NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

Technical Report 32-1381

Finite Element Formulation for Linear Thermoviscoelastic Materials

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Preface

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Abstract

The finite difference and finite element matrix equations are developed for linear thermoviscoelastic materials. The equations are derived for a general three-dimensional body, but are applicable upon trivial changes to one- and two-dimensional configurations. A brief statement of the thermoviscoelastic field equations is followed by the development of the finite difference equations in time and then by the finite element formulation in space. Some attention is given to the experimental determination of material properties and their use in analytical work. An expansion of the experimentally or analytically determined material property functions in terms of exponential series leads to recurrence matrix equations, eliminating the problem of recalculating at each time step the history of material response. As an example, the details of setting of the finite element equations are illustrated.

Finite Element Formulation for Linear Thermoviscoelastic Materials

I. Introduction

The upsurge of the finite element technique in structural and continuum mechanics during the last few years has given to the analyst a tool which provides the flexibility and the versatility necessary for the analysis of structural and continuum problems with complex boundary conditions and complex configurations. A vast literature exists about the finite element technique and its application, mainly to elastic and plastic static problems and to some steady-state dynamic problems (e.g., Refs. 1 and 2). However, the extension of this technique to viscoelastic problems without using the elastic-viscoelastic correspondence principle has been accomplished only in a few relatively simple cases (e.g., Ref. 3). In Ref. 4, a short description of problems in the stress analysis of linear thermoviscoelastic solid propellants is given with a review of some related recent literature. It has been concluded that only for rather special cases can the elastic-viscoelastic correspondence principle be invoked, i.e., when the material properties are independent of thermal changes. Since the properties of most viscoelastic materials are highly temperature-sensitive, it is concluded that the development of a general program should be based on the solution of integral equations in real time rather than on the elastic-viscoelastic correspondence principle. Such an approach using a finite difference technique in space and time has been applied recently in Ref. 5 to simple one-dimensional axisymmetrical problems.

In this report the finite difference equations in time and finite element matrix equations in space are developed for general linear thermoviscoelastic problems.

II. Governing Equations

A. General Assumptions

A region of space D + B with boundary B (Fig. 1) is assumed filled with a continuous medium of linear thermoviscoelastic material. The material may have a space- and time-dependent distribution of density and may be subjected at each point in D to externally imposed quasi-static inertial forces per unit volume $f_j(x_h, t)$, such as gravity forces, centrifugal forces, etc. In addition, a known space- and time-dependent thermal field (uncoupled theory)

$$\theta(x_h, t) = T(x_h, t) - T_0 \qquad (1)$$

exists in the medium in D + B, where $T(x_h, t)$ is the local instantaneous temperature, and T_0 is a conveniently selected reference temperature for which the material in D + B is completely relaxed.

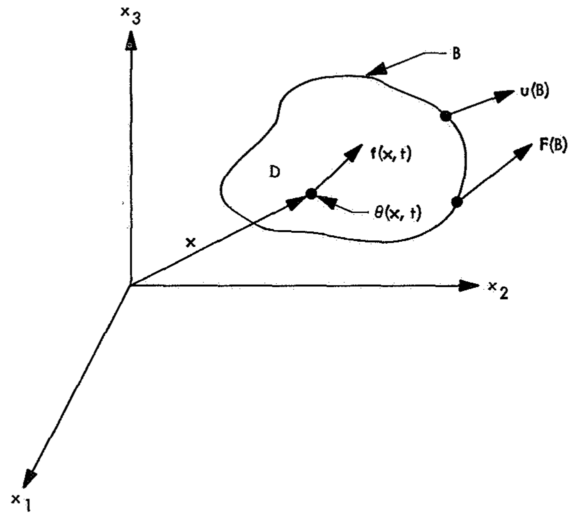

Fig. 1. Region D + B with prescribed body forces f, thermal field θ, boundary displacements U, and boundary tractions F referred to cartesian coordinates x_j

At each boundary point in B, the boundary displacements u_j (B) or the boundary tractions P_j (B) are prescribed. Thus, in cartesian component form, the following displacement boundary conditions

$$u_j = u_j(\mathrm{B}) \tag{2}$$

or, the following traction boundary conditions[1]

$$n_i \tau_{ij} = P_j(\mathrm{B}) \tag{3}$$

are valid at each point in B, where n_i denotes the components of the unit normal vector to B, u_j denotes the displacement components, and τ_{ij} denotes the components of the stress tensor in D, as x_h tends to a corresponding point in B.

The initial conditions are specified so that all mechanical field quantities, displacements, velocities, stresses, strains, etc., vanish for $t < 0$, and the temperature increase $\theta(x_h, t) = 0$ when $t = 0$.

B. Material Properties

The constitutive property of the material in the region D + B may be anisotropic, but is assumed to be linear. The coefficients of thermal expansion α'_{ij}, then represent a second-order symmetric tensor with six independent thermal expansion coefficients. The coefficients $\alpha'_{ij}(T')$ may be functions of temperature, in which case it is convenient to define average thermal expansion coefficients $\alpha_{ij}(T)$ by the relation

$$\alpha_{ij}(T) = \frac{1}{\theta}\int_{T_0}^{T} \alpha'_{ij}(T')\,dT' \tag{4}$$

The mechanical properties of the material are characterized by the general anisotropic time-dependent relaxation moduli $E_{ijkl}(t)$ of the fourth-order material property tensor, which has the following symmetry properties,

$$E_{ijkl} = E_{jikl} = E_{ijlk} = E_{klij} \tag{5}$$

and, therefore, has 21 independent components in the general case.

In the particular case of isotropy, the fourth-order tensor can be represented by two independent components, e.g., the bulk modulus K and the shear modulus G, while the thermal expansion is characterized by the single expansion coefficient α. The respective isotropic tensor components are then given by

$$E_{ijkl}(t) = \left\{ G(t)(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{kj}) + \left[K(t) - \frac{2}{3}G(t)\right]\delta_{ij}\delta_{kl} \right\} \tag{6}$$

and

$$\alpha_{ij}(T) = \delta_{ij}\,\alpha(T) \tag{7}$$

where δ_{ij} is the Kronecker delta characterized by

$$\delta_{ij} = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases} \tag{8}$$

A further assumption is that the material in the region D + B behaves in a thermorheologically simple way, showing, for changes of temperature, a pure shift in the characteristic functions, i.e., creep functions and relaxation moduli, when these are plotted against the logarithm of time. To express thermorheologically simple behavior analytically (Refs. 6 and 7), a "reduced time" ξ is introduced by

$$\xi(x_h, t) = \int_0^t \frac{d\tau}{a[T(x_h, \tau)]} \tag{9}$$

where $a(T)$ is an experimentally determined time-shift function (Ref. 8) of temperature T, only; its dependence on position x_h and time t is implicit through T, and is in

[1]The usual summation convention in tensor theory is used for double indices unless indicated otherwise.

many cases well described by the Williams–Landel–Ferry (WLF) equation

$$a(T) = \exp\left[-\frac{C_1(T-T_g)}{C_2+(T-T_g)}\right] \qquad (10)$$

where T_g is the glass transition temperature and C_1 and C_2 are constants.

Recent experimental work has shown that the shift function may also be a function of the applied stresses and the induced strains, and their time derivatives. A few simple examples of such nonlinear material behaviour have been discussed in Ref. 9. In the following development it is assumed that $a(T)$ is determined *a priori* and is only a function of temperature. Dependence on stresses and strains can be easily included, however, requiring only the step-by-step determination of the shift function after each time-step in the numerical computations, while the finite element formulation remains unchanged.

For each relaxation modulus the following relation then holds,

$$E^{T}_{ijkl}(t) = E^{T_0}_{ijkl}(\xi) \qquad (11)$$

which states that the relaxation moduli at an arbitrary temperature T corresponding to time t are expressed by their values at a reference temperature T_0 related to the new "reduced time" scale ξ (Fig. 2).

Similar to the well-known relationships between the various moduli for isotropic elastic materials, there exist corresponding relationships between the various moduli for isotropic linear viscoelastic materials. That is to say,

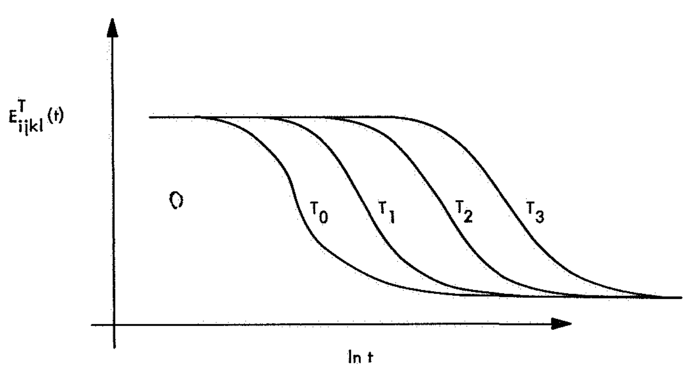

Fig. 2. Relaxation modulus as a function of time for different temperatures. If the relaxation modulus is plotted vs the reduced time ξ, all curves will fall upon the single curve for T_0

if any two of the seven characteristic functions of isotropic materials, i.e., the *extension modulus* $E(t)$, and *extension compliance* $F(t)$, the *shear modulus* $G(t)$ and *shear compliance* $J(t)$, the *bulk modulus* $K(t)$ and *bulk compliance* $H(t)$, and *Poisson's ratio* $\nu(t)$, are known from experiment, the others can be determined analytically.

In experimental work it is usually convenient to measure $F(t)$ and $\nu(t)$, where $F(t)$ gives the extension of a material specimen as a function of time under constant unit stress (Fig. 3). The tensile stress in the longitudinal direction results in a relative elongation of the longitudinal dimension $+\epsilon(t)$, and in a lateral contraction of the sample $-\epsilon_l(t)$, both of which are related through $\nu(t)$ in general by

$$-\epsilon_l(t) = \nu(t)\,\epsilon_0 + \int_0^t \nu(t-\tau)\frac{\partial\epsilon}{\partial\tau}\,d\tau \qquad (12)$$

If the initial step elongation ϵ_0 at $t = 0$ is held constant, Poisson's ratio is simply determined by

$$\nu(t) = \frac{-\epsilon_l(t)}{\epsilon_0} \qquad (13)$$

After $F(t)$ and $\nu(t)$ are determined, the compliances $J(t)$ and $H(t)$ can be computed using the following relations (Ref. 6):

$$J(t) = 2F(t)\,[1+\nu(0)] + 2\int_0^t F(\tau)\frac{\partial}{\partial\tau}\nu(t-\tau)\,d\tau \qquad (14)$$

$$H(t) = 3F(t)\,[1-2\nu(0)] - 6\int_0^t F(\tau)\frac{\partial}{\partial\tau}\nu(t-\tau)\,d\tau \qquad (15)$$

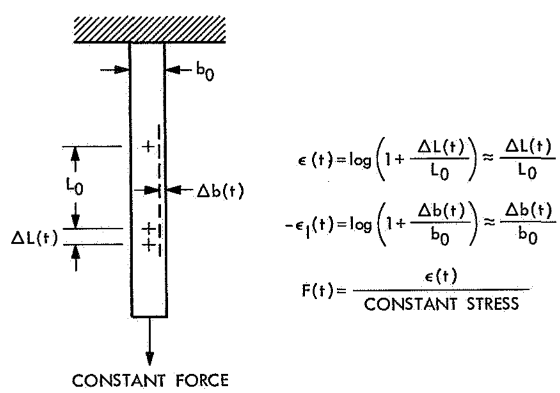

Fig. 3. Determination of longitudinal strains $\epsilon(t)$, lateral strains ϵ_l, and compliance $F(t)$

The shear modulus $G(t)$ and the bulk modulus $K(t)$ in Eq. (6) are then obtained from the following integral equations of the first kind:

$$\int_0^t J(t-\tau)\, G(\tau)\, d\tau = t \tag{16}$$

$$\int_0^t H(t-\tau)\, K(\tau)\, d\tau = t \tag{17}$$

In the isotropic case one has then, similar to Eq. (11), the relations

$$G^T(t) = G^{T_0}(\xi); \qquad K^T(t) = K^{T_0}(\xi) \tag{18}$$

For a discussion of various experimental techniques for the measurement of viscoelastic material properties, reference is made, for example, to Ref. 10.

C. Thermoviscoelastic Field Equations

The general field equations of quasi-static linear thermoviscoelasticity are

(1) equilibrium equations,

$$\sigma_{ij,j} + f_i = 0 \tag{19}$$

(2) strain-displacement equations,

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{20}$$

(3) and constitutive equations,

$$\sigma_{ij}(x_h, t) = \int_{-\infty}^{t} E_{ijkl}\,[\xi(x_h, t) - \xi'(x_h, \tau)] \times \frac{\partial}{\partial\tau}[\epsilon_{kl}(x_h, \tau) - \alpha_{kl}(x_h, \tau)\,\theta(x_h, \tau)]\, d\tau \tag{21}$$

where the argument $[\xi(x_h, t) - \xi'(x_h, \tau)]$ is obtained using Eq. (9). If one considers the assumed initial conditions and allows for the possibility of an initially induced step strain $\epsilon_{kl(0)}(x_h)$ at $t = 0$, then Eq. (21) becomes

$$\sigma_{ij} = E_{ijkl}(\xi)\,\epsilon_{kl(0)}(x_h) + \int_0^t E_{ijkl}(\xi - \xi') \times \frac{\partial}{\partial\tau}(\epsilon_{kl} - \alpha_{kl}\,\theta)\, d\tau \tag{22}$$

The principle of virtual work states that at any time-instant t, the work done by the internal stresses in D when going through the arbitrary virtual strains $\delta\epsilon_{ij}$ is equal to the work done by the body forces f_j in D when going through the corresponding virtual displacements δu_j, and by the boundary forces $P_j(\mathrm{B})$ when going through the corresponding virtual boundary displacements $\delta u_j(\mathrm{B})$. The principle of virtual work can be written immediately as

$$\int_{\mathrm{D}} \sigma_{ij}\,\delta\epsilon_{ij}\, d\mathrm{D} = \int_{\mathrm{D}} f_j\,\delta u_j\, d\mathrm{D} + \int_{\mathrm{B}} P_j(\mathrm{B})\,\delta u_j(\mathrm{B})\, d\mathrm{B} \tag{23}$$

Substituting Eq. (22) into Eq. (23) gives

$$\int_{\mathrm{D}} \delta\epsilon_{ij} \left\{ E_{ijkl}(\xi)\,\epsilon_{kl(0)} + \int_0^t E_{ijkl}(\xi - \xi')\frac{\partial}{\partial\tau}(\epsilon_{kl} - \alpha_{kl}\,\theta)\, d\tau \right\} d\mathrm{D} = \int_{\mathrm{D}} \delta u_j\, f_j\, d\mathrm{D} + \int_{\mathrm{B}} \delta u_j(\mathrm{B})\, P_j(\mathrm{B})\, d\mathrm{B} \tag{24}$$

D. Incremental Field Equations

Starting at $t = 0$, the time axis is subdivided into not necessarily equal intervals $\Delta t_{(m)}$ so that

$$t_{(m)} = \Delta t_{(m)} + t_{(m-1)} \tag{25}$$

From Eq. (9) one then obtains the corresponding "reduced time" intervals,

$$\xi_{(m)} = \Delta\xi_{(m)} + \xi_{(m-1)} \tag{26}$$

from

$$\Delta\xi_{(m)} = \int_{t_{(m-1)}}^{t_{(m)}} \frac{d\tau}{a(T)} \approx \frac{\Delta t_{(m)}}{a_{(m)}(T)} \tag{27}$$

The corresponding field quantities at time $t_{(m)}$ are defined in terms of those at time $t_{(m-1)}$ and the associated incremental quantities by

$$\left.\begin{aligned} \sigma_{ij(m)} &= \Delta\sigma_{ij(m)} + \sigma_{ij(m-1)} \\ f_{j(m)} &= \Delta f_{j(m)} + f_{j(m-1)} \\ u_{j(m)} &= \Delta u_{j(m)} + u_{j(m-1)} \\ \epsilon_{ij(m)} &= \Delta\epsilon_{ij(m)} + \epsilon_{ij(m-1)} \\ (\alpha_{ij}\,\theta)_{(m)} &= \Delta(\alpha_{ij}\,\theta)_{(m)} + (\alpha_{ij}\,\theta)_{(m-1)} \\ P_j(\mathrm{B})_{(m)} &= \Delta P_j(\mathrm{B})_{(m)} + P_j(\mathrm{B})_{(m-1)} \end{aligned}\right\} \tag{28}$$

With a substitution of the appropriate quantities of Eqs. (28) into Eqs. (19) and (20), it follows that the latter are identically satisfied also only for the increments, i.e.,

$$\Delta\sigma_{ij(m),j} + \Delta f_{i(m)} = 0 \tag{29}$$

$$\Delta\epsilon_{ij(m)} = \frac{1}{2}\left(\Delta u_{i(m),j} + \Delta u_{j(m),i}\right) \tag{30}$$

The virtual work equation, Eq. (23), at the time-instant $t_{(m)}$ is

$$\int_D \sigma_{ij(m)}\,\delta\Delta\epsilon_{ij(m)}\,dD = \int_D f_{j(m)}\,\delta\Delta u_{j(m)}\,dD + \int_B P_j(B)_{(m)}\,\delta\Delta u_j(B)_{(m)}\,dB \tag{31}$$

where for the arbitrary virtual strains and displacements the corresponding variations of the increments $\delta\Delta\epsilon_{ij(m)}$, $\delta\Delta u_{j(m)}$, and $\delta\Delta u_j(B)_{(m)}$ have been chosen.

Substituting for $\sigma_{ij(m)}$, $f_{j(m)}$ and $P_j(B)_{(m)}$ from Eqs. (28), Eq. (31) is identically satisfied also for the time-instant $t_{(m-1)}$ and independently also only for the mth increments.

The stress at $t_{(n)}$ follows from Eq. (22) as

$$\sigma_{ij(n)} = E_{ijkl}(\xi_{(n)})\,\epsilon_{kl(0)} + \int_0^{t_{(n)}} E_{ijkl}(\xi_{(n)} - \xi') \times \frac{\partial}{\partial\tau}(\epsilon_{kl} - \alpha_{kl}\,\theta)\,d\tau \tag{32}$$

With the following approximation at $t_{(m)}$,

$$\frac{\partial}{\partial\tau}(\epsilon_{kl} - \alpha_{kl}\,\theta)\,d\tau \approx \Delta\epsilon_{kl(m)} - \Delta(\alpha_{kl}\,\theta)_{(m)} \tag{33}$$

the integral in Eq. (32) can be approximated by a summation as follows:

$$\sigma_{ij(n)} \approx E_{ijkl}(\xi_{(n)})\,\epsilon_{kl(0)} + \sum_{m=1}^{m=n} E_{ijkl}(\xi_{(n)} - \xi_{(m-1)})\,[\Delta\epsilon_{kl(m)} - \Delta(\alpha_{kl}\,\theta)_{(m)}] \tag{34}$$

Now, with a substitution of Eq. (34) into Eq. (31), Eq. (31) can be rewritten for $t_{(n)}$ in the following form:

$$\int_B \delta\Delta u_{j(n)}(B)_{(n)}\,P_j(B)_{(n)}\,dB = \int_D \delta\Delta\epsilon_{ij(n)}\,[E_{ijkl}(\xi_{(n)})\,\epsilon_{kl(0)} + \sum_{m=1}^{m=n} E_{ijkl}(\xi_{(n)} - \xi_{(m-1)})\,\Delta\epsilon_{kl(m)}]\,dD - \int_D \delta\Delta\epsilon_{ij(n)} \sum_{m=1}^{m=n} E_{ijkl}(\xi_{(n)} - \xi_{(m-1)})\,\Delta(\alpha_{kl}\,\theta)_{(m)}\,dD - \int_D \delta\Delta u_{j(n)}\,f_{j(n)}\,dD \tag{35}$$

III. Element Equations

The virtual work relation of Eq. (35) is the starting point for the development of the finite element equations in this section. It is convenient to write Eq. (35) in matrix notation, using the following substitutions:

$$\delta\Delta u_{j(n)}, \delta\Delta\epsilon_{ij(n)} \rightarrow \{\delta\Delta\mathbf{u}_n\}^T, \{\delta\Delta\epsilon_n\}^T$$

$$P_j(B)_{(n)}, \epsilon_{kl(0)}, \Delta\epsilon_{kl(m)}, \Delta(\alpha_{kl}\,\theta)_{(m)}, f_{j(n)}, \sigma_{ij(n)} \rightarrow \{\mathbf{P}_n\}, \{\epsilon_0\}, \{\Delta\epsilon_m\}, \{\Delta(\boldsymbol{\alpha}\theta)_m\}, \{\mathbf{f}_n\}, \{\boldsymbol{\sigma}_{(n)}\} \tag{36}$$

$$E_{ijkl}(\xi_{(n)}), E_{ijkl}(\xi_{(n)} - \xi_{(m-1)}) \rightarrow [\mathbf{E}(\xi_n)], [\mathbf{E}(\xi_n - \xi_{m-1})] \tag{37}$$

Eq. (34) then becomes,

$$\{\boldsymbol{\sigma}_n\} = [\mathbf{E}(\xi_n)]\,\{\epsilon_0\} + \sum_{m=1}^{m=n} [\mathbf{E}(\xi_n - \xi_{m-1})]\,\{\Delta\epsilon_m - \Delta(\boldsymbol{\alpha}\theta)_m\} \tag{38}$$

and Eq. (35) becomes

$$\int_B \{\delta\Delta\mathbf{u}_n\}^T \{\mathbf{P}_n\}\, dB = \int_D \{\delta\Delta\epsilon_n\}^T \left([\mathbf{E}(\xi_n)]\{\epsilon_0\} + \sum_{m=1}^{m=n} [\mathbf{E}(\xi_n - \xi_{m-1})]\{\Delta\epsilon_m\}\right) dD$$

$$- \int_D \{\delta\Delta\epsilon_n\}^T \sum_{m=1}^{m=n} [\mathbf{E}(\xi_n - \xi_{m-1})]\{\Delta(\alpha\theta)_m\}\, dD - \int_D \{\delta\Delta\mathbf{u}_n\}^T \{\mathbf{f}_n\}\, dD \tag{39}$$

The entire domain D is now divided into subdomains (finite elements) which are connected to each other at nodal points (Fig. 4). In each subdomain a local rectangular coordinate system is conveniently located. For instance in the Ith subdomain D_I, located at vector position $\{\mathbf{r}^I\}$ with respect to the global coordinate system (x_1, x_2, x_3), the local coordinate system (z_1^I, z_2^I, z_3^I) is shown in Fig. 4. The local position vector, $\{\mathbf{z}^I\}$ and the global position vector $\{\mathbf{x}^I\}$ of a point in D_I are related to each other by a coordinate transformation of the form

$$\{\mathbf{x}^I\} = [\mathbf{C}^I]\{\mathbf{z}^I\} + \{\mathbf{r}^I\} \tag{40}$$

where $[\mathbf{C}^I]$ is the orthogonal coordinate transformation matrix involving the direction cosines between the coordinate axes. In many cases it is convenient to let the local and the global coordinate systems be the same; and the following developments will be restricted to these cases, with the realization that the generalization of the following results requires only elementary transformation involving $[\mathbf{C}^I]$ and $\{\mathbf{r}^I\}$ in Eq. (40).

In the Ith subdomain D_I, a displacement field is assumed of the form

$$\{\mathbf{u}^I(\mathbf{x}_h, t)\} = \sum_p \{\boldsymbol{\phi}^I_{(p)}(x_h)\}\, q^I_{(p)}(t) \tag{41}$$

where the $\boldsymbol{\phi}^I_{(p)}$ are assumed functions of position which are to be chosen so that compatibility at the boundary between adjacent elements is preserved. The time functions $q^I_{(p)}(t)$ are the unknown generalized coordinates of which there are as many associated with an element as there are nodal displacement degrees of freedom. The number of terms in the series in Eq. (41) is therefore a function of the type of element and its number of nodal points.

Equation (41) can be written in the form

$$\{\mathbf{u}^I\} = [\boldsymbol{\phi}^I]\{\mathbf{q}^I\} \tag{42}$$

and the corresponding increment at time $t_{(n)}$ is

$$\{\Delta\mathbf{u}^I_n\} = [\boldsymbol{\phi}^I]\{\Delta\mathbf{q}^I_n\} \tag{43}$$

The incremental nodal displacements of the Ith element are equal in number to the generalized coordinates and at $t_{(n)}$ are

$$\{\Delta\mathbf{U}^I_n\} = [\boldsymbol{\Phi}^I]\{\Delta\mathbf{q}^I_n\} \tag{44}$$

in which the matrix $[\boldsymbol{\Phi}^I]$ is formed by successively introducing each nodal point coordinate in Eq. (43). The matrix elements in Eq. (44) are, thus, not functions of x_h. Solving for $\{\Delta\mathbf{q}^I_n\}$ gives

$$\{\Delta\mathbf{q}^I_n\} = [\boldsymbol{\Phi}^I]^{-1}\{\Delta\mathbf{U}^I_n\} \tag{45}$$

With the use of Eqs. (30) and (41), the column of incremental strain components in the Ith element at $t_{(n)}$ can be expressed in terms of the generalized coordinates as follows,

$$\{\Delta\epsilon^I_n\} = [\boldsymbol{\psi}^I]\{\Delta\mathbf{q}^I_n\} \tag{46}$$

where the elements of the rectangular matrix $[\boldsymbol{\psi}^I]$ are functions of x_h involving the $\boldsymbol{\phi}^I_{(p)}$ and their derivatives.

With Eqs. (45) and (46) the strain increments, in terms of the increments of nodal displacements, become

$$\{\Delta\epsilon^I_n\} = [\boldsymbol{\psi}^I][\boldsymbol{\Phi}^I]^{-1}\{\Delta\mathbf{U}^I_n\} \tag{47}$$

It can be similarly shown that the initially induced strains $\{\epsilon^I_0\}$, in terms of the initially induced nodal displacements $\{\mathbf{U}^I_0\}$, become

$$\{\epsilon^I_0\} = [\boldsymbol{\psi}^I][\boldsymbol{\Phi}^I]^{-1}\{\mathbf{U}^I_0\} \tag{48}$$

When Eq. (45) and the transpose law for matrix products are used, the transpose of the variation of the incremental strain components in Eq. (46) becomes

$$\{\delta\Delta\epsilon^I_n\}^T = \{\delta\Delta\mathbf{U}^I_n\}^T [\boldsymbol{\Phi}^I]^{-1\,T} [\boldsymbol{\psi}^I]^T \tag{49}$$

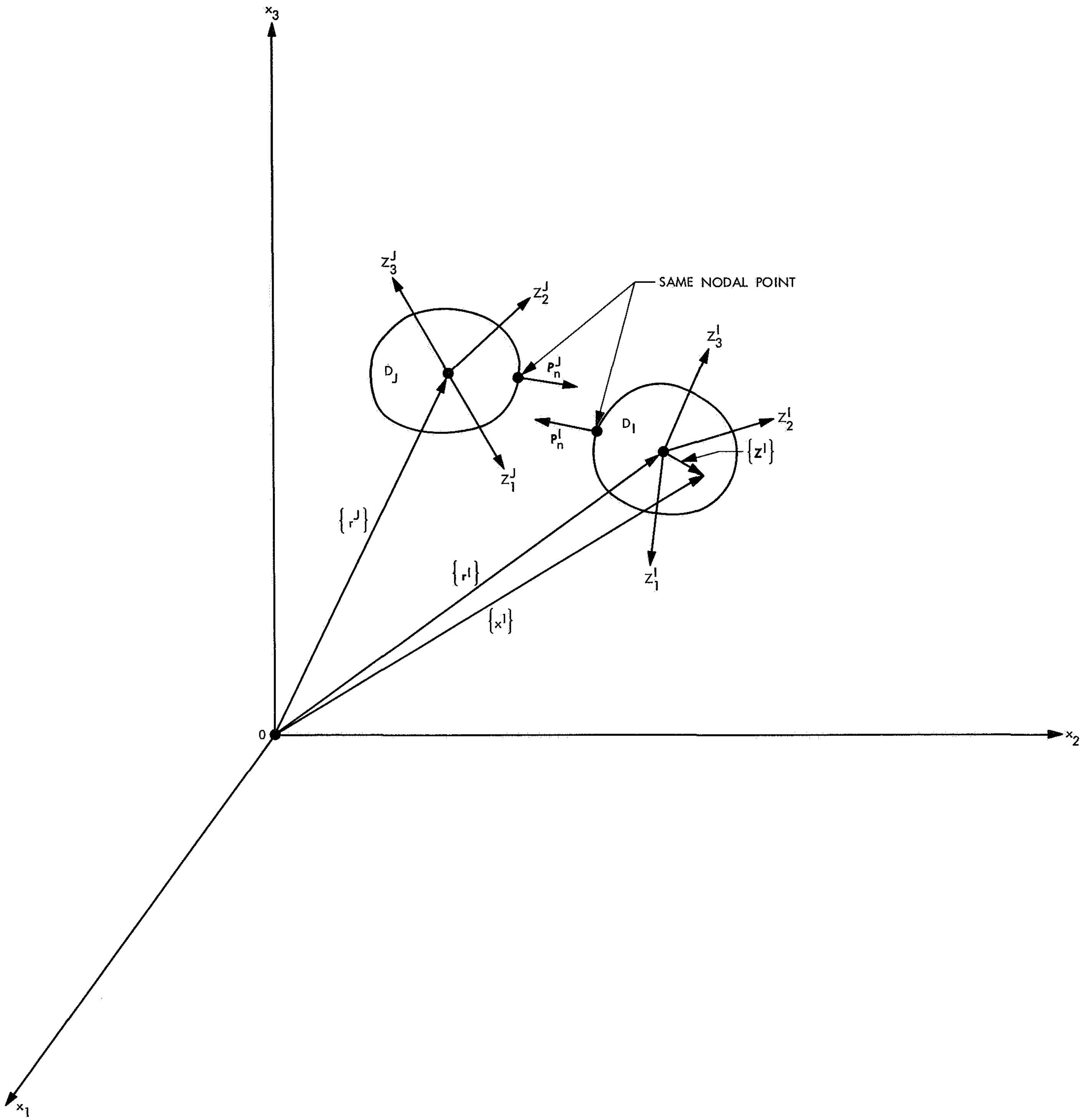

Fig. 4. Typical subdomains (finite elements) with local coordinate systems and nodal forces referred to the global coordinate system

If $\{P^I_n\}$ is the column matrix of the nodal force components for the Ith element (Fig. 4) then, from Eq. (39) with Eqs. (43), (47), (48), and (49), one obtains for the Ith element as the domain of integration

$$\{P^I_n\} = \int_{D_I} [\overset{T}{\Phi^I}]^{-1} [\overset{T}{\psi^I}] \sum_{m=1}^{m=n} [E^I(\xi_n - \xi_{m-1})] [\psi^I] [\Phi^I]^{-1} dD_I \{\Delta U^I_m\} + \int_{D_I} [\overset{T}{\Phi^I}]^{-1} [\overset{T}{\psi^I}] [E^I(\xi_n)] [\psi^I] [\Phi^I]^{-1} dD_I \{U^I_0\}$$
$$- \int_{D_I} [\overset{T}{\Phi^I}]^{-1} [\overset{T}{\psi^I}] \sum_{m=1}^{m=n} [E^I(\xi_n - \xi_{m-1})] \{\Delta(\alpha\theta)^I_m\} dD_I - \int_{D_I} [\overset{T}{\Phi^I}]^{-1} [\overset{T}{\varphi^I}] \{f^I_n\} dD_I \qquad (50)$$

The following quantities are defined:

$$[K^I_{n,m-1}] = \int_{D_I} [\overset{T}{\Phi^I}]^{-1} [\overset{T}{\psi^I}] [E^I(\xi_n - \xi_{m-1})] [\psi^I] [\Phi^I]^{-1} dD_I \qquad (51)$$

$$[K^I_n] = \int_{D_I} [\overset{T}{\Phi^I}]^{-1} [\overset{T}{\psi^I}] [E^I(\xi_n)] [\psi^I] [\Phi^I]^{-1} dD_I \qquad (52)$$

$$\{T^I_{n,m-1}\} = \int_{D_I} [\overset{T}{\Phi^I}]^{-1} [\overset{T}{\psi^I}] [E^I(\xi_n - \xi_{m-1})] \{\Delta(\alpha\theta)^I_m\} dD_I \qquad (53)$$

$$\{F^I_n\} = \int_{D_I} [\overset{T}{\Phi^I}]^{-1} [\overset{T}{\varphi^I}] \{f^I_n\} dD_I \qquad (54)$$

Interchanging integration and summation in Eq. (50), one obtains with Eqs. (51) to (54) the element equations in the following form:

$$\{P^I_n\} = [K^I_{n,n-1}] \{\Delta U^I_n\} + \sum_{m=1}^{m=n-1} [K^I_{n,m-1}] \{\Delta U^I_m\}$$
$$+ [K^I_n] \{U^I_0\} - \sum_{m=1}^{m=n} \{T^I_{n,m-1}\} - \{F^I_n\} \qquad (55)$$

At time $t = 0$, Eq. (55) becomes,

$$\{P^I_0\} = [K^I_0] \{U^I_0\} - \{F^I_0\} \qquad (56)$$

Equation (55) gives the unknown nodal force vector $\{P^I_n\}$ for the Ith finite element at time

$$t_{(n)} = \sum_{m=1}^{m=n} \Delta t_{(m)}$$

in terms of the *initial* element nodal displacements $\{U^I_0\}$, the subsequent element nodal displacement increments $\{\Delta U^I_m\}$ at time $t_{(m)}$, the thermal field quantities, and the body force quantities. With the exception of the element nodal displacements $\{U^I_0\}$ and $\{\Delta U^I_m\}$, all quantities on the right hand side in Eq. (55) are known initially at $t = 0$.

IV. System Equations

In the previous section, only a single finite element was considered. In this section an arbitrary number of N such elements will be assembled, constituting the entire system of elements into which the region D was subdivided. Thus, it follows that

$$D = \sum_{I=1}^{I=N} D_I \qquad (57)$$

The N sets of uncoupled element equations, Eq. (55), can be ordered and written in the following form:

$$\begin{Bmatrix} \{P^1_n\} \\ \vdots \\ \{P^I_n\} \\ \vdots \\ \{P^N_n\} \end{Bmatrix} = \begin{bmatrix} [K^1_{n,n-1}] & & & & \\ & \ddots & & & \\ & & [K^I_{n,n-1}] & & \\ & & & \ddots & \\ & & & & [K^N_{n,n-1}] \end{bmatrix} \begin{Bmatrix} \{\Delta U^1_n\} \\ \vdots \\ \{\Delta U^I_n\} \\ \vdots \\ \{\Delta U^N_n\} \end{Bmatrix} + \sum_{m=1}^{m=n-1} \begin{bmatrix} [K^1_{n,m-1}] & & & & \\ & \ddots & & & \\ & & [K^I_{n,m-1}] & & \\ & & & \ddots & \\ & & & & [K^N_{n,m-1}] \end{bmatrix} \begin{Bmatrix} \{\Delta U^1_m\} \\ \vdots \\ \{\Delta U^I_m\} \\ \vdots \\ \{\Delta U^N_m\} \end{Bmatrix}$$
$$+ \begin{bmatrix} [K^1_n] & & & & \\ & \ddots & & & \\ & & [K^I_n] & & \\ & & & \ddots & \\ & & & & [K^N_n] \end{bmatrix} \begin{Bmatrix} \{U^1_0\} \\ \vdots \\ \{U^I_0\} \\ \vdots \\ \{U^N_0\} \end{Bmatrix} - \sum_{m=1}^{m=n} \begin{Bmatrix} \{T^1_{n,m-1}\} \\ \vdots \\ \{T^I_{n,m-1}\} \\ \vdots \\ \{T^N_{n,m-1}\} \end{Bmatrix} - \begin{Bmatrix} \{F^1_n\} \\ \vdots \\ \{F^I_n\} \\ \vdots \\ \{F^N_n\} \end{Bmatrix} \qquad (58)$$

Compactly, this can be written

$$\{\mathbf{P}_n\} = [\mathbf{K}_{n,n-1}]\{\Delta\mathbf{U}_n\} + \sum_{m=1}^{m=n-1}[\mathbf{K}_{n,m-1}]\{\Delta\mathbf{U}_m\}$$

$$+ [\mathbf{K}_n]\{\mathbf{U}_0\} - \sum_{m=1}^{m=n}\{\mathbf{T}_{n,m-1}\} - \{\mathbf{F}_n\} \qquad (59)$$

To interconnect the discrete finite elements, equilibrium conditions and compatibility conditions are imposed at the nodal points. If the externally applied forces at the nodal points, Fig. 5, of the interconnected system, i.e., at the system nodal points, at time $t_{(n)}$ are put in the vector form, $\{\bar{\mathbf{P}}_n\}$, then equilibrium requires that at each system nodal point the sum of the internal forces, i.e., the nodal forces at the element nodal points interconnected at the system nodal points, equals the externally applied force at that point. To this end a connectivity matrix or assembly matrix $[\mathbf{A}]$ is formed, which, when premultiplying $\{\mathbf{P}_n\}$, gives $\{\bar{\mathbf{P}}_n\}$. Thus,

$$[\mathbf{A}]\{\mathbf{P}_n\} = \{\bar{\mathbf{P}}_n\}$$

The same relation, of course, also holds for time $t = 0$, i.e., (60)

$$[\mathbf{A}]\{\mathbf{P}_0\} = \{\bar{\mathbf{P}}_0\}$$

Here, the assembly matrix $[\mathbf{A}]$ is the so-called Boolean matrix consisting of only zero and unit elements corresponding to the interconnected elements and points.

It can now be shown that the incremental displacements at $t_{(n)}$ at the nodal points of the individual elements and of the assembly, i.e., the element nodal displacements $\{\Delta\mathbf{U}_n\}$, and the system nodal displacements $\{\Delta\bar{\mathbf{U}}_n\}$, respectively, are related as follows:

$$[\mathbf{A}]^T\{\Delta\bar{\mathbf{U}}_n\} = \{\Delta\mathbf{U}_n\}$$

And similarly at time $t = 0$, (61)

$$[\mathbf{A}]^T\{\bar{\mathbf{U}}_0\} = \{\mathbf{U}_0\}$$

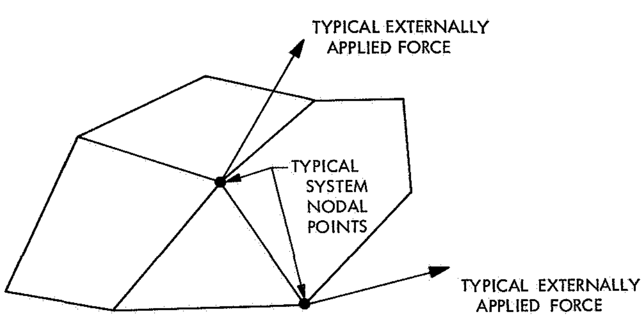

Fig. 5. Assembly of finite elements

Substituting Eqs. (61) into Eq. (59) and using both parts of Eq. (60) gives the forces applied externally to the system at the system nodal points,

$$\{\bar{\mathbf{P}}_n\} = [\mathbf{A}][\mathbf{K}_{n,n-1}][\mathbf{A}]^T\{\Delta\bar{\mathbf{U}}_n\}$$

$$+ \sum_{m=1}^{m=n-1}[\mathbf{A}][\mathbf{K}_{n,m-1}][\mathbf{A}]^T\{\Delta\bar{\mathbf{U}}_m\}$$

$$+ [\mathbf{A}][\mathbf{K}_n][\mathbf{A}]^T\{\bar{\mathbf{U}}_0\}$$

$$- \sum_{m=1}^{m=n}[\mathbf{A}]\{\mathbf{T}_{n,m-1}\} - [\mathbf{A}]\{\mathbf{F}_n\} \qquad (62)$$

A certain number of the system nodal displacements are restrained to follow a certain prescribed path in space and time (including zero displacements, as, for example, for unmovable supports). These displacement conditions must be imposed on the displacement vector,

$$\{\mathbf{U}_n\} = \{\mathbf{U}_0\} + \sum_{m=1}^{m=n}\{\Delta\mathbf{U}_n\} \qquad (63)$$

Thus, the values of the prescribed nodal displacements at each time-instant $t_{(n)}$ are known, hence, their increments are also known. The unrestraint system nodal displacements then become the unknowns. If $\{\Delta\bar{\bar{\mathbf{U}}}_n\}$ is the vector of *only the unknown* incremental system *nodal displacements,* then these are related to *all* the incremental system *nodal displacements,* including the prescribed ones, by a boundary condition matrix $[\mathbf{B}]$ in the following form:

$$[\mathbf{B}]^T\{\Delta\bar{\bar{\mathbf{U}}}_n\} = \{\Delta\bar{\mathbf{U}}_n\}$$

and for $t = 0$, (64)

$$[\mathbf{B}]^T\{\bar{\bar{\mathbf{U}}}_0\} = \{\bar{\mathbf{U}}_0\}$$

The forces of restraint which ensure the prescribed nodal displacements are unknown external nodal forces (supporting forces), while all the other nodal forces are known externally applied forces. If $\{\bar{\bar{\mathbf{P}}}_n\}$ is the vector of *all known forces* applied to the system nodal points, then these can be expressed, in terms of *all* the nodal forces including the *unknown* ones, in the following form:

$$[\mathbf{B}]\{\bar{\mathbf{P}}_n\} = \{\bar{\bar{\mathbf{P}}}_n\}$$

and for $t = 0$, (65)

$$[\mathbf{B}]\{\bar{\mathbf{P}}_0\} = \{\bar{\bar{\mathbf{P}}}_0\}$$

Substituting now Eqs. (64) into Eq. (62) and using Eq. (65) gives the vector of the known forces applied to the system nodal points,

$$\{\overline{\overline{\mathbf{P}}}_n\} = [\mathbf{B}]\,[\mathbf{A}]\,[\mathbf{K}_{n,n-1}]\,[\mathbf{A}]^T\,[\mathbf{B}]^T\,\{\Delta\overline{\overline{\mathbf{U}}}_n\} + \sum_{m=1}^{m=n-1}[\mathbf{B}]\,[\mathbf{A}]\,[\mathbf{K}_{n,m-1}]\,[\mathbf{A}]^T\,[\mathbf{B}]^T\,\{\Delta\overline{\overline{\mathbf{U}}}_m\}$$

$$+ [\mathbf{B}]\,[\mathbf{A}]\,[\mathbf{K}_n]\,[\mathbf{A}]^T\,[\mathbf{B}]^T\,\{\overline{\overline{\mathbf{U}}}_0\} - \sum_{m=1}^{m=n}[\mathbf{B}]\,[\mathbf{A}]\,\{\mathbf{T}_{n,m-1}\} - [\mathbf{B}]\,[\mathbf{A}]\,\{\mathbf{F}_n\} \qquad (66a)$$

for $n = 1, 2, 3, \cdots$.

Also,

$$\{\overline{\overline{\mathbf{P}}}_0\} = [\mathbf{B}]\,[\mathbf{A}]\,[\mathbf{K}_0]\,[\mathbf{A}]^T\,[\mathbf{B}]^T\,\{\overline{\overline{\mathbf{U}}}_0\} - [\mathbf{B}]\,[\mathbf{A}]\,\{\mathbf{F}_0\} \qquad (66b)$$

for $n = 0$, i.e., at $t = 0$.

Equation (66) can be solved for the unknown incremental system nodal displacements at $t_{(n)}$,

$$\{\Delta\overline{\overline{\mathbf{U}}}_n\} = \left[[\mathbf{B}]\,[\mathbf{A}]\,[\mathbf{K}_{n,n-1}]\,[\mathbf{A}]^T\,[\mathbf{B}]^T\right]^{-1}\left\{\{\overline{\overline{\mathbf{P}}}_n\} - \sum_{m=1}^{m=n-1}[\mathbf{B}]\,[\mathbf{A}]\,[\mathbf{K}_{n,m-1}]\,[\mathbf{A}]^T\,[\mathbf{B}]^T\,\{\Delta\overline{\overline{\mathbf{U}}}_m\}\right.$$

$$\left. - [\mathbf{B}]\,[\mathbf{A}]\,[\mathbf{K}_n]\,[\mathbf{A}]^T\,[\mathbf{B}]^T\,\{\overline{\overline{\mathbf{U}}}_0\} + \sum_{m=1}^{m=n}[\mathbf{B}]\,[\mathbf{A}]\,\{\mathbf{T}_{n,m-1}\} + [\mathbf{B}]\,[\mathbf{A}]\,\{\mathbf{F}_n\}\right\} \qquad (67)$$

The displacements at $t = 0$ follow from Eq. (67) as,

$$\{\overline{\overline{\mathbf{U}}}_0\} = \left[[\mathbf{B}]\,[\mathbf{A}]\,[\mathbf{K}_0]\,[\mathbf{A}]^T\,[\mathbf{B}]^T\right]^{-1}\left\{\{\overline{\overline{\mathbf{P}}}_0\} + [\mathbf{B}]\,[\mathbf{A}]\,\{\mathbf{F}_0\}\right\} \qquad (68)$$

and the following terms for $t = t_{(1)}, t_{(2)}, t_{(3)}, \cdots$ are, from Eq. (67),

$$\{\Delta\overline{\overline{\mathbf{U}}}_1\} = \left[[\mathbf{B}]\,[\mathbf{A}]\,[\mathbf{K}_{1,0}]\,[\mathbf{A}]^T\,[\mathbf{B}]^T\right]^{-1}$$

$$\times\left\{\{\overline{\overline{\mathbf{P}}}_1\} - [\mathbf{B}]\,[\mathbf{A}]\,[\mathbf{K}_1]\,[\mathbf{A}]^T\,[\mathbf{B}]^T\,\{\overline{\overline{\mathbf{U}}}_0\}\right.$$

$$\left. + [\mathbf{B}]\,[\mathbf{A}]\,\{\mathbf{T}_{1,0}\} + [\mathbf{B}]\,[\mathbf{A}]\,\{\mathbf{F}_1\}\right\} \qquad (69)$$

$$\{\Delta\overline{\overline{\mathbf{U}}}_2\} = \left[[\mathbf{B}]\,[\mathbf{A}]\,[\mathbf{K}_{2,1}]\,[\mathbf{A}]^T\,[\mathbf{B}]^T\right]^{-1}$$

$$\times\left\{\{\overline{\overline{\mathbf{P}}}_2\} - [\mathbf{B}]\,[\mathbf{A}]\,[\mathbf{K}_{2,0}]\,[\mathbf{A}]^T\,[\mathbf{B}]^T\,\{\Delta\overline{\overline{\mathbf{U}}}_1\}\right.$$

$$- [\mathbf{B}]\,[\mathbf{A}]\,[\mathbf{K}_2]\,[\mathbf{A}]^T\,[\mathbf{B}]^T\,\{\overline{\overline{\mathbf{U}}}_0\}$$

$$+ [\mathbf{B}]\,[\mathbf{A}]\,\{\mathbf{T}_{2,1}\} + [\mathbf{B}]\,[\mathbf{A}]\,\{\mathbf{T}_{2,0}\}$$

$$\left. + [\mathbf{B}]\,[\mathbf{A}]\,\{\mathbf{F}_2\}\right\} \qquad (70)$$

$$\{\Delta\overline{\overline{\mathbf{U}}}_3\} = \left[[\mathbf{B}]\,[\mathbf{A}]\,[\mathbf{K}_{3,2}]\,[\mathbf{A}]^T\,[\mathbf{B}]^T\right]^{-1}$$

$$\times\left\{\{\overline{\overline{\mathbf{P}}}_3\} - [\mathbf{B}]\,[\mathbf{A}]\,[\mathbf{K}_{3,1}]\,[\mathbf{A}]^T\,[\mathbf{B}]^T\,\{\Delta\overline{\overline{\mathbf{U}}}_2\}\right.$$

$$- [\mathbf{B}]\,[\mathbf{A}]\,[\mathbf{K}_{3,0}]\,[\mathbf{A}]^T\,[\mathbf{B}]^T\,\{\Delta\overline{\overline{\mathbf{U}}}_1\}$$

$$- [\mathbf{B}]\,[\mathbf{A}]\,[\mathbf{K}_3]\,[\mathbf{A}]^T\,[\mathbf{B}]^T\,\{\overline{\overline{\mathbf{U}}}_0\}$$

$$+ [\mathbf{B}]\,[\mathbf{A}]\,\{\mathbf{T}_{3,2}\} + [\mathbf{B}]\,[\mathbf{A}]\,\{\mathbf{T}_{3,1}\}$$

$$\left. + [\mathbf{B}]\,[\mathbf{A}]\,\{\mathbf{T}_{3,0}\} + [\mathbf{B}]\,[\mathbf{A}]\,\{\mathbf{F}_3\}\right\} \qquad (71)$$

Equation (67) is written in the form of a recursion equation, where the incremental displacements of the system at time $t_{(n)}$ are expressed in terms of all the previous incremental displacements, the applied loads, and the applied thermal field. Using Eq. (67), it is necessary at each incremental time-step to go back and start the computation af time $t = 0$, as is evident from the summation terms. This usually leads to excessive computational work if results for relatively long durations of time are sought. Such computational work can be reduced in many cases, however, if certain assumptions are made for the material property matrix [E], as will be shown in the next section.

After the displacement increments $\{\Delta\bar{\bar{\mathbf{U}}}_n\}$ and $\{\bar{\bar{\mathbf{U}}}_0\}$ are determined, the element nodal displacements $\{\Delta\mathbf{U}_n\}$ and $\{\mathbf{U}_0\}$ can be computed using Eqs. (64) and (61),

$$\left.\begin{aligned}\{\Delta\mathbf{U}_n\} &= [\overset{\mathrm{T}}{\mathbf{A}}]\,[\overset{\mathrm{T}}{\mathbf{B}}]\,\{\Delta\bar{\bar{\mathbf{U}}}_n\}\\ &\text{and for } t=0,\\ \{\mathbf{U}_0\} &= [\overset{\mathrm{T}}{\mathbf{A}}]\,[\overset{\mathrm{T}}{\mathbf{B}}]\,\{\Delta\bar{\bar{\mathbf{U}}}_0\}\end{aligned}\right\} \qquad (72)$$

To obtain the nodal displacements for the Ith element only, Eq. (72) is premultiplied by a rectangular matrix $[\mathbf{I}^I]$ which contains only unit or zero elements and which selects only the Ith element nodal displacements. Thus,

$$\left.\begin{aligned}\{\Delta\mathbf{U}_n^I\} &= [\mathbf{I}^I]\,[\overset{\mathrm{T}}{\mathbf{A}}]\,[\overset{\mathrm{T}}{\mathbf{B}}]\,\{\Delta\bar{\bar{\mathbf{U}}}_n\}\\ &\text{and for } t=0,\\ \{\mathbf{U}_0^I\} &= [\mathbf{I}^I]\,[\overset{\mathrm{T}}{\mathbf{A}}]\,[\overset{\mathrm{T}}{\mathbf{B}}]\,\{\bar{\bar{\mathbf{U}}}_0\}\end{aligned}\right\} \qquad (73)$$

Substituting now Eqs. (73) into Eqs. (47) and (48) gives the incremental strains in the Ith element at time $t_{(n)}$,

$$\left.\begin{aligned}\{\Delta\epsilon_n^I\} &= [\psi^I]\,[\boldsymbol{\Phi}^I]^{-1}\,[\mathbf{I}^I]\,[\overset{\mathrm{T}}{\mathbf{A}}]\,[\overset{\mathrm{T}}{\mathbf{B}}]\,\{\Delta\bar{\bar{\mathbf{U}}}_n\}\\ &\text{and at time } t=0,\\ \{\epsilon_0^I\} &= [\psi^I]\,[\boldsymbol{\Phi}^I]^{-1}\,[\mathbf{I}^I]\,[\overset{\mathrm{T}}{\mathbf{A}}]\,[\overset{\mathrm{T}}{\mathbf{B}}]\,\{\bar{\bar{\mathbf{U}}}_0\}\end{aligned}\right\} \qquad (74)$$

The stresses in the Ith element at $t = t_{(n)}$ are then obtained from Eq. (38) at time $t_{(n)}$,

$$\{\sigma_n^I\} = [\mathbf{E}^I(\xi_n)]\,[\psi^I]\,[\boldsymbol{\Phi}^I]^{-1}\,[\mathbf{I}^I]\,[\overset{\mathrm{T}}{\mathbf{A}}]\,[\overset{\mathrm{T}}{\mathbf{B}}]\,\{\bar{\bar{\mathbf{U}}}_0\} + \sum_{m=1}^{m=n}[\mathbf{E}^I(\xi_n-\xi_{m-1})]\left\{[\psi^I]\,[\boldsymbol{\Phi}^I]^{-1}\,[\mathbf{I}^I]\,[\overset{\mathrm{T}}{\mathbf{A}}]\,[\overset{\mathrm{T}}{\mathbf{B}}]\,\{\Delta\bar{\bar{\mathbf{U}}}_m\} - \{\Delta(\alpha\theta)_m^I\}\right\} \qquad (75)$$

and at time $t = 0$,

$$\{\sigma_0^I\} = [\mathbf{E}^I(0)]\,[\psi^I]\,[\boldsymbol{\Phi}^I]^{-1}\,[\mathbf{I}^I]\,[\overset{\mathrm{T}}{\mathbf{A}}]\,[\overset{\mathrm{T}}{\mathbf{B}}]\,\{\bar{\bar{\mathbf{U}}}_0\} \qquad (76)$$

Equations (75) and (76) provide the complete solution of the problem. Equation (76) is the elastic stress solution if the matrix $[\mathbf{E}^I(0)]$ is assumed to be the elastic material property matrix. With this assumption, the second of Eqs. (74) is the elastic strain solution and the second of Eqs. (73) the elastic displacement solution.

V. Viscoelastic Property Functions

It has been mentioned that the solution of viscoelastic problems as expressed by Eqs. (67) and (68) often requires extensive computational efforts because of the recurring summations starting at time $t = 0$ for each incremental time-step. In linear viscoelasticity it is now possible, however, to represent the components of the material property matrix in Eqs. (51), (52), and (53) by an exponential series of the following form:

$$\left.\begin{aligned}[\mathbf{E}(\xi_n-\xi_{m-1})] &= [\mathbf{E}_0] + \sum_{r=1}^{r=s}[\mathbf{E}_r]\exp\{-(\xi_n-\xi_{m-1})/\tau_r\}\\ &\text{and}\\ [\mathbf{E}(\xi_n)] &= [\mathbf{E}_0] + \sum_{r=1}^{r=s}[\mathbf{E}_r]\exp\{-\xi_n/\tau_r\}\end{aligned}\right\} \qquad (77)$$

In these expansions the coefficients $[\mathbf{E}_r]$ and the characteristic relaxation times τ_r are chosen so that experimental data or a particular discrete linear viscoelastic model are represented with sufficient accuracy.

Assuming that within an elementary domain, e.g., D_I, the thermal field is independent of the spacial coordinates, the first, second, third, and fourth term in Eq. (55) can then be represented, using Eqs. (51), (52), and (53), respectively, in the following form:

$$[\mathbf{K}_{n,n-1}^I]\,\{\Delta\mathbf{U}_n^I\} = \int_{D_I}[\overset{\mathrm{T}}{\boldsymbol{\Phi}^I}]^{-1}\,[\overset{\mathrm{T}}{\psi^I}]\,[\mathbf{E}_0^I]\,[\psi^I]\,[\boldsymbol{\Phi}^I]^{-1}\,dD_I\,\{\Delta\mathbf{U}_n^I\} + \sum_{r=1}^{r=s}\int_{D_I}[\overset{\mathrm{T}}{\boldsymbol{\Phi}^I}]^{-1}\,[\overset{\mathrm{T}}{\psi^I}]\,[\mathbf{E}_r^I]\,[\psi^I]\,[\boldsymbol{\Phi}^I]^{-1}\,dD_I\,\{\Delta\mathbf{U}_n^I\}\exp(-(\xi_n-\xi_{n-1})/\tau_r) \qquad (78)$$

$$\sum_{m=1}^{m=n-1} [\mathbf{K}^I_{n,m-1}]\{\Delta\mathbf{U}^I_m\} = \int_{D_I} [\Phi^I]^{-1^{T}} [\psi^I]^{T} [\mathbf{E}^I_0][\psi^I][\Phi^I]^{-1} d\mathrm{D}_I \sum_{m=1}^{m=n-1} \{\Delta\mathbf{U}^I_m\}$$

$$+ \sum_{r=1}^{r=s} \int_{D_I} [\Phi^I]^{-1^{T}} [\psi^I]^{T} [\mathbf{E}^I_r][\psi^I][\Phi^I]^{-1} d\mathrm{D}_I \sum_{m=1}^{m=n-1} \{\Delta\mathbf{U}^I_m\} \exp(-(\xi_n - \xi_{m-1})/\tau_r) \quad (79)$$

$$[\mathbf{K}^I_n]\{\mathbf{U}^I_0\} = \int_{D_I} [\Phi^I]^{-1^{T}} [\psi^I]^{T} [\mathbf{E}^I_0][\psi^I][\Phi^I]^{-1} d\mathrm{D}_i \{\mathbf{U}^I_0\}$$

$$+ \sum_{r=1}^{r=s} \int_{D_I} [\Phi^I]^{-1^{T}} [\psi^I]^{T} [\mathbf{E}^I_r][\psi^I][\Phi^I]^{-1} d\mathrm{D}_I \{\mathbf{U}^I_0\} \exp(-\xi_n/\tau_r) \quad (80)$$

$$\sum_{m=1}^{m=n} \{\mathbf{T}^I_{n,m-1}\} = \int_{D_I} [\Phi^I]^{-1^{T}} [\psi^I]^{T} [\mathbf{E}^I_0] d\mathrm{D}_I \sum_{m=1}^{m=n} \{\Delta(\alpha\theta)^I_m\}$$

$$+ \sum_{r=1}^{r=s} \int_{D_I} [\Phi^I]^{-1^{T}} [\psi^I]^{T} [\mathbf{E}^I_r] d\mathrm{D}_I \sum_{m=1}^{m=n} \{\Delta(\alpha\theta)^I_m\} \exp(-(\xi_n - \xi_{m-1})/\tau_r) \quad (81)$$

If one defines

$$\{\mathbf{W}^I_{n-1,r}\} = \sum_{m=1}^{m=n-1} \{\Delta\mathbf{U}^I_m\} \exp(-(\xi_n - \xi_{m-1})/\tau_r) \quad (82)$$

then one can write

$$\{\mathbf{W}^I_{n-1,r}\} = \{\Delta\mathbf{U}^I_{n-1}\} \exp(-(\xi_n - \xi_{n-2})/\tau_r) + \exp(-(\xi_n - \xi_{n-1})/\tau_r) \sum_{m=1}^{m=n-2} \{\Delta\mathbf{U}^I_m\} \times \exp(-(\xi_{n-1} - \xi_{m-1})/\tau_r) \quad (83)$$

which yields the recursion equation

$$\{\mathbf{W}^I_{n-1,r}\} = \{\Delta\mathbf{U}^I_{n-1}\} \exp(-(\xi_n - \xi_{n-2})/\tau_r) + \exp(-(\xi_n - \xi_{n-1})/\tau_r)\{\mathbf{W}^I_{n-2,r}\} \quad (84)$$

Similarly, if one defines

$$\{\Theta^I_{n,r}\} = \sum_{m=1}^{m=n} \{\Delta(\alpha\theta)^I_m\} \exp(-(\xi_n - \xi_{m-1})/\tau_r) \quad (85)$$

then one obtains the following recursion equation:

$$\{\Theta^I_{n,r}\} = \{\Delta(\alpha\theta)^I_n\} \exp(-(\xi_n - \xi_{n-1})/\tau_r) + \exp(-(\xi_n - \xi_{n-1})/\tau_r)\{\Theta^I_{n-1,r}\} \quad (86)$$

When one recognizes that

$$\sum_{m=1}^{m=n-1} \{\Delta\mathbf{U}^I_m\} = \{\{\mathbf{U}^I_{n-1}\} - \{\mathbf{U}^I_0\}\} \quad (87)$$

$$\sum_{m=1}^{m=n} \{\Delta(\alpha\theta)^I_m\} = \{(\alpha\theta)^I_n\} \quad (88)$$

and defines

$$[\mathbf{k}^I_0] = \int_{D_I} [\Phi^I]^{-1^{T}} [\psi^I]^{T} [\mathbf{E}^I_0][\psi^I][\Phi^I]^{-1} d\mathrm{D}_I \quad (89)$$

$$[\mathbf{k}^I_r] = \int_{D_I} [\Phi^I]^{-1^{T}} [\psi^I]^{T} [\mathbf{E}^I_r][\psi^I][\Phi^I]^{-1} d\mathrm{D}_I \quad (90)$$

$$[\mathbf{d}^I_0] = \int_{D_I} [\Phi^I]^{-1^{T}} [\psi^I]^{T} [\mathbf{E}^I_0] d\mathrm{D}_I \quad (91)$$

$$[\mathbf{d}^I_r] = \int_{D_I} [\Phi^I]^{-1^{T}} [\psi^I]^{T} [\mathbf{E}^I_r] d\mathrm{D}_I \quad (92)$$

then Eqs. (78) to (81) become, respectively,

$$[\mathbf{K}^I_{n,n-1}]\{\Delta\mathbf{U}^I_n\} = [\mathbf{k}^I_0]\{\Delta\mathbf{U}^I_n\} + \sum_{r=1}^{r=s} [\mathbf{k}^I_r]\{\Delta\mathbf{U}^I_n\} \exp(-(\xi_n - \xi_{n-1})/\tau_r) \quad (93)$$

$$\sum_{m=1}^{m=n-1} [\mathbf{K}^I_{n,m-1}]\{\Delta\mathbf{U}^I_m\} = [\mathbf{k}^I_0]\{\{\mathbf{U}^I_{n-1}\} - \{\mathbf{U}^I_0\}\} + \sum_{r=1}^{r=s} [\mathbf{k}^I_r]\{\mathbf{W}^I_{n-1,r}\} \quad (94)$$

$$[\mathbf{K}_n^I]\{\mathbf{U}_0^I\} = [\mathbf{k}_0^I]\{\mathbf{U}_0^I\} + \sum_{r=1}^{r=s}[\mathbf{k}_r^I]\{\mathbf{U}_0^I\}\exp(-\xi_n/\tau_r) \qquad (95)$$

and

$$\sum_{m=1}^{m=n}\{\mathbf{T}_{n,m-1}^I\} = [\mathbf{d}_0^I]\{(\alpha\theta)_n^I\} + \sum_{r=1}^{r=s}[\mathbf{d}_r^I]\{\Theta_{nr}^I\} \qquad (96)$$

When Eqs. (93) to (96) are substituted into Eq. (55), all subsequent developments in the previous sections apply as shown; however, it can be seen from Eqs. (94) to (96) that at each time-step the evaluation of the associated displacement increments requires information from only the immediate previous time-step by virtue of the recursion equations, (84) and (86).

VI. An Illustration

For the detailed determination of the matrices in Sections III and IV, it is well to illustrate here the matrix development steps for a simple but typical two-dimensional example.

A typical triangular element I is shown in Fig. 6. The displacement field in the element is assumed to be a linear function of space. Equation (41) then becomes, in this case,

$$\begin{Bmatrix} u_1^I \\ u_2^I \end{Bmatrix} = \begin{Bmatrix} q_{(1)}^I + x_1^I q_{(2)}^I + x_2^I q_{(3)}^I \\ q_{(4)}^I + x_1^I q_{(5)}^I + x_2^I q_{(6)}^I \end{Bmatrix} \qquad (97)$$

hence the rectangular matrix in Eqs. (42) and (43) is

$$[\phi^I] = \begin{bmatrix} 1 & x_1^I & x_2^I & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_1^I & x_2^I \end{bmatrix} \qquad (98)$$

Equation (43) is then

$$\begin{Bmatrix} \Delta u_{1(n)}^I \\ \Delta u_{2(n)}^I \end{Bmatrix} = \begin{bmatrix} 1 & x_1^I & x_2^I & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_1^I & x_2^I \end{bmatrix} \begin{Bmatrix} \Delta q_{(1)(n)}^I \\ \Delta q_{(2)(n)}^I \\ \Delta q_{(3)(n)}^I \\ \Delta q_{(4)(n)}^I \\ \Delta q_{(5)(n)}^I \\ \Delta q_{(6)(n)}^I \end{Bmatrix} \qquad (99)$$

Introducing successively each nodal coordinate from Fig. 6 into Eq. (99) gives the rectangular matrix in Eq. (44) as

$$[\Phi^I] = \begin{bmatrix} 1 & a_i^I & b_i^I & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & a_i^I & b_i^I \\ 1 & a_j^I & b_j^I & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & a_j^I & b_j^I \\ 1 & a_k^I & b_k^I & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & a_k^I & b_k^I \end{bmatrix} \qquad (100)$$

Similarly, substituting Eq. (99) into Eq. (30) and using Eq. (46) gives the rectangular matrix in Eq. (46) as

$$[\psi^I] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \tfrac{1}{2} & 0 & \tfrac{1}{2} & 0 \end{bmatrix} \qquad (101)$$

The matrices Eqs. (98), (100), and (101), together with the material matrix $[\mathbf{E}^I]$, the thermal matrix $\{\Delta(\alpha\theta)_m^I\}$, and the body force matrix $\{\mathbf{f}_n^I\}$ are sufficient to generate the element matrices Eqs. (51) to (56), hence, Eq. (58).

It is now necessary to specify the particular system (assembly of elements) under consideration and to give the loading and supporting conditions. In the system used here as an illustration (Fig. 7) 3 elements and 5 nodal points are shown as well as the loading and supporting conditions.

The construction of the assembly matrix [A] is accomplished by writing the equilibrium conditions at each nodal point. Thus, at time $t_{(n)}$,

$$\begin{Bmatrix} P_{1in}^{\mathrm{I}} \\ P_{2in}^{\mathrm{I}} \\ P_{1jn}^{\mathrm{I}} + P_{1in}^{\mathrm{II}} + P_{1in}^{\mathrm{III}} \\ P_{2jn}^{\mathrm{I}} + P_{2in}^{\mathrm{II}} + P_{2in}^{\mathrm{III}} \\ P_{1jn}^{\mathrm{III}} \\ P_{2jn}^{\mathrm{III}} \\ P_{1kn}^{\mathrm{III}} + P_{1jn}^{\mathrm{II}} \\ P_{2kn}^{\mathrm{II}} + P_{2jn}^{\mathrm{II}} \\ P_{1kn}^{\mathrm{II}} + P_{1kn}^{\mathrm{I}} \\ P_{2kn}^{\mathrm{II}} + P_{2kn}^{\mathrm{I}} \end{Bmatrix} = \begin{Bmatrix} \bar{P}_{1n}^{1} \\ \bar{P}_{2n}^{1} \\ 0 \\ 0 \\ \bar{P}_{1n}^{3} \\ \bar{P}_{2n}^{3} \\ 0 \\ 0 \\ \bar{P}_{1n}^{5} \\ \bar{P}_{2n}^{5} \end{Bmatrix} \qquad (102)$$

where the supporting forces at nodal points 1 and 3 are unknown while the applied force at nodal point 5 is known. From Eq. (102) the assembly matrix A in Eq. (60) can be constructed in the following form:

$$[\mathbf{A}] = \begin{bmatrix} 1\,0 & 0\,0 & 0\,0 & 0\,0 & 0\,0 & 0\,0 & 0\,0 & 0\,0 & 0\,0 \\ 0\,1 & 0\,0 & 0\,0 & 0\,0 & 0\,0 & 0\,0 & 0\,0 & 0\,0 & 0\,0 \\ 0\,0 & 1\,0 & 0\,0 & 1\,0 & 0\,0 & 0\,0 & 1\,0 & 0\,0 & 0\,0 \\ 0\,0 & 0\,1 & 0\,0 & 0\,1 & 0\,0 & 0\,0 & 0\,1 & 0\,0 & 0\,0 \\ 0\,0 & 0\,0 & 0\,0 & 0\,0 & 0\,0 & 0\,0 & 0\,0 & 1\,0 & 0\,0 \\ 0\,0 & 0\,0 & 0\,0 & 0\,0 & 0\,0 & 0\,0 & 0\,0 & 0\,1 & 0\,0 \\ 0\,0 & 0\,0 & 1\,0 & 0\,0 & 0\,0 & 1\,0 & 0\,0 & 0\,0 & 0\,0 \\ 0\,0 & 0\,0 & 0\,1 & 0\,0 & 0\,0 & 0\,1 & 0\,0 & 0\,0 & 0\,0 \\ 0\,0 & 0\,0 & 0\,0 & 0\,0 & 1\,0 & 0\,0 & 0\,0 & 0\,0 & 1\,0 \\ 0\,0 & 0\,0 & 0\,0 & 0\,0 & 0\,1 & 0\,0 & 0\,0 & 0\,0 & 0\,1 \end{bmatrix} \tag{103}$$

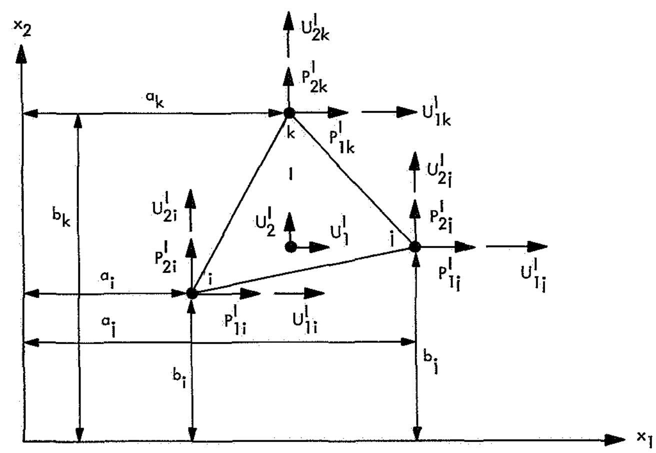

Fig. 6. Typical triangular two-dimensional element

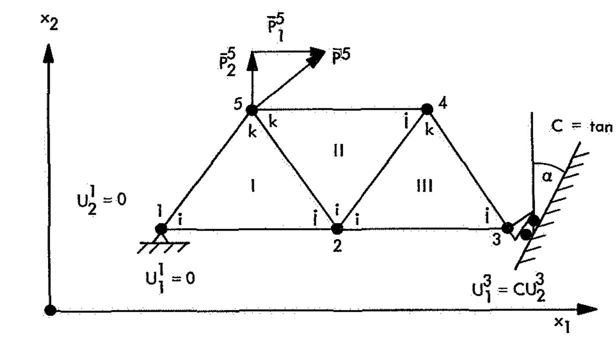

Fig. 7. Example of a two-dimensional continuum consisting of 3 elements with general boundary conditions

All the incremental system nodal displacements at time $t_{(n)}$ can be written in terms of only the unknown system nodal displacements (see Eq. 64) as follows:

$$\{\Delta\bar{\mathbf{U}}_n\} = \begin{Bmatrix} \Delta\bar{U}^1_{1n} \\ \Delta\bar{U}^1_{2n} \\ \Delta\bar{U}^2_{1n} \\ \Delta\bar{U}^2_{2n} \\ \Delta\bar{U}^3_{1n} \\ \Delta\bar{U}^3_{2n} \\ \Delta\bar{U}^4_{1n} \\ \Delta\bar{U}^4_{2n} \\ \Delta\bar{U}^5_{1n} \\ \Delta\bar{U}^5_{2n} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \Delta\bar{\bar{U}}^2_{1n} \\ \Delta\bar{\bar{U}}^2_{2n} \\ C\Delta\bar{\bar{U}}^3_{2n} \\ \Delta\bar{\bar{U}}^3_{2n} \\ \Delta\bar{\bar{U}}^4_{1n} \\ \Delta\bar{\bar{U}}^4_{2n} \\ \Delta\bar{\bar{U}}^5_{1n} \\ \Delta\bar{\bar{U}}^5_{2n} \end{Bmatrix} = [\mathbf{B}]^T \begin{Bmatrix} \Delta\bar{\bar{U}}^2_{1n} \\ \Delta\bar{\bar{U}}^2_{2n} \\ \Delta\bar{\bar{U}}^3_{2n} \\ \Delta\bar{\bar{U}}^4_{1n} \\ \Delta\bar{\bar{U}}^4_{2n} \\ \Delta\bar{\bar{U}}^5_{1n} \\ \Delta\bar{\bar{U}}^5_{2n} \end{Bmatrix} \tag{104}$$

The boundary condition matrix in Eq. (64) follows, then, as

$$[\mathbf{B}] = \begin{bmatrix} 0\,0 & 1\,0 & 0\,0 & 0\,0 & 0\,0 \\ 0\,0 & 0\,1 & 0\,0 & 0\,0 & 0\,0 \\ 0\,0 & 0\,0 & C\,1 & 0\,0 & 0\,0 \\ 0\,0 & 0\,0 & 0\,0 & 1\,0 & 0\,0 \\ 0\,0 & 0\,0 & 0\,0 & 0\,1 & 0\,0 \\ 0\,0 & 0\,0 & 0\,0 & 0\,0 & 1\,0 \\ 0\,0 & 0\,0 & 0\,0 & 0\,0 & 0\,1 \end{bmatrix} \tag{105}$$

The steps of constructing the matrices $[\phi^r]$, $[\Phi^r]$, $[\psi^r]$, [A] and [B] in a problem are basically always the same as illustrated in this simple example. Three-dimensional tetrahedron elements require exactly the same formulation, the only addition being that of one dimension. Extensions to shells with curved elements, elements with intermediate nodal points, cubic elements, etc., are straightforward.

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