n}\underline{\mathbf{u}},\xi_h^n\rangle_\eta+\langle d_{t_n}\underline{\mathbf{u}}-\partial_t\underline{\mathbf{u}}(t_n),\xi_h^n\rangle_\eta \\ & +C_0\langle\eta_h^n,\xi_h^n\rangle_\eta.\end{aligned} \tag{4.19}$$

Using the fact that $(a-b)a=\frac{a^2-b^2}{2}+\frac{(a-b)^2}{2}$, we rewrite (4.19) as

$$\begin{aligned} & \|\xi_h^n\|_{0,\eta}^2+(\Delta t_n)^2\left\|\frac{\xi_h^n-\xi_h^{n-1}}{\Delta t_n}\right\|_{0,\eta}^2+2\Delta t_na_\eta(\xi_h^n,\xi_h^n) \\ = & \|\xi_h^{n-1}\|_{0,\eta}^2+2\Delta t_n\langle d_{t_n}\mathbb{P}_s\underline{\mathbf{u}}-d_{t_n}\underline{\mathbf{u}},\xi_h^n\rangle_\eta \\ & +2\Delta t_n\langle d_{t_n}\underline{\mathbf{u}}-\partial_t\underline{\mathbf{u}}(t_n),\xi_h^n\rangle_\eta+2\Delta t_nC_0\langle\eta_h^n,\xi_h^n\rangle_\eta.\end{aligned} \tag{4.20}$$

Summing up (4.20), we get

$$\begin{aligned} & \|\xi_h^n\|_{0,\eta}^2+\sum_{i=1}^n(\Delta t_i)^2\left\|\frac{\xi_h^i-\xi_h^{i-1}}{\Delta t_i}\right\|_{0,\eta}^2+2\sum_{i=1}^n\Delta t_ia_\eta(\xi_h^i,\xi_h^i) \\ = & ||\xi_h^0||_{0,\eta}^2+2\sum_{i=1}^n\Delta t_i\langle d_{t_i}\mathbb{P}_s\underline{\mathbf{u}}-d_{t_i}\underline{\mathbf{u}},\xi_h^i\rangle_\eta \\ & +2\sum_{i=1}^n\Delta t_i\langle d_{t_i}\underline{\mathbf{u}}-\partial_t\underline{\mathbf{u}}(t_i),\xi_h^i\rangle_\eta+2\sum_{i=1}^nC_0\Delta t_i\langle\eta_h^i,\xi_h^i\rangle_\eta \\ = & R^0+R_1^n+R_2^n+R_3^n.\end{aligned} \tag{4.21}$$

For the second term on the right-hand side of (4.21), we have

$$
\begin{aligned}
|R_1^n| &= 2\left|\sum_{i=1}^n \Delta t_i \langle d_{t_i}\mathbb{P}_s\underline{\mathbf{u}} - d_{t_i}\underline{\mathbf{u}}, \xi_h^i\rangle_\eta\right| \\
&= 2\left|\sum_{i=1}^n \left\langle \int_{t_{i-1}}^{t_i} \partial_t \eta_h(t)\mathrm{d}t, \xi_h^i \right\rangle_\eta\right| \\
&\le 6\eta\|\eta_h\|^2_{H^1(0,t_n;\underline{\mathbf{L}}^2)} + \frac{1}{6}\sum_{i=1}^n \|\xi_h^i\|^2_{0,\eta}\Delta t_i \\
&\le Ch^{2(r+1)}(\|\underline{\mathbf{u}}\|^2_{H^1(0,t_n;\underline{\mathbf{H}}^{r+1})} \\
&\quad + \|p\|^2_{H^1(0,t_n;H^r(\Omega_c))}) + \frac{1}{6}\sum_{i=1}^n \|\xi_h^i\|^2_{0,\eta}\Delta t_i.
\end{aligned} \tag{4.22}
$$

From the fact that

$$
\partial_t f(t_i) - \frac{f(t_i) - f(t_{i-1})}{\Delta t_i} = \frac{1}{\Delta t_i}\int_{t_{i-1}}^{t_i} (t - t_{i-1})\partial_{tt} f(t)\mathrm{d}t, \tag{4.23}
$$

we have

$$
\begin{aligned}
|R_2^n| &= 2\left|\sum_{i=1}^n \Delta t_i \langle d_{t_i}\underline{\mathbf{u}} - \partial_t\underline{\mathbf{u}}(t_i), \xi_h^i\rangle_\eta\right| \\
&\le 6\eta(\Delta t_i)^2\|\partial_{tt}\underline{\mathbf{u}}\|^2_{L^2(0,t_n;\underline{\mathbf{L}}^2)} + \frac{1}{6}\sum_{i=1}^n \|\xi_h^i\|^2_{0,\eta}\Delta t_i.
\end{aligned} \tag{4.24}
$$

Using Proposition 4.3, we estimate R_3^n as

$$
\begin{aligned}
|R_3^n| &= \left|2\sum_{i=1}^n C_0\Delta t_i\langle \eta_h^i, \xi_h^i\rangle_\eta\right| \\
&\le 6\eta C_0^2 \sum_{i=1}^n \|\eta_h^i\|_0^2\Delta t_i + \frac{1}{6}\sum_{i=1}^n \|\xi_h^i\|^2_{0,\eta}\Delta t_i \\
&\le Ch^{2(r+1)}(\|\underline{\mathbf{u}}\|^2_{L^\infty(0,t_n;\underline{\mathbf{H}}^{r+1})} + \|p\|^2_{L^\infty(0,t_n;H^r(\Omega_c))}) \\
&\quad + \frac{1}{6}\sum_{i=1}^n \|\xi_h^i\|^2_{0,\eta}\Delta t_i.
\end{aligned} \tag{4.25}
$$

Insert the estimates (4.22), (4.24), and (4.25) into (4.21) and add $\sum_{i=1}^n 2\Delta t_i C_0 < \xi_h^i, \xi_h^i >_\eta$ to both sides to obtain the following estimate:

$$
\begin{aligned}
&\|\xi_h^n\|^2_{0,\eta} + \sum_{i=1}^n (\Delta t_i)^2 \left\|\frac{\xi_h^i - \xi_h^{i-1}}{\Delta t_i}\right\|^2_{0,\eta} + 2\sum_{i=1}^n \Delta t_i a_\eta(\xi_h^i, \xi_h^i) + 2\sum_{i=1}^n \Delta t_i C_0\langle \xi_h^i, \xi_h^i\rangle_\eta \\
&= R^0 + R_1^n + R_2^n + R_3^n + 2\sum_{i=1}^n \Delta t_i C_0 \langle \xi_h^i, \xi_h^i\rangle_\eta \\
&\le \|\xi_h^0\|^2_{0,\eta} + 6\eta(\Delta t_i)^2\|\partial_{tt}\underline{\mathbf{u}}\|^2_{L^2(0,t_n;\underline{\mathbf{L}}^2)} + \frac{1}{2}\sum_{i=1}^n \|\xi_h^i\|^2_{0,\eta}\Delta t_i + 2\sum_{i=1}^n \Delta t_i C_0\|\xi_h^i\|^2_{0,\eta} \\
&\quad + Ch^{2(r+1)}(\|\underline{\mathbf{u}}\|^2_{L^\infty(0,t_n;\underline{\mathbf{H}}^{r+1})} + \|p\|^2_{L^\infty(0,t_n;H^r(\Omega_c))}) \\
&\quad + Ch^{2(r+1)}(\|\underline{\mathbf{u}}\|^2_{H^1(0,t_n;\underline{\mathbf{H}}^{r+1})} + \|p\|^2_{H^1(0,t_n;H^r(\Omega_c))}).
\end{aligned} \tag{4.26}
$$

Using the Gronwall inequality (3.7), we obtain the following estimate:

$$\|\xi_h^n\|_{0,\eta}^2 + \sum_{i=1}^{n}(\Delta t_i)^2 \left\| \frac{\xi_h^i - \xi_h^{i-1}}{\Delta t_i} \right\|_{0,\eta}^2 + 2\sum_{i=1}^{n} \Delta t_i a_\eta(\xi_h^i, \xi_h^i) + 2\sum_{i=1}^{n} \Delta t_i C_0 \langle \xi_h^i, \xi_h^i \rangle_\eta$$
$$\le C\Big(\|\xi_h^0\|_{0,\eta}^2 + (\Delta t_i)^2 \|\partial_{tt}\underline{\mathbf{u}}\|_{L^2(0,T;\underline{\mathbf{L}}^2)}^2$$
$$+ h^{2(r+1)}(\|\underline{\mathbf{u}}\|_{L^\infty(0,T;\underline{\mathbf{H}}^{r+1})}^2 + \|p\|_{L^\infty(0,T;H^r(\Omega_c))}^2)$$
$$+ h^{2(r+1)}(\|\underline{\mathbf{u}}\|_{H^1(0,T;\underline{\mathbf{H}}^{r+1})}^2 + \|p\|_{H^1(0,T;H^r(\Omega_c))}^2)\Big). \tag{4.27}$$

Using the conditions in the theorem, the estimates in Proposition 4.3, and the estimate (4.27), we get the estimate (4.16). The proof is completed. □

5. Numerical experiments.

5.1. Direct solver. We use a direct solver for our numerical simulation; i.e., by adding up Darcy's equation and the transient Stokes equations, we obtain the following linear equation and solve it together with the incompressibility condition as a whole:

$$\int_{\Omega_c} \partial_t \mathbf{u}\mathbf{v} \mathrm{d}\Omega_c + \eta \int_{\Omega_m} \phi_t \psi \mathrm{d}\Omega_m + 2\nu \int_{\Omega c} \mathbb{D}\mathbf{u} : \mathbb{D}\mathbf{v} \mathrm{d}\Omega_c + \frac{\eta}{S} \int_{\Omega_m} (\mathbb{K}\nabla\phi)\cdot\nabla\psi \mathrm{d}\Omega_m$$
$$+ g\int_{\Gamma_{cm}} \phi \mathbf{v}\cdot\mathbf{n}_{cm} \mathrm{d}\Gamma_{cm} - \int_{\Omega_c} p\nabla\cdot\mathbf{v} \mathrm{d}\Omega_c - \frac{\eta}{S}\int_{\Gamma_{cm}} \mathbf{u}\cdot\mathbf{n}_{cm}\psi \mathrm{d}\Gamma_{cm}$$
$$+ \frac{\nu\alpha\sqrt{2}}{\sqrt{\mathrm{trace}(\mathbf{\Pi})}} \int_{\Gamma_{cm}} P_\tau(\mathbf{u} + \mathbb{K}\nabla\phi)\cdot\mathbf{v} \mathrm{d}\Gamma_{cm}$$
$$= \langle \mathbf{f}_1, \mathbf{w}\rangle_c + \langle f_2, \phi\rangle_m + g\int_{\Gamma_{cm}} z\mathbf{w}\cdot\mathbf{n}_{cm} \mathrm{d}\Gamma.$$

We set $\mathbf{\Pi}$ to be a constant matrix in the simulation; hence it could be taken out of the integral. The scalar parameter $\sqrt{d}$ in the Beavers–Joseph interface boundary condition has been changed to $\sqrt{2}$ here because we work with two-dimensional domains in our numerical simulation. The scaling parameter η is set to 1 for convenience. In fact, once we know that the fully discretized coupling problem has a unique solution after being scaled by η, the system without scaling should also have the same unique solution. From the viewpoint of linear algebra, the η simply plays a role of preconditioning in the following sense:

$$\begin{pmatrix} \eta\mathbf{A}_{Darcy} & \eta\mathbf{B}_{interface} \\ \mathbf{C}_{interface} & \mathbf{D}_{Stokes} \end{pmatrix} = \begin{pmatrix} \eta\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{A}_{Darcy} & \mathbf{B}_{interface} \\ \mathbf{C}_{interface} & \mathbf{D}_{Stokes} \end{pmatrix}.$$

Since we are using the direct solver, the issue of condition number does not concern us. However, as $h \to 0$, we need to use a large enough η to scale the system. Otherwise, the linear system may become very ill-conditioned and degenerate in the limit.

Quadratic elements are used for the Darcy problem and div-stable Taylor–Hood elements are used for the Stokes part. This is saying that the r in Proposition 3.1 equals 2.

5.2. Steady-state solution. We perform numerical experiments on the unit square, $(0,1)\times(-0.25, 0.75)$. Let $(0,1)\times\{0\}$ be the interface separating the conduit from above and the matrix from below. For simplicity, all the parameters such as $\mathbb{K}$, K, S, ν, α, ρ, and g are set to 1. The scaling factor, η, is also set to 1, as has been

TABLE 1

Convergence rate for steady-state problem. Taylor–Hood element is used in the Stokes region and quadratic element is used in the Darcy region. We obtain convergence rate of 2 for seminorm $|\underline{\mathbf{u}}|_1$ and p and 3 for $\|\underline{\mathbf{u}}\|_{\underline{\mathbf{L}}^2}$. The last row of convergence rate is obtained by linear least square fitting.

h	$\|\mathbf{u}-\mathbf{u}^h\|_0$	$\|\mathbf{u}-\mathbf{u}^h\|_1$	$\|p-p^h\|_0$	$\|\phi-\phi^h\|_0$	$\|\phi-\phi^h\|_1$
2^{-3}	2.827E-4	1.081E-2	8.565E-3	5.482E-4	3.117E-2
2^{-4}	3.628E-5	2.667E-3	1.929E-3	5.993E-5	7.782E-3
2^{-5}	4.606E-6	6.640E-4	4.654E-4	7.085E-6	1.944E-3
2^{-6}	5.800E-7	1.658E-4	1.152E-4	8.683E-7	4.859E-4
Rate of conv.	2.976	2.009	2.070	3.099	2.001

explained. We set the appropriate forcing term and Dirichlet boundary conditions such that the following solution to the steady-state Stokes–Darcy problem is exact:

$$\begin{cases} u &= x^2y^2+e^{-y}, \\ v &= -\frac{2}{3}xy^3+[2-\pi\sin(\pi x)], \\ p &= -[2-\pi\sin(\pi x)]\cos(2\pi y), \\ \phi &= [2-\pi\sin(\pi x)][-y+\cos(\pi(1-y))]. \end{cases}$$

Table 1 gives the convergence rate of the steady-state problem. The well-posedness of the steady-state Stokes–Darcy system with the Beavers–Joseph condition (time-independent case of (2.4)) is not discussed in this paper. In [2], it is shown that the well-posedness is obtainable for a small enough Beavers–Joseph parameter α and the rescaling technique is not useful for the time-independent case. The numerical results here attest that the steady-state problem may be solvable with reasonable accuracy even without theoretical guarantee of well-posedness. These numerical results are also useful in selecting the initial condition, $\underline{\mathbf{u}}_h^0$, for the time-dependent problem and help avoid computing the projection (4.2) for the initial condition (see the discussion in the next subsection).

5.3. Transient solution. With the same domain setting, we set the appropriate forcing term and Dirichlet boundary conditions such that the following solution to the transient Stokes–Darcy problem is exact:

$$\begin{cases} u &= [x^2y^2+e^{-y}]\cos(2\pi t), \\ v &= [-\frac{2}{3}xy^3+[2-\pi\sin(\pi x)]]\cos(2\pi t), \\ p &= -[2-\pi\sin(\pi x)]\cos(2\pi y)\cos(2\pi t), \\ \phi &= [2-\pi\sin(\pi x)][-y+\cos(\pi(1-y))]\cos(2\pi t). \end{cases}$$

We use the numerical solution computed in the steady-state case as the numerical initial condition, $\underline{\mathbf{u}}_h^0$. The requirement of the error $\underline{\mathbf{u}}_h^0-\mathbb{P}_s\underline{\mathbf{u}}(0)$ in Theorem 4.4 is satisfied because $\|\underline{\mathbf{u}}_h^0-\underline{\mathbf{u}}(0)\|_{\underline{\mathbf{L}}^2}\le C^*h^3$ (corresponding to $r=2$). Tables 2 and 3 summarize the convergence rates with different order combinations of h and Δt. In particular, Table 3 confirms the convergence rate estimation provided in Theorem 4.4.

Appendix. Well-posedness of the semidiscrete finite element approximation problem (3.8). From (3.8) we have that

$$\langle\partial_t\underline{\mathbf{u}}^h(t),\underline{\mathbf{v}}^h\rangle_\eta+a_\eta(\underline{\mathbf{u}}^h(t),\underline{\mathbf{v}}^h)=\langle\underline{\mathbf{F}}(t),\underline{\mathbf{v}}^h\rangle_\eta\quad\forall\underline{\mathbf{v}}^h\in\underline{\mathbf{V}}^h. \tag{A.1}$$

Table 2
Convergence rate with $\Delta t \sim h^2$.

Δt	h	$\|\mathbf{u}-\mathbf{u}^h\|_0$	$\|\mathbf{u}-\mathbf{u}^h\|_1$	$\|p-p^h\|_0$	$\|\phi-\phi^h\|_0$	$\|\phi-\phi^h\|_1$
2^{-6}	2^{-3}	1.114*E*-3	1.458*E*-2	1.563*E*-2	3.677*E*-3	4.314*E*-2
2^{-8}	2^{-4}	2.774*E*-4	3.741*E*-3	3.662*E*-3	9.053*E*-4	1.090*E*-2
2^{-10}	2^{-5}	6.949*E*-5	9.449*E*-4	8.928*E*-4	2.255*E*-4	2.733*E*-3
2^{-12}	2^{-6}	1.741*E*-5	2.372*E*-4	2.214*E*-4	5.632*E*-5	6.837*E*-4
Rate of conv.		2.000	1.981	2.046	2.010	1.994

Table 3
Convergence rate with $\Delta t \sim h^3$.

Δt	h	$\|\mathbf{u}-\mathbf{u}^h\|_0$	$\|\mathbf{u}-\mathbf{u}^h\|_1$	$\|p-p^h\|_0$	$\|\phi-\phi^h\|_0$	$\|\phi-\phi^h\|_1$
2^{-6}	2^{-3}	1.114*E*-3	1.458*E*-2	1.563*E*-2	3.677*E*-3	4.314*E*-2
2^{-9}	2^{-4}	1.426*E*-4	3.008*E*-3	2.181*E*-3	4.613*E*-4	9.262*E*-3
2^{-12}	2^{-5}	1.797*E*-5	6.921*E*-4	4.296*E*-4	5.726*E*-5	2.124*E*-3
2^{-15}	2^{-6}	2.254*E*-6	1.682*E*-4	1.064*E*-4	7.124*E*-6	5.079*E*-4
Rate of conv.		2.984	2.143	2.394	3.005	2.135

Equation (A.1) is a system of linear ODEs with constant coefficients. Thus, it has a unique solution $u^h(t) \in \underline{\mathbf{V}}^h$, $0 < t < T$. Let $\underline{\mathbf{v}}^h = \underline{\mathbf{u}}^h$ in (A.1), and use the Gårding inequality (2.6) to obtain

$$\frac{1}{2}\frac{d}{dt}\|\underline{\mathbf{u}}^h(t)\|_{0,\eta}^2 + C_1\|\underline{\mathbf{u}}^h(t)\|_1^2 \le \left(C_0 + \frac{1}{2}\right)\|\underline{\mathbf{u}}^h(t)\|_{0,\eta}^2 + \frac{1}{2}\|\underline{\mathbf{F}}(t)\|_{0,\eta}^2.$$

Integrating the above inequality, we obtain

$$\begin{aligned}&\|\underline{\mathbf{u}}^h(t)\|_{0,\eta}^2 + 2C_1\int_0^t \|\underline{\mathbf{u}}^h(t)\|_1^2 \mathrm{d}s \\ &\quad \le \|P^h\underline{\mathbf{u}}_0\|_{0,\eta}^2 + \int_0^t \|\underline{\mathbf{F}}(s)\|_{0,\eta}^2 \mathrm{d}s + (2C_0+1)\int_0^t \|\underline{\mathbf{u}}^h(s)\|_{0,\eta}^2 \mathrm{d}s.\end{aligned}$$

By the Gronwall inequality (3.7), we have that

$$\|\underline{\mathbf{u}}^h(t)\|_{0,\eta}^2 \le C(\|P^h\underline{\mathbf{u}}_0\|_{0,\eta}^2 + \|\underline{\mathbf{F}}\|_{L^2(0,T;\underline{\mathbf{L}}^2)}^2), \tag{A.2}$$

where C is a constant. This proves that $\underline{\mathbf{u}}^h \in L^2(0,T;\underline{\mathbf{V}}^h)$. By regarding $\underline{\mathbf{u}}^h$ as the solution to the ODE (A.1), it is easy to see that $\underline{\mathbf{u}}^h \in H^1(0,T;\underline{\mathbf{V}}^h)$. However, the norm is not uniformly bounded and depends on h.

Now consider the following equivalent form of (A.1):

$$\tilde{a}_\eta(\underline{\mathbf{u}}^h, \underline{\mathbf{v}}^h) = -\langle \partial_t \underline{\mathbf{u}}^h, \underline{\mathbf{v}}^h\rangle_\eta + C_0\langle \underline{\mathbf{u}}^h, \underline{\mathbf{v}}^h\rangle_\eta + \langle \underline{\mathbf{F}}, \underline{\mathbf{v}}^h\rangle_\eta \quad \forall \underline{\mathbf{v}}^h \in \underline{\mathbf{V}}^h, \tag{A.3}$$

where C_0 is a constant that appeared in (2.6) and $\tilde{a}$ is a bilinear form on $\underline{\mathbf{W}}^h \times \underline{\mathbf{W}}^h$ defined as

$$\tilde{a}_\eta(\underline{\mathbf{w}}^h, \underline{\mathbf{v}}^h) = a_\eta(\underline{\mathbf{w}}^h, \underline{\mathbf{v}}^h) + C_0\langle \underline{\mathbf{w}}^h, \underline{\mathbf{v}}^h\rangle_\eta \quad \forall \underline{\mathbf{w}}^h, \underline{\mathbf{v}}^h \in \underline{\mathbf{W}}^h.$$

Clearly, $\tilde{a}$ is coercive. From the estimate (A.2) and the inf-sup condition (3.1), we can find $p^h \in Q^h$ satisfying (3.8) and the estimate (see [5, Theorem I.4.1])

$$\|p^h(t)\|_0 \le C(h)(\|P^h\underline{\mathbf{u}}_0\|_{0,\eta} + \|\underline{\mathbf{F}}\|_{L^2(0,T;\underline{\mathbf{L}}^2)}),$$

where C is a constant dependent on h. This proves that $p^h \in L^2(0,T;Q^h)$.

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