Miloslav Feistauer; Jiří Felcman; Zdeněk Vlášek

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FINITE ELEMENT SOLUTION OF FLOWS THROUGH CASCADES OF PROFILES IN A LAYER OF VARIABLE THICKNESS

MILOSLAV FEISTAUER, JIŘÍ FELCMAN, ZDENĚK VLÁŠEK

Dedicated to Professor Jan Polášek on the occasion of his sixtieth birthday

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Summary. The paper is devoted to the numerical modelling of a subsonic irrotational non-viscous flow past a cascade of profiles in a variable thickness fluid layer. It leads to a nonlinear two-dimensional elliptic problem with nonstandard nonhomogeneous boundary conditions. The problem is discretized by the finite element method. Both theoretical and practical questions of the finite element implementation are studied: convergence of the method, numerical integration, iterative methods for the solution of the discrete problem and the algorithmization of the finite element solution. Some numerical results obtained by a multi-purpose program written by authors are presented.

Key words: cascade of profiles, subsonic flow, stream function, nonlinear second-order elliptic problem, variational formulation, weak solution, finite element approximation, numerical integration, iterative solution of a nonlinear algebraic system.

AMS classification: 65 N 30, 76—08, 76 B 05, 76 N 10.

INTRODUCTION

In [5] we studied several boundary value problems for a stream function that describe stationary irrotational non-viscous flows through cascades of blades on an arbitrary surface of revolution in a variable thickness layer. The results were then extended in [14] to cover also quasistationary flows through cascades of rotor blades.

Here we shall deal with the finite element solution of these interesting and topical problems. We describe the discretization process, prove the convergence of the method and deal with some aspects of algorithmization. We also present some numerical results. Our theoretical investigations yield a contribution to the convergence results obtained by a series of authors and treated e.g. in [2] or [23]. The paper represents an extension of these results to nonlinear boundary value problems with nonstandard nonhomogeneous boundary conditions.

1. CONTINUOUS PROBLEM

Let us consider a rotating blade row inserted into an axially symmetric channel. We shall study a flow past this blade row in a layer of variable thickness, i.e. in the space between two close axially symmetric stream surfaces $\mathscr{S}_1$ and $\mathscr{S}_2$.

By introducing convenient coordinates x_1, x_2 on $\mathscr{S}_1$ (cf. [5, 12]), this surface and its intersections with the blades can be conformally transformed into the plane (x_1, x_2), where we get a domain $\Omega \subset \mathbf{R}_2$. The boundary $\partial\Omega$ of Ω is formed by two straight lines

$$K_i = \{(x_1, x_2); x_1 = d_i, x_2 \in R_1\} \quad i = 1, 2, \quad d_1 < d_2 \tag{1.1}$$

and by an infinite number of disjoint simple closed curves C_k, $k = 0, \pm 1, \pm 2, \ldots$, periodically spaced in the direction of x_2 with a period $\tau > 0$ (see Fig. 1). The curves C_k are in the strip $\mathscr{P}$ between the lines K_1 and K_2, and form the so-called *cascade of profiles*, the lines K_1 and K_2 represent the *inlet* and the *outlet* of the cascade, respectively. The domain Ω is periodic in the direction of x_2 with the period τ:

$$(x_1, x_2) \in \bar{\Omega} \Leftrightarrow (x_1, x_2 + \tau) \in \bar{\Omega}\,. \tag{1.2}$$

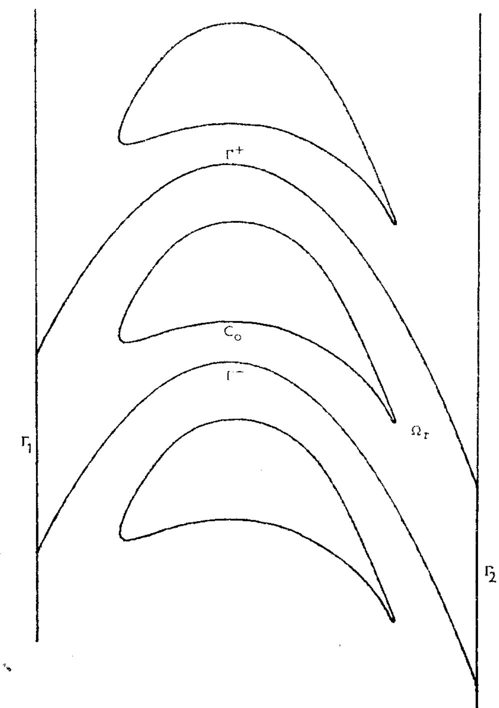

Fig. 1

1.1. Classical formulation

Quasistationary, irrotational, non-viscous, incompressible or subsonic compressible flow field through the cascade of blades in the fluid layer between the surfaces $\mathscr{S}_1$, $\mathscr{S}_2$ is modelled via the stream function by the equation of the form

$$\sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left(b(x, (\nabla\psi)^2) \frac{\partial\psi}{\partial x_i}\right) = \omega \frac{\partial r^2}{\partial x_1}(x), \tag{1.3}$$

considered in the domain Ω. $\omega \in \mathbf{R}_1$ denotes the angular velocity of the rotating blade row. The function b is given by the dependence of the density on the velocity. Moreover, both b and r depend on the geometry of the fluid layer. (Cf. [10, 14].)

Similarly as in [5] we meet several types of boundary conditions.

First we assume that the velocity field is also τ-periodic, which means

$$\psi(x_1, x_2 + \tau) = \psi(x_1, x_2) + Q, \quad (x_1, x_2) \in \bar{\Omega} \tag{1.4}$$

(periodicity condition) with $Q \in R_1$ given.

On the inlet or outlet we often consider the conditions

$$\psi(d_i, x_2) = \Psi_i(x_2) + q_i, \quad x_2 \in \mathbf{R}_1, \tag{1.5}$$

where Ψ_i are given by

$$\Psi_i(x_2) = \int_0^{x_2} \varphi_i(\xi)\, d\xi, \quad x_2 \in \mathbf{R}_1, \quad i = 1, 2. \tag{1.6}$$

The functions φ_i are τ-periodic in $\mathbf{R}_1$ and

$$Q = \int_0^{\tau} \varphi_1(\xi)\, d\xi = \int_0^{\tau} \varphi_2(\xi)\, d\xi. \tag{1.7}$$

Hence,

$$\Psi_i(x_2 + \tau) = \Psi_i(x_2) + Q, \quad x_2 \in \mathbf{R}_1, \quad i = 1, 2. \tag{1.8}$$

$q_i \in \mathbf{R}_1$ can be unknown.

As another possibility we use the condition

$$\left[b(\cdot, (\nabla\psi)^2) \frac{\partial\psi}{\partial n}\right](d_i, x_2) = -m_i(x_2) + (-1)^i\, \omega r^2(d_i, x_2), \tag{1.9}$$

$$x_2 \in \mathbf{R}_1, \quad i = 1 \quad \text{or} \quad i = 2;$$

m_i: $\mathbf{R}_1 \to \mathbf{R}_1$ is a given τ-periodic function. Finally,

$$\frac{1}{\tau}\int_{x_2}^{x_2+\tau} \left[b(\cdot, (\nabla\psi)^2) \frac{\partial\psi}{\partial n}\right](d_i, \xi)\, d\xi = -\bar{\mu}_i + (-1)^i\, \omega r^2(d_i, x_2), \tag{1.10}$$

$$x_2 \in \mathbf{R}_1, \quad i = 1 \quad \text{or} \quad i = 2.$$

$\bar{\mu}_i \in \mathbf{R}_1$ is a given constant. Here $\partial/\partial n$ denotes the derivative with respect to the unit outer normal $\boldsymbol{n} = (n_1, n_2)$ to $\partial\Omega$.

On the profiles we have

$$(1.11) \qquad \psi \mid C_k = q_0 + kQ\,, \quad k = 0, \pm 1, \pm 2, \ldots$$

($q_0 \in \mathbf{R}_1$ can be unknown), and either

$$(1.12) \qquad \int_{C_k} b(\cdot, (\nabla\psi)^2) \frac{\partial \psi}{\partial n} \,\mathrm{d}S = -\gamma + \omega \int_{C_k} r^2 n_1 \,\mathrm{d}S\,, \quad k = 0, \pm 1, \pm 2, \ldots$$

with $\gamma \in \mathbf{R}_1$ given, or

$$(1.13) \qquad \frac{\partial \psi}{\partial n}(z_k) = 0\,, \quad k = 0, \pm 1, \pm 2\,, \ldots.$$

Here, $z_k = z_0 + (0, k\tau) \in C_k$ are given trailing stagnation points. For the explanation of the physical meaning of these conditions, see [5, 10, 12, 14, 15].

From the point of view of technical practice it is convenient to consider the classical formulation of the following boundary value problems:

I) Let τ-periodic functions φ_1, φ_2: $\mathbf{R}_1 \to \mathbf{R}_1$ satisfying (1.7) be given, let Q and Ψ_1, Ψ_2 satisfy (1.6) and (1.8).

Problem (PSIω.1.1). For given constants $\bar{\mu}_1, \bar{\mu}_2 \in \mathbf{R}_1$ find $\psi \in C^2(\bar{\Omega})$ and constants q_1, q_2 satisfying the equation (1.3) in Ω and the conditions a) (1.4), b) (1.11) with $q_0 = 0$, c) (1.5) and (1.10) for $i = 1, 2$.

Problem (PSIω.1.2). Given $\bar{\mu}_1, \gamma \in \mathbf{R}_1$, find $\psi \in C^2(\bar{\Omega})$ and constants q_0, q_1 satisfying the equation (1.3) in Ω and the conditions a) (1.4), b) (1.11) (with q_0 unknown), c) (1.5) for $i = 1, 2$ with q_1 unknown and $q_2 = 0$, d) (1.10) for $i = 1$, e) (1.12).

Problem (PSI.ω.1.3). Given $\bar{\mu}_1 \in \mathbf{R}_1$ and trailing stagnation points $z_k = z_0 + (0, k\tau) \in C_k$, find $\psi \in C^2(\bar{\Omega})$ and constants q_0, q_1 satisfying (1.3) in Ω and a) (1.4), b) (1.11) (with q_0 unknown), c) (1.5) for $i = 1, 2$ with q_1 unknown and $q_2 = 0$, d) (1.10) for $i = 1$, e) (1.13).

II) Let a constant $Q \in \mathbf{R}_1$ and τ-periodic functions m_1, m_2: $\mathbf{R}_1 \to \mathbf{R}_1$ be given.

Problem (PSIω.2.1). Find $\psi \in C^2(\bar{\Omega})$ satisfying the equation (1.3) in Ω and the conditions a) (1.4), b) (1.11) with $q_0 = 0$ and c) (1.9) for $i = 1, 2$.

At the end of this section, let us describe the mathematical properties of the functions b and r.

1.1.1 Properties of b **and** r. 1) The function $b = b(x, \eta)$ $(x \in \mathscr{P} = \{(x_1, x_2);\ x_1 \in (d_1, d_2),\ x_2 \in \mathbf{R}_1\},\ \eta \geq 0)$ is continuous in $\overline{\mathscr{P}} \times \langle 0, +\infty)$, r and $\partial r/\partial x_1$ are continuous in $\overline{\mathscr{P}}$. The function b has continuous derivatives $\partial b/\partial \eta$ and $\partial b/\partial x_i$, $i = 1, 2$, in $\overline{\mathscr{P}} \times \langle 0, +\infty)$.

2) There exist positive constants c_1, c_2, c_3, c_4 such that

$$(1.14) \qquad c_1 \leqq b \leqq c_2 \quad \text{in} \quad \overline{\mathscr{P}} \times \langle 0, +\infty),$$

$$(1.15) \qquad 0 \leqq \frac{\partial b}{\partial \eta} \leqq c_3\,, \quad \left|\frac{\partial b}{\partial x_i}\right| \leqq c_3 \quad (i = 1, 2) \quad \text{in} \quad \overline{\mathscr{P}} \times \langle 0, +\infty)\,,$$

$$(1.16) \qquad \left|\xi \frac{\partial b}{\partial \eta}(x, \xi^2)\right| , \quad \left|\xi^2 \frac{\partial b}{\partial \eta}(x, \xi^2)\right| \leqq c_4$$

$$\forall x \in \overline{\mathscr{P}} , \quad \forall \xi \in \mathbf{R}_1 .$$

3) If $\alpha_1 \in \mathbf{R}_1$, $x \in \overline{\mathscr{P}}$, then the function $\xi b(x, \alpha_1^2 + \xi^2)$ of the variable ξ is increasing in $\mathbf{R}_1$.

4) b and r are τ-periodic in the direction of x_2:

$$(1.17) \qquad b(x_1, x_2 + \tau, \eta) = b(x_1, x_2, \eta) ,$$

$$r(x_1, x_2 + \tau) = r(x_1, x_2) , \quad \forall (x_1, x_2) \in \overline{\mathscr{P}} , \quad \forall \eta \geqq 0 .$$

(See e.g. [5] or [15].) ∎

Similarly as in [5, 8, 15], using the Mean Value Theorem, we can prove

1.1.2. Lemma. *Let $x \in \mathscr{P}$, ξ, $\tilde{\xi}$, $\vartheta \in \mathbf{R}_2$. Then*

$$(1.18) \qquad [b(x, \tilde{\xi}^2)\, \tilde{\xi} - b(x, \xi^2)\, \xi] \, . \, (\tilde{\xi} - \xi) \geqq c_1(\tilde{\xi} - \xi)^2$$

and

$$(1.19) \qquad |[b(x, \tilde{\xi}^2)\, \tilde{\xi} - b(x, \xi^2)\, \xi] \, . \, \vartheta| \leqq K|\tilde{\xi} - \xi| \, |\vartheta|$$

with $K = c_2 + 2c_4$, where c_1, c_2, c_4 are the constants from 1.1.1. ∎

1.2. Variational formulation and weak solution

Let $\Omega_\tau \subset \Omega$ be a curved strip of a width τ in the x_2-direction cut from the domain Ω. Its boundary $\partial\Omega_\tau$ consists of two components – the profile C_0 (inner component) and the union $\Gamma_1 \cup \Gamma_2 \cup \Gamma^- \cup \Gamma^+$ (outer component), where $\Gamma_i \subset K_i$ is a segment of the length τ, Γ^- is a piecewise linear arc and $\Gamma^+ = \{(x_1, x_2 + \tau); (x_1, x_2) \in \Gamma^-\}$. The initial points of Γ^- and Γ^+ lie on K_1, their terminal points lie on K_2 and all the other points are elements of Ω. See Fig. 1.

Let us assume that the profile C_0 is "sufficiently smooth" so that $\partial\Omega_\tau$ is Lipschitz-continuous and it is possible to define one-dimensional Lebesgue measure on $\partial\Omega_\tau$ (see [20]). By Ω_τ^* we denote the bounded domain with $\partial\Omega_\tau^* = \Gamma_1 \cup \Gamma_2 \cup \Gamma^- \cup \Gamma^+$.

Let $\psi \in C^2(\overline{\Omega})$ be a solution of the equation (1.3). Let us multiply this equation by an arbitrary function $v \in C^\infty(\overline{\Omega}_\tau)$ and integrate over the domain Ω_τ. If we use Green's theorem, we get

$$(1.20) \quad \int_{\partial\Omega_\tau} b \frac{\partial\psi}{\partial n} v \, dS - \int_{\Omega_\tau} b \, \nabla\psi \, . \, \nabla v \, dx = \omega \int_{\partial\Omega_\tau} r^2 n_1 v \, dS - \omega \int_{\Omega_\tau} r^2 \frac{\partial v}{\partial x_1} \, dx .$$

By a suitable choice of test functions v we get variational formulations of the problems formulated in Section 1.1.

In what follows we shall work with the well-known Hilbert spaces $L_2((0, \tau))$, $L_2(\Omega_\tau)$, $H^1(\Omega_\tau)$, $H_0^1(\Omega_\tau)$, $H^2(\Omega_\tau)$, $L_2(\partial\Omega_\tau)$ and $H^{3/2}(\Omega_\tau) = W_2^{3/2}(\Omega_\tau)$ (see e.g. [20, 18, 16, 2] and also [5]). Let us put

(1.21)
$$\|v\|_{0,\partial\Omega_\tau} = \left(\int_{\partial\Omega_\tau} v^2 \, dS\right)^{1/2}, \quad v \in L_2(\partial\Omega_\tau),$$
$$\|v\|_{0,\Omega_\tau} = \left(\int_{\Omega_\tau} v^2 \, dx\right)^{1/2}, \quad v \in L_2(\Omega_\tau),$$
$$|v|_{1,\Omega_\tau} = \left(\int_{\Omega_\tau} (\nabla v)^2 \, dx\right)^{1/2}, \quad v \in H^1(\Omega_\tau),$$
$$\|v\|_{1,\Omega_\tau} = (\|v\|^2_{0,\Omega_\tau} + |v|^2_{1,\Omega_\tau})^{1/2}, \quad v \in H^1(\Omega_\tau),$$
$$\|v\|_{3/2,\Omega_\tau} = \left(\|v\|^2_{1,\Omega_\tau} + \sum_{i=1}^{2} \int_{\Omega_\tau}\int_{\Omega_\tau} \frac{\left|\frac{\partial v}{\partial x_i}(x) - \frac{\partial v}{\partial x_i}(y)\right|^2}{|x-y|^3} \, dx \, dy\right)^{1/2}, \quad v \in H^{3/2}(\Omega_\tau),$$
$$|v|_{2,\Omega_\tau} = \left(\int_{\Omega_\tau} \sum_{i,j=1}^{2} \left(\frac{\partial^2 v}{\partial x_i \, \partial x_j}\right)^2 dx\right)^{1/2}, \quad v \in H^2(\Omega_\tau),$$
$$\|v\|_{2,\Omega_\tau} = (\|v\|^2_{0,\Omega_\tau} + |v|^2_{1,\Omega_\tau} + |v|^2_{2,\Omega_\tau})^{1/2}, \quad v \in H^2(\Omega_\tau).$$

It is known that $\|\cdot\|_{0,\partial\Omega_\tau}$, $\|\cdot\|_{0,\Omega_\tau}$, $\|\cdot\|_{1,\Omega_\tau}$, $\|\cdot\|_{3/2,\Omega_\tau}$ and $\|\cdot\|_{2,\Omega_\tau}$ are norms in the spaces $L_2(\partial\Omega_\tau)$, $L_2(\Omega_\tau)$, $H^1(\Omega_\tau)$, $H^{3/2}(\Omega_\tau)$ and $H^2(\Omega_\tau)$, respectively. $|\cdot|_{1,\Omega_\tau}$ is a seminorm in $H^1(\Omega_\tau)$ and a norm in $H^1_0(\Omega_\tau)$, equivalent to the norm $\|\cdot\|_{1,\Omega_\tau}$. Similar spaces will also be considered over other domains with Lipschitz-continuous boundaries.

For simplicity, let us denote

(1.22)
$$W = H^1(\Omega_\tau)$$

and define the form $a\colon W \times W \to \mathbf{R}_1$:

(1.23)
$$a(\psi, v) = \int_{\Omega_\tau} b(\cdot, (\nabla\psi)^2) \, \nabla\psi \, . \, \nabla v \, dx, \quad \psi, v \in W,$$

linear with respect to v. Following the results from [5] or [15], we can show that the problems (PSIω.1.1), (PSIω.1.2) ant (PSIω.2.1) are formally equivalent to the following *variational weak formulation*: Find $\psi\colon \bar{\Omega}_\tau \to \mathbf{R}_1$ such that

(1.24)
$$\text{a)} \ \psi \in W, \qquad \text{b)} \ \psi - \psi^* \in V,$$
$$\text{c)} \ a(\psi, v) = \mu(v) \quad \forall v \in V.$$

Here, $V \subset W$ is a convenient closed subspace of $W = H^1(\Omega_\tau)$. μ is a continuous linear functional defined on the space V (i.e., $\mu \in V^*$, where V^* denotes the dual to V) and $\psi^* \in W$ is a suitable function. The function ψ with the properties (1.24,a–c) is called a *weak solution of the problem.*

In what follows we shall specify V, μ and ψ^* for the problems (PSIω.1.1), (PSIω.1.2) and (PSIω.2.1):

1.2.1. Problem (PSIω.1.1). We put

(1.25)
$$\mathscr{V} = \{v \in C^\infty(\bar{\Omega}_\tau);\ v \mid \Gamma_i = \text{const}, \quad i = 1, 2,$$
$$v \mid C_0 = 0,\ v(x_1, x_2 + \tau) = v(x_1, x_2) \ \forall (x_1, x_2) \in \Gamma^-\},$$

(1.26) $$V = \{v \in W;\ v \mid \Gamma_i = \text{const},\ i = 1, 2,\ v \mid C_0 = 0,\ v(x_1, x_2 + \tau) = v(x_1, x_2) \text{ for almost every } (x_1, x_2) \in \Gamma^-\}$$,[1])

(1.27) $$\mu(v) = -\tau \sum_{i=1}^{2} \bar{\mu}_i\, v \mid \Gamma_i + \mu_1(v), \quad v \in V$$

with

(1.28) $$\mu_1(v) = \omega \int_{\Omega_\tau} r^2 \frac{\partial v}{\partial x_1}\, dx .$$

We see that $r \in L_4(\Omega_\tau)$ is a sufficient condition under which μ_1 is a continuous linear functional defined on V. $\psi^* \in W$ is a function satisfying the conditions

(1.29) a) $\psi^*(x_1, x_2 + \tau) = \psi^*(x_1, x_2) + Q, \quad (x_1, x_2) \in \Gamma^-$,
b) $\psi^* \mid \Gamma_i = \Psi_i, \quad i = 1, 2,$ c) $\psi^* \mid C_0 = 0$. ■

1.2.2. Problem (PSIω.1.2). In this case we have

(1.25)* $$\mathscr{V} = \{v \in C^\infty(\bar{\Omega}_\tau);\ v \mid C_0 = \text{const},\ v \mid \Gamma_1 = \text{const},\ v \mid \Gamma_2 = 0,\ v(x_1, x_2 + \tau) = v(x_1, x_2)\ \forall (x_1, x_2) \in \Gamma^-\},$$

(1.26)* $$V = \{v \in W;\ v \mid C_0 = \text{const},\ v \mid \Gamma_1 = \text{const},\ v \mid \Gamma_2 = 0,\ v(x_1, x_2 + \tau) = v(x_1, x_2) \text{ for almost every } (x_1, x_2) \in \Gamma^-\},$$

(1.27)* $$\mu(v) = -\tau \bar{\mu}_1 v \mid \Gamma_1 - \gamma v \mid C_0 + \mu_1(v), \quad v \in V.$$

$\psi^* \in W$ satisfies the conditions (1.29,a–c). ■

1.2.3. Problem (PSIω.2.1). Let $m_i\colon \mathbf{R}_1 \to \mathbf{R}_1$ be τ-periodic and $m_i \mid (0, \tau) \in L_2((0, \tau))$. We put

(1.25)** $$\mathscr{V} = \{v \in C^\infty(\bar{\Omega}_\tau);\ v \mid C_0 = 0,\ v(x_1, x_2 + \tau) = v(x_1, x_2)\ \forall (x_1, x_2) \in \Gamma^-\},$$

(1.26)** $$V = \{v \in W;\ v \mid C_0 = 0,\ v(x_1, x_2 + \tau) = v(x_1, x_2) \text{ for almost every } (x_1, x_2) \in \Gamma^-\},$$

(1.27)** $$\mu(v) = -\sum_{i=1}^{2} \int_{\Gamma_i} m_i v\, dS + \mu_1(v), \quad v \in V.$$

$\psi^* \in W$ is a function satisfying the conditions

(1.29)** a) $\psi^*(x_1, x_2 + \tau) = \psi^*(x_1, x_2) + Q, \quad (x_1, x_2) \in \Gamma^-$,
b) $\psi^* \mid C_0 = 0$. ■

Let us summarize the results obtained in [5].

1.2.4. Theorem. 1) *V is a closed subspace of $W = H^1(\Omega_\tau)$ with the norm $|\cdot|_{1,\Omega_\tau}$, equivalent to the norm $\|\cdot\|_{1,\Omega_\tau}$.*
2) *The set $\mathscr{V}$ is dense in V. For an arbitrary $v \in V$ and $\varepsilon > 0$ there exists $w_{v,\varepsilon} \in \mathscr{V}$ such that $|w_{v,\varepsilon} - v|_{1,\Omega_\tau} < \varepsilon$.*

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[1]) The concept "almost every $x \in \Gamma^-$" is considered here in the sense of the one-dimensional measure on $\partial\Omega_\tau$.

3) *There exist constants* α, $K > 0$ *such that*

$$\alpha|\psi_1 - \psi_2|^2_{1,\Omega_\tau} \leqq a(\psi_1, \psi_1 - \psi_2) - a(\psi_2, \psi_1 - \psi_2) \quad \forall \psi_1, \psi_2 \in W, \tag{1.30}$$

$$|a(\psi_1, v) - a(\psi_2, v)| \leqq K\|\psi_1 - \psi_2\|_{1,\Omega_\tau} \|v\|_{1,\Omega_\tau} \quad \forall \psi_1, \psi_2, v \in W. \tag{1.31}$$

4) *There exists* $\psi^* \in C^\infty(\bar{\Omega}_\tau)$ satisfying (1.29,a–b)**. *If for* $i = 1, 2$, φ_i: $\mathbf{R}_1 \to \mathbf{R}_1$ *are* τ*-periodic functions,* $\varphi_i \mid (0, \tau) \in L_2((0, \tau))$ *and* φ_i, Ψ_i $(i = 1, 2)$, Q *satisfy* (1.6)–(1.8), *then there exists a function* $\psi^* \in H^{3/2}(\Omega_\tau)$ *satisfying the conditions* (1.29,a–c). *Moreover, if* φ_i $(i = 1, 2)$ *are* β*-Hölder-continuous in* $\langle 0, \tau + \varepsilon\rangle$ *with* $\beta \in (\frac{1}{2}, 1\rangle$ *and* $\varepsilon > 0$, *then* $\psi^* \in H^2(\Omega_\tau)$. *In both cases* (1.29,a–c) *and* (1.29,a–b)**, ψ^* *can be chosen equal to zero in a certain neighbourhood of* C_0. *Hence, it can be extended onto* $\bar{\Omega}^*_\tau$ *so that* $\psi^* \in H^{3/2}(\Omega^*_\tau)$ *or even* $\psi^* \in H^2(\Omega^*_\tau)$.

5) *The problem* (1.24,a–c) *has exactly one solution (that does not depend on the choice of* $\psi^* \in W$ *satisfying the conditions* (1.29,a–c) *or* (1.29,a–b)**). ■

Because of the discrete trailing conditions (1.13), the problem (PSIω.1.3) has no weak formulation of the form (1.24,a–c), which would be the basis for the application of the finite element method to its numerical solution. In this case we have to consider solutions sufficiently smooth. According to [6, 7] we can reformulate the stream function problem in the following way.

1.2.5. "Variational formulation" of the problem (PSIω.1.3). Let the curve C_0 and the τ-periodic functions φ_i: $\mathbf{R}_1 \to \mathbf{R}_1$ be sufficiently smooth, e.g. $C_0 \in C^{2,\alpha}$, $\varphi_i \in C^{1,\alpha}(\mathbf{R}_1)$, $\alpha \in (0, 1)$. Then there exists $\psi^* \in C^{2,\alpha}(\bar{\Omega}_\tau)$ satisfying the conditions (1.29,a–c) and equal to zero in a neighbourhood $\mathscr{U}(C_0)$ of C_0. Then, of course,

$$\frac{\partial \psi^*}{\partial n}(z_0) = 0. \tag{1.32}$$

Under the notation

$$V = \{v \in C^1(\bar{\Omega}_\tau);\ v \mid C_0 = 0,\ v \mid \Gamma_2 = 0,\ v \mid \Gamma_1 = \text{const},\ v(x_1, x_2 + \tau) = v(x_1, x_2)\ \forall (x_1, x_2) \in \Gamma^-\}, \tag{1.33}$$

$$\tilde{V} = \{v \in C^2(\bar{\Omega}_\tau);\ v \mid C_0 = \text{const},\ v \mid \Gamma_1 = \text{const},\ v \mid \Gamma_2 = 0,\ \frac{\partial v}{\partial n}(z_0) = 0,\ v(x_1, x_2 + \tau) = v(x_1, x_2)\ \forall (x_1, x_2) \in \Gamma^-\}, \tag{1.34}$$

$$\mu(v) = -\tau\bar{\mu}_1\, v \mid \Gamma_1 + \mu_1(v), \quad v \in V \tag{1.35}$$

(μ_1 is defined by (1.28)), the problem (PSIω.1.3) is equivalent to finding ψ: $\bar{\Omega}_\tau \to \mathbf{R}_1$ such that

$$\text{a)}\ \psi \in C^2(\bar{\Omega}_\tau), \quad \text{b)}\ \psi - \psi^* \in \tilde{V}, \quad \text{c)}\ a(\psi, v) = \mu(v)\ \ \forall v \in V. \tag{1.36}$$ ■

We see that (1.36,a–c) is not a variational formulation in the usual sense because of the discrete trailing condition. Moreover, $\tilde{V} \neq V$. However, it is important that the solution ψ satisfies the integral identity (1.36,c) which is the basis for the application of the finite element method. The solvability of the problem (PSIω.1.3) will be studied in the forthcoming paper [9] (cf. also [3]).

2. FINITE ELEMENT SOLUTION

2.1. Discretization of the problem

Let C_0 be approximated by a simple closed piecewise linear curve C_{0h} so that the domain Ω_τ can be replaced by a polygonal domain $\Omega_{\tau h}$ with $\partial\Omega_{\tau h} = C_{0h} \cup \Gamma_1 \cup \cup \Gamma^- \cup \Gamma_2 \cup \Gamma^+$. Let $\mathscr{T}_h$ be a triangulation of $\Omega_{\tau h}$ with the usual properties, i.e., $T \in \mathscr{T}_h$ are closed triangles and

(2.1) a) $\bar{\Omega}_{\tau h} = \bigcup_{T \in \mathscr{T}_h} T$,

b) if $T_1, T_2 \in \mathscr{T}_h$, then either $T_1 \cap T_2 = \emptyset$ or T_1 and T_2 have a common side or T_1 and T_2 have a common vertex.

Further, denoting by $\sigma_h = \{P_1, \ldots, P_N\}$ the set of all vertices of $\mathscr{T}_h$, we assume that

(2.2) a) $\sigma_h \cap \partial\Omega_{\tau h} \subset \partial\Omega_\tau$, $\sigma_h \subset \bar{\Omega}_\tau$,

b) $P_j = (x_1, x_2) \in \sigma_h \cap \Gamma^- \Leftrightarrow P_j^\tau = (x_1, x_2 + \tau) \in \sigma_h \cap \Gamma^+$.

We denote by $h(T)$ the length of the longest side and by $\Theta(T)$ the smallest angle of the triangle $T \in \mathscr{T}_h$ and put $h = \max_{T \in \mathscr{T}_h} h(T)$, $\Theta_h = \min_{T \in \mathscr{T}_h} \Theta(T)$. We shall say that the system $\mathscr{S} = \{\mathscr{T}_h\}_{h \in (0,h_0)}$ $(h_0 > 0)$ of triangulations is *regular*, if $\Theta_h \geqq \Theta > 0$ for all $\mathscr{T}_h \in \mathscr{S}$ and Θ does not depend on h.

The approximate solution ψ_h will be sought in the finite dimensional space of conforming piecewise linear elements $W_h \subset H^1(\Omega_{\tau h})$:

(2.3) $$W_h = \{v_h;\ v_h \in C(\bar{\Omega}_{\tau h}),\ v_h \text{ is affine on each } T \in \mathscr{T}_h\}.$$

Let us define the operator of the Lagrange interpolation r_h: $H^1(\Omega_{\tau h}) \cap C(\bar{\Omega}_{\tau h}) \to W_h$ by

(2.4) $$r_h v \in W_h, \quad v \in H^1(\Omega_{\tau h}) \cap C(\bar{\Omega}_{\tau h}),$$
$$r_h v(P_j) = v(P_j) \quad \forall P_j \in \sigma_h.$$

The *discrete stream function problem* can be written down quite analogously as the continuous problem (1.24,a–c): we seek a function ψ_h: $\bar{\Omega}_{\tau h} \to \mathbf{R}_1$ satisfying the conditions

(2.5) a) $\psi_h \in W_h$, b) $\psi_h - \psi_h^* \in V_h$, c) $a_h(\psi_h, v_h) = \mu_h(v_h) \quad \forall v_h \in V_h$.

$\psi_h^* \in W_h$ is an approximate analogue of the function ψ^*, V_h, a_h, μ_h are suitable approximations of V, a, μ, respectively. In all cases we shall define a_h: $H^1(\Omega_{\tau h}) \times \times H^1(\Omega_{\tau h}) \to \mathbf{R}_1$ by

(2.6) $$a_h(\psi, v) = \int_{\Omega_{\tau h}} b(\cdot, (\nabla\psi)^2)\, \nabla\psi \,.\, \nabla v \,\mathrm{d}x\,, \quad \psi, v \in H^1(\Omega_{\tau h}).$$

The definition of V_h, a_h and μ_h for the particular problems will be introduced in 2.1.2–2.1.4. In 2.1.6 we shall speak about the discretization of the problem (PSIω.1.3) with the trailing conditions.

On the basis of Lemma 1.1.2 we can easily prove the following theorem on the properties of the form a_h:

2.1.1. Theorem. 1) *If* $\psi \in H^1(\Omega_{\tau h})$, *then the mapping* "$v \in H^1(\Omega_{\tau h}) \to a_h(\psi, v) \in \mathbf{R}_1$" *is a continuous linear functional on* $H^1(\Omega_{\tau h})$.

2) *There exist constants* α *and* $K > 0$ *independent of* h *such that*

$$\alpha|\psi_1 - \psi_2|^2_{1,\Omega_{\tau h}} \leqq a_h(\psi_1, \psi_1 - \psi_2) - a_h(\psi_2, \psi_1 - \psi_2) \quad \forall \psi_1, \psi_2 \in H^1(\Omega_{\tau h}) \tag{2.7}$$

(*uniform strong monotony*) *and*

$$|a_h(\psi_1, v) - a_h(\psi_2, v)| \leqq K\|\psi_1 - \psi_2\|_{1,\Omega_{\tau h}} \|v\|_{1,\Omega_{\tau h}} \quad \forall \psi_1, \psi_2, v \in H^1(\Omega_{\tau h}) \tag{2.8}$$

(*uniform Lipschitz continuity*). ∎

2.1.2. Discretization of the problem (PSIω.1.1). In this case we put

$$V_h = \{v_h \in W_h;\ v_h \mid \Gamma_i = \text{const},\ i = 1, 2,\ v_h \mid C_{0h} = 0,\ v_h(P_j^\tau) = v_h(P_j)\ \forall P_j \in \sigma_h \cap \Gamma^-\}, \tag{2.9}$$

and

$$\mu_h(v_h) = \mu_{1h}(v_h) + \mu_{2h}(v_h), \quad v_h \in V_h, \tag{2.10}$$

where

$$\text{a)} \quad \mu_{1h}(v_h) = \omega \int_{\Omega_{\tau h}} r^2 \frac{\partial v_h}{\partial x_1}\, dx, \tag{2.11}$$

$$\text{b)} \quad \mu_{2h}(v_h) = -\tau \sum_{i=1}^{2} \bar{\mu}_i v_h \mid \Gamma_i .$$

Let $\psi_h^* \in W_h$ be a function satisfying the conditions

$$\text{a)} \quad \psi_h^*(P_j^\tau) = \psi_h^*(P_j) + Q \quad \forall P_j \in \sigma_h \cap \Gamma^-, \tag{2.12}$$

$$\text{b)} \quad \psi_h^*(P_j) = \Psi_i(P_j) \quad \forall P_j \in \sigma_h \cap \Gamma_i,\quad i = 1, 2, \qquad \text{c)} \quad \psi_h^* \mid C_{0h} = 0 .$$

On the basis of the assertion 4) from 1.2.4 and the inclusions $H^{3/2}(\Omega_\tau^*) \subset C(\bar{\Omega}_\tau^*)$, $H^2(\Omega_\tau^*) \subset C(\bar{\Omega}_\tau^*)$ (see [18, 20]), we can put

$$\psi_h^* = r_h \psi^* .$$

Another simple example of $\psi_h^* \in W_h$ with the properties (2.12,a–c) is defined by the conditions

$$\text{a)} \quad \psi_h^*(P_j) = 0 \quad \text{for} \quad P_j \in \sigma_h \cap (\Omega_{\tau h} \cup C_{0h} \cup (\Gamma^- - (\Gamma_1 \cup \Gamma_2))), \tag{2.13}$$

$$\text{b)} \quad \psi_h^*(P_j) = Q \quad \text{for} \quad P_j \in \sigma_h \cap (\Gamma^+ - (\Gamma_1 \cup \Gamma_2)),$$

$$\text{c)} \quad \psi_h^*(P_j) = \Psi_i(P_j) \quad \text{for} \quad P_j \in \sigma_h \cap \Gamma_i,\quad i = 1, 2 .$$

∎

2.1.3. Discretization of the problem (PSIω.1.2). We put

$$(2.9)^* \qquad V_h = \{v_h \in W_h;\ v_h \mid C_{0h} = \text{const},\ v_h \mid \Gamma_1 = \text{const},\ v_h \mid \Gamma_2 = 0,\ v_h(P_j^\tau) = v_h(P_j)\ \forall P_j \in \sigma_h \cap \Gamma^-\},$$

μ_h is given by (2.10), μ_{1h} is determined by (2.11,a) and

$$(2.11)^* \qquad \mu_{2h}(v_h) = -\tau\bar{\mu}_1 v_h \mid \Gamma_1 - \gamma v_h \mid C_0\,, \quad v_h \in V_h\,.$$

The function ψ_h^* is defined by (2.12,a–c). ■

2.1.4. Discretization of the problem (PSIω.2.1):

$$(2.9)^{**} \qquad V_h = \{v_h \in W_h;\ v_h \mid C_{0h} = 0,\ v_h(P_j^\tau) = v_h(P_j)\ \forall P_j \in \sigma_h \cap \Gamma^-\}\,.$$

The function μ_h is given by (2.10) with μ_{1h} determined in (2.11,a) and

$$(2.11)^{**} \qquad \mu_{2h}(v_h) = -\sum_{i=1}^{2}\int_{\Gamma_i} m_i v_h \, dS\,, \quad v_h \in V_h\,.$$

$\psi_h^* \in W_h$ is a function satisfying the conditions

$$(2.12)^{**} \qquad \text{a)}\ \ \psi_h^*(P_j^\tau) = \psi_h^*(P_j) + Q\,, \quad \forall P_j \in \sigma_h \cap \Gamma^-\,,$$
$$\text{b)}\ \ \psi_h^* \mid C_{0h} = 0\,.$$

Again, we can put $\psi_h^* = r_h\psi^*$. Another possibility used in practical computations is defined by

$$(2.13)^{**} \qquad \text{a)}\ \ \psi_h^*(P_j) = 0 \quad \text{for} \quad P_j \in \sigma_h - \Gamma^+\,,$$
$$\text{b)}\ \ \psi_h^*(P_j) = Q \quad \textit{for} \quad P_j \in \sigma_h \cap \Gamma^+\,. \qquad ■$$

It is easy to see that μ_h is a linear functional on the finite dimensional space V_h. Moreover, $\|\cdot\|_{1,\Omega_{\tau h}}$ and $|\cdot|_{1,\Omega_{\tau h}}$ are equivalent norms on V_h. The norm $|\cdot|_{1,\Omega_{\tau h}}$ is induced by the scalar product $(u_h, v_h)_{1,\Omega_{\tau h}} = \int_{\Omega_{\tau h}} \nabla u_h \cdot \nabla v_h \, dx$. On the basis of these results and of Theorem 2.1.1 we shall prove the solvability of the problem (2.5,a–c).

2.1.5. Theorem. *The problem* (2.5,a–c) *with a_h defined by* (2.6) *and V_h, μ_h, ψ_h^* from* 2.1.2 *or* 2.1.3 *or* 2.1.4 *has exactly one solution ψ_h. This solution does not depend on the choice of ψ_h^* with the properties* (2.12,a–c) *or* (2.12,a–b)**.

Proof. Theorem 2.1.1 implies that for fixed $\psi_h \in W_h$ the mapping "$v_h \in V_h \to a_h(\psi_h, v_h) \in \mathbf{R}_1$" is a continuous linear functional defined on V_h. Hence, there exists a mapping T_h: $V_h \to V_h$ and an element $\tilde{\mu}_h \in V_h$ defined by the relations

$$(2.14) \qquad \text{a)}\ \ (T_h(u_h), v_h)_{1,\Omega_{\tau h}} = a_h(\psi_h^* + u_h, v_h) \quad \forall u_h, v_h \in V_h\,,$$
$$\text{b)}\ \ (\tilde{\mu}_h, v_h)_{1,\Omega_{\tau h}} = \mu_h(v_h) \quad \forall v_h \in V_h\,.$$

We see that the problem (2.5,a–c) is equivalent to the operator equation

$$(2.15) \qquad T_h(u_h) = \tilde{\mu}_h$$

for an unknown $u_h \in V_h$. In virtue of Theorem 2.1.1 and of the equivalence of the norms $\|\cdot\|_{1,\Omega_{\tau h}}$ and $|\cdot|_{1,\Omega_{\tau h}}$, T_h is strongly monotone and Lipschitz-continuous. This implies that there exists $\nu > 0$ such that the operator

$$F_\nu(u_h) = u_h - \nu(T_h(u_h) - \tilde{\mu}_h)$$

is contractive and thus, it possesses a unique fixed point $u_h \in V_h$ which is a unique solution of (2.15) (cf. e.g. [5]). Moreover, this solution can be found as the limit of a sequence defined in the following way:

$$(2.16) \qquad \begin{aligned} &u_h^0 \in V_h \quad \text{is an arbitrary initial approximation,} \\ &u_h^{m+1} = F_\nu(u_h^m)\,, \quad \text{if} \quad m \geqq 0\,. \end{aligned}$$

The function $\psi_h = \psi_h^* + u_h$ is a solution of the problem (2.5,a–c).

In order to complete the proof, let us consider two functions $\psi_{h1}^*, \psi_{h2}^* \in W_h$ satisfying the conditions (2.12,a–c) or (2.12,a–b)** for the problems (PSIω.1.1), (PSIω.1.2) or for the problem (PSIω.2.1), respectively. Then $\psi_{h1}^* - \psi_{h2}^* \in V_h$. If $u_{hi} \in V_h$ and

$$(2.17)_i \qquad a_h(\psi_{hi}^* + u_{hi}, v_h) = \mu_h(v_h) \quad \forall v_h \in V_h\,, \quad i = 1, 2\,,$$

then $\psi_{hi} = \psi_{hi}^* + u_{hi}$, $i = 1, 2$, are two solutions of the problem (2.5,a–c). Let us subtract the equations $(2.17)_i$, $i = 1, 2$, where we substitute $v_h := \psi_{h1} - \psi_{h2} = \psi_{h1}^* - \psi_{h2}^* + u_{h1} - u_{h2} \in V_h$ and use (2.7):

$$0 = a_h(\psi_{h1}, \psi_{h1} - \psi_{h2}) - a_h(\psi_{h2}, \psi_{h1} - \psi_{h2}) \geqq \alpha|\psi_{h1} - \psi_{h2}|_{1,\Omega_{\tau h}}^2\,.$$

This immediately yields the equality $\psi_{h1} = \psi_{h2}$, which we wanted to prove. ■

2.1.6. For completeness we shall also describe the *discretization of the problem* (PSIω.1.3), even if its theoretical study will be carried out in a separate paper. Here we briefly summarize the results from [6, 7, 10]. Let us assume that there exists a triangle $T_0 \in \mathscr{T}_h$ with vertices $P_{i_0*} = z_0$ and $P_{i_0} \in \Omega_{\tau h}$ and with the side $S_0 = P_{i_0*}P_{i_0}$ normal to C_0.

The test functions $v \in V$ with V defined by (1.33) are approximated by v_h from the finite-dimensional space

$$(2.18) \qquad \begin{aligned} V_h = \{v_h \in W_h;\ & v_h \mid C_{0h} = 0,\ v_h \mid \Gamma_2 = 0,\ v_h \mid \Gamma_1 = \text{const},\\ & v_h(P_j^\tau) = v_h(P_j)\ \forall P_j \in \sigma_h \cap \Gamma^-\}\,. \end{aligned}$$

If we discretize the trailing condition (1.13) by the finite-difference equation

$$(2.19) \qquad \frac{\psi_h(P_{i_0*}) - \psi_h(P_{i_0})}{|P_{i_0*} - P_{i_0}|} = 0$$

and take into account the condition that the stream function is constant on the profile C_0, then we get the conditions

$$(2.20) \qquad \psi_h(P_j) = q_0 = \psi_h(P_{i_0}) \quad \textit{for all} \quad P_j \in \sigma_h \cap C_{0h}\,.$$

Let $\psi_h^* \in W_h$ satisfy (2.13,a–c). Then the approximate solution of the problem (PSIω.1.3) is sought in the form $\psi_h = \psi_h^* + u_h$, $u_h \in \tilde{V}_h$, where

$$(2.21) \qquad \begin{aligned} \tilde{V}_h = \{v_h \in W_h;\ & v_h \mid C_{0h} \cup S_0 = \text{const},\ v_h \mid \Gamma_1 = \text{const},\ v_h \mid \Gamma_2 = 0\,,\\ & v_h(P_j^\tau) = v_h(P_j)\ \forall P_j \in \sigma_h \cap \Gamma^-\} \end{aligned}$$

is a finite-dimensional space approximating $\tilde{V}$.

We define the approximate solution of the problem (PSIω.1.3) as a function ψ_h satisfying

(2.22) $$\text{a) } \psi_h \in W_h\,, \quad \text{b) } \psi_h - \psi_h^* \in \tilde{V}_h\,,$$
$$\text{c) } a_h(\psi_h, v_h) = \mu_h(v_h) \quad \forall v_h \in V_h.$$

The functional μ_h is defined by (2.10) with μ_{1h} from (2.11,a) and

(2.23) $$\mu_{2h}(v_h) = -\tau\bar{\mu}_1 v_h \mid \Gamma_1\,, \quad v_h \in V_h\,.$$ ■

2.2. Convergence of the finite element method

We shall study the error estimate $\|\psi - \psi_h\|_{1,\Omega_{\tau h}}$ and the convergence of the finite element method in case of the problems (PSIω.1.1) and (PSIω.2.1). The investigation of these questions for the problem (PSIω.1.2) with a given velocity circulation is connected with some technical difficulties and, therefore, a separate paper [8] is devoted to it.

A quite different approach must be applied to the study of solvability of the continuous and discrete problems (PSIω.1.3) with the trailing conditions, and to the investigation of convergence of the finite element method in this case. It will be the subject-matter of the paper [9].

Let us consider a domain $\tilde{\Omega}_\tau$ and a constant $h_0 > 0$ such that

(2.24) $$\partial\tilde{\Omega}_\tau = \tilde{C}_0 \cup \Gamma_1 \cup \Gamma_2 \cup \Gamma^- \cup \Gamma^+\,,$$
$$\Omega_\tau, \Omega_{\tau h} \subset \tilde{\Omega}_\tau \quad \forall h \in (0, h_0)\,,$$

where $\tilde{C}_0$ is a simple closed curve, $\tilde{C}_0 \in C^2$ and $\tilde{C}_0 \subset \operatorname{Int} C_0$ ($\operatorname{Int} C_0$ is the bounded component of $\mathbf{R}_2 - C_0$). Hence, $\partial\tilde{\Omega}_\tau$ is Lipschitz-continuous.

Following the results from [20], Ch. 2 we can choose $\tilde{\Omega}_\tau$ in such a way that the solution ψ of the problem (1.24,a–c) possesses an extension $\tilde{\psi}$ from Ω_τ onto $\tilde{\Omega}_\tau$, $\tilde{\psi} \in H^1(\tilde{\Omega}_\tau)$. Moreover, if C_0 is sufficiently smooth (e.g. $C_0 \in C^2$) and $\psi \in H^2(\Omega_\tau)$, then $\tilde{\psi} \in H^2(\tilde{\Omega}_\tau)$. In what follows, for simplicity, we shall denote this extension again by the symbol ψ.

In the study of convergence we shall need the uniform equivalence of the norms $|\cdot|_{1,\Omega_{\tau h}}$ and $\|\cdot\|_{1,\Omega_{\tau h}}$ on the space V_h.

2.2.1. Lemma. *Let V_h be defined by (2.9) or (2.9)**. Then there exists a constant $c > 0$ such that*

(2.25) $$\|v_h\|_{1,\Omega_{\tau h}} \leqq c|v_h|_{1,\Omega_{\tau h}} \quad \forall v_h \in V_h\,, \quad \forall h \in (0, h_0)\,.$$

Proof. Let us define the space $\hat{V} = \{v \in H^1(\tilde{\Omega}_\tau);\ v \mid \tilde{C}_0 = 0\}$. Since $\partial\tilde{\Omega}_\tau$ is Lipschitz-continuous and the one-dimensional Lebesgue measure (defined on $\partial\tilde{\Omega}_\tau$) of $\tilde{C}_0$ is positive, the well-known Fridrichs inequality holds ([20]):

$$\int_{\tilde{\Omega}_\tau} v^2\,dx \leqq \hat{c}\int_{\tilde{\Omega}_\tau} (\nabla v)^2\,dx \quad \forall v \in \hat{V}$$

with a constant $\hat{c} > 0$ independent of $v \in \hat{V}$. Hence, there exists $c > 0$ such that

(2.26) $$\|v\|_{1,\tilde{\Omega}_\tau} \leqq c|v|_{1,\tilde{\Omega}_\tau} \quad \forall v \in \hat{V}\,.$$

Let $h \in (0, h_0)$ and $v_h \in V_h$. Let us denote by $v_h^e: \tilde{\Omega}_\tau \to \mathbf{R}_1$ the extension of v_h defined by

$$\text{(2.27)} \qquad \text{a)} \quad v_h^e \mid \Omega_{\tau h} = v_h\,, \qquad \text{b)} \quad v_h^e \mid \tilde{\Omega}_\tau - \Omega_{\tau h} = 0\,.$$

It is not difficult to show that $v_h^e \in \hat{V}$ and

$$\text{(2.28)} \qquad \begin{aligned} |v_h|_{1,\Omega_{\tau h}} &= |v_h^e|_{1,\tilde{\Omega}_\tau}\,, \\ \|v_h\|_{1,\Omega_{\tau h}} &= \|v_h^e\|_{1,\tilde{\Omega}_\tau}\,. \end{aligned}$$

Together with (2.26) this implies that

$$\|v_h\|_{1,\Omega_{\tau h}} = \|v_h^e\|_{1,\tilde{\Omega}_\tau} \leqq c|v_h^e|_{1,\tilde{\Omega}_\tau} = c|v_h|_{1,\Omega_{\tau h}}\,,$$

which proves (2.25). ■

Our further considerations will be based on the following abstract error estimate.

2.2.2. Theorem. *For each $h \in (0, h_0)$ let the following assumptions be satisfied:*

1) *$W_h \subset H^1(\Omega_{\tau h})$ is a finite-dimensional space, $V_h \subset W_h$ is its subspace, $\psi_h^* \in W_h$,*

$$\text{(2.29)} \qquad X_h = \psi_h^* + V_h = \{\Phi_h = \psi_h^* + v_h;\, v_h \in V_h\}\,.$$

μ_h, l_h: $V_h \to \mathbf{R}_1$ are continuous linear functionals.

2) *a_h: $H^1(\Omega_{\tau h}) \times H^1(\Omega_{\tau h}) \to \mathbf{R}_1$ is a form satisfying the conditions* (2.7) *and* (2.8).

3) *$\psi \in H^1(\Omega_{\tau h})$ and $\psi_h \in X_h$ satisfy the relations*

$$\text{(2.30)} \qquad a_h(\psi, v_h) = \mu_h(v_h) + l_h(v_h) \quad \forall v_h \in V_h$$

and

$$\text{(2.31)} \qquad a_h(\psi_h, v_h) = \mu_h(v_h) \quad \forall v_h \in V_h\,,$$

respectively.

4) *The condition* (2.25) *is satisfied.*

Then there exist constants A_1, $A_2 > 0$ such that for each $h \in (0, h_0)$

$$\text{(2.32)} \qquad \|\psi - \psi_h\|_{1,\Omega_{\tau h}} \leqq A_1 \|l_h\|_{1,\Omega_{\tau h}}^* + A_2 \inf_{\Phi_h \in X_h} \|\psi - \Phi_h\|_{1,\Omega_{\tau h}}\,,$$

where

$$\text{(2.33)} \qquad \|l_h\|_{1,\Omega_{\tau h}}^* = \sup_{\substack{v_h \in V_h \\ v_h \neq 0}} \frac{|l_h(v_h)|}{\|v_h\|_{1,\Omega_{\tau h}}}\,.$$

For the proof see [8]. ■

a) Polygonal domain

First, we shall deal with the case when the domain Ω_τ is polygonal. Then for $h_0 > 0$ sufficiently small and $h \in (0, h_0)$ we have $C_{0h} = C_0$, $\Omega_{\tau h} = \Omega_\tau$, $V_h \subset V$, $a_h = a$, $\mu_h = \mu$ and $l_h \equiv 0$.

2.2.3. Theorem. *Let $\mathscr{S} = \{\mathscr{T}_h\}_{h \in (0, h_0)}$ be a regular system of triangulations of the polygonal domain Ω_τ, $\psi^* \in H^2(\Omega_\tau)$. Let ψ be the solution of the problem* (1.24,a–c) *with ψ^*, V and μ given in* 1.2.1 *or* 1.2.3 *for the problem* (PSIω.1.1) or (PSIω.2.1), *respectively. Further, let ψ_h be the solution of the discrete problem*

(2.5,a – c), *where* ψ_h^*, V_h *and* $\mu_h = \mu$ *are defined in* 2.1.2 *or* 2.1.4 *in case of the problem* (PSIω.1.1) *or* (PSIω.2.1), *respectively. Then*

$$(2.34) \qquad \lim_{h\to 0} \|\psi - \psi_h\|_{1,\Omega_{\tau h}} = 0\,.$$

Proof. It is easy to verify that the assumptions of Theorem 2.2.2 are satisfied. Since $l_h \equiv 0$, we have

$$(2.35) \qquad \|\psi - \psi_h\|_{1,\Omega_\tau} \leqq A_2 \inf_{\Phi_h \in X_h} \|\psi - \Phi_h\|_{1,\Omega_\tau}\,,$$

as follows from (2.32). Further, if we take into account the assertion 5) of Theorem 1.2.4 and Theorem 2.1.5, we can write $\psi = \psi^* + u$, $u \in V$ and $\psi_h = \psi_h^* + u_h$, where $\psi_h^* = r_h\psi^*$ and $u_h \in V_h$. It is evident that $\Phi_h = \psi_h^* + v_h \in X_h$ for all $v_h \in V_h$. Hence,

$$(2.36) \qquad \|\psi - \psi_h\|_{1,\Omega_\tau} \leqq A_2(\|\psi^* - \psi_h^*\|_{1,\Omega_\tau} + \inf_{v_h \in V_h} \|u - v_h\|_{1,\Omega_\tau})\,.$$

Let $\varepsilon > 0$ be arbitrary. In virtue of general approximation properties of the finite element spaces,

$$(2.37) \qquad \|v - r_h v\|_{1,\Omega_\tau} \leqq ch\|v\|_{2,\Omega_\tau} \quad \text{for} \quad v \in H^2(\Omega_\tau)$$

(see [2]). Hence, there exists $h_1 \in (0, h_0)$ such that

$$(2.38) \qquad \|\psi^* - \psi_h^*\| \leqq \varepsilon/3\,, \quad \text{if} \quad h \in (0, h_1)\,.$$

Since the set $\mathscr{V} \subset C^\infty(\bar{\Omega}_\tau)$ is dense in V, there exists $\tilde{u} \in \mathscr{V}$ satisfying the inequality

$$(2.39) \qquad \|\tilde{u} - u\|_{1,\Omega_\tau} \leqq \varepsilon/3\,.$$

From (2.37) we get $\tilde{h}_1 \in (0, h_1)$ such that

$$(2.40) \qquad \|\tilde{u} - r_h\tilde{u}\|_{1,\Omega_\tau} \leqq \varepsilon/3\,, \quad \text{if} \quad h \in (0, \tilde{h}_1).$$

Of course $r_h\tilde{u} \in V_h$. Now, (2.36)–(2.40) yield the estimate

$$\|\psi - \psi_h\|_{1,\Omega_\tau} \leqq A_2(\|\psi^* - \psi_h^*\|_{1,\Omega_\tau} + \|u - \tilde{u}\|_{1,\Omega_\tau} + \|\tilde{u} - r_h\tilde{u}\|_{1,\Omega_\tau}) \leqq A_2\varepsilon$$

for all $h \in (0, \tilde{h}_1)$, which already implies the assertion (2.34). ■

2.2.4. Remark. Similarly as in [8], it is sufficient to assume that $\psi^* \in H^{3/2}(\Omega_\tau)$. since then

$$(2.41) \qquad \|\psi^* - r_h\psi^*\|_{1,\Omega_\tau} \leqq c\|\psi^*\|_{3/2,\Omega_\tau}\, h^{1/2}\,.$$ ■

b) Nonpolygonal domain

Now let us assume that the profile C_0 is smooth enough and hence, the domain Ω_τ is not polygonal. Let us suppose that $C_0 \in C^2$. Following the regularity results derived in [21] for the solutions of elliptic problems and using the above mentioned possibility of the extension of ψ from Ω_τ onto $\tilde{\Omega}_\tau$, we shall make the assumption that the weak solution of the problem (1.24,a – c) satisfies the condition

$$(2.42) \qquad \psi \in H^2(\tilde{\Omega}_\tau)\,.$$

Here $\tilde{\Omega}_\tau$ satisfies the relations (2.24). Of course, we shall assume that the system of triangulations $\{\mathscr{T}_h\}_{h\in(0,h_0)}$ has the properties introduced in Section 2.1. Under the above assumptions the following result was proved in [8]:

2.2.5. Lemma. There exists a constant $c > 0$ such that

(2.43) $$\operatorname{meas}[(\Omega_\tau - \Omega_{\tau h}) \cup (\Omega_{\tau h} - \Omega_\tau)] \leqq ch^2 \quad \text{for} \quad h \in (0, h_0). \qquad \blacksquare$$

On the basis of Theorem 2.2.2 we shall prove the error estimate of the finite element method.

2.2.6. Theorem. *Let $\{\mathcal{T}_h\}_{h\in(0,h_0)}$ be a regular system of triangulations with the properties from Section* 2.1 *and let* (2.24) *be valid. If $\psi \in H^2(\tilde{\Omega}_\tau)$ is a solution of the problem* (1.24,a–c) *and ψ_h is a solution of the corresponding discrete problem* (2.5,a–c) *and $r^2 \in H^1(\tilde{\Omega}_\tau)$, then there exist constants h_1, $c > 0$, independent of h, such that*

(2.44) $$\|\psi - \psi_h\|_{1,\Omega_{\tau h}} \leqq ch \quad \textit{for all} \quad h \in (0, h_1).$$

Proof. Let us prove this theorem e.g. for the problem (PSIω.1.1). If we put

(2.45) $$f = -\sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left(b(\cdot, (\nabla\psi)^2)\frac{\partial\psi}{\partial x_i}\right) + \omega \frac{\partial r^2}{\partial x_1},$$

then from the assumption (2.42) and from the properties of b and r we conclude that $f \in L_2(\tilde{\Omega}_\tau)$. With the use of Green's theorem we get from (1.24,a–c) that $f = 0$ almost everywhere in Ω_τ and ψ satisfies the conditions (1.4), (1.11) with $q_0 = 0$, (1.5) and (1.10) for $i = 1, 2$ (restricted to $\bar{\Omega}_\tau$).

Let $v_h \in V_h$. Then

(2.46) $$\int_{\Omega_{\tau h} - \Omega_\tau} f v_h \, dx = \int_{\Omega_{\tau h}} f v_h \, dx =$$

$$= -\int_{\Omega_{\tau h}} \left[\sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left(b(\cdot, (\nabla\psi)^2)\frac{\partial\psi}{\partial x_i}\right) - \omega \frac{\partial r^2}{\partial x_1}\right] v_h \, dx =$$

$$= \int_{\Omega_{\tau h}} \left[b(\cdot, (\nabla\psi)^2)\, \nabla\psi \,.\, \nabla v_h - \omega r^2 \frac{\partial v_h}{\partial x_1}\right] dx -$$

$$- \sum_{i=1}^{2} v_h \mid \Gamma_i \int_{\Gamma_i} \left[b(\cdot, (\nabla\psi)^2)\frac{\partial\psi}{\partial n} - \omega r^2 n_1\right] dS$$

(cf. the definition (2.9) of the space V_h). In view of (1.10), (2.6), (2.10), (2.11) and (2.46), we have

$$\int_{\Omega_{\tau h} - \Omega_\tau} f v_h \, dx = a_h(\psi, v_h) - \mu_h(v_h).$$

Hence,

(2.47) $$a_h(\psi, v_h) = \mu_h(v_h) + l_h(v_h) \quad \forall v_h \in V_h,$$

where

(2.48) $$l_h(v_h) = \int_{\Omega_{\tau h} - \Omega_\tau} f v_h \, dx.$$

It is evident that l_h is a continuous linear functional on V_h.

Now, let us apply Theorem 2.2.2 whose assumptions are obviously satisfied. In virtue of (2.32) it is sufficient to estimate

$$\inf_{\Phi_h \in X_h} \|\psi - \Phi_h\|_{1,\Omega_{\tau h}} \quad \text{and} \quad \|l_h\|^*_{1,\Omega_{\tau h}} .$$

Concerning the first term, we use the fact that $r_h\psi \in X_h$ and hence,

$$(2.49) \qquad \inf_{\Phi_h \in X_h} \|\psi - \Phi_h\|_{1,\Omega_{\tau h}} \leqq \|\psi - r_h\psi\|_{1,\Omega_{\tau h}} \leqq ch\|\psi\|_{2,\Omega_{\tau h}} \leqq ch\|\psi\|_{2,\tilde{\Omega}_\tau}$$

(cf. (2.37)). Let us derive the estimate of $\|l_h\|^*_{1,\Omega_{\tau h}}$. If $v_h \in V_h$, then

$$(2.50) \qquad |l_h(v_h)| \leqq \int_{\Omega_{\tau h} - \Omega_\tau} |fv_h| \, \mathrm{d}x \leqq \|f\|_{0,\Omega_{\tau h} - \Omega_\tau} \|v_h\|_{0,\Omega_{\tau h} - \Omega_\tau} .$$

Since $\|f\|_{0,\Omega_{\tau h} - \Omega_\tau} \leqq \|f\|_{0,\tilde{\Omega}_\tau}$, it remains to find the estimate of $\|v_h\|_{0,\Omega_{\tau h} - \Omega_\tau}$. In Lemma 2.2.7 we shall prove that

$$(2.51) \qquad \|v_h\|_{0,\Omega_{\tau h} - \Omega_\tau} \leqq \tilde{c}h^2\|v_h\|_{1,\Omega_{\tau h}}$$

with $\tilde{c}$ independent of h. From this, (2.32), (2.49) and (2.50) we can immediately conclude that the error estimate (2.44) holds. ∎

2.2.7. Lemma. There exists a constant $\tilde{c} > 0$ such that (2.51) holds for each $v_h \in V_h$ and $h \in (0, h_0)$.

Proof. The set $\Omega_{\tau h} - \bar{\Omega}_\tau$ can be written as a union of disjoint open sets $\mathscr{S}^h_r$, $r = 1, \ldots, k_h$, with $\partial\mathscr{S}^h_r = \sum^h_r \cup S^h_r$, where $S^h_r \subset C_{0h}$ is a side of a triangle adjacent to $\partial\Omega_{\tau h}$, and $\sum^h_r \subset C_0$. In each $\mathscr{S}^h_r$ we can introduce local coordinates x_r, y_r; y_r is measured along S^h_r and x_r is measured in the normal direction to S^h_r. Then $\mathscr{S}^h_r = \{(x_r, y_r);\, y_r \in (0, s^h_r),\ 0 < x_r < \alpha^h_r(y_r)\}$, where $\alpha^h_r\colon \langle 0, s^h_r\rangle \to \langle 0, +\infty)$. We have

$$(2.52) \qquad \int_{\Omega_{\tau h} - \Omega_\tau} v_h^2 \, \mathrm{d}x = \sum_{\mathscr{S}_r{}^h \subset \Omega_{\tau h} - \Omega_\tau} \int_{\mathscr{S}_r{}^h} v_h^2 \, \mathrm{d}x$$

and

$$(2.53) \qquad \int_{\mathscr{S}_r{}^h} v_h^2 \, \mathrm{d}x = \int_0^{s_r{}^h} \left(\int_0^{\alpha_r{}^h(y_r)} v_h^2(x_r, y_r) \, \mathrm{d}x_r \right) \mathrm{d}y_r .$$

Since $v_h(0, y_r) = 0$ for all $y_r \in \langle 0, s^h_r\rangle$ and $0 \leqq x_r \leqq \alpha^h_r(y_r) \leqq \text{const} \,.\, h^2$ (see [8]), we can write

$$v_h^2(x_r, y_r) = \left[\int_0^{x_r} \frac{\partial v_h(\xi, y_r)}{\partial x_r} \, \mathrm{d}\xi \right]^2 \leqq \text{const} \,.\, h^2 \int_0^{\alpha_r{}^h(y_r)} \left(\frac{\partial v_h(\xi, y_r)}{\partial x_r} \right)^2 \mathrm{d}\xi .$$

By substituting into (2.53) we get the inequality

$$\int_{\mathscr{S}_r{}^h} v_h^2 \, \mathrm{d}x \leqq \text{const} \,.\, h^4 \int_0^{s_r{}^h} \left(\int_0^{\alpha_r{}^h(y_r)} \left(\frac{\partial v_h(x_r, y_r)}{\partial x_r} \right)^2 \mathrm{d}x_r \right) \mathrm{d}y_r \leqq \text{const} \,.\, h^4 \int_{\mathscr{S}_r{}^h} (\nabla v_h)^2 \, \mathrm{d}x .$$

This and (2.52) imply

$$\int_{\Omega_{\tau h} - \Omega_\tau} v_h^2 \, \mathrm{d}x \leqq \text{const} \,.\, h^4 \int_{\Omega_{\tau h} - \Omega_\tau} (\nabla v_h)^2 \, \mathrm{d}x \leqq \text{const} \,.\, h^4 \int_{\Omega_{\tau h}} (\nabla v_h)^2 \, \mathrm{d}x .$$

This already yields (2.51). ∎

3. CALCULATION OF THE APPROXIMATE SOLUTION

In this section we shall deal with the solution of the discrete problems (2.5,a–c) (i.e., the problems without trailing conditions) and (2.22,a–c) (the problems with trailing conditions).

3.1. The system of algebraic equations equivalent to the discrete problem

First, let us study the problem (2.5,a–c). It is known that in the space W_h there exists a basis formed by the functions w_i, $i = 1, \ldots, N$, such that

$$(3.1)\qquad \begin{aligned} w_i(P_i) &= 1, \quad i = 1, \ldots, N, \\ w_i(P_j) &= 0, \quad i, j = 1, \ldots, N, \quad i \neq j. \end{aligned}$$

If $v_h \in W_h$, then

$$(3.2)\qquad v_h = \sum_{i=1}^{N} v_h(P_i)\, w_i .$$

Let us denote by $\{w_i^*\}_{i=1}^n$ ($n = \dim V_h < N$) a basis in V_h. Since $V_h \subset W_h$, it is evident that each w_i^* can be written as a linear combination of functions w_j. E.g., in the space V_h defined by (2.9) (for the problem (PSIω.1.1)) we consider the basis consisting of the following elements:

(3.3)

a) w_i for $P_i \in \sigma_h \cap \Omega_{\tau h}$,

b) $w_i + w_j$ for $P_i = P_j^\tau \in \sigma_h \cap (\Gamma^+ - (\Gamma_1 \cup \Gamma_2))$, $P_j \in \sigma_h \cap (\Gamma^- - (\Gamma_1 \cup \Gamma_2))$

c) $\sum_{P_j \in \sigma_h \cap \Gamma_i} w_j$, $i = 1, 2$.

To prove that these functions form a basis in V_h, it is sufficient to notice that in view of (3.2) and (2.9), for $v_h \in V_h$ we have

$$(3.4)\qquad v_h = \sum_{i=1}^{N} v_h(P_i)\, w_i =$$

$$= \sum_{P_j \in \sigma_h \cap \Omega_{\tau h}} v_h(P_j)\, w_j + \sum_{\substack{P_j \in \sigma_h \cap (\Gamma^- - (\Gamma_1 \cup \Gamma_2)) \\ P_i = P_j^\tau}} v_h(P_i)\,(w_i + w_j) + \sum_{i=1}^{2} v_h \mid \Gamma_i \sum_{P_j \in \sigma_h \cap \Gamma_i} w_j ,$$

and that $\sigma_h - C_{0h}$ is a sum of disjoint sets $\{P_j\}_{P_j \in \sigma_h \cap \Omega_{\tau h}}$, $\{P_j, P_j^\tau\}_{P_j \in \sigma_h \cap (\Gamma^- - (\Gamma_1 \cup \Gamma_2))}$ and $\{P_j;\, P_j \in \sigma_h \cap \Gamma_i\}$, $i = 1, 2$. From (3.4) we see that every $v_h \in V_h$ can be written in the form

$$(3.5)\qquad v_h = \sum_{i=1}^{n} v_i w_i^* ,$$

where the coefficients v_i are uniquely determined. The coefficients at the basis functions of the type (3.3,a) or (3.3,b) or (3.3,c) are equal to $v_h(P_i)$, $P_i \in \sigma_h \cap \Omega_{\tau h}$ or $v_h(P_i)$, $P_i \in \sigma_h \cap (\Gamma^- - (\Gamma_1 \cup \Gamma_2))$ or $v_h \mid \Gamma_i$, $i = 1, 2$, respectively.

Similarly we can proceed in the case of the problems (PSIω.1.2) or (PSIω.2.1). We let the details to the reader.

If we seek an approximate solution in the form

$$\psi_h = \psi_h^* + u_h\,, \quad u_h \in V_h \tag{3.6}$$

(cf. the proof of Theorem 2.1.5), then

$$\psi_h = \psi_h^* + \sum_{j=1}^{n} u_j w_j^*\,, \quad u_j \in \mathbf{R}_1\,, \quad j = 1, \ldots, n\,. \tag{3.7}$$

Hence, the problem (2.5,a–c) can be written in the equivalent form

$$a_h(\psi_h^* + \sum_{j=1}^{n} u_j w_j^*, w_i^*) = \mu_h(w_i^*)\,, \quad i = 1, \ldots, n \tag{3.8}$$

or, in view of (2.6),

$$\int_{\Omega_{\tau h}} b(\cdot, (\nabla\psi_h^* + \sum_{s=1}^{n} u_s \nabla w_s^*)^2) \sum_{j=1}^{n} u_j \nabla w_j^* \cdot \nabla w_i^* \,\mathrm{d}x = \tag{3.9}$$

$$= \mu_h(w_i^*) - \int_{\Omega_{\tau h}} b(\cdot, (\nabla\psi_h^* + \sum_{s=1}^{n} u_s \nabla w_s^*)^2) \nabla\psi_h^* \cdot \nabla w_i^* \,\mathrm{d}x\,, \quad i = 1, \ldots, n\,.$$

This is obviously a system of n equations for unknown values $u_j \in \mathbf{R}_1$, $j = 1, \ldots, n$.

If we put $\bar{u} = (u_1, \ldots, u_n)^T$, $\boldsymbol{\Phi}(\bar{u}) = (\boldsymbol{\Phi}_1(\bar{u}), \ldots, \boldsymbol{\Phi}_n(\bar{u}))^T$ and $\mathbb{A}(\bar{u}) = (a_{ij}(\bar{u}))_{i,j=1}^n$ with

$$\boldsymbol{\Phi}_i(\bar{u}) = \mu_h(w_i^*) - \int_{\Omega_{\tau h}} b(\cdot, (\nabla\psi_h^* + \sum_{s=1}^{n} u_s \nabla w_s^*)^2) \nabla\psi_h^* \cdot \nabla w_i^* \,\mathrm{d}x\,, \tag{3.10}$$

$$a_{ij}(\bar{u}) = \int_{\Omega_{\tau h}} b(\cdot, (\nabla\psi_h^* + \sum_{s=1}^{n} u_s \nabla w_s^*)^2) \nabla w_i^* \cdot \nabla w_j^* \,\mathrm{d}x\,,$$

then (3.9) can be written in the matrix form

$$\mathbb{A}(\bar{u})\,\bar{u} = \boldsymbol{\Phi}(\bar{u})\,. \tag{3.11}$$

From the properties of the function b, from Theorem 2.1.5 and from (3.10) we easily derive the following properties of this system:

3.1.1. Theorem. 1) *The system* (3.11) *has a unique solution which determines a unique solution of* (2.5,a–c) *via the formula* (3.7).

2) *For each* $\bar{u} \in \mathbf{R}_n$ *the matrix* $\mathbb{A}(\bar{u})$ *is symmetric and positive definite.*

3) *The functions* $\boldsymbol{\Phi}_i(\bar{u})$ *and* $a_{ij}(\bar{u})$ *are bounded and continuously differentiable in* $\mathbf{R}_n$.

4) *If b depends on x only (the flow is incompressible), then the system* (3.11) *is linear.* ■

Now let us briefly mention the construction of the system (3.11). For the purpose of iterative solution of this system, which will be described in Section 3.2, it is necessary to be able to calculate the matrix $\mathbb{A}(\bar{u})$ and the vector $\boldsymbol{\Phi}(\bar{u})$ for an arbitrary $\bar{u} \in \mathbf{R}_n$.

If we use (2.1,a) and the fact that $\nabla v_h \mid T = \text{const}$ for each $T \in \mathcal{T}_h$ and each $v_h \in W_h$, then we can write

$$a_h(\psi_h, v_h) = \sum_{T \in \mathcal{T}_h} \nabla\psi_h \mid T \,.\, \nabla v_h \mid T \int_T b(\cdot, (\nabla\psi_h \mid T)^2)\,\mathrm{d}x \tag{3.12}$$

$$\text{for} \quad \psi_h \in W_h\,, \quad v_h \in V_h\,.$$

Further, if $\psi_h \in W_h$ is a function determined by (3.7) with given $u_1, \ldots, u_n$, then

$$a_{ij}(\bar{u}) = \sum_{T \in \mathcal{T}_h} \nabla w_i^* \mid T \,.\, \nabla w_j^* \mid T \int_T b(\cdot, (\nabla\psi_h \mid T)^2)\,\mathrm{d}x\,, \tag{3.13}$$

$$\Phi_i(\bar{u}) = \mu_{2h}(w_i^*) + \omega \sum_{T \in \mathcal{T}_h} \frac{\partial v_h}{\partial x_1} \mid T \int_T r^2\,\mathrm{d}x -$$

$$- \sum_{T \in \mathcal{T}_h} \nabla\psi_h^* \mid T \,.\, \nabla w_i^* \mid T \int_T b(\cdot, (\nabla\psi_h \mid T)^2)\,\mathrm{d}x\,,$$

$$i, j = 1, \ldots, n\,, \quad \bar{u} = (u_1, \ldots, u_n)^T\,.$$

In general, it is impossible to calculate the integrals in (3.12) or (3.13) exactly and we must use suitable numerical quadratures. The integral of a function f over a triangle T can be evaluated e.g. with the use of the formula

$$\int_T f\,\mathrm{d}x \approx f(x_T)\,\text{meas}\,(T) \tag{3.14}$$

or

$$\int_T f\,\mathrm{d}x \approx \tfrac{1}{3}(f(P_i) + f(P_j) + f(P_k))\,\text{meas}\,(T)\,, \tag{3.15}$$

where x_T is the centre of the triangle T, meas (T) is its measure and P_i, P_j, P_k its vertices.

Also the integrals from the definition (2.11)** of the functional μ_{2h} arising in the problem (PSIω.2.1) should be approximated. We use the following possibility: If $S \subset \Gamma_k$ $(k = 1, 2)$ is a side given by vertices P_i, P_j of a triangle $T \in \mathcal{T}_h$ adjacent to Γ_k, then we write

$$\int_S m_k w_i\,\mathrm{d}S \approx |P_i - P_j|\,(\tfrac{1}{3}(m_k(P_i) - m_k(P_j)) + \tfrac{1}{2}m_k(P_j))\,. \tag{3.16}$$

This formula is obtained by linear approximation of the function m_k on the side S and it uses the fact that $w_i(P_i) = 1$, $w_i(P_j) = 0$.

Then, instead of the problem (2.5,a–c) we get a modified problem equivalent to a new system of equations of the form (3.11). However, the elements of the matrix $\mathbb{A}(\bar{u})$ and the components of the vector $\Phi(\bar{u})$ are calculated from the formulae (3.13), where the integrals are replaced by the above mentioned quadratures.

The question how the numerical integration affects the resulting error of the method can be studied quite analogously as in [8]. We shall not deal with this problem here.

3.1.2. Some remarks to the problem (PSIω.1.3). Let us return to part 2.1.6 devoted to the discretization of the problem (PSIω.1.3). The following assertions hold:

1) The space V_h defined in (2.18) has a basis $\{w_k^*\}$ which consists of the following elements:

(3.17)
$$\begin{aligned} &\text{a)} \quad w_i \quad \text{for} \quad P_i \in \sigma_h \cap \Omega_{\tau h}\,, \\ &\text{b)} \quad w_i + w_j \quad \text{for} \quad P_i \in \sigma_h \cap (\Gamma^- - (\Gamma_1 \cup \Gamma_2)),\ P_j = P_i^\tau\,, \\ &\text{c)} \quad \sum_{P_j \in \sigma_h \cap \Gamma_1} w_j\,. \end{aligned}$$

2) In the space $\tilde{V}_h$ defined by (2.21) we have a basis $\{\tilde{w}_k^*\}$ formed by

(3.18)
$$\begin{aligned} &\text{a)} \quad w_i \quad \text{for} \quad P_i \in \sigma_h \cap \Omega_{\tau h} - \{P_{i_0}\}\,, \\ &\text{b)} \quad w_i + w_j \quad \text{for} \quad P_i \in \sigma_h \cap (\Gamma^- - (\Gamma_1 \cup \Gamma_2))\,, \quad P_j = P_i^\tau\,, \\ &\text{c)} \quad \sum_{P_j \in \sigma_h \cap \Gamma_1} w_j\,, \\ &\text{d)} \quad \sum_{P_j \in \sigma_h \cap C_{0h} \cup \{P_{i_0}\}} w_j\,. \end{aligned}$$

Proof of these assertion is a consequence of (3.2), the definitions of the spaces V_h, $\tilde{V}_h$ and the fact of the disjoint decomposition of $\sigma_h - \Gamma_2$. ∎

We see that the bases defined in (3.17,a–c) and (3.18,a–d) have the same number of elements equal to $n = \operatorname{card}(\sigma_h \cap \Omega_{\tau h}) + \operatorname{card}[\sigma_h \cap (\Gamma^- - (\Gamma_1 \cup \Gamma_2))] + 1$ $(<N)$.

The sought approximate solution will be expressed as

(3.19)
$$\psi_h = \psi_h^* + \sum_{j=1}^{n} u_j \tilde{w}_j^*\,.$$

In view of the discrete formulation (2.22,a–c), the relations (2.10), (2.11,a) (2.23) and (2.6), we get a system of algebraic equations for unknown values $u_1, \ldots, u_n$:

(3.20)
$$\begin{aligned} &\int_{\Omega_{\tau h}} b(\cdot, (\nabla\psi_h^* + \sum_{s=1}^{n} u_s \nabla\tilde{w}_s^*)^2) \sum_{j=1}^{n} u_j \nabla\tilde{w}_j^* \cdot \nabla w_i^* \,dx = \\ &= \mu_h(w_i^*) - \int_{\Omega_{\tau h}} b(\cdot, (\nabla\psi_h^* + \sum_{s=1}^{n} u_s \nabla\tilde{w}_s^*)^2) \nabla\psi_h^* \cdot \nabla w_i^* \,dx\,, \\ &\qquad i = 1, \ldots, n\,. \end{aligned}$$

This system has the form (3.11), where the elements of $\mathbb{A}(\bar{u})$ and the components of $\boldsymbol{\Phi}(\bar{u})$ are defined by the relations

(3.21)
$$\begin{aligned} \Phi_i(\bar{u}) &= \mu_h(w_i^*) - \int_{\Omega_{\tau h}} b(\cdot, (\nabla\psi_h^* + \sum_{s=1}^{n} u_s \nabla\tilde{w}_s^*)^2) \nabla\psi_h^* \cdot \nabla w_i^* \,dx\,, \\ a_{ij}(\bar{u}) &= \int_{\Omega_{\tau h}} b(\cdot, (\nabla\psi_h^* + \sum_{s=1}^{n} u_s \nabla\tilde{w}_s^*)^2) \nabla w_i^* \cdot \nabla\tilde{w}_j^* \,dx\,. \end{aligned}$$

Since $V_h \neq \tilde{V}_h$, the matrix $\mathbb{A}(\bar{u})$ is not symmetric. Nevertheless, we have

3.1.3. Theorem. 1) *The functions $\Phi_i(\bar{u})$ and $a_{ij}(\bar{u})$ are bounded and continuously differentiable in $\mathbf{R}_n$.*

2) *If b depends on x only, then the system considered is linear.*

3) *If all angles of all $T \in \mathcal{T}_h$ are less than or equal to 90° and the triangulation $\mathcal{T}_h$ is sufficiently fine, then $\mathbb{A}(\bar{u})$ is irreducibly diagonally dominant (see e.g.* [1]) *for each $\bar{u} \in \mathbf{R}_n$.*

Proof of assertions 1), 2) follows immediately from the properties of the function b. For the proof of 3) see [15]. (Similar results can also be found in [13].) ∎

If the flow is incompressible, then it follows from the assertion 3) that the linear system (3.11) is uniquely solvable under the above mentioned assumptions.

3.2. Iterative solution of the system (3.11)

The system (3.11) is linear for an incompressible flow and can be solved by the successive over-relaxation method (SOR). Its convergence follows from Theorem 3.1.1 (the assertion 2)). If we consider trailing conditions, then in view of assertion 3) from Theorem 3.1.3, the Gauss-Seidel method converges. However, the matrix $\mathbb{A}$ differs from a symmetric one only slightly. This implies the conjecture that also the SOR method could be convergent and even faster than the Gauss-Seidel method. This has also been confirmed by a series of numerical experiments.

In the nonlinear compressible case we applied two methods:

3.2.1. Generalized relaxation method. We write $\mathbb{A} = \mathbb{A}_L + \mathbb{D} + \mathbb{A}_U$, where the matrix $\mathbb{A}_L$ is strictly lower triangular, $\mathbb{D}$ diagonal and $\mathbb{A}_U$ strictly upper triangular. We construct a sequence $\{\bar{u}^k\}_{k \geqq 0}$ by the process

(3.22)
$$\begin{aligned} &\text{a)} \quad \bar{u}^0 \in \mathbf{R}_n \quad \text{is a convenient initial approximation,} \\ &\text{b)} \quad \mathbb{A}_L(\bar{u}^k)\, \bar{u}^{k+1} + \mathbb{D}(\bar{u}^k)\, \bar{u}^{k+1/2} + \mathbb{A}_U(\bar{u}^k)\, \bar{u}^k = \Phi(\bar{u}^k)\,, \\ &\text{c)} \quad \bar{u}^{k+1} = \bar{u}^k + \nu(\bar{u}^{k+1/2} - \bar{u}^k)\,, \quad k \geqq 0\,, \quad \nu \in (0, 1\rangle\,. \end{aligned}$$

This method was originally proposed by M. Feistauer and used successfully in calculations of subsonic flows by the finite difference method (see e.g. [4]). The convergence of this method can be proved under relatively restrictive assumptions. Numerical experiments show that it converges for the problems both without and with trailing conditions if $\nu = 1$, the initial approximation represents the incompressible flow and the calculated stream field is subsonic. The optimal relaxation parameter can be found only experimentally. However, the convergence of this method seems to be faster in connection with the finite-difference method. This is the reason why we applied also another method. ∎

3.2.2. Steepest-descent method with preconditioning. This method is inspired by the iterative process (2.16) used in the proof of the solution of the discrete problem (2.5,a–c). With respect to the definition of the operators T_h and F_ν, (2.16) can be written in the form

$$(3.23)\qquad \begin{gathered} u_h^0 \in V_h\,, \\ (u_h^{k+1}, w_i^*) = (u_h^k, w_i^*) - \nu[a_h(u_h^k + \psi_h^*, w_i^*) - \mu_h(w_i^*)]\,, \\ i = 1, \ldots, n\,, \quad k \geqq 0\,, \quad \nu > 0\,. \end{gathered}$$

Here $(\cdot, \cdot)$ denotes a scalar product on V_h. In order to get good convergence of the process (3.23), it is convenient to choose this scalar product in some relationship with the form a_h. We put, e.g.,

$$(3.24)\qquad (u_h, v_h) = (u_h, v_h)_b = \int_{\Omega_{\tau h}} b(\cdot, 0)\, \nabla u_h \,.\, \nabla v_h \,\mathrm{d}x\,, \quad u_h, v_h \in V_h\,.$$

It follows from the properties of the function b that (3.24) really determines a scalar product on V_h.

Now we see that (3.23) represents an iterative process, which can be written in the following way:

$$(3.25)\qquad \begin{gathered} \bar{u}^0 \in \mathbf{R}_n\,, \\ \mathbb{B}\xi^{k+1} = -(\mathbb{A}(\bar{u}^k)\,\bar{u}^k - \Phi(\bar{u}^k))\,, \\ \bar{u}^{k+1} = \bar{u}^k + \nu\xi^{k+1}\,, \quad k \geqq 0\,. \end{gathered}$$

$\nu > 0$ is a suitable parameter and $\mathbb{B}$ is the matrix with elements

$$(3.26)\qquad \mathrm{b}_{ij} = \sum_{T \in \mathcal{T}_h} \nabla w_i^* \mid T \,.\, \nabla w_j^* \mid T \int_T b(x, 0)\,\mathrm{d}x\,, \quad i, j = 1, \ldots, n\,.$$

Usually we choose the initial approximation $\bar{u}^0$ as a solution of an incompressible flow.

In view of the results from Section 2, we know that there exists $\tilde{\nu} > 0$ such that the process (3.25) converges for $\nu \in (0, \tilde{\nu})$. The choice of the optimal value of ν can be carried out only experimentally. The estimate of the value $\tilde{\nu}$ is determined by the constants α, K from (2.7) and (2.8), respectively.

The method described represents the steepest descent method with preconditioning for the minimization of the functional Φ_h, defined by

$$(3.27)\qquad \Phi_h(u_h) = \int_{\Omega_h} F(\cdot, (\nabla(\psi_h^* + u_h))^2)\,\mathrm{d}x - \mu_h(u_h)\,, \qquad u_h \in V_h\,,$$

$$F(x, \eta) = \tfrac{1}{2}\int_0^\eta b(x, t)\,\mathrm{d}t\,, \quad \eta \geqq 0\,,$$

on the space V_h. From this point of view, other iterative methods arise as possible ones – e.g. the steepest descent method with an optimal step size in every iteration or the well-known conjugate gradient method. We shall not go into details. Cf. e.g. [17]. ■

4. EXAMPLES

Based on the theory presented, a system of FORTRAN programs was written which allows the finite element solution of irrotational cascade flows on an arbitrary surface of revolution in a layer of variable thickness of an arbitrary form.

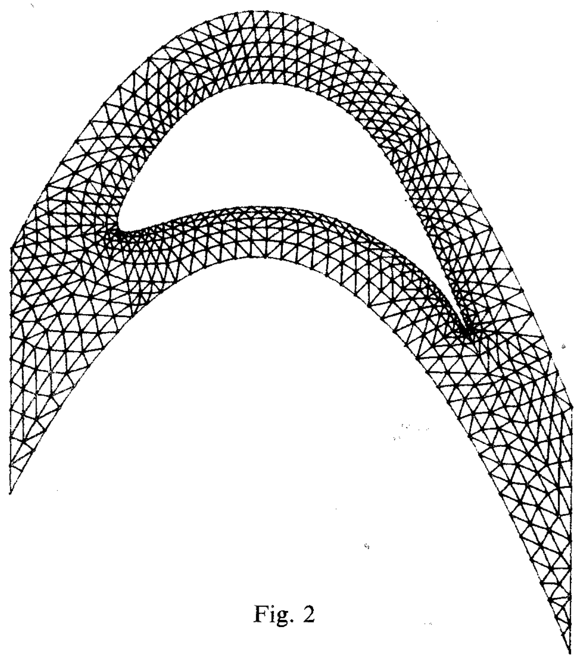


Fig. 2

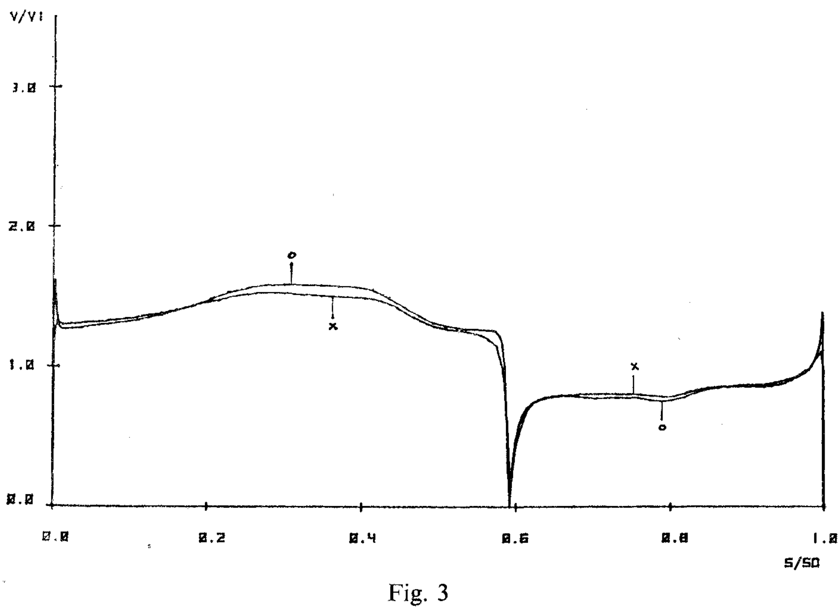


Fig. 3

Here we shall present some calculated stream fields past a cascade of profiles. The domain Ω_τ is drawn in Fig. 1. Fig. 2 represents the triangulation. It consists of 685 vertices and 1160 triangles. In all the examples considered the iterative processes (SOR in linear cases, generalized relaxation method in case of a nonlinear subsonic flow) were stopped when the relative error in the stream function was less than 10^{-4}.

The first example concerns an incompressible flow with the inlet angle 57°. The resulting outlet angle is $-64°39'$. We have used the trailing condition. In Fig. 3 we see the comparison of the velocity distributions (i.e. $V/V1$-distributions, where V is the velocity and $V1$ is the inlet velocity) round the profile calculated by the finite element method (denoted by $\times$) and by the integral equation method (denoted by $\bigcirc$).

Figs. 4–6 represent the streamlines, lines of constant velocity and velocity vectors drawn at the vertices of the triangulation.

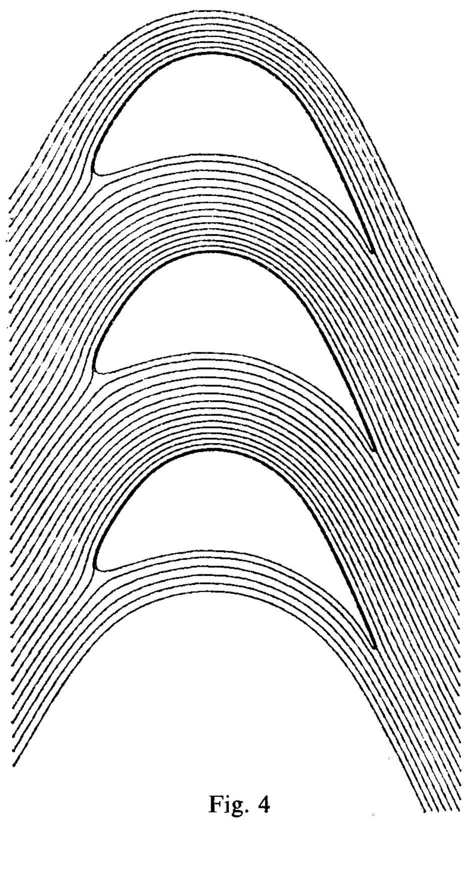
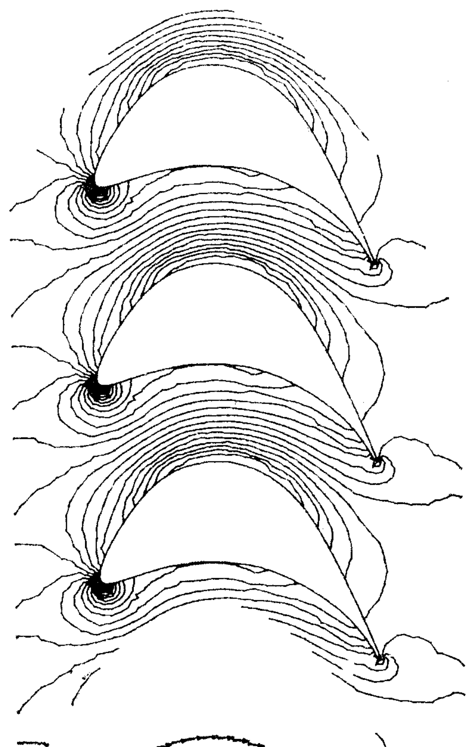


Fig. 4

Fig. 5



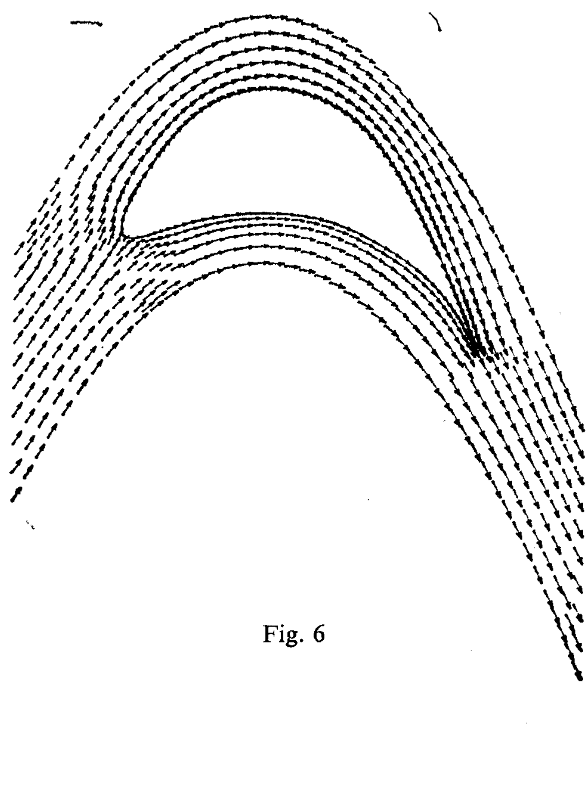


Fig. 6

Fig. 7 represents the velocity distribution on the profile calculated for the compressible fluid with the use of the method (3.22,a – c). For the inlet Mach number $M = 0.7$ this iterative process converged in 50 iterations.

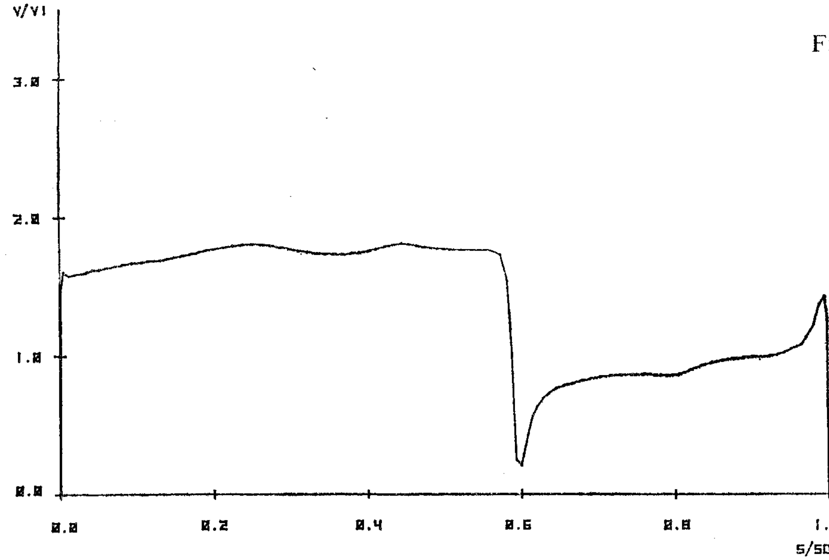


Fig. 7

As the third example we consider an incompressible cascade flow in a fluid layer whose thickness decreases linearly from its inlet value $h_1 > 0$ to the outlet thickness $h_2 = h_1/4$.

In Fig. 8 the velocity distribution is drawn and Fig. 9 represents the streamlines.

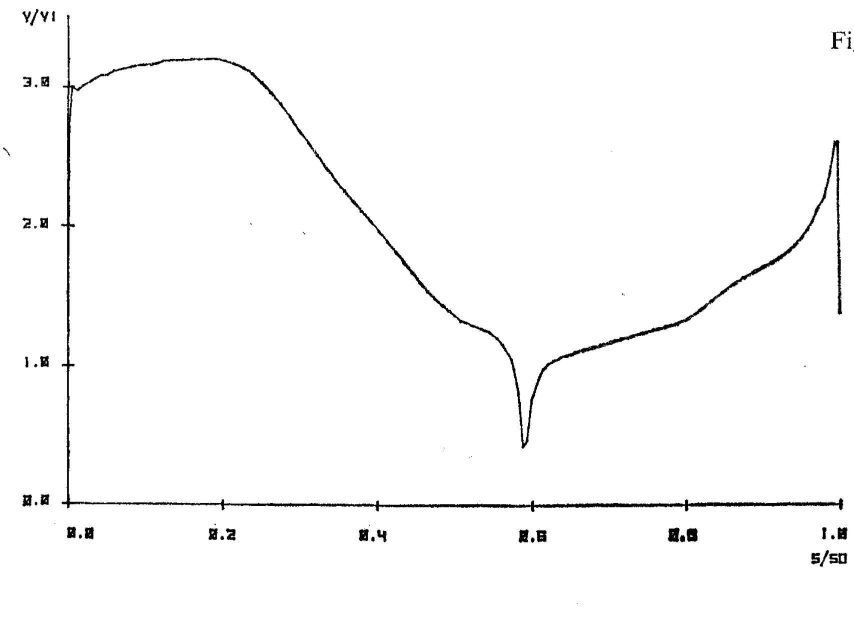


Fig. 8

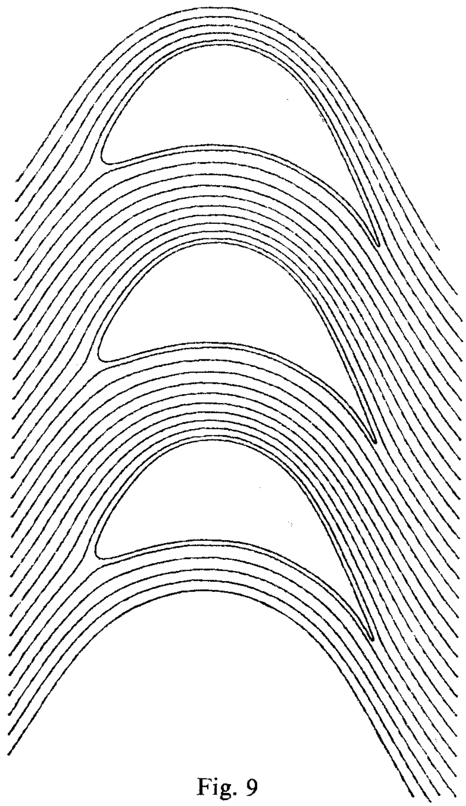


Fig. 9

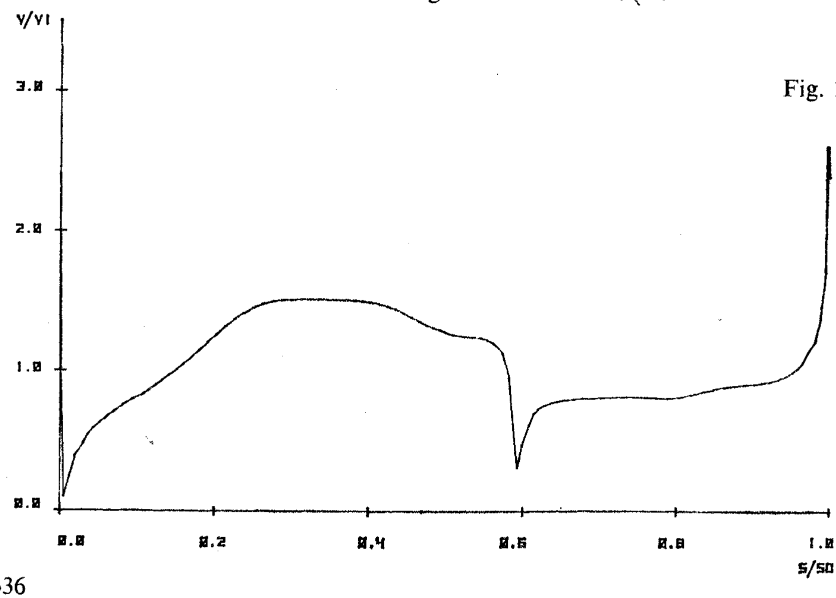


Fig. 10

In all the above examples we considered the trailing condition. In the following example we do not use this condition but we assume that beside the inlet angle also the outlet angle $-45°$ is given. We consider again an incompressible plane flow. The results drawn in Fig. 10 (velocity distribution) and Fig. 11 (streamlines) show that the outlet angle is not well determined. (It ought to be equal to $-64°39'$ as in the first example.) This last example demonstrates the necessity to use the trailing conditions.

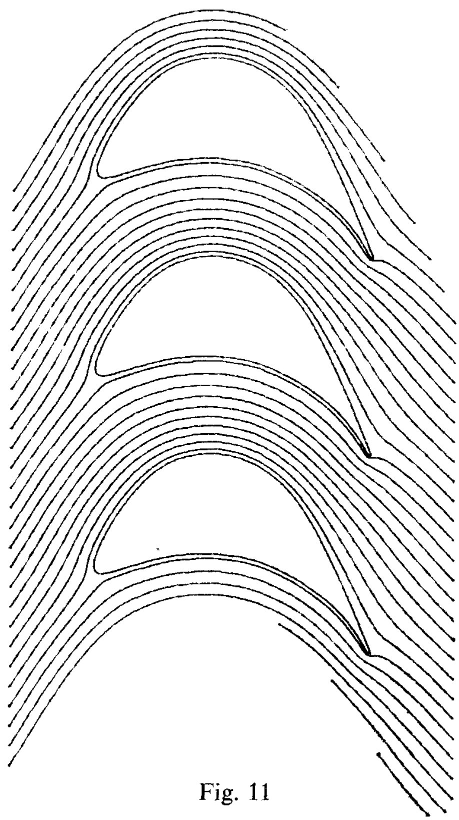


Fig. 11

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Souhrn

ŘEŠENÍ PROUDĚNÍ PROFILOVÝMI MŘÍŽEMI VE VRSTVĚ PROMĚNNÉ TLOUŠŤKY METODOU KONEČNÝCH PRVKŮ

MILOSLAV FEISTAUER, JIŘÍ FELCMAN, ZDENĚK VLÁŠEK

V předloženém článku se zabýváme numerickým modelováním nevířivých stacionárních nebo kvazistacionárních proudových polí profilovými mřížemi ve vrstvě proměnné tloušťky na osově symetrické proudoploše. Tento problém, velmi důležitý pro konstruktéry lopatkových strojů, je formulován jako nelineární okrajová úloha eliptického typu pro proudovou funkci s nestandardními nehomogenními podmínkami a je diskretizován metodou konečných prvků. Článek je věnován jak teoretickým, tak praktickým aspektům této metody: jde zejména o konvergenci metody, numerickou integraci, iterační metody pro řešení nelineárního diskrétního problému a algoritmizaci. Jsou též uvedeny ukázky numerických výsledků získaných mnohoúčelovým programem, který autoři vytvořili.

Резюме

РЕШЕНИЕ ТЕЧЕНИЙ РЕШЕТКАМИ ПРОФИЛЕЙ В СЛОЕ ПЕРЕМЕННОЙ ТОЛЬЩИНЫ МЕТОДОМ КОНЕЧНЫХ ЭЛЕМЕНТОВ

MILOSLAV FEISTAUER, JIŘÍ FELCMAN, ZDENĚK VLÁŠEK

Работа посвящена численному моделлированию дозвуковых стационарных или квазистационарных безвихревых течений решетками профилей в слое переменной тольщины на осесимметричной поверхности тока. Эта проблема, очень важная для конструкторов лопаточных машин, формулирована как нелинейная задача эллиптического типа для функции тока с нестандартными неоднородными краевыми условиями и дискретизирована методом конечных элементов. Статья посвящена теоретическим и практическим аспектам метода: изучены сходимость метода, численное интегрирование, итерационные методы для решения нелинейной дискретной задачи и алгорифмизация. Приведены некоторые результаты численных расчетов по авторами составленной программе.

Authors' address: RNDr. *Miloslav Feistauer*, CSc., RNDr. *Jiří Felcman*, RNDr. *Zdeněk Vlášek*, CSc., Matematicko-fyzikální fakulta UK, Malostranské n. 25, 118 00 Praha 1.