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FINITE ELEMENT SOLUTION OF THE FUNDAMENTAL EQUATIONS OF SEMICONDUCTOR DEVICES II*

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Abstract. In part I of the paper (see Zlámal [13]) finite element solutions of the nonstationary semiconductor equations were constructed. Two fully discrete schemes were proposed. One was nonlinear, the other partly linear. In this part of the paper we justify the nonlinear scheme. We consider the case of basic boundary conditions and of constant mobilities and prove that the scheme is unconditionally stable. Further, we show that the approximate solution, extended to the whole time interval as a piecewise linear function, converges in a strong norm to the weak solution of the semiconductor equations. These results represent an extended and corrected version of results announced without proof in Zlámal [14].

Keywords: semiconductor devices, finite element method, fully discrete approximate solution, convergence

MSC 2000: 65N30, 65N12

1. Introduction

We consider the case of constant mobilities,

$$\mu_n = \text{const} > 0, \quad \mu_p = \text{const} > 0.$$

For simplicity, we restrict ourselves to two dimensions. After scaling introduced in part I of the paper (see [13]), the nonstationary semi-conductor equations assume

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the form (see I, (1.5)–(1.7))

$$-\Delta\psi = \alpha(p - n + N(x)) \quad \text{in } \Omega \ \ \forall t \in (0,T), \tag{1.1}$$

$$\frac{\partial n}{\partial t} = \delta_n \nabla \cdot (\nabla n - n\nabla\psi) - R(n,p) \quad \text{in } Q, \tag{1.2}$$

$$\frac{\partial p}{\partial t} = \delta_p \nabla \cdot (\nabla p + p\nabla\psi) - R(n,p) \quad \text{in } Q, \tag{1.3}$$

$$R(n,p) = \frac{np-1}{n+p+2}, \quad Q = \Omega \times (0,T).$$

Here all quantities are dimensionless, $\delta_s = \gamma_s \mu_s$ $(s = n, p)$, γ_s are positive constants introduced in part I, $x = (x_1, x_2)$ and the boundary Γ of the bounded domain Ω is a polygon, i.e. the union of a finite number of linear segments Γ_j, $1 \leqslant j \leqslant J$ (Γ_{j+1} follows Γ_j according to the positive orientation). We also fix a partition of the set $\{1, \dots, J\}$ into two subsets $\mathcal{D}$ and $\mathcal{N}$ and denote $\Gamma^1 = \bigcup\limits_{j\in\mathcal{D}} \Gamma_j$, $\Gamma^2 = \bigcup\limits_{j\in\mathcal{N}} \Gamma_j$. The boundary conditions are Dirichlet nonhomogeneous and Neumann homogeneous boundary conditions

$$\psi\big|_{\Gamma^1} = \psi^*\big|_{\Gamma^1}, \ \ n\big|_{\Gamma^1} = n^*\big|_{\Gamma^1}, \ \ p\big|_{\Gamma^1} = p^*\big|_{\Gamma^1}, \ \ n^*, p^* > 0 \text{ on } \Gamma^1, \tag{1.4}$$

$$\frac{\partial\psi}{\partial\nu}\Big|_{\Gamma^2} = \frac{\partial n}{\partial\nu}\Big|_{\Gamma^2} = \frac{\partial p}{\partial\nu}\Big|_{\Gamma^2} = 0, \tag{1.5}$$

ν is the unit outward normal to Q and we assume

$$\Gamma^1 \neq \emptyset.$$

In addition, we have the initial condition

$$n\big|_{t=0} = n^0(x), \quad p\big|_{t=0} = p^0(x) \text{ in } \Omega, \quad n^0(x), p^0(x) > 0 \text{ on } \overline{\Omega} \tag{1.6}$$

and the requirement

$$n(x,t) > 0, \quad p(x,t) > 0 \quad \text{on } \overline{Q}. \tag{1.7}$$

The weak formulation of the problem reads as follows (it differs somewhat from that introduced in part I, Remark 1.2):

P r o b l e m P. Given $\psi^* \in H^{1,\infty}(\Omega)$, $n^*, p^* \in H^{1,q}(\Omega)$, $q > 2$, $n^*\big|_{\Gamma^1} > 0$, $p^*\big|_{\Gamma^1} > 0$ and N measurable and bounded on $\overline{\Omega}$, $n^0, p^0 \in H^{1,q}(\Omega)$, $q > 2$, $n^0 > 0$ and

$p^0 > 0$ on $\bar{\Omega}$, find ψ, n, p such that $\psi - \psi^* \in L^\infty(0,T; V \cap H^{1,\infty}(\Omega))$, $n - n^*$, $p - p^* \in L^2(0,T;V)$ and

$$\forall t \in (0,T) \ d(\psi, v) = \alpha(p - n + N(x), v) \ \forall v \in V, \tag{1.8}$$

$$\left.\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}(n,v) + \delta_n \bar{\nu}^2(\psi; n, v) + (R(n,p), v) = 0 &\quad (1.9)\\ &\frac{\mathrm{d}}{\mathrm{d}t}(p,v) + \delta_n \bar{\pi}^2(\psi; p, v) + (R(n,p), v) = 0 &\quad (1.10)\end{aligned}\right\} \text{ in } \mathscr{D}'((0,T)) \ \forall v \in V,$$

$$n\big|_{t=0} = n^0, \ \ p\big|_{t=0} = p^0 \quad \text{in } \Omega, \tag{1.11}$$

$$\forall t \in [0,T] \ \ n \geqslant 0, \ \ p \geqslant 0 \quad \text{a.e.} \quad \text{in } \Omega. \tag{1.12}$$

Here $(\cdot,\cdot)$ denotes the scalar product in $L_2(\Omega)$, $V = \{v\colon v \in H^1(\Omega), \ v\big|_{\Gamma^1} = 0\}$ and the form $\mathrm{d}(\psi, v)$ has the same meaning as in part I, (1.18), whereas $\bar{\nu}^2(\psi; n, v)$ and $\bar{\pi}^2(\psi, p, v)$ arise by dividing the original ones by mobilities, i.e.

$$d(\psi, v) = \int_\Omega \nabla\psi \cdot \nabla v \,\mathrm{d}x, \quad \bar{\nu}^2(\psi; n, v) = \int_\Omega (\nabla n - n\nabla\psi) \cdot \nabla v \,\mathrm{d}x, \tag{1.13}$$
$$\bar{\pi}^2(\psi; p, v) = \int_\Omega (\nabla p + p\nabla\psi) \cdot \nabla v \,\mathrm{d}x.$$

We will prove later that if ψ, n, p, belonging to the spaces introduced above, satisfy (1.8)–(1.10), then $n, p \in C([0,T]; L^2(\Omega))$, hence the initial condition (1.11) makes sense. We also prove that under a condition on $\partial\Omega$ introduced later the problem P has at most one solution.

In the sequel, we assume that all requirements concerning the data introduced in the definition of the problem P are satisfied (if more assumptions are needed, they will be explicitly introduced).

2. A fully discrete approximate solution

We consider a family $\{T_h\}$ of triangulations of $\bar{\Omega}$. If an element K belongs to T_h, then h_K denotes the greatest side of K and, of course, we have the requirement

$$h = \max_{K \in T_h} h_K \to 0.$$

In the paper we need the following assumption.

A^0: The family $\{T_h\}$ fulfills the minimum angle condition, i.e. if θ_h is the minimum angle of all angles of T_h, then $\theta_h \geqslant \theta_0 > 0$, and it is of acute type, i.e. any angle of any triangle from $\bigcup_h T_h$ is not greater than $\frac{1}{2}\pi$.

R e m a r k 2.1. If J is the Jacobian matrix of the linear mapping which maps a given triangle K on the reference triangle $\widehat{K}$, then

$$Ch_K^2 \leqslant |\det J| \leqslant C^{-1} h_K^2 \quad \forall K \in \bigcup_h T_h.$$

These bounds are very simple consequences of the minimum angle condition (see Zlámal [15]). We will use them implicitly at several places.

By W_h we denote the space

$$W_h = \{v\colon\ v \in C(\overline{\Omega}),\ \ v \text{ is a linear polynomial on each } K \in T_h\},$$

$V_h \subset W_h$ is the space

$$V_h = \{v\colon\ v \in W_h, v\big|_{\Gamma^1} = 0\}.$$

Let Ψ, N, P, v belong to to W_h. By $\Psi_j, \ldots$ we denote the value of $\psi, \ldots$ at the node x^j. The discrete analogues of the forms $\bar{\nu}^2(\psi; n, v)$ and $\bar{\pi}^2(\psi; p, v)$ are

$$\text{(2.1)}\quad \bar{\nu}_h^2(\Psi; N, v) = \sum_{K\in T_h} \sum_{r=j,k,m} v_r \Big(\int_K (J^T)^{-1} D^K J^T \nabla N \cdot \nabla v^r \,\mathrm{d}x - \int_K N_r \nabla\Psi \cdot \nabla v^r \,\mathrm{d}x \Big),$$

$$\text{(2.2)}\quad \bar{\pi}_h^2(\Psi; P, v) = \sum_{K\in T_h} \sum_{r=j,k,m} v_r \Big(\int_K (J^T)^{-1} B^K J^T \nabla P \cdot \nabla v^r \,\mathrm{d}x + \int_K P_r \nabla\Psi \cdot \nabla v^r \,\mathrm{d}x \Big),$$

J is the Jacobian matrix of the mapping which maps K on $\widehat{K}$ in such a way that the node x^r is mapped to the vertex $(0, 0)$ in the reference plane (see I, p. 33), v_r is the value $v(x^r)$, v^r is the basis function associated with the node x^r and B^K, D^K are the matrices

$$\text{(2.3)}\quad \left.\begin{aligned} B^K &= \operatorname{diag}(B(\Psi_1 - \Psi_2), B(\Psi_1 - \Psi_3)),\\ &\quad B(\xi) = \xi(\mathrm{e}^{\xi} - 1)^{-1}\\ D^K &= \operatorname{diag}(D(\Psi_1 - \Psi_2), D(\Psi_1 - \Psi_3)),\\ &\quad D(\xi) = \mathrm{e}^{\xi} B(\xi) = B(-\xi) \end{aligned}\right\} -\infty < \xi < \infty.$$

Here Ψ_1, Ψ_2, Ψ_3 stand for the local notation of the values of Ψ at the vertices x^j, x^k, x^m such that $\Psi_1 = \Psi_j$, $\Psi_2 = \Psi_k$, $\Psi_3 = \Psi_m$. All factors in the first terms on the right-hand sides of (2.1), (2.2) depend on the mapping chosen and we have two such mappings. Nevertheless, the integrands are the same, i.e. the forms are uniquely determined. (2.1) and (2.2) are definitions equivalent to those introduced in part I, (2.18) and (2.19) (of course, we have to put $\mu_s = 1$, $s = n, p$ in these equations).

The $L^2(\Omega)$-scalar product $(\cdot,\cdot)$ will be approximated by $(\cdot,\cdot)_h$ defined in part I, (3.3) ($(u,v)_h = \sum_{j=1}^{q} m_j u_j v_j$, $m_j > 0$). We consider an equally spaced partition of the interval $[0,T]$: $t_i = i\Delta t$, $i = 0,\dots,r$, $r = \frac{T}{\Delta t}$. By f_I we understand, as in part I, the interpolate of a given function f.

Now, we can introduce the fully discrete approximate solution Ψ^i, N^i, P^i, $i = 0,\dots,r$. The defining equations are

$$d(\Psi^i, v) = \alpha(P^i - N^i + N_I, v)_h, \tag{2.4}$$

$$\left.\begin{aligned} &(\Delta N^i, v)_h + \delta_n \Delta t \bar{\nu}_h^2(\Psi^i; N^i, v) + \Delta t (R^i, v)_h = 0 \quad (2.5)\\ &(\Delta P^i, v)_h + \delta_p \Delta t \bar{\pi}_h^2(\Psi^i; P^i, v) + \Delta t (R^i, v)_h = 0 \quad (2.6) \end{aligned}\right\} \ \forall v \in V_h,\ \ i = 1,\dots,r,$$

$$\Psi^i_j = \psi^*(x^j),\ \ N^i_j = n^*(x^j),\ \ P^i_j = p^*(x^j)\ \ \forall x^j \in \Gamma^1, \tag{2.7}$$

$$N^0 = n^0_I,\ \ P^0 = p^0_I, \tag{2.8}$$

$$N^i > 0,\ \ P^i > 0 \quad \text{on } \overline{\Omega}, \quad i = 1,\dots,r. \tag{2.9}$$

Here $\Delta N^i = N^i - N^{i-1}$, $\Delta P^i = P^i - P^{i-1}$ and $R^i = R(N^i, P^i)$.

3. Stability, existence, uniqueness

We introduce three more assumptions.

A^1: The measure of the angles (lying inside Ω) of the polygonal boundary Γ is smaller than π at vertices where two sides of Γ^1 or Γ^2 meet and smaller than $\frac{1}{2}\pi$ at vertices where a side of Γ^1 and a side of Γ^2 meet,

A^2: $\psi^* \in H^{2,q}(\Omega)$, $\quad N \in H^{1,q}(\Omega)$, $\quad q > 2$,

A^3: $\psi^* \in H^{2,q}(\Omega)$, $\quad N \in H^{1,q}(\Omega)$, $\quad q > 2$, $\quad n^*, p^*, n^0, p^0 \in H^2(\Omega)$, $\quad \psi^*\big|_{\Gamma_j} \in H^{2,q}(\Gamma_j)$ $\forall j \in \mathcal{D}$.

Remark 3.1. Consider the boudary value problem

$$\begin{cases} -\Delta u = f \quad \text{in } \Omega, \quad f \in L^{q_0}(\Omega), \\ u\big|_{\Gamma_j} = g_j,\ \ j \in \mathcal{D}, \quad g_j \in H^{2-1/q_0, q_0}(\Gamma_j), \\ g_j(\sigma_j) = g_{j+1}(\sigma_j) \ \text{ if } j, j+1 \in \mathcal{D}, \\ \frac{\partial u}{\partial \nu}\big|_{\Gamma_j} = g_j,\ \ j \in \mathcal{N},\ \ g_j \in H^{1-1/q_0, q_0}(\Gamma_j). \end{cases} \tag{3.1}$$

Here σ_j is the vertex of Γ where Γ_j and Γ_{j+1} meet. Then, if the condition A^1 is satisfied and $q_0 \in [2, 2+\varepsilon]$ where ε is a sufficiently small positive number, the operator

$$T_{0,q_0}\colon\ u \to \Big\{-\Delta u;\ u\big|_{\Gamma_j},\ \ j\in\mathcal{D};\ \ \frac{\partial u}{\partial \nu}\Big|_{\Gamma_j},\ \ j\in\mathcal{N}\Big\}$$

from $H^{2,q_0}(\Omega)$ into the subspace of $L^{q_0}(\Omega)\times\prod\limits_{j\in\mathcal{D}} H^{2-1/q_0,q_0}(\Gamma_j)\times\prod\limits_{j\in\mathcal{N}} H^{1-1/q_0,q_0}(\Gamma_j)$ defined by the consistency condition (3.1) is an isomorphism (see Grisvald [7], p. 209–210). Therefore T_{0,q_0}^{-1} is bounded, i.e. we have

$$\|u\|_{H^{2,q_0}(\Omega)} \leqslant C\Big(\|f\|_{L^{q_0}(\Omega)} + \sum_{j\in\mathcal{D}}\|g_j\|_{H^{2-1/q_0,q_0}(\Gamma_j)} + \sum_{j\in\mathcal{N}}\|g_j\|_{H^{1-1/q_0,q_0}(\Gamma_j)}\Big), \quad 2\leqslant q_0\leqslant 2+\varepsilon. \tag{3.2}$$

Theorem 3.1. *If A^0 is fulfilled, the scheme* (2.4)–(2.8) *with the requirement* (2.9) *is unconditionally stable in the following sense: for an arbitrary h and for Δt sufficintly small, $\Delta t\leqslant\Delta t_0$ where Δt_0 does not depend on h, we have*

$$\max_{1\leqslant i\leqslant r}\{\|N^i\|_{L^2(\Omega)}, \|P^i\|_{L^2(\Omega)}, \|\Psi^i\|_{H^1(\Omega)}\} + \Big\{\Delta t\sum_{i=1}^{r}(\|N^i\|^2_{H^1(\Omega)} + \|P^i\|^2_{H^1(\Omega)})\Big\}^{1/2} \leqslant C(\|N^0\|_{L^2(\Omega)} + \|P^0\|_{L^2(\Omega)} + 1). \tag{3.3}$$

In (3.3) and in the sequel, C denotes a positive constant, not necessarily the same at any two places, which does not depend on h_K, h, Δt and on the index i. It is also independent of the parameter θ introduced in the proof of Theorem 4.1, with an exception mentioned later.

Proof. a) We consider the first term in (2.1), i.e.

$$A(\Psi;N,v) = \sum_{k\in T_h}\sum_{r=j,k,m} v_r\int_K (J^T)^{-1}D^K J^T\nabla N\cdot\nabla v^r\,\mathrm{d}x. \tag{3.4}$$

In this paragraph, the coordinates are denoted by x, y. Let (x_r,y_r), $r=j,k,m$, be the vertices of the element $K\in T_h$. After elementary, though not short computation, we come to the following expression for the form A:

$$\begin{aligned} A(\Psi;N,v) = \sum_{K\in T_h}[&\alpha_j^K(N_m-N_k)(B_{km}v_m - B_{mk}v_k)\\ &+\alpha_k^K(N_j-N_m)(B_{mj}v_j - B_{jm}v_m)\\ &+\alpha_m^K(N_k-N_j)(B_{jk}v_k - B_{kj}v_j)], \end{aligned} \tag{3.5}$$

where

$$
(3.6)\qquad \begin{cases} \alpha_j^K = & \dfrac{1}{4\,\mathrm{area}(K)}[(x_j - x_k)(x_j - x_m) + (y_j - y_k)(y_j - y_m)], \\ \alpha_k^K = & \dfrac{1}{4\,\mathrm{area}(K)}[(x_k - x_m)(x_k - x_j) + (y_k - y_m)(y_k - y_j)], \\ \alpha_m^K = & \dfrac{1}{4\,\mathrm{area}(K)}[(x_m - x_j)(x_m - x_k) + (y_m - y_j)(y_m - y_k)], \end{cases}
$$

$$
(3.7)\qquad B_{rs} = B(\Psi_r - \Psi_s), \quad r, s = j, k, m.
$$

The coefficients α_r^K can be expressed in a different form. Let us denote by θ_r the measure of the angle of K lying at the vertex (x_r, y_r). The square brackets on the right-hand sides of (3.6) are scalar products of sides of K considered as vectors and we easily derive

$$
(3.8)\qquad \alpha_r^K = \frac{1}{2}\cot\theta_r, \quad r = j, k, m.
$$

From the acuteness and from the minimum angle condition it follows immediately that

$$
(3.9)\qquad 0 \leqslant \alpha_r^K \leqslant \frac{1}{2}\cot\theta_0 = C, \quad r = j, k, m, \;\; \forall K \in \bigcup_h T_h.
$$

The function $B(\xi)$ has the property

$$
B(-\xi) = B(\xi) + \xi \quad \forall \xi \in (-\infty, \infty).
$$

Cosequently,

$$
B_{rs}v_s - B_{sr}v_r = B_{rs}(v_s - v_r) + (\Psi_s - \Psi_r)v_r
$$

and also

$$
B_{rs}v_s - B_{sr}v_r = B_{sr}(v_s - v_r) + (\Psi_s - \Psi_r)v_s,
$$

hence

$$
B_{rs}v_s - B_{sr}v_r = \frac{1}{2}(B_{rs} + B_{sr})(v_s - v_r) + \frac{1}{2}(\Psi_s - \Psi_r)(v_r + v_s).
$$

Introducing coefficients

$$
(3.10)\qquad b_{rs} = b_{sr} = \frac{1}{2}(B_{rs} + B_{sr}),
$$

we get

$$A(\Psi; N, v) = a(\Psi; N, v) + \frac{1}{2} b(\Psi; N, v), \tag{3.11}$$

$$\begin{aligned} a(\Psi; N, v) = \sum_{K \in T_h} [&\alpha_j^K b_{mk}(N_m - N_k)(v_m - v_k) \\ &+ \alpha_k^K b_{mj}(N_j - N_m)(v_j - v_m) \\ &+ \alpha_m^K b_{kj}(N_k - N_j)(v_k - v_j)], \end{aligned} \tag{3.12}$$

$$\begin{aligned} b(\Psi; N, v) = \sum_{K \in T_h} [&\alpha_j^K (\Psi_m - \Psi_k)(N_m - N_k)(v_m + v_k) \\ &+ \alpha_k^K (\Psi_j - \Psi_m)(N_j - N_m)(v_j + v_m) \\ &+ \alpha_m^K (\Psi_k - \Psi_j)(N_k - N_j)(v_k + v_j)]. \end{aligned} \tag{3.13}$$

Evidently, the form $a(\Psi; N, v)$ is symmetric with respect to N and v. Further

$$\begin{aligned} &a(\Psi; v, v) \geqslant \mathrm{d}(v, v), \\ &\quad \mathrm{d}(\Psi, v) = a(0; \Psi, v) \\ &\qquad = \sum_{K \in T_h} \big(\alpha_j^K (\Psi_m - \Psi_k)(v_m - v_k) + \alpha_k^K (\Psi_j - \Psi_m)(v_j - v_m) \\ &\qquad\quad + \alpha_m^K (\Psi_k - \Psi_j)(v_k - v_j)\big). \end{aligned} \tag{3.14}$$

The equality follows from $a(0; w, v) = A(0; w, v) = \mathrm{d}(w, v)$ (set $D^K = I$ in (3.4)), the inequality from (3.9) and from the properties of the function $\chi(\xi) = \frac{1}{2}(B(\xi) + B(-\xi)) = \frac{1}{2}\xi\frac{\mathrm{e}^{\xi}+1}{\mathrm{e}^{\xi}-1}$. We have $\chi(0) = 1$, $\chi(\xi) = \chi(-\xi)$, $\chi(\infty) = \infty$ and $\chi'(\xi) \geqslant 0$ $\forall \xi \in [0, \infty)$. Therefore,

$$\chi(\xi) \geqslant 1 \qquad \forall \xi \in (-\infty, \infty)$$

and hence $b_{rs} \geqslant 1$.

In the sequel, by Nv ($N, v \in W_h$) we always mean the function $w \in W_h$ such that

$$w(x^j) \equiv w_j = N(x^j) v(x^j) \equiv N_j v_j$$

and by w^2 the function ww. Then

$$\sum_{K \in T_h} \sum_{r = j,k,m} v_r \int_K N_r \nabla\Psi \cdot \nabla v^r \, \mathrm{d}x = \sum_{K \in T_h} \int_K \nabla\Psi \cdot \nabla(Nv) = \mathrm{d}(\Psi, Nv). \tag{3.15}$$

Hence, by (2.1), (3.4) and (3.11) we obtain

$$\bar{\nu}_h^2(\Psi; N, v) = a(\Psi; N, v) + \frac{1}{2} b(\Psi; N, v) - \mathrm{d}(\Psi, Nv). \tag{3.16}$$

Denoting

$$c(\Psi;N,v) = -\frac{1}{2}b(\Psi;N,v) + \mathrm{d}(\Psi,Nv) \tag{3.17}$$

and taking into account (3.14), we conclude that

$$\bar{\nu}_h^2(\Psi;N,v) = a(\Psi;N,v) - c(\Psi;N,v), \tag{3.18}$$

$$\begin{aligned} c(\Psi;N,v) = \frac{1}{2}\sum_{K\in T_h} &[\alpha_j^K(\Psi_m-\Psi_k)(N_m+N_k)(v_m-v_k) \\ &+ \alpha_k^K(\Psi_j-\Psi_m)(N_j+N_m)(v_j-v_m) \\ &+ \alpha_m^K(\Psi_k-\Psi_j)(N_k+N_j)(v_k-v_j)]. \end{aligned} \tag{3.19}$$

In a similar way we derive

$$\bar{\pi}_h^2(\Psi;N,v) = a(\Psi;P,v) + c(\Psi;P,v) \tag{3.20}$$

and, moreover,

$$c(\Psi;w,w) = \frac{1}{2}d(\Psi,w^2). \tag{3.21}$$

b) First, let us note that Theorem 3.1.6 of Ciarlet ([3], p. 124), and Sobolev's imbedding theorems yield

$$\|\tau_I\|_{L^\infty(\Omega)} \leqslant C\|\tau\|_{H^{1,q}(\Omega)} \ \ \forall \tau \in H^{1,q}(\Omega), \quad q>2. \tag{3.22}$$

We use the following notation: $\sigma=\psi_I^*$, $\omega=n_I^*$, $\varrho=p_I^*$, $\overline{N}^i=N_i-\omega$, $\overline{P}^i=P^i-\varrho$, $\|v\|_h^2=(v,v)_h$ $\forall v\in W_h$. Setting $v=\Psi^i-\psi_I^*$ in (2.4), we easily obtain

$$\|\Psi^i\|_V \leqslant C(\|\overline{N}^i\|_h + \|\overline{P}^i\|_h + 1), \quad \|v\|_V^2 = d(v,v) \ \ \forall v\in V. \tag{3.23}$$

From (2.5) and (2.6) we obtain

$$\begin{aligned} &\delta_n^{-1}(\Delta\overline{N}^i,\overline{N}^i)_h + \Delta t\, a(\Psi^i;N^i,N^i) \\ &\quad = \Delta t\big(a(\Psi^i;N^i,\omega) + c(\Psi^i;\overline{N}^i,\overline{N}^i) + c(\Psi^i;\omega,\overline{N}^i) - \delta_n^{-1}(R^i,\overline{N}^i)_h\big), \end{aligned} \tag{3.24}$$

$$\begin{aligned} &\delta_p^{-1}(\Delta\overline{P}^i,\overline{P}^i)_h + \Delta t\, a(\Psi^i;P^i,P^i) \\ &\quad = \Delta t\big(a(\Psi^i;P^i,\varrho) - c(\Psi^i;\overline{P}^i,\overline{P}^i) - c(\Psi^i;\varrho,\overline{P}^i) - \delta_p^{-1}(R^i,\overline{P}^i)_h\big). \end{aligned} \tag{3.25}$$

We have to estimate the terms on the right-hand sides of (3.24) and (3.25). Since the function $\chi(\xi)$ is bounded by

$$\frac{1}{2}|\xi| < \chi(\xi) < 1 + \frac{1}{2}|\xi| \quad \forall \xi \neq 0, \tag{3.26}$$

it follows from (3.12), (3.9) and (3.23) that

$$
\begin{aligned}
|a(\Psi^i; N^i, \omega)| \leqslant & \sum_{K\in T_h} (\alpha_j^K |N_m^i - N_k^i|\,|\omega_m - \omega_k| + \ldots) \\
& + \|\omega\|_{L^\infty(\Omega)} \sum_{K\in T_h} (\alpha_j^K |\Psi_m^i - \Psi_k^i|\,|N_m^i - N_k^i| + \ldots) \\
& \leqslant \|\omega\|_V \, \|N^i\|_V + \|\omega\|_{L^\infty(\Omega)} \|\Psi^i\|_V \|N^i\|_V \\
& \leqslant C\|N^i\|_V (1 + \|\Psi^i\|_V) \leqslant C\|N^i\|_V (\|\overline{N}^i\|_h + \|\overline{P}^i\|_h + 1) \\
& \leqslant \frac{1}{4}\|N^i\|_V^2 + C(\|\overline{N}^i\|_h^2 + \|\overline{P}^i\|_h^2 + 1).
\end{aligned}
$$

Further, by (3.19),

$$
\begin{aligned}
|c(\Psi^i; \omega, \overline{N}^i)| &\leqslant C\|\omega\|_{L^\infty(\Omega)} \|\Psi^i\|_V \|\overline{N}^i\|_V \\
&\leqslant C(\|\overline{N}^i\|_h + \|\overline{P}^i\|_h + 1)\|\overline{N}^i\|_V \\
&\leqslant C(\|\overline{N}^i\|_h^2 + \|\overline{P}^i\|_h^2 + 1) + \frac{1}{4}\|N^i\|_V^2.
\end{aligned}
$$

By (3.21) and (2.4), we obtain

$$
\begin{aligned}
c(\Psi^i; \overline{N}^i, \overline{N}^i) - c(\Psi^i; \overline{P}^i, \overline{P}^i) &= -\frac{1}{2} d(\Psi^i, (\overline{P}^i)^2 - (\overline{N}^i)^2) \\
&= -\frac{1}{2}\alpha(P^i - N^i + N_I, (\overline{P}^i)^2 - (\overline{N}^i)^2)_h \\
&= -\frac{1}{2}\alpha(P^i - N^i, (P^i)^2 - (N^i)^2)_h - \frac{1}{2}\alpha(N_I, (\overline{P}^i)^2 - (\overline{N}^i)^2)_h \\
&\quad - \frac{1}{2}\alpha(P^i - N^i, -2\varrho P^i + \varrho^2 + 2\omega N^i - \omega^2)_h.
\end{aligned}
$$

Since

$$
\big(P^i - N^i, (P^i)^2 - (N^i)^2\big)_h = \sum_{j=1}^{q} m_j (P_j^i - N_j^i)^2 (P_j^i + N_j^i) \geqslant 0, \tag{3.27}
$$

it follows easily that

$$
c(\Psi^i; \overline{N}^i, \overline{N}^i) - c(\Psi^i; \overline{P}^i, \overline{P}^i) \leqslant C(\|\overline{N}^i\|_h^2 + \|\overline{P}^i\|_h^2 + 1). \tag{3.28}
$$

Finally, by (3.13) from part I, we derive $|R^i| \leqslant \frac{1}{2} + N^i + P^i \leqslant C + |\overline{N}^i| + |\overline{P}^i|$ and

$$
|(R^i, \overline{N}^i)_h| \leqslant C(\|\overline{N}^i\|_h^2 + \|\overline{P}^i\|_h^2 + 1).
$$

Adding (3.24) and (3.25) and using the above estimates and the inequality

$$a(\Psi; v, v) \geqslant d(v, v)$$

(see (3.14)), we get

$$\delta_n^{-1}(\Delta\overline{N}^i, \overline{N}^i)_h + \delta_p^{-1}(\Delta\overline{P}^i, \overline{P}^i)_h + \frac{1}{2}\Delta t(\|N^i\|_V^2 + \|P^i\|_V^2) \leqslant C\Delta t(\|\overline{N}^i\|_h^2 + \|\overline{P}^i\|_h^2 + 1).$$

Summing from $i = 1$ to $i = m$ yields

$$\begin{aligned}(3.29)\quad & \delta_n^{-1}\|\overline{N}^m\|_h^2 + \delta_p^{-1}\|\overline{P}^m\|_h^2 + \Delta t\sum_{i=1}^{m}(\|N^i\|_V^2 + \|P^i\|_V^2) \\ & \leqslant \delta_n^{-1}\|\overline{N}^0\|_h^2 + \delta_p^{-1}\|\overline{P}^0\|_h^2 + C + C\Delta t\sum_{i=1}^{m}(\|\overline{N}^i\|_h^2 + \|\overline{P}^i\|_h^2).\end{aligned}$$

For Δt such that $\delta_n^{-1} - C\Delta t \geqslant \frac{1}{2}\delta_n^{-1}$ and $\delta_p^{-1} - C\Delta t \geqslant \frac{1}{2}\delta_p^{-1}$, i.e. for $\Delta t \leqslant \frac{1}{2C}\min(\delta_n^{-1}, \delta_p^{-1}) = \Delta t_0$, we easily derive

$$\delta_n^{-1}\|\overline{N}^m\|_h^2 + \delta_p^{-1}\|\overline{P}^m\|_h^2 \leqslant C(\|\overline{N}^0\|_h^2 + \|\overline{P}^0\|_h^2 + 1) + C\Delta t\sum_{i=1}^{m-1}(\|\overline{N}^i\|_h^2 + \|\overline{P}^i\|_h^2)$$

and the discrete Gronwall inequality implies

$$\|\overline{N}^i\|_h \leqslant C(\|\overline{N}^0\|_h + \|\overline{P}^0\|_h + 1), \;\; \|\overline{P}^i\|_h \leqslant C(\|\overline{N}^0\|_h + \|\overline{P}^0\|_h + 1), \;\; i = 1, \dots, r,$$

while (3.29) yields

$$\Delta t\sum_{i=1}^{r}(\|N^i\|_v^2 + \|P^i\|_v^2) \leqslant C(\|\overline{N}^0\|_h^2 + \|\overline{P}^0\|_h^2 + 1).$$

Since $\|v\| \leqslant \|v\|_h \leqslant C\|v\| \; \forall v \in W_h$ (see Raviart [12]) and since we have proved (3.23), the estimate (3.3) is proved. □

Remark 3.2. Here we use the letter K for "stiffness" matrices. Therefore, we denote the element by e instead of by K. We derive easily for the general case of nonconstant mobilities that

$$(3.30)\quad \nu_h^2(\Psi; N, v) = \sum_{e\in T_h}(\mathbf{v}^e)^T K_n^e(\boldsymbol{\Psi}^e)\mathbf{N}^e, \quad \pi_h^2(\Psi; P, v) = \sum_{e\in T_h}(\mathbf{v}^e)^T K_p^e(\boldsymbol{\Psi}^e)\mathbf{P}^e,$$

where

$$\mathbf{v}^e = (v_j, v_k, v_m)^T, \qquad \mathbf{\Psi}^e = (\Psi_j, \Psi_k, \Psi_m)^T,$$
$$\mathbf{N}^e = (N_j, N_k, N_m)^T, \quad \mathbf{P}^e = (P_j, P_k, P_m)^T,$$

(3.31) $$K_n^e(\Psi^e) = \mu_n^e(K_1^e - \frac{1}{2}K_2^e), \quad K_p^e(\Psi^e) = \mu_p^e(K_1^e + \frac{1}{2}K_2^e).$$

Here $\mu_s^e = \mu_s(x^e, \|\nabla\Psi\|)$, $(s = n, p)$, x^e is the center of gravity of e (see part I) and

$$K_1^e = \{k_{s,r}^1\}_{r,s=1}^3, \;\; k_{rs}^1 = k_{sr}^1, \;\; K_2^e = \{k_{rs}^2\}_{r,s=1}^3,$$
$$k_{11}^1 = \alpha_k^e b_{jm} + \alpha_m^e b_{kj}, \;\; k_{12}^1 = k_{21}^1 = -\alpha_m^e b_{jk}, \;\; k_{13}^1 = k_{31}^1 = -\alpha_k^e b_{jm},$$
$$k_{22}^1 = \alpha_m^e b_{kj} + \alpha_j^e b_{mk}, \;\; k_{23}^1 = k_{32}^1 = -\alpha_j^e b_{km}, \;\; k_{33}^1 = \alpha_j^e b_{mk} + \alpha_k^e b_{jm},$$
$$k_{11}^2 = \alpha_k^e(\Psi_j - \Psi_m) + \alpha_m^e(\Psi_j - \Psi_k), \;\; k_{12}^2 = -\alpha_m^e(\Psi_k - \Psi_j), \;\; k_{13}^2 = \alpha_k^e(\Psi_j - \Psi_m),$$
$$k_{21}^2 = -k_{12}^2, \;\; k_{22}^2 = \alpha_m^e(\Psi_k - \Psi_j) + \alpha_j^e(\Psi_k - \Psi_m), \;\; k_{23}^2 = -\alpha_j^e(\Psi_m - \Psi_k),$$
$$k_{31}^2 = -k_{13}^2, \;\; k_{32}^2 = -k_{23}^2, \;\; k_{33}^2 = \alpha_j^e(\Psi_m - \Psi_k) + \alpha_k^e(\Psi_m - \Psi_j).$$

The coefficients α_r^e and b_{rs} are given by (3.6) or (3.8) and by (3.10), respectively. For the forms $\nu_h^1(\Psi; W, v)$ and $\pi_h^1(\Psi; Z, v)$ one can derive

(3.32) $$\nu_h^1(\Psi; W, v) = \sum_{e \in T_h} \mu_n^e[\alpha_j^e \beta_{mk}^n (W_m - W_k)(v_m - v_k) + \ldots],$$

(3.33) $$\pi_h^1(\Psi; Z, v) = \sum_{e \in T_h} \mu_p^e[\alpha_j^e \beta_{mk}^p (Z_m - Z_k)(v_m - v_k) + \ldots],$$

(3.34) $$\begin{cases} \beta_{rs}^n = -\dfrac{\Psi_s - \Psi_r}{\mathrm{e}^{-\Psi_s} - \mathrm{e}^{-\Psi_r}} = \left(\displaystyle\int_0^1 \exp[-\Psi_r - (\Psi_s - \Psi_r)\xi]\,\mathrm{d}\xi\right)^{-1} \\ \qquad = \mathrm{e}^{\Psi_s} B(\Psi_s - \Psi_r), \\ \beta_{rs}^p = \dfrac{\Psi_s - \Psi_r}{\mathrm{e}^{\Psi_s} - \mathrm{e}^{\Psi_r}} = \left(\displaystyle\int_0^1 \exp[\Psi_r + (\Psi_s - \Psi_r)\xi]\,\mathrm{d}\xi\right)^{-1} \\ \qquad = \mathrm{e}^{-\Psi_r} B(\Psi_s - \Psi_r), \end{cases}$$

(3.35) $$\begin{cases} \nu_h^1(\Psi, W, v) = \displaystyle\sum_{e \in T_h} (\mathbf{v}^e)^T K_n^e(\mathbf{\Psi}^e)\mathbf{W}^e, \\ \pi_h^1(\Psi; Z, v) = \displaystyle\sum_{e \in T_h} (\mathbf{v}^e)^T K_p^e(\mathbf{\Psi}^e)\mathbf{Z}^e, \\ \quad K_n^e(\mathbf{\Psi}^e) = \mu_n^K K_3^e, \;\; K_p^e(\mathbf{\Psi}^e) = \mu_p^K K_4^e, \end{cases}$$

(3.36) $$K_3^e = \{k_{rs}^3\}_{r,s=1}^3, \;\; k_{rs}^3 = k_{sr}^3, \;\; K_4^e = \{k_{rs}^4\}_{r,s=1}^3, \;\; k_{rs}^4 = k_{sr}^4$$

and the formulas for k^3_{rs} and k^4_{rs} are the same as for k^1_{rs} with the exception that we replace the coefficients b^n_{rs} by β^n_{rs} and β^p_{rs}, respectively. Lemma 3.1 from part I follows directly from the expressions (3.32), (3.33) and from (3.9).

Formulas (3.30) and (3.35) show that the global "stiffness" matrices can be derived from "element stiffness" matrices by the well known assembly procedure always used in finite element computations.

Before passing to the next theorem we introduce some lemmas.

Lemma 3.1. *Let the family $\{T_h\}$ satisfy the minimum angle condition, let* A^1 *be satisfied and $\psi^* \in H^{2,q}(\Omega)$, $F \in W_h$. If $\Psi \in W_h$ is the solution of*

$$\begin{aligned}\Psi(x^j) &= \psi^*(x^j) \ \ \forall x^j \in \Gamma^1,\\ \mathrm{d}(\Psi, v) &= (F, v)_h \ \ \forall v \in V_h,\end{aligned}$$

then Ψ belongs to $H^{1,p}(\Omega)$, $1 \leqslant p < \infty$ if $q = 2$, $p = \infty$ if $q > 2$, and

$$\|\Psi\|_{H^{1,p}(\Omega)} \leqslant C(\|F\|_{L^{q_0}(\Omega)} + \|\psi^*\|_{H^{2,q_0}(\Omega)}), \tag{3.37}$$

where $q_0 = 2 + \varepsilon$, $\varepsilon = 0$ if $q = 2$ and ε is positive and sufficiently small if $q > 2$.

The proof of Lemma 3.1 requires two more lemmas.

Lemma 3.2. *Let the family $\{T_h\}$ satisfy the minimum angle condition. If $F \in W_h$ and $1 < q \leqslant \infty$ then there exists $G \in L^q(\Omega)$ such that*

$$\begin{aligned}(F, v)_h &= (G, v) \ \ \forall v \in W_h,\\ \|G\|_{L^q(\Omega)} &\leqslant C\|F\|_{L^q(\Omega)}.\end{aligned}$$

P r o o f. Let $q < \infty$. Then

$$\begin{aligned}|(F, v)_h| &= \Big|\sum_{K \in T_h} \frac{1}{3}\,\mathrm{area}(K) \sum_{r=j,k,m} F_r v_r\Big|\\ &\leqslant C \sum_{K\in T_h} h_K^2 \Big\{\sum_{r=j,k,m} |F_r|^q\Big\}^{1/q}\Big\{\sum_{r=j,k,m} |v_r|^{q'}\Big\}^{1/q'}\\ &\leqslant C \sum_{K\in T_h} \Big\{\sum_{r=j,k,m} h_K^2|F_r|^q\Big\}^{1/q}\Big\{\sum_{r=j,k,m} h_K^2|v_r|^{q'}\Big\}^{1/q'}.\end{aligned}$$

Now, let $\mathbf{w}$ be the vector $(w_j, w_k, w_m)^T$ and consider the functional $\Phi(\mathbf{w}) = \int_{\widehat{K}} |\widehat{w}|^q \,\mathrm{d}\xi$ ($\widehat{w}$ is the linear polynomial assuming the values w_r ($r = j, k, m$) at the nodes of the reference triangle $\widehat{K}$). Denoting by $\mathbf{y}$ the vector $\frac{1}{\|\mathbf{w}\|_q}\mathbf{w}$ we have $\|\mathbf{y}\|_q = 1$ and

$$\frac{\Phi(\mathbf{w})}{\|\mathbf{w}\|_q^q} = \Phi(\mathbf{y}) \geqslant \min_{\|\mathbf{y}\|=1} \Phi(\mathbf{y}) = C > 0.$$

Hence

$$\sum_{r=j,k,m} h_K^2 |F_r|^q \leqslant C h_K^2 \int_{\widehat{K}} |\widehat{F}|^q \, \mathrm{d}\xi \leqslant C \int_K |F|^q \, \mathrm{d}x \tag{3.38}$$

and

$$|(F,v)_h| \leqslant C \sum_{K \in T_h} \|F\|_{L^q(K)} \|v\|_{L^{q'}(K)} \leqslant C \|F\|_{L^q(\Omega)} \|v\|_{L^{q'}(\Omega)}.$$

Now $(F,v)_h$ is a linear bounded functional on $W_h \subset L^{q'}(\Omega)$. Almost the same argument proves this assertion for $q = \infty$. By the Hahn-Banach theorem, we can extend this functional onto $L^{q'}(\Omega)$ and the norm is preserved. Hence, keeping the same notation, we have $\|(F,\cdot)_h\|_* \leqslant C\|F\|_{L^q(\Omega)}$. On the other hand, the extension has a unique representation (G,v), where $G \in L^q(\Omega)$ and $\|G\|_{L^q(\Omega)} = \|(F,\cdot)_h\|_* \leqslant C\|F\|_{L^q(\Omega)}$ (see, e.g., Kufner, John, Fučík [8]). □

Lemma 3.3. *Let the family $\{T_h\}$ satisfy the minimum angle condition. If $\tau \in H^{2,q}(\Omega)$, $1 < q \leqslant \infty$, then there exists $g \in L^q(\Omega)$ such that*

$$d(\tau - \tau_I, v) = (g, v) \ \ \forall v \in W_h,$$
$$\|g\|_{L^q(\Omega)} \leqslant C\|\tau\|_{H^{2,q}(\Omega)}.$$

P r o o f. We have

$$|d(\tau - \tau_I, v)| \leqslant \sum_{K \in T_h} \|\nabla(\tau - \tau_I)\|_{L^q(K)} \|\nabla v\|_{L^{q'}(K)}.$$

It is easy to show that

$$\|\nabla v\|_{L^{q'}(K)}^{q'} \leqslant C h_K^{-q'} \|v\|_{L^{q'}(K)}.$$

Therefore

$$|d(\tau - \tau_I, v)| \leqslant C \sum_{K \in T_h} h_K^{-1} \|\nabla(\tau - \tau_I)\|_{L^q(K)} \|v\|_{L^{q'}(K)}.$$

Since $\|\nabla(\tau - \tau_I)\|_{L^q(K)} \leqslant C h_K \|\tau\|_{H^{2,q}(K)}$ (see [3], Theorem 3.1.6) we get

$$d(\tau - \tau_I, v)| \leqslant C \sum_{K \in T_h} \|\tau\|_{H^{2,q}(K)} \|v\|_{L^{q'}(K)} \leqslant C \|\tau\|_{H^{2,q}(\Omega)} \|v\|_{L^{q'}(\Omega)}$$

and the proof can be completed in the same way as the proof of the preceding lemma. □

Proof of Lemma 3.1. Let φ be the solution of the problem

$$-\Delta\varphi = G + g + \Delta\psi^*, \quad \varphi\big|_{\Gamma^1} = 0, \quad \frac{\partial\varphi}{\partial\nu}\Big|_{\Gamma^2} = -\frac{\partial\psi^*}{\partial\nu}\Big|_{\Gamma^2}.$$

We have $\varphi \in V$, $d(\varphi, v) = (G + g, v) - d(\psi^*, v)$ $\forall v \in V$. On the other hand, $\Psi - \Psi_I^* \in V_h$ and

$$\begin{aligned} d(\Psi - \psi_I^*, v) &= (F, v)_h - d(\psi_I^*, v) \\ &= (G, v) + d(\psi^* - \psi_I^*, v) - d(\psi^*, v) \\ &= (G + g, v) - d(\psi^*, v) \ \ \forall v \in V_h. \end{aligned}$$

Therefore $\Psi - \psi_I^*$ is the Ritz projection of φ. According to the theorem by Rannacher and Scott ([11], p. 438), it follows that $\|\Psi - \psi_I^*\|_{H^{1,p}(\Omega)} \leqslant C\|\varphi\|_{H^{1,p}(\Omega)}$ for $2 \leqslant p \leqslant \infty$ (the theorem is proved for the case $\Gamma^2 = \emptyset$; however, under the condition A^1 the operator $u \to \{-\Delta u;\ \frac{\partial u}{\partial\nu}\big|_{\Gamma_j},\ j \in \mathcal{N}\}$ from the space $\{u\colon u \in H^{2,q_0}(\Omega),\ u|_{\Gamma^1} = 0\}$ into the space $L^{q_0}(\Omega) \times \prod\limits_{j\in\mathcal{N}} H^{1-1/q_0,q_0}(\Gamma_j)$ is an isomorphism for $q_0 \in [2, 2+\varepsilon]$ (see Remark 3.1) and this is the reason why the proof is valid for the case $\Gamma^1 \neq \emptyset$, $\Gamma^2 \neq \emptyset$, too). Sobolev imbedding theorems yield

$$\|\Psi - \psi_I^*\|_{H^{1,p}(\Omega)} \leqslant C\|\varphi\|_{H^{2,q_0}(\Omega)}, \quad q_0 = 2 + \varepsilon,$$

where $\varepsilon = 0$ if $p < \infty$ and ε is positive and sufficiently small if $p = \infty$. Consequently, (3.2) gives

$$\begin{aligned} \|\Psi\|_{H^{1,p}(\Omega)} &\leqslant C\Bigg(\|F\|_{L^{q_0}(\Omega)} + \|\psi^*\|_{H^{2,q_0}(\Omega)} + \sum_{j\in\mathcal{N}}\Big\|\frac{\partial\psi^*}{\partial\nu}\Big\|_{H^{1-1/q_0,q_0}(\Omega)}\Bigg) \\ &\leqslant C(\|F\|_{L^{q_0}(\Omega)} + \|\psi^*\|_{H^{2,q_0}(\Omega)}). \end{aligned}$$

Here we have used an inequality which holds for $u \in H^{2,p}(\Omega)$, $p > 1$ and for each j, $1 \leqslant j \leqslant J$:

$$\|u\|_{H^{2-1/p,p}(\Gamma_j)} + \Big\|\frac{\partial u}{\partial\nu}\Big\|_{H^{1-1/p,p}(\Gamma_j)} \leqslant C\|u\|_{H^{2,p}(\Omega)}. \tag{3.38'}$$

(Concerning the proof, we consider Calderon's extension $\widetilde{u}$ of u; see, e.g., Nečas [10], Theorem 3.10 and Remark 3.5, p. 80–81). Then $\|\widetilde{u}\|_{H^{2,p}(E^2)} \leqslant C\|u\|_{H^{2,p}(\Omega)}$. Let Ω^1 be a domain from $C^{1,1}$ such that $\Gamma_j \subset \partial\Omega^1$ and $\Omega \subset \Omega^1$. Then

$$\begin{aligned} \|u\|_{H^{2-1/p,p}(\Gamma_j)} + \Big\|\frac{\partial u}{\partial\nu}\Big\|_{H^{1-1/p,p}(\Gamma_j)} &= \|\widetilde{u}\|_{H^{2-1/p,p}(\Gamma_j)} + \Big\|\frac{\partial\widetilde{u}}{\partial\nu}\Big\|_{H^{1-1/p,p}(\Gamma_j)} \\ &\leqslant \|\widetilde{u}\|_{H^{2-1/p,p}(\partial\Omega^1)} + \Big\|\frac{\partial\widetilde{u}}{\partial\nu}\Big\|_{H^{1-1/p,p}(\partial\Omega^1)} \\ &\leqslant C\|\widetilde{u}\|_{H^{2,p}(\Omega^1)} \leqslant C\|\widetilde{u}\|_{H^{2,p}(E^2)} \leqslant C\|u\|_{H^{2,p}(\Omega)}. \end{aligned}$$

The second inequality follows from the theorem on traces; see, e.g., [10], Theorem 5.5, p. 99. □

Theorem 3.2. *Let* A^0 *be fulfilled. Then there exist* Ψ^i, N^i, P^i, $i = 1, \ldots, r$, *satisfying* (2.4)–(2.9). *If, in addition,* A^1 *and* A^2 *are fulfilled and* Δt *is sufficiently small,* $\Delta t \leqslant \Delta t_0$ *where* Δt_0 *does not depend on* h*, then these solutions are unique.*

P r o o f. a) The existence of Ψ^i, N^i, P^i, $i = 1, \ldots, r$, can be proved by elementary degree theory in the same way as we proved the existence theorem in part I (see Theorem 3.1). The proof makes use of two things: of the a priori estimate (3.1) and of the maximum principle (Lemma 3.1 from part I).

b) As far as uniqueness concerns, we prove first another a priori estimate. To this end we set $v = (N^i)^3 - \omega^3$ and $v = (P^i)^3 - \varrho^3$ in (2.5) and in (2.6), respectively (we recall that if $w \in W_h$, then w^m is the function $\sum\limits_{r=1}^{q} w_r^m v^r \in W_h$, $m = 2, 3, \ldots$). We get

$$\begin{aligned}(3.39)\quad &\delta_n^{-1}(\Delta N^i, (N^i)^3)_h + \Delta t \bar{\nu}_h^2(\Psi^i; N^i, (N^i)^3)\\ &= \delta_n^{-1}(\Delta N^i, \omega^3) + \Delta t\big(\bar{\nu}_h^2(\Psi^i; N^i, \omega^3) - \delta_n^{-1}(R^i, (N^i)^3)_h + \delta_n^{-1}(R^i, \omega^3)_h\big),\end{aligned}$$

$$\begin{aligned}(3.39')\quad &\delta_p^{-1}(\Delta P^i, (P^i)^3)_h + \Delta t \bar{\pi}_h^2(\Psi^i; P^i, (P^i)^3)\\ &= \delta_p^{-1}(\Delta P^i, \varrho^3)_h + \Delta t\big(\bar{\pi}_h^2(\Psi^i; P^i, \varrho^3) - \delta_p^{-1}(R^i, (P^i)^3)_h + \delta_p^{-1}(R^i, \varrho^3)_h\big).\end{aligned}$$

We estimate the second terms on the left-hand sides of the above equations. For $w \in W_h$, $w \geqslant 0$, we have (see (3.12), (3.13), (3.14) and (3.26))

$$\begin{aligned}&\frac{1}{2}a(\Psi; w, w^3) + \frac{1}{2}b(\Psi; w, w^3) - \frac{1}{4}d(\Psi, w^4)\\ &= \frac{1}{4}\sum_{K \in T_h}\{\alpha_j^K[2b_{mk}(w_m - w_k)(w_m^3 - w_k^3) + 2(\Psi_m - \Psi_k)(w_m - w_k)(w_m^3 + w_k^3)\\ &\quad - (\Psi_m - \Psi_k)(w_m^4 - w_k^4)] + \ldots\}\\ &\geqslant \frac{1}{4}\sum_{K \in T_h}\{\alpha_j^K[|\Psi_m - \Psi_k|(w_m - w_k)^2(w_m^2 + w_m w_k + w_k^2)\\ &\quad + (\Psi_m - \Psi_k)(w_m - w_k)^2(w_m^2 - w_k^2)] + \ldots\}.\end{aligned}$$

If $\Psi_m - \Psi_k \geqslant 0$, then $|\Psi_m - \Psi_k|(w_m^2 + w_m w_k + w_k^2) + (\Psi_m - \Psi_k)(w_m^2 - w_k^2) = (\Psi_m - \Psi_k)(2w_m^2 + w_m w_k) \geqslant 0$.

If $\Psi_m - \Psi_k < 0$, then $|\Psi_m - \Psi_k|(w_m^2 + w_m w_k + w_k^2) + (\Psi_m - \Psi_k)(w_m^2 - w_k^2) = (\Psi_k - \Psi_m)(w_m w_k + 2w_k^2) \geqslant 0$.

Therefore

$$\frac{1}{2}a(\Psi; w, w^3) + \frac{1}{2}b(\Psi; w, w^3) - \frac{1}{4}d(\Psi, w^4) \geqslant 0 \qquad \forall w \in W_h,\ w \geqslant 0,$$

and

$$\underline{\pi}_h^2(\Psi; w, w^3) \geqslant \frac{1}{2}a(\Psi; w, w^3) - \frac{3}{4}d(\Psi, w^4) \quad \forall w \in W_h,\ w \geqslant 0. \tag{3.40}$$

The same argument gives

$$\overline{\pi}_h^2(\Psi; w, w^3) \geqslant \frac{1}{2}a(\Psi; w, w^3) + \frac{3}{4}d(\Psi, w^4) \quad \forall w \in W_h,\ w \geqslant 0. \tag{3.41}$$

From these inequalities we obtain, by adding (3.39) and (3.39′) and taking into account (3.16) and (3.20), that

$$\begin{aligned} (3.42)\quad & \delta_n^{-1}(\Delta N^i, (N^i)^3)_h + \delta_p^{-1}(\Delta P^i, (P^i)^3)_h + \frac{1}{2}\Delta t[a(\Psi^i; N^i, (N^i)^3) \\ &\qquad + a(\Psi^i; P^i, (P^i)^3) + \frac{3}{2}d(\Psi^i, (P^i)^4 - \varrho^4 - (N^i)^4 + \omega^4)] \\ &\leqslant \delta_n^{-1}(\Delta N^i, \omega^3)_h + \delta_p^{-1}(\Delta P^i, \varrho^3)_h + \Delta t(X^i + Y^i), \\ X^i &= a(\Psi^i; N^i, \omega^3) + \frac{1}{2}b(\Psi^i; N^i, \omega^3) - d(\Psi^i, N^i\omega^3 - \omega^4) \\ &\qquad - \frac{1}{4}d(\Psi^i, \omega^4) - \delta_n^{-1}(R^i, (N^i)^3)_h + \delta_n^{-1}(R^i, \omega^3)_h, \\ Y^i &= a(\Psi^i; P^i, \varrho^3) - \frac{1}{2}b(\Psi^i; P^i, \varrho^3) + d(\Psi^i; P^i\varrho^3 - \varrho^4) \\ &\qquad + \frac{1}{4}d(\Psi^i, \varrho^4) - \delta_p^{-1}(R^i, (P^i)^3)_h + \delta_p^{-1}(R^i, \varrho^3)_h. \end{aligned}$$

Now, we estimate some terms in (3.42). Since $a(\Psi; w, w^3) \geqslant 0$ for $w \in W_h$, we get from (2.4)

$$\begin{aligned} & a(\Psi^i; N^i, (N^i)^3) + a(\Psi^i; P^i, (P^i)^3) + \frac{3}{2}d(\Psi^i, (P^i)^4 - \varrho^4 - (N^i)^4 + \omega^4) \\ &\geqslant \frac{3}{2}\alpha(P^i - N^i, (P^i)^4 - (N^i)^4)_h + \frac{3}{2}\alpha(P^i - N^i, -\varrho^4 + \omega^4)_h \\ &\quad + \frac{3}{2}\alpha(N_I, (P^i)^4 - (N^i)^4)_h + \frac{3}{2}\alpha(N_I, -\varrho^4 + \omega^4)_h \\ &\geqslant \frac{3}{2}\alpha(P^i - N^i, -\varrho^4 + \omega^4)_h + \frac{3}{2}\alpha(N_I, (P^i)^4 - (N^i)^4)_h + \frac{3}{2}\alpha(N_I, -\varrho^4 + \omega^4)_h \\ &\geqslant -C(\|N^i\|_{4,h}^4 + \|P^i\|_{4,h}^4 + 1), \qquad \|w\|_{4,h}^4 = \sum_{j=1}^{q} m_j w_j^4. \end{aligned}$$

In the last estimate we have used the estimates (3.3), (3.22) and the fact that

$$(P^i - N^i, (P^i)^4 - (N^i)^4)_h = \sum_{j=1}^{q} m_j (P^i_j - N^i_j)^2 (P^i_j + N^i_j)((P^i_j)^2 + (N^i_j)^2) \geqslant 0$$

for nonnegative N^i and P^i.

In the same way as before we find that, due to (3.3) and (3.23),

$$\begin{aligned} |a(\Psi^i; N^i, \omega^3| &\leqslant 3\|\omega\|^2_{L^\infty(\Omega)} \|\omega\|_V \|N^i\|_V + \|\omega\|^3_{L^\infty(\Omega)} \|\Psi^i\|_V \|N^i\|_V \\ &\leqslant C\|N^i\|_V \leqslant C(\|N^i\|^2_V + 1), \\ |a(\Psi^i; P^i, \varrho^3)| &\leqslant C(\|P^i\|^2_V + 1). \end{aligned}$$

Further

$$\begin{aligned} |b(\Psi^i; N^i, \omega^3)| &\leqslant 2\|\omega\|^3_{L^\infty(\Omega)} \|\Psi^i\|_V \|N^i\|_V \leqslant C(\|N^i\|^2_V + 1), \\ |b(\Psi^i; P^i, \varrho^3)| &\leqslant C(\|P^i\|^2_V + 1), \\ |d(\Psi^i, N^i\omega^3 - \omega^4)| &= |\alpha(P^i - N^i + N_I, N^i\omega^3 - \omega^4)_h| \\ &\leqslant C(\|N^i\|^2 + \|P^i\|^2 + 1) \leqslant C, \\ |d(\Psi^i, \omega^4)| &\leqslant 4\|\omega\|^3_{L^\infty(\Omega)} \|\Psi^i\|_V \|\omega\|_V \leqslant C, \\ |d(\Psi^i, P^i\varrho^3 - \varrho^4)| &\leqslant C, \\ |d(\Psi^i, \varrho^4)| &\leqslant C, \\ |(R^i, (N^i)^3)_h| &\leqslant C(\|N^i\|^4_{4,h} + \|P^i\|^4_{4,h} + 1), \\ |(R^i, (P^i)^3)_h| &\leqslant C(\|N^i\|^4_{4,h} + \|P^i\|^4_{4,h} + 1), \\ |(R^i, \omega^3)_h| &\leqslant C(1 + \|N^i\|_h + \|P^i\|_h) \leqslant C, \\ |(R^i, \varrho^3)_h| &\leqslant C. \end{aligned}$$

All these estimates and (3.42) lead to

$$\begin{aligned} &\delta_n^{-1}(\Delta N^i, (N^i)^3)_h + \delta_p^{-1}(\Delta P^i, (P^i)^3)_h \\ &\qquad \leqslant \delta_n^{-1}(\Delta N^i, \omega^3)_h + \delta_p^{-1}(\Delta P^i, \varrho^3)_h + C\Delta t \\ &\qquad\quad + C\Delta t(\|N^i\|^2_V + \|P^i\|^2_V) + C\Delta t(\|N^i\|^4_{4,h} + \|P^i\|^4_{4,h}). \end{aligned}$$

Since

$$\left|\sum_{i=1}^{m} (\Delta N^i, \omega^3)_h\right| = |(N^m, \omega^3)_h - (N^0, \omega^3)_h| \leqslant C,$$

we get, using (3.3),

$$\sum_{i=1}^{m}[\delta_n^{-1}(\Delta N^i,(N^i)^3)_h+\delta_p^{-1}(\Delta P^i,(P^i)^3)_h]$$
$$\leqslant C + C\Delta t\sum_{i=1}^{m}(\|N^i\|_{4,h}^4+\|P^i\|_{4,h}^4),\quad m\leqslant r.$$

Young's inequality $ab\leqslant \frac{a^p}{p}+\frac{a^{p'}}{p'}$ for $p=4$ gives $(\Delta N^i,(N^i)^3)_h=\|N^i\|_{4,h}^4-(N^{i-1},(N^i)^3)_h\geqslant \frac14(\|N^i\|_{4,h}^4-\|N^{i-1}\|_{4,h}^4)$ and again for $\Delta t\leqslant\Delta t_0$ it follows from the discrete Gronwall inequality that

$$\max_{1\leqslant i\leqslant r}\{\|N^i\|_{4,h},\|P^i\|_{4,h}\}\leqslant C(\|N^0\|_{4,h}+\|P^0\|_{4,h}+1).$$

For $v\in W_h$, $v\geqslant 0$, it is easy to show that the norms $\|\cdot\|_{4,h}$ and $\|\cdot\|_{L^4(\Omega)}$ are equivalent uniformly with respect to h on the space W_h. Therefore

$$\max_{1\leqslant i\leqslant r}\{\|N^i\|_{L^4(\Omega)},\|P^i\|_{L^4(\Omega)}\}\leqslant C(\|N^0\|_{L^4(\Omega)}+\|P^0\|_{L^4(\Omega)}+1). \tag{3.43}$$

Remark 3.3. Before proceeding to the next paragraph, notice that in the proof of (3.43) we have not used the assumptions A^1 and A^2. An inequality for the $\|\cdot\|_{L^n(\Omega)}$-norm of the form (3.43) is true for $n=3,4,\ldots$ However, we need just (3.43) for proving another a priori estimate playing an important role in our proof of the next theorem.

c) Now we use the assumptions A^1 and A^2. From (2.4), (3.37) and (3.43) it follows ($q_0=2+\varepsilon$, $\varepsilon>0$ and sufficiently small) that

$$\begin{aligned}\max_{1\leqslant i\leqslant r}\|\Psi^i\|_{H^{1,\infty}(\Omega)}&\leqslant C\max_{1\leqslant i\leqslant r}(\|N^i\|_{L^{q_0}(\Omega)}+\|P^i\|_{L^{q_0}(\Omega)}+1)\\&\leqslant C(\|N^0\|_{L^4(\Omega)}+\|P^0\|_{L^4(\Omega)}+1).\end{aligned}\tag{3.44}$$

d) Let $\overline{\Psi}^i$, $\overline{N}^i$, $\overline{P}^i$ and $\overline{\overline{\Psi}}^i$, $\overline{\overline{N}}^i$, $\overline{\overline{P}}^i$, $i=1,\ldots,r$, be two solutions of the problem (2.4)–(2.7), (2.9) with initial values $\overline{N}^0$, $\overline{P}^0$ and $\overline{\overline{N}}^0$, $\overline{\overline{P}}^0$, respectively, such that

$$\|\overline{N}^0\|_{L^4(\Omega)}+\|\overline{P}^0\|_{L^4(\Omega)}\leqslant C,\quad \|\overline{\overline{N}}^0\|_{L^4(\Omega)}+\|\overline{\overline{P}}^0\|_{L^4(\Omega)}\leqslant C. \tag{3.45}$$

Hence, by (3.43) and (3.44), it follows that

$$\begin{aligned}&\max_{1\leqslant i\leqslant r}(\|\overline{N}^i\|_{L^4(\Omega)}+\|\overline{P}^i\|_{L^4(\Omega)}+\|\overline{\Psi}^i\|_{H^{1,\infty}(\Omega)})\leqslant C,\\&\max_{1\leqslant i\leqslant r}(\|\overline{\overline{N}}^i\|_{L^4(\Omega)}+\|\overline{\overline{P}}^i\|_{L^4(\Omega)}+\|\overline{\overline{\Psi}}^i\|_{H^{1,\infty}(\Omega)})\leqslant C.\end{aligned}\tag{3.46}$$

We set $\widetilde{\Psi}^i = \overline{\overline{\Psi}}{}^i - \overline{\Psi}^i$, $\widetilde{N}^i = \overline{\overline{N}}{}^i - \overline{N}^i$, $\widetilde{P}^i = \overline{\overline{P}}{}^i - \overline{P}^i$, $i = 0, \ldots, r$. We consider the variational form (2.5) of the continuity equation for $\overline{N}^i$ and $\overline{\overline{N}}{}^i$, subtract these equations and afterwards choose $v = \widetilde{N}^i$. We do the same thing for (2.6) and add the result to the preceding one, getting

$$
\begin{aligned}
(3.47)\quad & \delta_n^{-1}(\Delta\widetilde{N}^i, \widetilde{N}^i)_h + \delta_p^{-1}(\Delta\widetilde{P}^i, \widetilde{P}^i)_h + \Delta t\big(a(\overline{\overline{\Psi}}{}^i; \widetilde{N}^i, \widetilde{N}^i) + a(\overline{\overline{\Psi}}{}^i; \widetilde{P}^i, \widetilde{P}^i)\big) \\
&= \Delta t\big(-a(\overline{\overline{\Psi}}{}^i; \overline{N}^i, \widetilde{N}^i) + a(\overline{\Psi}^i; \overline{N}^i, \widetilde{N}^i) + c(\overline{\overline{\Psi}}{}^i; \widetilde{N}^i, \widetilde{N}^i) + c(\overline{\overline{\Psi}}{}^i; \overline{N}^i, \widetilde{N}^i) \\
&\quad - c(\overline{\Psi}^i; \overline{N}^i, \widetilde{N}^i) - a(\overline{\overline{\Psi}}{}^i; \overline{P}^i, \widetilde{P}^i) + a(\overline{\Psi}^i; \overline{P}^i, \widetilde{P}^i) - c(\overline{\overline{\Psi}}{}^i; \widetilde{P}^i, \widetilde{P}^i) \\
&\quad - c(\overline{\overline{\Psi}}{}^i; \overline{P}^i, \widetilde{P}^i) + c(\overline{\Psi}^i; \overline{P}^i, \widetilde{P}^i) - (\overline{\overline{R}}{}^i - \overline{R}^i, \delta_n^{-1}\widetilde{N}^i + \delta_p^{-1}\widetilde{P}^i)_h\big), \\
&\quad 1 \leqslant i \leqslant r.
\end{aligned}
$$

The third term on the left-hand side of (3.47) is bounded from below, due to (3.14), as follows:

$$a(\overline{\overline{\Psi}}{}^i; \widetilde{N}^i, \widetilde{N}^i) + a(\overline{\overline{\Psi}}{}^i; \widetilde{P}^i, \widetilde{P}^i) \geqslant \|\widetilde{N}^i\|_V^2 + \|\widetilde{P}^i\|_V^2.$$

We estimate the terms on the right-hand side of (3.47). Since $|\chi'(\xi)| \leqslant \frac{1}{2}$ for all $\xi \in (-\infty, \infty)$, we have

$$
\begin{aligned}
&|a(\overline{\overline{\Psi}}{}^i; \overline{N}^i, \widetilde{N}^i) - a(\overline{\Psi}^i; \overline{N}^i, \widetilde{N}^i)| \\
&\qquad = \Big|\sum_{K\in T_h} \big(\alpha_j^K(\bar{\bar{b}}_{mk} - \bar{b}_{mk})(\overline{N}_m^i - \overline{N}_k^i)(\widetilde{N}_m^i - \widetilde{N}_k^i) + \ldots\big)\Big| \\
&\qquad \leqslant \frac{1}{2}\sum_{K\in T_h} \big(\alpha_j^K |\widetilde{\Psi}_m^i - \widetilde{\Psi}_k^i|\,|\overline{N}_m^i - \overline{N}_k^i|\,|\widetilde{N}_m^i - \widetilde{N}_k^i| + \ldots\big).
\end{aligned}
$$

Evidently, $|\widetilde{\Psi}_s^i - \widetilde{\Psi}_r^i| \leqslant Ch_K\|\widetilde{\Psi}^i\|_{H^{1,\infty}(K)}$, $r, s = j, k, m$. Thus, using (3.9), the inequality from Remark 2.1 and (3.3), we obtain (notice that $v_j^2 + v_k^2 + v_m^2 \leqslant c\int_{\widehat{K}} \widehat{v}^2\,\mathrm{d}\xi$)

$$
\begin{aligned}
&|a(\overline{\overline{\Psi}}{}^i; \overline{N}^i, \widetilde{N}^i) - a(\overline{\Psi}^i; \overline{N}^i, \widetilde{N}^i)| \\
&\qquad \leqslant C\|\widetilde{\Psi}^i\|_{H^{1,\infty}(\Omega)} \sum_{K\in T_h} h_K\big(\alpha_j^K|\overline{N}_m^i - \overline{N}_k^i|\,|\widetilde{N}_m^i - \widetilde{N}_k^i| + \ldots\big) \\
&\qquad \leqslant C\|\widetilde{\Psi}^i\|_{H^{1,\infty}(\Omega)} \Big\{\sum_{K\in T_h} h_K^2\big((\overline{N}_j^i)^2 + (\overline{N}_k^i)^2 + (\overline{N}_m^i)^2\big)\Big\}^{1/2} \\
&\qquad\qquad \times \Big\{\sum_{K\in T_h} \big(\alpha_j^K(\widetilde{N}_m^i - \widetilde{N}_k^i)^2 + \ldots\big)\Big\}^{1/2} \\
&\qquad \leqslant C\|\widetilde{\Psi}^i\|_{H^{1,\infty}(\Omega)}\|\overline{N}^i\|\,\|\widetilde{N}^i\|_V \leqslant C\|\widetilde{\Psi}^i\|_{H^{1,\infty}(\Omega)}\|\widetilde{N}^i\|_V.
\end{aligned}
$$

Similarly,

$$|a(\overline{\overline{\Psi}}{}^i; \overline{P}^i, \widetilde{P}^i) - a(\overline{\Psi}^i; \overline{P}^i, \widetilde{P}^i)| \leqslant C\|\widetilde{\Psi}^i\|_{H^{1,\infty}(\Omega)}\|\widetilde{P}^i\|_V.$$

In the same way we find, with regard to (3.46),

$$
\begin{aligned}
&|c(\overline{\overline{\Psi}}^i;\widetilde{N}^i,\widetilde{N}^i)| \leqslant C\|\overline{\overline{\Psi}}^i\|_{H^{1,\infty}(\Omega)}\|\widetilde{N}^i\|_h\|\widetilde{N}^i\|_V \leqslant C\|\widetilde{N}^i\|_h^2 + \frac{1}{4}\|\widetilde{N}^i\|_V^2,\\
&|c(\overline{\overline{\Psi}}^i;\widetilde{P}^i,\widetilde{P}^i)| \leqslant C\|\widetilde{P}^i\|_h^2 + \frac{1}{4}\|\widetilde{P}^i\|_V^2,\\
&|c(\overline{\overline{\Psi}}^i;\overline{N}^i,\widetilde{N}^i) - c(\overline{\Psi}^i;\overline{N}^i,\widetilde{N}^i)| \leqslant C\|\widetilde{\Psi}^i\|_{H^{1,\infty}(\Omega)}\|\widetilde{N}^i\|_V,\\
&|c(\overline{\overline{\Psi}}^i;\overline{P}^i,\widetilde{P}^i) - c(\overline{\Psi}^i;\overline{P}^i,\widetilde{P}^i)| \leqslant C\|\widetilde{\Psi}^i\|_{H^{1,\infty}(\Omega)}\|\widetilde{P}^i\|_V
\end{aligned}
$$

and, due to (3.13) from part I,

$$
|(\overline{\overline{R}}^i - \overline{R}^i, \delta_n^{-1}\widetilde{N}^i + \delta_p^{-1}\widetilde{P}^i)_h| \leqslant C(\|\widetilde{N}^i\|_h^2 + \|\widetilde{P}^i\|_h^2).
$$

Thus, it follows from (3.47), that

$$
\begin{aligned}
&\delta_n^{-1}(\Delta\widetilde{N}^i,\widetilde{N}^i)_h + \delta_p^{-1}(\Delta\widetilde{P}^i,\widetilde{P}^i)_h + \frac{3}{4}\Delta t(\|\widetilde{N}^i\|_V^2 + \|\widetilde{P}^i\|_V^2)\\
&\qquad \leqslant C\Delta t\|\widetilde{\Psi}^i\|_{H^{1,\infty}(\Omega)}(\|\widetilde{N}^i\|_V + \|\widetilde{P}^i\|_V) + C\Delta t(\|\widetilde{N}^i\|_h^2 + \|\widetilde{P}^i\|_h^2).
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
(3.48)\qquad &\delta_n^{-1}(\Delta\widetilde{N}^i,\widetilde{N}^i)_h + \delta_p^{-1}(\Delta\widetilde{P}^i,\widetilde{P}^i)_h + \frac{1}{2}\Delta t(\|\widetilde{N}^i\|_V^2 + \|\widetilde{P}^i\|_V^2)\\
&\qquad \leqslant C\Delta t(\|\widetilde{\Psi}^i\|_{H^{1,\infty}(\Omega)}^2 + \|\widetilde{N}^i\|_h^2 + \|\widetilde{P}^i\|_h^2).
\end{aligned}
$$

Since $\widetilde{\Psi}^i$ satisfies

$$
(3.48')\qquad d(\widetilde{\Psi}^i, v) = \alpha(\widetilde{P}^i - \widetilde{N}^i, v)_h \ \ \forall v \in V_h, \quad \widetilde{\Psi}^i(x^j) = 0 \ \ \forall x^j \in \Gamma^1,
$$

we have by Lemma 3.1

$$
\|\widetilde{\Psi}^i\|_{H^{1,\infty}(\Omega)} \leqslant C(\|\widetilde{N}^i\|_{L^{q_0}(\Omega)} + \|\widetilde{P}^i\|_{L^{q_0}(\Omega)}), \quad q_0 = 2+\varepsilon, \ \ \varepsilon > 0.
$$

We apply the inequality (see, e.g., Gilbarg, Trudinger [5], p. 139)

$$
(3.49)\quad \|u\|_{L^q(\Omega)} \leqslant \|u\|_{L^p(\Omega)}^{\lambda}\|u\|_{L^r(\Omega)}^{1-\lambda} \ \ \forall u \in L^r(\Omega), \quad p \leqslant q \leqslant r, \quad \frac{1}{q} = \frac{\lambda}{p} + \frac{1-\lambda}{r},
$$

for $p=2$, $q=q_0$, $r=2(q_0-1) \Rightarrow \lambda = \frac{1}{q_0}$. Hence by the Sobolev imbedding theorem

$$
\|\widetilde{N}^i\|_{L^{q_0}(\Omega)} \leqslant \|\widetilde{N}^i\|^{\lambda}\|\widetilde{N}^i\|_{L^{2(q_0-1)}(\Omega)}^{1-\lambda} \leqslant C\|\widetilde{N}^i\|^{\lambda}\|\widetilde{N}^i\|_V^{1-\lambda}.
$$

Squaring and using Young's inequality (with $p = \frac{1}{\lambda}$) we get

$$C\|\widetilde{\psi}^i\|^2_{H^{1,\infty}(\Omega)} \leqslant \frac{1}{4}(\|\widetilde{N}^i\|^2_V + \|\widetilde{P}^i\|^2_V) + C(\|\widetilde{N}^i\|^2_h + \|\widetilde{P}^i\|^2_h).$$

From (3.48) it follows that

$$\delta_n^{-1}(\Delta\widetilde{N}^i, \widetilde{N}^i)_h + \delta_p^{-1}(\Delta\widetilde{P}^i, \widetilde{P}^i)_h + \frac{1}{4}\Delta t(\|\widetilde{N}^i\|^2_V + \|\widetilde{P}^i\|^2_V)$$
$$\leqslant C\Delta t(\|\widetilde{N}^i\|^2_h + \|\widetilde{P}^i\|^2_h), \quad i = 1, \ldots, r.$$

By summing up we obtain

$$\delta_n^{-1}\|\widetilde{N}^m\|^2_h + \delta_p^{-1}\|\widetilde{P}^m\|^2_h + \frac{1}{2}\Delta t\sum_{i=1}^{m}(\|\widetilde{N}^i\|^2_V + \|\widetilde{P}^i\|^2_V)$$
$$\leqslant \delta_n^{-1}\|\widetilde{N}^0\|^2_h + \delta_p^{-1}\|\widetilde{P}^0\|^2_h + C\Delta t\sum_{i=1}^{m}(\|\widetilde{N}^i\|^2_h + \|\widetilde{P}^i\|^2_h)$$

and for $\Delta t \leqslant \Delta t_0 = \frac{1}{2C}\min(\delta_n^{-1}, \delta_p^{-1})$ we have

$$\delta_n^{-1}\|\widetilde{N}^m\|^2_h + \delta_p^{-1}\|\widetilde{P}^m\|^2_h + \Delta t\sum_{i=1}^{m}(\|\widetilde{N}^i\|^2_V + \|\widetilde{P}^i\|^2_V)$$
$$\leqslant C(\|\widetilde{N}^0\|^2_h + \|\widetilde{P}^0\|^2_h) + C\Delta t\sum_{i=1}^{m-1}(\|\widetilde{N}^i\|^2_h + \|\widetilde{P}^i\|^2_h).$$

The discrete Gronwall inequality gives

$$\max_{1\leqslant i\leqslant r}(\|\overline{\overline{N}}^i - \overline{N}^i\| + \|\overline{\overline{P}}^i - \overline{P}^i\|) + \left\{\Delta t\sum_{i=1}^{r}(\|\overline{\overline{N}}^i - \overline{N}^i\|^2_{H^1(\Omega)}\right.$$
$$\left.+\|\overline{\overline{P}}^i - \overline{P}^i\|^2_{H^1(\Omega)})\right\}^{1/2} \leqslant C(\|\overline{\overline{N}}^0 - \overline{N}^0\| + \|\overline{\overline{P}}^0 - \overline{P}^0\|). \tag{3.50}$$

Further, (3.48′), (3.37) and (3.50) imply

$$\max_{1\leqslant i\leqslant r}\|\overline{\overline{\Psi}}^i - \overline{\Psi}^i\|_{H^{1,q}(\Omega)} + \left\{\Delta t\sum_{i=1}^{r}\|\overline{\overline{\Psi}}^i - \overline{\Psi}_i\|^2_{H^{1,\infty}(\Omega)}\right\}^{1/2}$$
$$\leqslant C(\|\overline{\overline{N}}^0 - \overline{N}^0\| + \|\overline{\overline{P}}^0 - \overline{P}^0\|), \quad 1 \leqslant q \leqslant \infty. \tag{3.51}$$

If $\overline{\overline{N}}^0 = \overline{N}^0$, $\overline{\overline{P}}^0 = \overline{P}^0$, then $\overline{\overline{\Psi}}^i = \overline{\Psi}^i$, $\overline{\overline{N}}^i = \overline{N}^i$, $\overline{\overline{P}}^i = \overline{P}^i$ for $i = 1, \ldots, r$, which proves the uniqueness. □

We state the main result of this paragraph as

Lemma 3.4. *Let* A^0, A^1, A^2 *and* (3.45) *be fulfilled. If* Δt *is sufficiently small,* $\Delta t \leqslant \Delta t_0$ *where* Δt_0 *does not depend on* h, *then the differences* $\overline{\overline{N}}^i - \overline{N}^i$, $\overline{\overline{P}}^i - \overline{P}^i$ *and* $\overline{\overline{\Psi}}^i - \overline{\Psi}^i$ *satisfy the inequalities* (3.50) *and* (3.51), *respectively. Here* $\overline{\Psi}^i$, $\overline{N}^i$, $\overline{P}^i$ *and* $\overline{\overline{\Psi}}^i$, $\overline{\overline{N}}^i$, $\overline{\overline{P}}^i$ *satisfy* (2.4)–(2.7) *and* (2.9).

4. Convergence

We extend the approximate solution piecewise linearly onto the interval [0,T]:

$$\begin{aligned} N^\delta &= N^{i-1} + \frac{t - t_{i-1}}{\Delta t}\Delta N^i, \\ P^\delta &= P^{i-1} + \frac{t - t_{i-1}}{\Delta t}\Delta P^i, \\ \Psi^\delta &= \Psi^{i-1} + \frac{t - t_{i-1}}{\Delta t}\Delta \Psi^i \quad \text{ on } \quad [t_{i-1}, t_i],\ \ i = 1, \ldots, r. \end{aligned}$$

Here $\delta = (h, \Delta t)$ and Ψ^0 is uniquely defined by $d(\Psi^0, v) = \alpha(P^0 - N^0 + N_I, v)_h \ \ \forall v \in V_h$, $\Psi^0(x^j) = \psi^*(x^j) \ \ \forall x^j \in \Gamma^1$.

Lemma 4.1. *Let* A^0, A^1 *and* A^2 *be fulfilled. If* n^0 *and* p^0 *satisfy the corresponding boundary conditions* (1.4) *and* (1.5), $n^0, p^0 \in H^2(\Omega)$ *and* $\Delta t \geqslant ch^2$, *then there exists a triple* n, p, ψ, $\psi - \psi^* \in C([0,T]; V \cap H^{1,\infty}(\Omega))$, $n - n^*, p - p^* \in L^\infty(0,T;V)$, $n, p \in C([0,T]; L^q(\Omega))$ *for any number* $q \geqslant 1$, *such that for* $\delta \to 0$ *we have*

$$\begin{aligned} &\|\psi - \Psi^\delta\|_{C([0,T];H^{1,\infty}(\Omega))} \to 0, \\ &\|n - N^\delta\|_{C([0,T];L^q(\Omega))} \to 0, \\ &\|p - P^\delta\|_{C([0,T];L^q(\Omega))} \to 0. \end{aligned} \tag{4.1}$$

The triple n, p, ψ *is the unique solution of the problem* P *(see Introduction).*

Remark 4.1. The assumption that n^0 and p^0 satisfy the boundary conditions is restrictive. It will be removed in Theorem 4.1 at the cost that the convergence theorem will be weaker.

Proof. a) We use the compactness method (see Lions [9] and the references given there). We show that from any sequence $\{\Psi^{\delta j}, N^{\delta j}, P^{\delta, j}\}$ of the family

$\{\Psi^\delta, N^\delta, P^\delta\}$ with $\delta_j \to 0$ one can choose a subsequence $\{\Psi^{\delta j(\nu)}, N^{\delta j(\nu)}, P^{\delta j(\nu)}\}$ such that for any q, $1 \leqslant q < \infty$,

$$\begin{aligned}
&\|\psi - \Psi^{\delta j(\nu)}\|_{C([0,T];H^{1,\infty}(\Omega))} \to 0\\
&\|n - N^{\delta j(\nu)}\|_{C([0,T];L^q(\Omega))} \to 0,\\
&\|p - P^{\delta j(\nu)}\|_{C([0,T];L^q(\Omega))} \to 0 \quad \text{if } \nu \to \infty,
\end{aligned}$$

and that ψ, n, p is a solution of the problem P. If this problem has a unique solution then (4.1) follows.

b) We derive some more a priori estimates. In these estimates, i.e. in (4.5), (4.6) and (4.11), the constant C may depend on a parameter θ to be introduced later. More exactly, it will depend or not depend on θ according to whether n^0 and p^0 depend or do not depend on θ. However, the result is the same: the convergence (4.1).

From (3.44) it follows that

$$\max_{1\leqslant i\leqslant r} \|\Psi^i\|_{H^{1,\infty}(\Omega)} \leqslant C. \tag{4.2}$$

We set $v = \Delta N^i$ in (2.5) which is legitimate even for $i = 1$. Introducing a form $a_1(\Psi; N, v) = a(\Psi; N, v) - d(N, v)$, i.e.

$$a_1(\Psi; N, v) = \sum_{K\subset T_h} [\alpha_j^K (b_{mk} - 1)(N_m - N_k)(v_m - v_k) + \ldots], \tag{4.3}$$

we get, due to (3.17) and (3.18),

$$\begin{aligned}
\delta_n^{-1}\|\Delta N^i\|_h^2 + \Delta t d(N^i, \Delta N^i) = {} & -\Delta t\Big(a_1(\Psi^i; N^i, \Delta N^i) + \frac{1}{2} b(\Psi^i; N^i, \Delta N^i)\\
& - d(\Psi^i; N^i, \Delta N^i) + \delta_n^{-1}(R^i, \Delta N^i)_h\Big).
\end{aligned} \tag{4.4}$$

We estimate the terms on the right-hand side of (4.4). Since $|\chi'(\xi)| \leqslant \frac{1}{2}$, we have $|\chi(\xi) - 1| \leqslant \frac{1}{2}|\xi|$ and by (4.2),

$$\begin{aligned}
&|a_1(\Psi^i; N^i, v)|\\
&\quad \leqslant C\|\Psi^i\|_{H^{1,\infty}(\Omega)} \sum_{K\in T_h} h_K(\alpha_j^K |N_m^i - N_k^i|\,|v_m - v_k| + \ldots)\\
&\quad \leqslant C\Big\{\sum_{K\in T_h} [\alpha_j^K (N_m^i - N_k^i)^2 + \ldots\Big\}^{1/2} \Big\{\sum_{K\in T_h} h_K^2(v_j^2 + v_k^2 + v_m^2)\Big\}^{1/2}\\
&\quad \leqslant C\|N^i\|_V \|v\|_h.
\end{aligned}$$

Consequently,

$$|\Delta t\, a_1(\Psi^i; N^i, \Delta N^i)| \leqslant \frac{1}{8}\delta_n^{-1}\|\Delta N^i\|_h^2 + C\Delta t^2\|N^i\|_V^2.$$

Similarly,

$$\begin{aligned}|\Delta t\, b(\Psi^i; N^i, \Delta N^i)| &\leqslant C\Delta t\|\Psi^i\|_{H^{1,\infty}(\Omega)}\|N^i\|_V\|\Delta N^i\|_h\\ &\leqslant \frac{1}{8}\delta_n^{-1}\|\Delta N^i\|_h^2 + C\Delta t^2\|N^i\|_V^2.\end{aligned}$$

By (3.43), the third term is bounded by

$$\begin{aligned}|\Delta t\, d(\Psi^i; N^i, \Delta N^i)| &= \alpha|\Delta t(P^i - N^i + N_I, N^i\Delta N^i)_h|\\ &\leqslant C\Delta t\|N^i\|_h\|\Delta N^i\|_h + C\Delta t(\|N^i\|_{h,4}^2 + \|P^i\|_{h,4}^2)\|\Delta N^i\|_h\\ &\leqslant C\Delta t^2 + \frac{\delta_n^{-1}}{8}\|\Delta N^i\|_h^2\end{aligned}$$

and

$$\begin{aligned}\delta_n^{-1}\Delta t|(R^i, \Delta N^i)_h| &\leqslant C\Delta t(1 + \|N^i\|_h + \|P^i\|_h)\|\Delta N^i\|_h\\ &\leqslant C\Delta t^2 + \frac{1}{8}\delta_n^{-1}\|\Delta N^i\|_h^2.\end{aligned}$$

Summing up (4.4) for $i = 1, \ldots, m \leqslant r$ we arrive at the inequality

$$\begin{aligned}&\delta_n^{-1}\sum_{i=1}^m\|\Delta N^i\|_h^2 + \frac{1}{2}\Delta t\|N^m\|_V^2 + \frac{1}{2}\Delta t\sum_{i=1}^m\|\Delta N^i\|_V^2\\ &\qquad\leqslant \frac{1}{2}\Delta t\|N^0\|_V^2 + C\Delta t + \frac{1}{2}\delta_n^{-1}\sum_{i=1}^m\|\Delta N^i\|_h^2,\end{aligned}$$

i.e.

$$\delta_n^{-1}\sum_{i=1}^m\|\Delta N^i\|_h^2 + \Delta t\biggl(\|N^m\|_h^2 + \sum_{i=1}^m\|\Delta N^i\|_V^2\biggr) \leqslant C\Delta t.$$

Two a priori bounds, needed in the sequel, are

$$\Delta t\sum_{i=1}^r\left\|\frac{\Delta N^i}{\Delta t}\right\|^2 \leqslant C, \qquad \max_{1\leqslant i\leqslant r}\|N^i\|_{H^1(\Omega)} \leqslant C. \tag{4.5}$$

In the same way we get

$$\Delta t\sum_{i=1}^r\left\|\frac{\Delta P^i}{\Delta t}\right\|^2 \leqslant C, \qquad \max_{1\leqslant i\leqslant r}\|P^i\|_{H^1(\Omega)} \leqslant C. \tag{4.6}$$

c) We subtract from (2.5) the equation corresponding to $t = t_{i-1}$ and choose $v = \Delta N^i$. We obtain

$$
\begin{aligned}
(4.7)\qquad & \delta_n^{-1}(\Delta N^i - \Delta N^{i-1}, \Delta N^i)_h + \Delta t\, d(\Delta N^i, \Delta N^i) \\
&= -\Delta t\big(a_1(\Psi^i; N^i, \Delta N^i) - a_1(\Psi^{i-1}; N^i, \Delta N^i) \\
&\quad + a_1(\Psi^{i-1}; N^i, \Delta N^i) - a_1(\Psi^{i-1}, N^{i-1}, \Delta N^i) \\
&\quad - c(\Psi^i; N^i, \Delta N^i) + c(\Psi^{i-1}; N^i, \Delta N^i) \\
&\quad - c(\Psi^{i-1}; N^i, \Delta N^i) + c(\Psi^{i-1}; N^{i-1}, \Delta N^i) \\
&\quad + \delta_n^{-1}(R^i - R^{i-1}, \Delta N^i)_h\big), \qquad 2 \leqslant i \leqslant r.
\end{aligned}
$$

We estimate the terms on the right-hand side of the above equations. First, $d(\Delta\Psi^i, v) = \alpha(\Delta P^i - \Delta N^i, v)_h \ \forall v \in V_h$ and $\Delta\Psi^i = 0$ on Γ^1. By (3.37),

$$
(4.8)\qquad \|\Delta\Psi^i\|_{H^{1,\infty}(\Omega)} \leqslant C(\|\Delta N^i\|_{L^{q_0}(\Omega)} + \|\Delta P^i\|_{L^{q_0}(\Omega)}), \quad q_0 = 2+\varepsilon,\ \varepsilon > 0.
$$

From (4.2), (4,5), (4.8) and from the Sobolev imbedding theorem we conclude that

$$
\begin{aligned}
&|a_1(\Psi^i; N^i, \Delta N^i) - a_1(\Psi^{i-1}; N^i, \Delta N^i)| \\
&\quad = \Big|\sum_{K\in T_h} [\alpha_j^K (b_{mk}^i - b_{mk}^{i-1})(N_m^i - N_k^i)(\Delta N_m^i - \Delta N_k^i) + \ldots]\Big| \\
&\quad \leqslant C\|\Delta\Psi^i\|_{H^{1,\infty}(\Omega)} \sum_{K\in T_h} h_K[\alpha_j^K |N_m^i - N_k^i|\, |\Delta N_m^i - \Delta N_k^i| + \ldots] \\
&\quad \leqslant Ch\|\Delta\Psi^i\|_{H^{1,\infty}(\Omega)} \|N^i\|_V \|\Delta N^i\|_V \leqslant Ch[\|\Delta N^i\|_V^2 + \|\Delta P^i\|_V^2], \\
&|a_1(\Psi^{i-1}; N^i, \Delta N^i) - a_1(\Psi^{i-1}; N^{i-1}, \Delta N^i)| \\
&\quad \leqslant Ch\|\Psi^{i-1}\|_{H^{1,\infty}(\Omega)} \|\Delta N^i\|_V^2 \leqslant Ch\|\Delta N^i\|_V^2.
\end{aligned}
$$

Further, by (4.8) and (3.49),

$$
\begin{aligned}
&|c(\Psi^i; N^i, \Delta N^i) - c(\Psi^{i-1}; N^i, \Delta N^i)| \leqslant C\|\Delta\Psi^i\|_{H^{1,\infty}(\Omega)} \|N^i\|\, \|\Delta N^i\|_V \\
&\leqslant C(\|\Delta N^i\|_{L^{q_0}(\Omega)} + \|\Delta P^i\|_{L^{q_0}(\Omega)})\|\Delta N^i\|_V \leqslant C(\|\Delta N^i\|^\lambda \|\Delta N^i\|_V^{2-\lambda}) \\
&\qquad + C(\|\Delta P^i\|^\lambda \|\Delta P^i\|_V^{1-\lambda} \|\Delta N^i\|_V), \quad \lambda = \frac{1}{q_0}.
\end{aligned}
$$

By Young's inequality with $p = \frac{2}{\lambda}$ and by Schwartz inequality and Young's inequality with $p = \frac{1}{\lambda}$, we get

$$
\begin{aligned}
&|c(\Psi^i; N^i, \Delta N^i) - c(\Psi^{i-1}; N^i, \Delta N^i)| \\
&\leqslant C(\|\Delta N^i\|^2 + \|\Delta P^i\|^2) + \frac{1}{16}(\|\Delta N^i\|_V^2 + \|\Delta P^i\|_V^2).
\end{aligned}
$$

Finally,

$$\begin{aligned}
&|c(\Psi^{i-1}; N^i, \Delta N^i) - c(\Psi^{i-1}; N^{i-1}, \Delta N^i)| \\
&\qquad \leqslant C\|\Psi^{i-1}\|_{H^{1,\infty}(\Omega)} \sum_{K\in T_h} h_K(\alpha_j^K |\Delta N_m^i + \Delta N_k^i|\, |\Delta N_m^i - \Delta N_k^i| + \ldots) \\
&\qquad \leqslant C\|\Delta N^i\|\, \|\Delta N^i\|_V \leqslant C\|\Delta N^i\|^2 + \frac{1}{16}\|\Delta N^i\|_V^2
\end{aligned}$$

and

$$|(R^i - R^{i-1}, \Delta N^i)_h| \leqslant C(\|\Delta N^i\|^2 + \|\Delta P^i\|^2).$$

From all these estimates and from (4.7), summing up by parts, we obtain that

$$\begin{aligned}
&\frac{1}{2}\delta_n^{-1}\|\Delta N^r\|_h^2 + \Delta t \sum_{i=2}^{r} \|\Delta N^i\|_V^2 \\
&\qquad \leqslant \frac{1}{2}\delta_n^{-1}\|\Delta N^1\|_h^2 + \Delta t\Big(Ch + \frac{1}{8}\Big) \sum_{i=2}^{r} (\|\Delta N^i\|_V^2 + \|\Delta P^i\|_V^2) + C\Delta t^2.
\end{aligned}$$

An inequality of the same type can be derived for P^i. Combining the two inequalities we get for h sufficiently small, $h \leqslant h_0$ where h_0 does not depend on Δt, that

$$\begin{aligned}
&\delta_n^{-1}\|\Delta N^r\|_h^2 + \delta_p^{-1}\|\Delta P^r\|_h^2 + \Delta t \sum_{i=1}^{r} (\|\Delta N^i\|_V^2 + \|\Delta P^i\|_V^2) \\
&\qquad \leqslant \delta_n^{-1}\|\Delta N^1\|_h^2 + \delta_p^{-1}\|\Delta P^1\|_h^2 + \Delta t(\|\Delta N^1\|_V^2 + \|\Delta P^1\|_V^2) + C\Delta t^2.
\end{aligned} \tag{4.9}$$

We will prove that

$$\delta_n^{-1}\|\Delta N^1\|_h^2 + \delta_p^{-1}\|\Delta P^1\|_h^2 + \Delta t(\|\Delta N^1\|_V^2 + \|\Delta P^1\|_V^2) \leqslant C\Delta t^2. \tag{4.10}$$

Then (4.9) implies the last desirable a priori estimate

$$\Delta t \sum_{i=1}^{r} \Big(\Big\|\frac{\Delta N^i}{\Delta t}\Big\|_{H^1(\Omega)}^2 + \Big\|\frac{\Delta P^i}{\Delta t}\Big\|_{H^1(\Omega)}^2 \Big) \leqslant C. \tag{4.11}$$

Concerning (4.10) we choose $i = 1$ and $v = \Delta N^1$ in (2.5). We get

$$\begin{aligned}
\delta_n^{-1}\|\Delta N^1\|_h^2 + \Delta t\, d(N^1, \Delta N^1) = &-\Delta t\Big(a_1(\Psi^1; N^1, \Delta N^1) + \frac{1}{2}b(\Psi^1; N^1, \Delta N^1) \\
&- d(\Psi^1; N^1, \Delta N^1) + \delta_n^{-1}(R^1, \Delta N^1)_h\Big).
\end{aligned}$$

It follows easily as above that

$$
\begin{aligned}
|a_1(\Psi^1; N^1, \Delta N^1)| &\leqslant C\|\Psi^1\|_{H^{1,\infty}(\Omega)}\|N^1\|_V\|\Delta N^1\| \leqslant C\|\Delta N^1\|_h,\\
|b(\Psi^1; N^1, \Delta N^1)| &\leqslant C\|\Psi^1\|_{H^{1,\infty}(\Omega)}\|N^1\|_V\|\Delta N^1\| \leqslant C\|\Delta N^1\|_h,\\
|d(\Psi^1, N^1, \Delta N^1)| &\leqslant |\alpha(P^1 - N^1 + N_I, N^1\Delta N^1)_h|\\
&\leqslant C(\|N^1\|_{4,h}^2 + \|P^1\|_{4,h}^2 + \|N^1\|_h)\|\Delta N^1\|_h \leqslant C\|\Delta N^1\|_h,\\
|(R^1, \Delta N^1)_h| &\leqslant C(1 + \|N^1\|_h + \|P^1\|_h)\|\Delta N^1\|_h \leqslant C\|\Delta N^1\|_h.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\delta_n^{-1}\|\Delta N^1\|_h^2 + \Delta t\|\Delta N^1\|_V^2 \leqslant\; &-\Delta t\, d(n^0, \Delta N^1) + \Delta t\, d(n_0 - N^0, \Delta N^1)\\
&+ C\Delta t\|\Delta N^1\|_h.
\end{aligned}
$$

Now, since $\frac{\partial n^0}{\partial \nu}\big|_{\Gamma^2} = 0$, we have $d(n^0, \Delta N^1) = -(\Delta n^0, \Delta N^1)$ (Δn^0 means, of course, $\frac{\partial^2 n^0}{\partial x^2} + \frac{\partial^2 n^0}{\partial y^2}$). Further,

$$
|d(n^0 - N^0, \Delta N^1)| \leqslant Ch\|n^0\|_{H^2(\Omega)}\,\|\Delta N^1\|_V \leqslant Ch\|\Delta N^1\|_V,
$$

and thus

$$
\begin{aligned}
\delta_n^{-1}\|\Delta N^1\|_h^2 + \Delta t\|\Delta N^1\|_V^2 &\leqslant C\Delta t\|\Delta N^1\|_h + C\Delta t h\|\Delta N^1\|_V\\
&\leqslant C\Delta t^2 + \frac{1}{2}\delta_n^{-1}\|\Delta N^1\|_h^2 + C\Delta t h^2 + \frac{1}{2}\Delta t\|\Delta N^1\|_V^2.
\end{aligned}
$$

Consequently, $\delta_n^{-1}\|\Delta N^1\|_h^2 + \Delta t\|\Delta N^1\|_V^2 \leqslant C\Delta t^2$.

d) In this paragraph we prove uniqueness. In fact, we prove inequalities from which uniqueness follows immediately and which are counterparts to (3.50) and (3.51). First, we write (1.9) and (1.10) in an operator form. For a given $\psi \in L^\infty(0,T; H^{1,\infty}(\Omega))$ and $n, p \in L^2(0,T; H^1(\Omega))$, $n \geqslant 0$, $p \geqslant 0$ a.e. in Ω, we define operators $A, B \in L^2(0,T; V')$ and $R \in L^2(0,T; L^2)$ by

$$
\text{(4.12)}\qquad \begin{cases} \langle A, z\rangle_V = \bar\nu^2(\psi; n, z),\;\; \langle B, z\rangle_V = \bar\pi^2(\psi; p, z) \;\; \forall z \in L^2(0,T; V),\\ \langle R, z\rangle \;\;\, = \big(R(n,p), z\big) \;\; \forall z \in L^2\big(0,T; L^2(\Omega)\big). \end{cases}
$$

Evidently, $|\bar\nu^2(\psi; n, z)| \leqslant (\|n\|_V + \|\psi\|_{H^{1,\infty}(\Omega)}\|n\|)\|z\|_V$, hence $\|A\|_{L^2(0,T;V')} \leqslant C(\|\psi\|_{L^\infty(0,T;H^{1,\infty}(\Omega))} + 1)\|n\|_{L^2(0,T;H^1(\Omega))}$ and, as $|R(n,p)| \leqslant \frac{1}{2} + n + p$ for $n, p \geqslant 0$, we have $\|R\|_{L^2(0,T;L^2(\Omega))} \leqslant C(\|n\|_{L^2(0,T;L^2(\Omega))} + \|p\|_{L^2(0,T;L^2(\Omega))} + 1)$. V is a Hilbert space which is dense in $L^2(\Omega)$ with a continuous imbedding. We identify $L^2(\Omega)$ with its dual by means of the scalar product $(\cdot,\cdot)$. Then $L^2(\Omega)$ can be identified with a subspace of V' and the following dense and continuous inclusions hold: $V \subset L^2(\Omega) \subset V'$. Furthermore, the operation $\langle \cdot, \cdot \rangle_V$, expressing the duality between V and V', is an extension of the scalar product $(\cdot, \cdot)$. The reflexivity of V yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(n, v) = \frac{\mathrm{d}}{\mathrm{d}t}(n - n^*, v) &= \frac{\mathrm{d}}{\mathrm{d}t}\langle n - n^*, v\rangle_V \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\langle v, n - n^*\rangle_{V'} \quad \forall v \in V \quad \text{in } \mathscr{D}'((0,T)).\end{aligned}$$

The same is true for p.

By a lemma of Temam (see Girault, Raviart [6], p. 149, (1.3)) n and p have weak derivatives n' and p', respectively, which satisfy

$$\delta_n^{-1} n' = -A - \delta_n^{-1} R \qquad \delta_p^{-1} p' = -B - \delta_p^{-1} R. \tag{4.13}$$

Hence $n', p' \in L^2(0,T;V')$ and consequently (see [6], Theorem 1.1, p. 151) $n - n^*$, $p - p^*$ and also n, p belong to $C([0,T]; L^2(\Omega))$. We see that the initial condition (1.11) makes sense.

Now, let (ψ_j, n_j, p_j), $j = 1, 2$, be two solutions of the problem P with the same boundary conditions and with initial conditions $n_j(0) = n_j^0$, $p_j(0) = p_j^0$, $n_j^0, p_j^0 \in L^2(\Omega)$, $j = 1, 2$. We assume that

$$\|\psi_j\|_{L^\infty(0,T;H^{1,\infty}(\Omega))} \leqslant C, \quad \|n_j\|_{L^\infty(0,T;L^2(\Omega))} + \|p_j\|_{L^\infty(0,T;L^2(\Omega))} \leqslant C \tag{4.14}$$

and recall that C is a constant which does not depend on any parameter. We set $\psi_* = \psi_2 - \psi_1$, $n_* = n_2 - n_1$, $p_* = p_2 - p_1$. We have $\psi_* \in L^\infty(0,T; V \cap H^{1,\infty}(\Omega))$, $n_*, p_* \in L^2(0,T;V)$ and (4.13) implies

$$\begin{aligned}\delta_n^{-1}\langle n_*', z\rangle_V &= -\langle A_2 - A_1, z\rangle_V - \delta_n^{-1}(R_2 - R_1, z) \\ \delta_p^{-1}\langle p_*', z\rangle_V &= -\langle B_2 - B_1, z\rangle_V - \delta_p^{-1}(R_2 - R_1, z)\end{aligned} \qquad \forall z \in L^2(0,T;V).$$

Choosing $z = n_*$ and $z = p_*$, adding and applying Green's formula (see [6], Theorem 1.1, p. 151) we get

$$\begin{aligned}&\frac{1}{2}(\delta_n^{-1}\|n_*(t)\|^2 + \delta_p^{-1}\|p_*(t)\|^2) \\ &\quad = \frac{1}{2}(\delta_n^{-1}\|n_*^0\|^2 + \delta_p^{-1}\|p_*^0\|^2) - \int_0^t (\langle A_2 - A_1, n_*\rangle_V + \langle B_2 - B_1, p_*\rangle_V)\,\mathrm{d}\tau \\ &\qquad - \int_0^t (R_2 - R_1, \delta_n^{-1} n_* + \delta_p^{-1} p_*)\,\mathrm{d}\tau.\end{aligned}$$

We estimate the last two terms. We have, with regard to (4.14),

$$
\begin{aligned}
\langle A_2 - A_1, n_* \rangle_V &= \|n_*\|_V^2 - \int_\Omega (n_2 \nabla\psi_2 - n_1 \nabla\psi_1) \cdot \nabla n_* \, \mathrm{d}x \\
&= \|n_*\|_V^2 - \int_\Omega n_* \nabla\psi_2 \cdot \nabla n_* \, \mathrm{d}x - \int_\Omega n_1 \nabla\psi_* \cdot \nabla n_* \, \mathrm{d}x \\
&\geqslant \|n_*\|_V^2 - (\|\psi_2\|_{H^{1,\infty}(\Omega)} \|n_*\| + \|n_1\| \, \|\psi_*\|_{H^{1,\infty}(\Omega)}) \, \|n_*\|_V \\
&\geqslant \frac{3}{4}\|n_*\|_V^2 - C\|\psi_*\|_{H^{1,\infty}(\Omega)}^2 - C\|n_*\|^2.
\end{aligned}
$$

Since

$$
d(\psi_*, v) = \alpha(p_* - n_*, v) \;\; \forall v \in V \quad \text{and} \quad \psi_*\big|_{\Gamma_1} = 0, \tag{4.15}
$$

it follows from (3.2) that $\|\psi_*\|_{H^{1,\infty}(\Omega)} \leqslant C(\|n_*\|_{L^{q_0}(\Omega)} + \|p_*\|_{L^{q_0}(\Omega)})$, $q_0 = 2 + \varepsilon$, $\varepsilon > 0$, and using (3.49) we derive as in Section 3 that

$$
C\|\psi_*\|_{H^{1,\infty}(\Omega)}^2 \leqslant \frac{1}{8}(\|n_*\|_V^2 + \|p_*\|_V^2) + C(\|n_*\|^2 + \|p_*\|^2).
$$

Hence

$$
\langle A_2 - A_1, n_* \rangle \geqslant \frac{3}{4}\|n_*\|_V^2 - \frac{1}{8}(\|n_*\|_V^2 + \|p_*\|_V^2) - C(\|n_*\|^2 + \|p_*\|^2).
$$

In the same way we obtain

$$
\langle B_2 - B_1, p_* \rangle \geqslant \frac{3}{4}\|p_*\|_V^2 - \frac{1}{8}(\|n_*\|_V^2 + \|p_*\|_V^2) - C(\|n_*\|^2 + \|p_*\|^2)
$$

and, moreover,

$$
|(R_2 - R_1, \delta_n^{-1} n_* + \delta_p^{-1} p_*)| \leqslant C(\|n_*\|^2 + \|p_*\|^2).
$$

Therefore

$$
\begin{aligned}
&\delta_n^{-1}\|n_*(t)\|^2 + \delta_p^{-1}\|p_*(t)\|^2 + \int_0^t (\|n_*(\tau)\|_V^2 + \|p_*(\tau)\|_V^2) \, \mathrm{d}\tau \\
&\leqslant \delta_n^{-1}\|n_*^0\|^2 + \delta_p^{-1}\|p_*^0\|^2 + C\int_0^t (\|n_*(\tau)\|^2 + \|p_*(\tau)\|^2) \, \mathrm{d}\tau
\end{aligned}
$$

and the Gronwall inequality gives

$$
\begin{aligned}
&\|n_2 - n_1\|_{C([0,T];L^2(\Omega))} + \|n_2 - n_1\|_{L^2(0,T;H^1(\Omega))} \\
&\quad + \|p_2 - p_1\|_{C([0,T];L^2(\Omega))} + \|p_2 - p_1\|_{L^2(0,T;H^1(\Omega))} \\
&\leqslant C(\|n_2^0 - n_1^0\| + \|p_2^0 - p_1^0\|).
\end{aligned} \tag{4.16}
$$

From (4.16) and (3.2) it follows that

$$\|\psi_2 - \psi_1\|_{C([0,T];H^2(\Omega))} + \|\psi_2 - \psi_1\|_{L^2(0,T;H^{1,\infty}(\Omega))} \leqslant C(\|n_2^0 - n_1^0\| + \|p_2^0 - p_1^0\|). \tag{4.17}$$

If $n_1^0 = n_2^0$ and $p_1^0 = p_2^0$ then (4.16) and (4.17) imply $\psi_1 = \psi_2$, $n_1 = n_2$, $p_1 = p_2$ on $\Omega \times (0,T)$, which proves the uniqueness of the problem P provided A^1 is satisfied.

Remark 4.2. If we say "uniqueness of the problem P", we mean, of course, that the data satisfy only the assumptions introduced in the definiton of problem P. In particular, n^0 and p^0 need not satisfy the boundary conditions. We did not use this assumption in deriving (4.16).

We state the result of this paragraph as

Lemma 4.2. *Let* A^1 *and* (4.14) *be fulfilled. Then the differences* $n_2 - n_1$, $p_2 - p_1$ *and* $\psi_2 - \psi_1$ *satisfy the inequalities* (4.16) *and* (4.17), *respectively. Here* ψ_j, n_j, p_j, $j = 1,2$, *are two solutions of the problem* P *with the same boundary data and with the initial data* n_j^0, p_j^0.

e) We need one more lemma.

Lemma 4.3. *Let the family* $\{T_h\}$ *satisfy the minimum angle condition. If* $u \in W_h$ *and* $q \geqslant 2$ *then there exists* $G \in L^q(\Omega)$ *such that*

$$(u,v) - (u,v)_h = (G,v) \;\; \forall v \in W_h, \quad \|G\|_{L^q(\Omega)} \leqslant Ch^{2/q}\|\nabla u\|_{L^2(\Omega)}. \tag{4.18}$$

Remark 4.3. For $q = 2$ the result was proved by Ciavaldini ([4], p. 470).

Proof. One proves easily that

$$\left|\int_{\widehat{K}} \widehat{f}\,\mathrm{d}\xi - I(\widehat{f})\right| \leqslant C|\widehat{f}|_{2,\widehat{K}}$$

where $I(\widehat{f}) = \frac{1}{6}(f_1 + f_2 + f_3)$ (f_j are the values of f at the vertices of $\widehat{K}$) is the approximate value of $\int_{\widehat{K}} \widehat{f}\,\mathrm{d}\xi$ and $|u|_{m,R} = \Big(\sum_{|\alpha|=m} \int_R |D^\alpha u|^2\,\mathrm{d}\xi\Big)^{1/2}$, $m = 1,2,\ldots$

Let $w \in W_h$ and consider $\widehat{f} = \widehat{u}\widehat{w}$. Then $|\widehat{f}|_{2,\widehat{K}} \leqslant C|\widehat{u}|_{1,\widehat{K}}|\widehat{w}|_{1,\widehat{K}}$ and therefore

$$
\begin{aligned}
|(u,w)-(u,w)_h| &= \Big|\sum_{K\in T_h} 2\,\mathrm{area}(K)\Big(\int_{\widehat{K}} \widehat{u}\widehat{w}\,\mathrm{d}\xi - I(\widehat{u}\widehat{w})\Big)\Big| \\
&\leqslant C\sum_{K\in T_h} h_K^2|\widehat{u}|_{1,\widehat{K}}|\widehat{w}|_{1,\widehat{K}} \\
&\leqslant C\Big\{\sum_{K\in T_h}|u|_{1,K}^2\Big\}^{1/2}\Big\{\sum_{K\in T_h} h_K^4(w_j^2+w_k^2+w_m^2)\Big\}^{1/2}.
\end{aligned}
$$

We make use of the inequality

$$
a^2+b^2+c^2 \leqslant (a^{q'}+b^{q'}+c^{q'})^{2/q'}, \quad a,b,c \geqslant 0.
$$

(To prove it consider the function $F(a,b,c) = (a^{q'}+b^{q'}+c^{q'})^{2/q'}$. Set $a=\alpha\xi$, $b=\alpha\eta$, $c=\alpha\zeta$, $\alpha^2 = a^2+b^2+c^2$. Then $\xi^2+\eta^2+\zeta^2 = 1$. Evidently, $\xi^2+\eta^2+\zeta^2 \leqslant \xi^{q'}+\eta^{q'}+\zeta^{q'}$, hence $F(\xi,\eta,\zeta)$ is bounded from below by 1 and, as $F(a,b,c) = \alpha^2F(\xi,\eta,\zeta)$, the above inequality follows.) We get

$$
|(u,v)-(u,w)_h| \leqslant C\|\nabla u\|_{L^2(\Omega)}\Big\{\sum_{K\in T_h} h_K^4(|w_j|^{q'}+|w_k|^{q'}+|w_m|^{q'})^{2/q'}\Big\}^{1/2}.
$$

By (3.38),

$$
\begin{aligned}
|(u,w)-(u,w)_h| &\leqslant C\|\nabla u\|_{L^2(\Omega)}\Big\{\sum_{K\in T_h} h_K^{4/q}\|w\|_{L^{q'}(K)}^2\Big\}^{1/2} \\
&\leqslant Ch^{2/q}\|\nabla u\|_{L^2(\Omega)}\Big\{\sum_{K\in T_h}\|w\|_{L^{q'}(K)}^2\Big\}^{1/2}.
\end{aligned}
$$

Now, we chose $w = \|v\|_{L^{q'}(\Omega)}^{-1}v$. Since $\|w\|_{L^{q'}(\Omega)} = 1$, we have $\|w\|_{L^{q'}(K)} \leqslant 1$ $\forall K \in \bigcup_h T_h$, hence $\|w\|_{L^{q'}(K)}^2 \leqslant \|w\|_{L^{q'}(K)}^{q'}$ and further

$$
|(u,w)-(u,w)_h| \leqslant Ch^{2/q}\|\nabla u\|_{L^2(\Omega)}\Big\{\sum_{K\in T_h}\|w\|_{L^{q'}(K)}^{q'}\Big\}^{1/2} = Ch^{2/q}\|\nabla u\|_{L^2(\Omega)}
$$

and

$$
|(u,v)-(u,v)_h| \leqslant Ch^{2/q}\|\nabla u\|_{L^2(\Omega)}\|v\|_{L^{q'}(\Omega)}.
$$

The proof can be completed in the same way as that of Lemma 3.2.

f) From (4.5), (4.6) and (4.11) it follows that

$$\|N^\delta\|_{C([0,T];H^1(\Omega))} \leqslant C, \qquad \Big\|\frac{\partial}{\partial t}N^\delta\Big\|_{L^2(0,T;H^1(\Omega))} \leqslant C, \tag{4.19}$$

$$\|P^\delta\|_{C([0,T];H^1(\Omega))} \leqslant C, \qquad \Big\|\frac{\partial}{\partial t}P^\delta\Big\|_{L^2(0,T;H^1(\Omega))} \leqslant C. \tag{4.20}$$

We shall consider sequences of functions from the families $\{N^\delta\}$ and $\{P^\delta\}$. We leave out subscripts and use always the same notation $\{N^\delta\}$ and $\{P^\delta\}$ for these subsequences, and δ is always such that $\delta \to 0$.

The family $\{N^\delta\}$ is bounded in $C([0,T];H^1(\Omega))$. It is also equicontinuous since

$$\|N^\delta(t_2) - N^\delta(t_1)\|_{H^1(\Omega)} = \Big\|\int_{t_1}^{t_2} \frac{\partial}{\partial t}N^\delta(\tau)\,\mathrm{d}\tau\Big\|_{H^1(\Omega)} \leqslant C|t_1 - t_2|^{1/2}$$

due to (4.19). As the imbedding of $H^1(\Omega)$ into $L^q(\Omega)$ is compact for any $1 \leqslant q < \infty$, the set $\{N^\delta(t)\}$ is relatively compact in $L^q(\Omega)$ for any $t \in [0,T]$. The assumptions of the Arzelà-Ascoli theorem for functions with values in a Banach space are satisfied (see, e.g., [8], p. 42), therefore there exists a subsequence (still denoted by N^δ) such that

$$\|n - N^\delta\|_{C([0,T];L^q(\Omega))} \to 0$$

where $n \in C([0,T];L^q(\Omega))$. The function n does not depend on q. If, namely, n_j, $j = 1,2$, correspond to q_j, $q_1 < q_2$, then

$$\begin{aligned}
&\|n_2 - n_1\|_{C([0,T];L^{q_1}(\Omega))} \\
&\leqslant \|n_2 - N^\delta\|_{C([0,T];L^{q_1}(\Omega))} + \|n_1 - N^\delta\|_{C([0,T];L^{q_1}(\Omega))} \\
&\leqslant C(\|n_2 - N^\delta\|_{C([0,T];L^{q_2}(\Omega))} + \|n_1 - N^\delta\|_{C([0,T];L^{q_1}(\Omega))}) \to 0.
\end{aligned}$$

Due to (4.19) and (3.2) we have also $\|N^\delta - n_I^*\|_{L^\infty(0,T;V)} \leqslant C$. Consider the family $\{\Phi^\delta\}$,

$$\langle \Phi^\delta, z\rangle_{L^1(0,T;V)} = \int_0^T d(N^\delta - n_I^*, z)\,\mathrm{d}t \qquad \forall z \in L^1(0,T;V).$$

It is a bounded linear functional on $L^1(0,T;V)$. Consequently, there is a subsequence $\{\Phi^\delta\}$ such that Φ^δ converges weakly* to an element of $L^\infty(0,T;V')$, i.e.

$$\langle \Phi^\delta, z\rangle_{L^1(0,T;V)} \to \langle \Phi, z\rangle_{L^1(0,T;V)} \ \ \forall z \in L^1(0,T;V), \quad \Phi \in L^\infty(0,T;V')$$

(see, e.g., Céa [2], p. 26). We have

$$\langle \Phi, z\rangle_{L^1(0,T;V)} = \int_0^T \langle \varphi, z\rangle_V \ \mathrm{d}t, \quad \varphi \in L^\infty(0,T;V')$$

(see, e.g., [8], p. 125). According to the Lax-Milgram theorem, there exists $\bar{\bar{n}} \in L^\infty(0,T;V)$ such that

$$\langle \varphi, v\rangle_V = d(\bar{\bar{n}}, v) \ \ \forall v \in V.$$

If we define $\bar{n} = \bar{\bar{n}} + n^*$, then $\bar{n} - n^* \in L^\infty(0,T;V)$ and, since $\int_0^T d(n^* - n_I^*, z)\,\mathrm{d}t \to 0$ if $h \to 0$,

$$\int_0^T d(N^\delta, z)\,\mathrm{d}t \to \int_0^T d(\bar{n}, z)\,\mathrm{d}t \ \ \forall z \in L^1(0,T;V).$$

We want to prove that $\bar{n} = n$. The set $\{N^\delta - n_I^*\}$ is bounded in $L^2(0,T;V)$. Therefore (see e.g. [2], p. 24) there exists a subsequence, denoted again in the same way, such that $N^\delta - n_I^*$ converges weakly to a function $\widetilde{n} \in L^2(0,T;V)$. We choose the functional $F(w) = \int_o^T d(w,z)\,\mathrm{d}t$ which is a linear bounded functional on $L^2(0,T;V)$. Then $\int_0^T d(N^\delta - n_I^*, z)\,\mathrm{d}t \to \int_0^T d(\widetilde{n}, z)\,\mathrm{d}t$ and $\int_0^T d(N^\delta, z)\,\mathrm{d}t \to \int_0^T d(\widetilde{n} + n^*, z)\,\mathrm{d}t$. Hence $\bar{n} = \widetilde{n} + n^*$. Further, we consider the functional $F(w) = \int_0^T (w,z)\,\mathrm{d}t$ on the same space. Again, $F \in (L^2(0,T;V))' \Rightarrow \int_0^T (N^\delta - n_I^*, z)\,\mathrm{d}t \to \int_0^T (\widetilde{n}, z)\,\mathrm{d}t$. But $\int_0^T (N^\delta - N_I^*, z)\,\mathrm{d}t \to \int_0^T (n - n^*, z)\,\mathrm{d}t$, thus $n - n^* = \widetilde{n} = \bar{n} - n^*$ and $\bar{n} = n$.

Summing up, there exists a function $n \in C([0,T]; L^q(\Omega))$ such that $n - n^* \in L^\infty(0,T;V)$ and

$$\|n - N^\delta\|_{C([0,T];L^q(\Omega))} \to 0, \tag{4.21}$$
$$\int_0^T d(N^\delta, z)\,\mathrm{d}t \to \int_0^T d(n, z)\,\mathrm{d}t \ \ \forall z \in L^1(0,T;V).$$

Similarly, there exists $p \in C([0,T]; L^q(\Omega))$ such that $p - p^* \in L^\infty(0,T;V)$ and

$$\|p - P^\delta\|_{C([0,T];L^q(\Omega))} \to 0, \tag{4.22}$$
$$\int_0^T d(P^\delta, z)\,\mathrm{d}t \to \int_0^T d(p, z)\,\mathrm{d}t \ \ \forall z \in L^1(0,T;V).$$

As $\|n^0 - N^0\| + \|p^0 - P^0\| \to 0$ if $h \to 0$, we see that n and p satisfy the initial condition (1.11). Let $E(t)$ be the subset of Ω where $n(t) < 0$ and $v^*(t) = n(t)$ on $E(t)$, $v^*(t) = 0$ on $\Omega - E(t)$. Then $v^*(t) \in L^2(\Omega)$, $((n(t) - N^\delta(t), v^*(t)) \to 0$ if $\delta \to 0$ for each $t \in [0,T]$ and $(n(t) - N^\delta(t), v^*(t)) = (\|n(t)\|^2_{L^2(E(t))} - N^\delta(t), n(t))_{L^2(E(t))} > \|n(t)\|^2_{L^2(E(t))}$. Thus $\|n(t)\|_{L^2(E(t))} = 0 \Rightarrow \operatorname{meas} E(t) = 0$. The requirement (1.12) is fulfilled.

g) We define a function ψ by (1.8), where n and p are the functions from (4.21), (4.22), and by the requirement $\psi - \psi^* \in V$. Evidently, $\psi - \psi^* \in C([0,T]; V \cap H^{1,\infty}(\Omega))$. We show that the first assertion of (4.1) is true. To this end, let us first consider the function G^δ determined by

$$(G^\delta, v) = (P^\delta, v) - (P^\delta, v)_h - [(N^\delta, v) - (N^\delta, v)_h] + (N_I, v) - (N_I, v)_h \ \ \forall v \in W_h.$$

Due to Lemma 4.3, (4.19) and (4.20) we have

$$\|G^\delta\|_{L^{q_0}(\Omega)} \leqslant Ch^{2/q_0}, \qquad q_0 = 2+\varepsilon, \quad \varepsilon > 0.$$

Here and in the sequel, the constant C does not depend on t (nor on the parameters mentioned at the beginning of the third section). Let F^δ be defined by

$$(F^\delta, v) = (P^\delta - N^\delta + N_I - G^\delta, v) = (P^\delta - N^\delta + N_I, v)_h \ \ \forall v \in W_h$$

and $Z \in W_h$ by

$$d(Z, v) = 0 \ \ \forall v \in V_h, \qquad Z\big|_{\Gamma^1} = \psi_I^*\big|_{\Gamma^1}.$$

Then

$$\Psi^\delta - Z \in V_h, \quad d(\Psi^\delta - Z, v) = \alpha(F^\delta, v) \ \ \forall v \in V_h.$$

Let φ^δ be the solution of the problem

$$-\Delta\varphi^\delta = \alpha(F^\delta, v) \ \ \forall v \in V, \quad \varphi^\delta\big|_{\Gamma^1} = \frac{\partial\varphi^\delta}{\partial\nu}\Big|_{\Gamma^2} = 0.$$

The equivalent variational formulation is

$$\varphi^\delta \in V, \quad d(\varphi^\delta, v) = \alpha(F^\delta, v) \ \ \forall v \in V.$$

Therefore $\Psi^\delta - Z$ is the Ritz projection of φ^δ, $\Psi^\delta - Z = R(\varphi^\delta)$. To use this fact we must bring φ^δ in connection with the function ψ. Let z be the function defined by

$$-\Delta z = 0, \quad z\big|_{\Gamma^1} = \psi^*\big|_{\Gamma^1}, \quad \frac{\partial z}{\partial\nu}\Big|_{\Gamma^2} = 0.$$

Then

$$-\Delta(\psi - z - \varphi^\delta) = \alpha(p - n + N - F^\delta), \quad (\psi - z - \varphi^\delta)\big|_{\Gamma^1} = \frac{\partial(\psi - z - \varphi^\delta)}{\partial\nu}\Big|_{\Gamma^2} = 0,$$

hence by the Sobolev imbedding theorem and (3.2)

$$\begin{aligned}\|\psi - z - \varphi^\delta\|_{C([0,T];H^{1,\infty}(\Omega))} &\leqslant C\|\psi - z - \varphi^\delta\|_{C([0,T];H^{2,q_0}(\Omega))} \\ &\leqslant C\|p - n + N - F^\delta\|_{C([0,T];L^{q_0}(\Omega))} = o(1).\end{aligned}$$

Now, $\psi - \Psi^\delta = \psi - z - \varphi^\delta + z - Z + \varphi^\delta - \varphi_I^\delta - R(\varphi^\delta - \varphi_I^\delta)$, consequently, due to the Rannacher-Scott theorem (see [11], p. 438),

$$\begin{aligned}\|\psi - \Psi^\delta\|_{C([0,T];H^{1,\infty}(\Omega))} &\leqslant o(1) + \|z - Z\|_{H^{1,\infty}(\Omega)} + C\|\varphi^\delta - \varphi_I^\delta\|_{C([0,T];H^{1,\infty}(\Omega))} \\ &\leqslant o(1) + \|z - Z\|_{H^{1,\infty}(\Omega)} + Ch^{1-2/q_0}\|\varphi^\delta\|_{C([0,T];H^{2,q_0}(\Omega))} \\ &\leqslant o(1) + \|z - Z\|_{H^{1,\infty}(\Omega)} + Ch^{1-2/q_0},\end{aligned}$$

i.e.

$$(4.22') \qquad \|\psi - \Psi^\delta\|_{C([0,T];H^{1,\infty}(\Omega))} \leqslant o(1) + \|z - Z\|_{H^{1,\infty}(\Omega)}.$$

Concerning the term $\|z - Z\|_{H^{1,\infty}(\Omega)}$, we define ζ by

$$-\Delta\zeta = 0, \quad \zeta\big|_{\Gamma_1} = \psi_I^*\big|_{\Gamma^1}, \quad \frac{\partial\zeta}{\partial\nu}\Big|_{\Gamma^2} = 0.$$

Then

$$\zeta - \psi_I^* \in V, \qquad d(\zeta - \psi_I^*, v) = -d(\psi_I^*, v) \ \ \forall v \in V.$$

Further,

$$Z - \psi_I^* \in V_h, \qquad d(Z - \psi_I^*, v) = -d(\psi_I^*, v) \ \ \forall v \in V_h$$

so that $Z - \psi_I^* = R(\zeta - \psi_I^*)$. We have

$$\zeta - Z = \zeta - \psi_I^* - (\zeta_I - \psi_I^*) + R(\zeta_I - \psi_I^*) - R(\zeta - \psi_I^*) = \zeta - \zeta_I - R(\zeta - \zeta_I),$$

consequently,

$$\|\zeta - Z\|_{H^{1,\infty}(\Omega)} \leqslant C\|\zeta - \zeta_I\|_{H^{1,\infty}(\Omega)} \leqslant Ch^{1-2/q_0}\|\zeta\|_{H^{2,q_0}(\Omega)} \leqslant Ch^{1-2/q_0}$$

and, by (3.2),

$$\begin{aligned}\|z - Z\|_{H^{1,\infty}(\Omega)} &\leqslant \|z - \zeta\|_{H^{1,\infty}(\Omega)} + o(1) \leqslant C\|z - \zeta\|_{H^{2,q_0}(\Omega)} + o(1) \\ &\leqslant C\sum_{j\in\mathcal{D}} \|\psi^* - \psi_I^*\|_{H^{2-1/q_0,q_0}(\Gamma_j)} + o(1).\end{aligned}$$

If we show that

$$(4.23) \qquad \|\psi^* - \psi_I^*\|_{H^{2-1/q_0,q_0}(\Gamma_j)} = o(1)$$

for all $j \in \mathcal{D}$, the convergence of Ψ^δ to ψ in the $C([0,T];H^{1,\infty}(\Omega))$-norm will follow from (4.22′).

Since Γ_j is a segment, it is sufficient to show that

$$\|f - f_I\|_{H^{2-1/q_0,q_0}(\Omega_0)} = o(1) \ \text{ if } f \in H^{2,q_0}(\Omega_0),$$

where $\Omega_0 = (0,1)$ and the elements K covering $\overline{\Omega}_0$ are subintervals of $[0,1]$.

We use the notation $\omega = f - f_I$. Then (see, e.g. [10], p. 81),

$$\|\omega\|^{q_0}_{H^{2-1/q_0,q_0}(\Omega_0)} = \|\omega\|^{q_0}_{H^{1,q_0}(\Omega_0)} + \int_{\Omega_0}\int_{\Omega_0} \frac{|\omega'(x)-\omega'(y)|^{q_0}}{|x-y|^{q_0}}\,\mathrm{d}x\,\mathrm{d}y$$
$$= \sum_K \|\omega\|^{q_0}_{H^{1,q_0}(K)} + \sum_K\sum_K \int_K\int_K \frac{|\omega'(x)-\omega'(y)|^{q_0}}{|x-y|^{q_0}}\,\mathrm{d}x\,\mathrm{d}y.$$

The integrals over the segments K will be transformed on $\widehat{K} = [0,1]$. We get

$$\|\omega\|^{q_0}_{H^{2-1/q_0}(\Omega_0)} \leqslant \sum_K h_K^{2(1-q_0)} \|\widehat{\omega}\|^{q_0}_{H^{1,q_0}(\widehat{K})}$$
$$+ \sum_K\sum_K h_K^{2(1-q_0)} \int_{\widehat{K}}\int_{\widehat{K}} \frac{|\widehat{\omega}'(\xi)-\widehat{\omega}'(\eta)|^{q_0}}{|\xi-\eta|^{q_0}}\,\mathrm{d}\xi\,\mathrm{d}\eta$$
$$\leqslant \sum_K h_K^{2(1-q_0)} \|\widehat{\omega}\|^{q_0}_{H^{2-1/q_0,q_0}(\widehat{K})}.$$

Now, we use the interpolation inequality

$$\|\cdot\|_{H^{s^*,p^*}} \leqslant C\|\cdot\|^{1-\theta}_{H^{s_0,p_0}}\|\cdot\|^{\theta}_{H^{s_1,p_1}}, \quad s_0 \neq s_1, \quad 1 < p_0, p_1 < \infty,$$
$$s^* = (1-\theta)s_0 + \theta s_1, \quad \frac{1}{p^*} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$$

(see Berg and Löfström [1], p. 153, formula (7)), where we choose $s^* = 2 - \frac{1}{q_0}$, $p^* = q_0$, $s_0 = 1$, $s_1 = 2$, $p_0 = p_1 = p^* = q_0$. We have

$$\|\omega\|^{q_0}_{H^{2-1/q_0,q_0}(\Omega_0)} \leqslant C\sum_K h_K^{2(1-q_0)} \|\widehat{\omega}\|_{H^{1,q_0}(\widehat{K})} \|\widehat{\omega}\|^{q_0-1}_{H^{2,q_0}(\widehat{K})}$$
$$\leqslant C\sum_K h_K^{2(1-q_0)} |\widehat{f}|^{q_0}_{H^{2,q_0}(\widehat{K})}$$
$$\leqslant C\sum_K h_K^{2(1-q_0)} h_K^{2q_0-1} |f|^{q_0}_{H^{2,q_0}(K)}$$
$$\leqslant Ch|f|^{q_0}_{H^{2,q_0}(\Omega_0)},$$

where

$$|\cdot|_{H^{m,p}} = \sum_{|\alpha|=m} \|D^\alpha(\cdot)\|_{L^p}, \quad m = 1,2,\ldots$$

and

$$\|f - f_I\|_{H^{2-1/q_0,q_0}(\Omega_0)} \leqslant Ch^{1/q_0}.$$

Remark 4.4. If $\psi^* \in H^{2,q}(\Omega)$, $q > 2$, then $\psi^*\big|_{\Gamma_j} \in H^{2-1/q,q}(\Gamma_j)$. We can choose ε so small that $q > q_0 = 2+\varepsilon$. One can expect that this is sufficient for (4.23) to hold, i.e., the assumption $\psi^*\big|_{\Gamma_j} \in H^{2,q_0}(\Gamma_j)$ for all $j \in \mathcal{D}$ is superfluous.

h) It remains to prove that the functions n and p satisfy (1.9) and (1.10), respectively. We restrict ourselves to (1.9). Consider a function $\varphi(t) \in \mathcal{D}((0,T))$ and define

$$\varphi_{\Delta t} = \varphi^i \quad \text{in} \quad (t_{i-1}, t_i], \quad i = 1, \dots, r.$$

For a given $v \in V$ we choose $\{v_h\}$ such that $v_h \in V_h$ and

$$\|v - v_h\|_V \to 0 \quad \text{if} \quad h \to 0.$$

We set $v = v_h \varphi^i$ in (2.5) and sum up. We get

$$\sum_{i=1}^{r} (\Delta N^i, v_h)_h \varphi^i + \delta_n \Delta t \sum_{i=1}^{r} \bar{\nu}_h^2(\Psi^i; N^i, v_h)\varphi^i + \Delta t \sum_{i=1}^{r} (R^i, v_h)_h \varphi^i = 0. \tag{4.24}$$

Concerning the first term we have

$$\begin{aligned}
\sum_{i=1}^{r} (\Delta N^i, v_h)_h \varphi^i &= \int_0^T \Big(\frac{\partial}{\partial t} N^\delta, v_h\Big)_h \varphi_{\Delta t} \, \mathrm{d}t \\
&= \int_0^T \Big(\frac{\partial}{\partial t} N^\delta, v_h\Big)_h (\varphi_{\Delta t} - \varphi) \, \mathrm{d}t + \int_0^T \Big(\frac{\partial}{\partial t} N^\delta, v_h - v\Big)_h \varphi \, \mathrm{d}t \\
&\quad + \int_0^T \Big[\Big(\frac{\partial}{\partial t} N^\delta, v\Big)_h - \Big(\frac{\partial}{\partial t} N^\delta, v\Big)\Big] \varphi \, \mathrm{d}t + \int_0^T \Big(\frac{\partial}{\partial t} N^\delta, v\Big) \varphi \, \mathrm{d}t.
\end{aligned}$$

The first three integrals on the right-hand side converge to zero if $\delta \to 0$, the last converges as follows:

$$\int_0^T \Big(\frac{\partial}{\partial t} N^\delta, v\Big) \varphi \, \mathrm{d}t = \int_0^T \frac{\partial}{\partial t} (N^\delta, v) \varphi \, \mathrm{d}t = -\int_0^T (N^\delta, v) \varphi' \, \mathrm{d}t \to -\int_0^T (n, v) \varphi' \, \mathrm{d}t,$$

thus

$$\sum_{i=1}^{r} (\Delta N^i, v_h)_h \varphi^i \to -\int_0^T (n, v) \varphi' \, \mathrm{d}t \quad \text{if} \quad \delta \to 0. \tag{4.25}$$

Concerning the second term in (4.24) we have

$$\begin{aligned}
\Delta t \sum_{i=1}^{r} \bar{\nu}_h^2(\Psi^i; N^i, v_h)\varphi^i &= \Delta t \sum_{i=1}^{r} d(N^i, v_h)\varphi^i + \Delta t \sum_{i=1}^{r} a_1(\Psi^i; N^i, v_h)\varphi^i \\
&\quad - \Delta t \sum_{i=1}^{r} c(\Psi^i; N^i, v_h)\varphi^i.
\end{aligned}$$

The sum $\Delta t \sum\limits_{i=1}^{r} d(N^i, v_h)\varphi^i$ can be expressed in the form

$$\begin{aligned}\Delta t \sum_{i=1}^{r} d(N^i, v_h)\varphi^i &= \sum_{i=1}^{r} \int_{t_{i-1}}^{t_i} d(N^i, v_h)\varphi_{\Delta t}\,\mathrm{d}t \\ &= \sum_{i=1}^{r} \int_{t_{i-1}}^{t_i} d(N^i, v_h - v)\varphi_{\Delta t}\,\mathrm{d}t + \sum_{i=1}^{r} \int_{t_{i-1}}^{t_i} d(N^i, v)(\varphi_{\Delta t} - \varphi)\,\mathrm{d}t \\ &\quad + \sum_{i=1}^{r} \int_{t_{i-1}}^{t_i} \frac{t_i - t}{\Delta t} d(\Delta N^i, v)\varphi\,\mathrm{d}t + \int_0^T d(N^\delta, v)\varphi\,\mathrm{d}t.\end{aligned}$$

Due to (4.19), the first three terms on the right-hand side converge to zero, whereas

$$\int_0^T d(N^\delta, v)\varphi\,\mathrm{d}t \to \int_0^T d(n, v)\varphi\,\mathrm{d}t$$

due to (4.21). Further

$$\left|\Delta t \sum_{i=1}^{r} a_1(\Psi^i; N^i, v_h)\varphi^i\right| \leqslant C\Delta t \|\Psi^i\|_{H^{1,\infty}(\Omega)} h \sum_{i=1}^{r} \|N^i\|_V \|v_h\|_V \leqslant Ch \to 0$$

and

$$\begin{aligned}\Delta t \sum_{i=1}^{r} c(\Psi^i; N^i, v_h)\varphi^i &= \sum_{i=1}^{r} \int_{t_{i-1}}^{t_i} c(\Psi^i; N^i, v_h)\varphi_{\Delta t}\,\mathrm{d}t \\ &= \sum_{i=1}^{r} \int_{t_{i-1}}^{t_i} c(\Psi^i; N^i, v_h)(\varphi_{\Delta t} - \varphi)\,\mathrm{d}t \\ &\quad + \sum_{i=1}^{r} \int_{t_{i-1}}^{t_i} \frac{t_i - t}{\Delta t} c(\Psi^i; \Delta N^i, v_h)\varphi\,\mathrm{d}t \\ &\quad + \sum_{i=1}^{r} \int_{t_{i-1}}^{t_i} \frac{t_i - t}{\Delta t} c(\Delta\Psi^i; N^\delta, v_h)\varphi\,\mathrm{d}t \\ &\quad + \int_0^T \left(c(\Psi^\delta; N^\delta, v_h) - \int_\Omega N^\delta \nabla\Psi^\delta \cdot \nabla v_h\,\mathrm{d}x \right)\varphi\,\mathrm{d}t \\ &\quad + \int_0^T \int_\Omega N^\delta \nabla\Psi^\delta \cdot \nabla(v_h - v)\varphi\,\mathrm{d}t \\ &\quad + \int_0^T \int_\Omega N^\delta \nabla\Psi^\delta \cdot \nabla v\varphi\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Since $|c(\Psi; N, w)| \leqslant C\|\Psi\|_{H^{1,\infty}(\Omega)} \|N\|_V \|w\|_V$ for $\Psi, n, w \in W_h$, we find easily by means of (4.2), (4.8) and (4.19) that the first three terms on the right-hand side converge to zero if $\delta \to 0$. Concerning the integrand of the fourth term we have (we denote by v_n, $n = 1, \ldots, q$, the values of v_h at the nodes)

$$
\begin{aligned}
&c(\Psi^\delta; N^\delta, v_h) - \int_\Omega N^\delta \nabla \Psi^\delta v_h \,\mathrm{d}x \\
&\quad = \sum_{K \in T_h} \Big[\alpha_j^K (\Psi_m^\delta - \Psi_k^\delta) \frac{1}{2} (N_m^\delta + N_k^\delta)(v_m - v_k) + \ldots \Big] \\
&\qquad - \sum_{K \in T_h} 2\,\mathrm{area}(K) \nabla \Psi^\delta \cdot \nabla v_h) \int_{\widehat{K}} \widehat{N}^\delta \,\mathrm{d}\xi.
\end{aligned}
$$

On the one hand (see (3.14)),

$$
d(\Psi^\delta, v_h) = \sum_{K \in T_h} [\alpha_j^K (\Psi_m^\delta - \Psi_k^\delta)(v_m - v_k) + \ldots],
$$

on the other hand,

$$
d(\Psi^\delta, v_h) = \sum_{K \in T_h} \int_K \nabla \Psi^\delta \cdot \nabla v_h \,\mathrm{d}x = \sum_{K \in T_h} \mathrm{area}(K) \nabla \Psi^\delta \cdot \nabla v_h.
$$

Therefore, $\mathrm{area}(K) \nabla \Psi^\delta \cdot \nabla v_h = \alpha_j^K (\Psi_m^\delta - \Psi_k^\delta)(v_m - v_k) + \ldots$, and

$$
\begin{aligned}
&c(\Psi^\delta; N^\delta, v_h) - \int_\Omega N^\delta \nabla \Psi^\delta \cdot \nabla v_h \,\mathrm{d}x \\
&\quad = \sum_{K \in T_h} \Big\{ \alpha_j^K (\Psi_m^\delta - \psi_k^\delta)(v_m - v_k) \Big[\frac{1}{2} (N_m^\delta + N_k^\delta) - 2 \int_{\widehat{K}} \widehat{N} \,\mathrm{d}\xi \Big] + \ldots \Big\} \\
&\quad = \sum_{K \in T_h} \Big\{ \alpha_j^K (\Psi_m^\delta - \Psi_k^\delta)(v_m - v_k) \frac{1}{6} \Big[(N_m - N_j) + (N_k - N_j) \Big] + \ldots \Big\}.
\end{aligned}
$$

Consequently,

$$
\left| c(\Psi^\delta; N^\delta, v_h) - \int_\Omega N^\delta \nabla \Psi^\delta \cdot \nabla v_h \,\mathrm{d}x \right| \leqslant Ch \|\Psi^\delta\|_{H^{1,\infty}(\Omega)} \|v_h\|_V \|N^\delta\|_V \leqslant Ch \to 0.
$$

The fifth term converges to zero and, since

$$
\begin{aligned}
&\int_0^T \int_\Omega (N^\delta \nabla \Psi^\delta - n \nabla \psi) \cdot \nabla v \,\mathrm{d}x \,\mathrm{d}t \\
&\quad = \int_0^T \varphi \int_\Omega N^\delta \nabla N (\Psi^\delta - \psi) \cdot \nabla v \,\mathrm{d}x \,\mathrm{d}t + \int_0^T \varphi \int_\Omega (N^\delta - n) \nabla \psi \cdot \nabla v \,\mathrm{d}x \,\mathrm{d}t \to 0
\end{aligned}
$$

due to (4.1) and (4.19), we see that

$$\Delta t \sum_{i=1}^{r} \bar{\nu}_h^2(\Psi^i; N^i, v_h)\varphi^i \to \int_0^T \bar{\nu}^2(\psi; n, v)\varphi \,\mathrm{d}t \quad \text{if} \quad \delta \to 0. \tag{4.26}$$

Finally, denoting $R^\delta = R(N^\delta, P^\delta)$, we have

$$\begin{aligned}
\Delta t \sum_{i=1}^{r} (R^i, v_h)_h \varphi^i &= \sum_{i=1}^{r} \int_{t_{i-1}}^{t_i} (R^i, v_h)_h \varphi_{\Delta t} \,\mathrm{d}t \\
&= \sum_{i=1}^{r} \int_{t_{i-1}}^{t_i} (R^i, v_h)_h (\varphi_{\Delta t} - \varphi) \,\mathrm{d}t \\
&\quad + \sum_{i=1}^{r} \int_{t_{i-1}}^{t_i} (R^i - R^\delta, v_h)_h \varphi \,\mathrm{d}t + \sum_{i=1}^{r} \int_{t_{i-1}}^{t_i} (R^\delta - R, v_h)_h \varphi \,\mathrm{d}t \\
&\quad + \sum_{i=1}^{r} \int_{t_{i-1}}^{t_i} \big((R, v_h)_h - (R, v_h)\big)\varphi \,\mathrm{d}t + \sum_{i=1}^{r} \int_{t_{i-1}}^{t_i} (R, v_h - v)\varphi \,\mathrm{d}t \\
&\quad + \int_0^T (R, v)\varphi \,\mathrm{d}t.
\end{aligned}$$

All sums on the right-hand side converge to zero, hence

$$\Delta t \sum_{i=1}^{r} (R^i, v_h)\varphi_i \to \int_0^T (R(n,p), v)\varphi \,\mathrm{d}t. \tag{4.27}$$

From (4.24), (4.25), (4.26) and (4.27) it follows that

$$-\int_0^T (n, v)\varphi' \,\mathrm{d}t + \delta_n \int_0^T \bar{\nu}^2(\psi; n, v)\varphi \,\mathrm{d}t + \int_0^T (R, v)\varphi \,\mathrm{d}t = 0$$

and the proof of Lemma 4.1 is complete. □

Theorem 4.1. *Let* A^0, A^1 *and* A^3 *be fulfilled and* $\Delta t \geqslant ch^2$. *If* ψ, n, p *is the (unique) solution of the problem* P, *then*

$$\|\psi - \Psi^\delta\|_{C([0,T];H^{1,q}(\Omega))} \to 0, \quad \|\psi - \Psi^\delta\|_{L^2(0,T;H^{1,\infty}(\Omega))} \to 0, \tag{4.28}$$

$$\|n - N^\delta\|_{C([0,T];L^2(\Omega))} \to 0, \qquad \|p - P^\delta\|_{C([0,T];L^2(\Omega))} \to 0 \tag{4.29}$$

for any $1 \leqslant q < \infty$ *as* $\delta \to 0$.

P r o o f. Let Ω_θ be the polygon lying in Ω with sides parallel to the sides of Ω at the distance θ, where $\theta > 0$ is sufficiently small. One can construct easily a function $\omega(x, \theta)$ with the following properties:

a) $\omega(x;\theta)=0$ on $\overline{\Omega}-\Omega_{1/2\theta}$,
b) $\omega(x;\theta)=1$ on $\overline{\Omega}_\theta$,
c) $\omega\in C^1(\overline{\Omega})\cap H^2(\Omega)$, $|D^\alpha\omega|\leqslant C\theta^{-1}$, $|\alpha|=1$.

Let z^j, $j=1,2$, be the solutions of the boundary value problems

$$-\Delta z^j=0 \text{ in } \Omega,\quad z^1\big|_{\Gamma^1}=n^*\big|_{\Gamma^1},\quad z^2\big|_{\Gamma^1}=p^*\big|_{\Gamma^1},\quad \frac{\partial z^j}{\partial\nu}\Big|_{\Gamma^2}=0.$$

By virtue of Remark 3.1 and the inequality (3.38′) we have $z^j\in H^2(\Omega)$. Consider the functions

$$n^0(x;\theta)=z^1(x)+\omega(x;\theta)\big(n^0(x)-z^1(x)\big),$$
$$p^0(x,\theta)=z^2(x)+\omega(x;\theta)\big(p^0(x)-z^2(x)\big).$$

We have

$$s^0(\theta)\in H^2(\Omega),\quad s^0\big|_{\Gamma^1}=s^*\big|_{\Gamma^1},\quad \frac{\partial s^0(\theta)}{\partial\nu}\Big|_{\Gamma^2}=0,\qquad (4.30)$$
$$s^0(x;\theta)=s^0(x) \text{ on } \overline{\Omega}_\theta,\quad s=n,p.$$

Further,

$$\|s^0(\theta)\|_{C(\bar{\Omega})}\leqslant C,\quad \|s^0-s^0(\theta)\|\leqslant C\theta^{1/2},\quad \|s^0(\theta)\|_{H^{1,q}(\Omega)}\leqslant C\theta^{-(1-1/q)},\qquad (4.31)$$
$$s=n,p,\quad q>2.$$

(Of course, the constant C does not depend on θ.)

Denote by $\psi(\theta)$, $n(\theta)$, $p(\theta)$ the solution of the problem P with the initial values $n^0(\theta)$, $p^0(\theta)$ (the boundary conditions (1.4) and (1.5) remain) and by $\Psi^\delta(\theta)$, $N^\delta(\theta)$, $P^\delta(\theta)$ the corresponding fully discrete approximate solution. By (4.30) and by Lemma 4.1, for each (sufficiently small) θ we have

$$\|\psi(\theta)-\Psi^\delta(\theta)\|_{C([0,T];H^{1,\infty}(\Omega))}\to 0,\qquad (4.32)$$
$$\|n(\theta)-N^\delta(\theta)\|_{C([0,T];L^2(\Omega))}\to 0,$$
$$\|p(\theta)-P^\delta(\theta)\|_{C([0,T];L^2(\Omega))}\to 0\quad\text{as }\delta\to 0.$$

It follows from (4.31) and from Lemmas 4.2 and 3.4 that

$$\|n-n(\theta)\|_{C([0,T];L^2(\Omega))}\leqslant C(\|n^0-n^0(\theta)\|+\|p^0-p^0(\theta)\|)\qquad (4.33)$$

and

$$\begin{aligned}\|N^\delta(\theta)-N^\delta\|_{C([0,T];L^2(\Omega))} &\leqslant C\big(\|(n^0-n^0(\theta))_I\|+\|(p^0-p^0(\theta))_I\|\big)\\ &\leqslant C(\|n^0-n^0(\theta)\|+\|n^0-n^0_I\|+\|n^0(\theta)-n^0(\theta)_I\|\\ &\qquad+\|p^0-p^0(\theta)\|+\|n^0-p^0_I\|+\|p^0(\theta)-p^0(\theta_I)\|).\end{aligned}$$

We have

$$\begin{gathered}\|s^0-s^0_I\|\leqslant Ch^2\|s^0\|_{H^2(\Omega)}\leqslant Ch^2,\\ \|s^0(\theta)-s^0(\theta)_I\|\leqslant Ch\|s^0(\theta)\|_{H^{1,q}(\Omega)}\leqslant Ch\theta^{-(1-1/q)}\quad\text{for}\quad q>2,\ \ s=n,p\end{gathered}$$

(see [3], Theorem 3.1.6, p. 124). Hence, choosing e.g. $q=3$ in the above estimates, we see that

$$\|N^\delta(\theta)-N^\delta\|_{C([0,T];L^2(\Omega))}\leqslant C\big(\|n^0-n^0(\theta)\|+\|p^0-p^0(\theta)\|+Ch\theta^{-2/3}\big). \tag{4.34}$$

Now, due to (4.33) and (4.34) we have

$$\begin{aligned}&\|n-N^\delta\|_{C([0,T];L^2(\Omega))}\\ &\quad\leqslant\|n-n(\theta)\|_{C([0,T];L^2(\Omega))}+\|n(\theta)-N^\delta(\theta)\|_{C([0,T];L^2(\Omega))}\\ &\qquad+\|N^\delta(\theta)-N^\delta\|_{C([0,T];L^2(\Omega))}\\ &\quad\leqslant C_1(\|n-n^0(\theta)\|+\|p-p^0(\theta)\|)\\ &\qquad+C_2h\theta^{-2/3}+\|n(\theta)-N^\delta(\theta)\|_{C([0,T];L^2(\Omega))}.\end{aligned}$$

To prove that $\|n-N^\delta\|_{C([0,T];L^2(\Omega))}\to 0$, let ε_0 be an arbitrary positive number. The second inequality in (4.31) guarantees the existence of θ_0 so small that $C_1(\|n-n^0(\theta_0)\|+\|p^0-p(\theta_0)\|)<\frac12\varepsilon_0$. From (4.32) it follows that for $|\delta|\equiv h+\Delta t$ sufficiently small, $|\delta|<\delta_0$, we have $C_2h\theta_0^{-2/3}+\|n(\theta_0)-N^\delta(\theta_0)\|_{C([0,T];L^2(\Omega))}<\frac12\varepsilon_0$. Hence $\|n-N^\delta\|_{C([0,T];L^2\Omega))}<\varepsilon_0$ for $|\delta|<\delta_0$. In the same way we prove the second part of (4.29). (4.28) follows in a similar way by virtue of Lemmas 4.2, 4.1 and 3.4. □

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