Finite elements wavelets on manifolds

Preprint No. 1199, Department of Mathematics, University of Utrecht, June 2001. Submitted to IMA J. Numer. Anal.

FINITE ELEMENT WAVELETS ON MANIFOLDS

HOANG NGUYEN AND ROB STEVENSON

Abstract. We construct locally supported, continuous wavelets on manifolds Γ that are given as the closure of a disjoint union of general smooth parametric images of an n-simplex. The wavelets are proven to generate Riesz bases for Sobolev spaces $H^s(\Gamma)$ when $s \in (-1, \frac{3}{2})$, if not limited by the global smoothness of Γ. These results generalize the findings from [DSt99], where it was assumed that each parametrization has a constant Jacobian determinant. The wavelets can be arranged to satisfy the cancellation property of in principal any order, except for wavelets with supports that extend to different patches, which generally satisfy the cancellation property of only order 1.

1. Introduction

This paper deals with the construction of wavelets on Hölder continuous piecewise smooth compact manifolds. As main application we have in mind the numerical solution of operator equations, in particular boundary integral equations. Essential requirements on the wavelets are then that they are locally supported, generate a Riesz basis for a relevant Sobolev space giving uniformly well-conditioned stiffness matrices, and furthermore that they have sufficiently many vanishing moments, or more generally cancellation properties, allowing for sparse but sufficiently accurate approximations of these matrices. For a thorough treatment of these topics, we refer to [Dah97, Sch98, Coh00].

As shown in [Dah96], the key to get such wavelets is to search them as L^2-stable bases of the subspaces generating L^2-biorthogonal multi-level space decompositions of two multiresolution analyses that satisfy Jackson and Bernstein estimates. Aiming at constructing wavelets on general polygonal domains, in [DSt99, Ste00] for both multiresolution analyses we used continuous Lagrange finite element type spaces. Having constructed once and for all some local bases on a reference element, which determine the order of the wavelets, the number of vanishing moments as well as the availability of locally supported dual wavelets, the concept of affine equivalence was applied to obtain explicit simple formulas for the wavelets in terms of the local topology of the mesh.

Other constructions of wavelets in (two-dimensional P_1) finite element spaces can be found in [KO95, FQ99, FQ00, CES00, HM00]. Alternative approaches to construct wavelet bases on non-tensor product domains or manifolds are based on domain decomposition like techniques, cf. [DSch99a, CTU99, DSch99b].

Date: June 18, 2001.

1991 *Mathematics Subject Classification.* 42C40, 65T60, 65N30, 65R20.

Key words and phrases. finite elements, wavelets, Riesz bases, vanishing moments, boundary integral equations.

As shown in [DSt99], our finite element wavelet construction can immediately be generalized to a restrictive class of manifolds consisting of a number of patches, where each patch can be described by a parametrization having a constant Jacobian determinant. Examples of such patches include parts of hyperplanes, spheres or cylinders. The point that hampers an application to general descriptions is that in case of non-constant Jacobian determinants, L^2-orthogonality between two functions on the reference element generally does not imply orthogonality between their push-forwards with respect to the canonical L^2-scalar product on the manifold.

To circumvent this problem an approach followed in the literature, e.g. in [DSch99a, CTU99, FQ99, FQ00], is to consider space decompositions that are biorthogonal with respect to a modified L^2-scalar product constructed by ignoring the Jacobian determinants. A somewhat hidden problem with this approach is that if the Jacobian determinants have jumps over the interfaces between patches, then the resulting wavelets cannot yield Riesz bases of Sobolev spaces $H^s(\Gamma)$ for $s \leq -\frac{1}{2}$. Another disadvantage is that in this case wavelets with supports that extend to different patches have no cancellation properties, except when patchwise cancellation properties are realized as in [CTU99].

Assuming that each patch is described by a smooth parametrization, the approach followed in this paper is to ignore the Jacobian determinant for constructing wavelets with supports inside one patch; whereas for wavelets with supports that extend to more than one patches the Jacobian determinants are taken into account in the sense that they are approximated by piecewise constants. The resulting wavelets span spaces which approximate the biorthogonal complements with respect to the canonical L^2-scalar product. Using a perturbation argument, we prove that the wavelets generate Riesz bases for $H^s(\Gamma)$ when $s \in (-1, \frac{3}{2})$, which interval safely includes the case $s = -\frac{1}{2}$ interesting for applications. Depending on the local bases applied on the reference element, wavelets with supports inside one patch satisfy the cancellation property of in principal arbitrary order, whereas wavelets with supports that extend to more than one patches satisfy the cancellation property of at least order one, and in some cases even of the same order as the wavelets with supports inside one patch. The wavelets can be implemented as efficiently as in the domain case.

The following notations will be used in this paper. In order to avoid the repeated use of generic but unspecified constants, by $C \lesssim D$ we mean that C can be bounded by a multiple of D, independently of parameters which C and D may depend on. Obviously, $C \gtrsim D$ is defined as $D \lesssim C$, and $C \eqsim D$ as $C \lesssim D$ and $C \gtrsim D$.

For some countable collection Φ of functions in a separable Hilbert space H with scalar product $\langle\ ,\ \rangle$ and norm $\|\ \|$, and for $\mathbf{c} = (c_\phi)_{\phi\in\Phi}$ a vector of scalars, with $\mathbf{c}^T\Phi$ we will mean the expansion $\sum_{\phi\in\Phi} c_\phi \phi$. We always consider spaces of scalar vectors as being equipped with scalar product $\langle \mathbf{c}, \mathbf{d}\rangle_{\ell^2} = \sum_{\phi\in\Phi} c_\phi \overline{d_\phi}$ and norm $\|\mathbf{c}\|_{\ell^2} = \langle \mathbf{c}, \mathbf{c}\rangle_{\ell^2}^{\frac{1}{2}}$, and consequently, the spaces of possibly infinite matrices as being equipped with the corresponding operator norm. For $x \in H$, with $\langle \Phi, x\rangle$ and $\langle x, \Phi\rangle$ we will mean the column- and row-vectors with coefficients $\langle \phi, x\rangle$ and $\langle x, \phi\rangle$, $\phi \in \Phi$. More generally, when $\tilde{\Phi}$ is another countable collection in H, with $\langle \Phi, \tilde{\Phi}\rangle$ is meant the matrix $(\langle \phi, \tilde{\phi}\rangle)_{\phi\in\Phi, \tilde{\phi}\in\tilde{\Phi}}$. A collection Φ is called a

Riesz system when $\|\mathbf{c}^T\Phi\| \eqsim \|\mathbf{c}\|_{\ell^2}$, and Φ is called a *Riesz basis* when it is in addition a basis for H.

2. Biorthogonal space decompositions

Continuing work from [DSt99, Ste00], we construct biorthogonal finite element type wavelets on manifolds. In [DSt99] it was assumed that the manifold is given as a disjoint union of images of parametric mappings, where each of them has a *constant Jacobian determinant*. Here and in the next sections, we will show how the construction can be generalized to general descriptions that may not satisfy this condition.

Our starting point is the standard closed reference n-simplex

$$\boldsymbol{T} = \{\lambda \in I\!R^{n+1} : \sum_{\ell=1}^{n+1} \lambda_\ell = 1, \lambda_\ell \geq 0\}.$$

The intersection of $\boldsymbol{T}$ with any lower dimensional coordinate plane will be called a *face* of $\boldsymbol{T}$. To avoid some technical complications, we will always assume that $n \leq 3$. We fix a refinement, sometimes called a triangulation, of $\boldsymbol{T}$ into 2^n congruent subsimplices $\boldsymbol{T}_1, \ldots, \boldsymbol{T}_{2^n}$, each of them determined by some ordered set of vertices.

For any closed n-simplex T, let $\lambda_T(z) \in [0,1]^{n+1}$ denote the barycentric coordinates of $z \in T$ with respect to the ordered set of vertices of T. Above dyadic refinement of $\boldsymbol{T}$ induces such a refinement of T into 2^n congruent subsimplices $(\lambda_T^{-1} \circ \lambda_{\boldsymbol{T}_k}^{-1} \circ \lambda_T)(T)$. The barycenter $\lambda_T^{-1}(\frac{1}{n+1}, \ldots, \frac{1}{n+1})$ of T will be denoted by $\zeta(T)$.

Starting with a collection τ_0, consisting of one n-simplex $T_0 \subset I\!R^n$ only, we obtain an infinite sequence of collections of simplices $(\tau_j)_{j\geq 0}$ by defining τ_{j+1} as the collection of all simplices that arise by applying above refinement to all simplices from τ_j.

We consider compact n-dimensional manifolds $\Gamma \subset I\!R^{n'}$. We assume that either $\Gamma \in C^{m,0}$ for some $1 \leq m \in I\!N$, or $\Gamma \in C^t$ for some $0 < t \notin I\!N$, which means that for $s \in [0,m]$ or $s \in [0,t)$, the Sobolev spaces $H^s(\Gamma)$ can be defined in the usual way using a partition of unity relative to some atlas. For s in above range, $H^{-s}(\Gamma)$ will be understood as the dual of $H^s(\Gamma)$.

We will assume that Γ is given as $\Gamma = \cup_{i=1}^p \overline{\Gamma_i}$, where $\Gamma_i = \kappa_i(T_0^{\rm int})$, with $\kappa_i : I\!R^n \to I\!R^{n'}$ being some smooth regular parametrization, and $T_0^{\rm int}$ the interior of T_0. We assume that for $1 \leq i \neq \breve{\imath} \leq p$, the intersection $\overline{\Gamma_i} \cap \overline{\Gamma_{\breve{\imath}}}$ is either empty, or there exists a permutation $\pi : I\!R^{n+1} \to I\!R^{n+1}$ such that

$$\pi \circ \lambda_{T_0} \circ \kappa_i^{-1} = \lambda_{T_0} \circ \kappa_{\breve{\imath}}^{-1} \text{ on } \overline{\Gamma_i} \cap \overline{\Gamma_{\breve{\imath}}}. \tag{2.1}$$

Remark 2.1. We assume here that Γ is given as a disjoint union of parametric images of an n-simplex. Alternatively, the wavelet construction outlined below can also be carried out, even requiring a few technicalities less, when instead an n-cube is taken as reference domain.

With μ being the induced Lebesgue measure on Γ, we have

$$\int_\Gamma u\overline{v}d\mu = \sum_{i=1}^p \int_{T_0} u(\kappa_i(z))\,\overline{v(\kappa_i(z))}\,|\partial\kappa_i(z)|\,dz,$$

where $|\partial\kappa_i(z)|$ are the Jacobian determinants. Besides μ, for $j_0 \in I\!N$ we will make use of auxiliary measures μ_{j_0} on Γ defined by $d\mu_{j_0}(x) = m_{j_0}(x)d\mu(x)$, where

$$m_{j_0}(x) = |\partial\kappa_i(\zeta(T))||\partial\kappa_i(\kappa_i^{-1}(x))|^{-1} \quad \text{if } x \in \kappa_i(T^{\text{int}}),\, T \in \tau_{j_0}, \tag{2.2}$$

giving

$$\int_\Gamma u\overline{v}d\mu_{j_0} = \sum_{i=1}^p \sum_{T\in\tau_{j_0}} |\partial\kappa_i(\zeta(T))| \int_T u(\kappa_i(z))\,\overline{v(\kappa_i(z))}\,dz.$$

All these measures are uniformly equivalent to μ, in the sense that for $\nu = \mu$ or $\nu = \mu_{j_0}$ the space $L^2(\Gamma)$ of ν-measurable functions u on Γ with $\int_\Gamma |u|^2 d\nu < \infty$ is the same, and all norms $(\int_\Gamma |u|^2 d\nu)^{\frac{1}{2}}$ are uniformly equivalent. The notation $\|u\|_{L^2(\Gamma)}$ will stand for any of these norms of u. With $\langle u, v\rangle_\nu$ we will mean $\int_\Gamma u\overline{v}d\nu$, where for notational convenience we suppress the fact that it concerns an L^2-scalar product on Γ.

The smoothness of the κ_i shows that

$$\sup_{1\leq i\leq p,\, T\in\tau_{j_0},\, z,\breve{z}\in T} \big|\,|\partial\kappa_i(z))| - |\partial\kappa_i(\breve{z}))|\,\big| \lesssim 2^{-j_0}, \tag{2.3}$$

and so

$$|\langle u, v\rangle_\mu - \langle u, v\rangle_{\mu_{j_0}}| \lesssim 2^{-j_0}\|u\|_{L^2(\Gamma)}\|v\|_{L^2(\Gamma)} \qquad (u, v \in L^2(\Gamma)). \tag{2.4}$$

Let $\boldsymbol{V}$ be some finite dimensional space of continuous functions on the reference n-simplex $\boldsymbol{T}$, which is *refinable* in the sense that

$$\boldsymbol{V} \subset \boldsymbol{V}^{(r)} := \{u \in C(\boldsymbol{T}) : u \circ \lambda_{\boldsymbol{T}_k}^{-1} \in \boldsymbol{V},\, 1 \leq k \leq 2^n\}. \tag{$\mathcal{R}$}$$

Apart from this 'primal' space $\boldsymbol{V}$, we consider a 'dual' space $\tilde{\boldsymbol{V}}$ that is also refinable, with

$$\dim\boldsymbol{V} = \dim\tilde{\boldsymbol{V}}.$$

We assume that for some $d, \tilde{d} \geq 2$,

$$\boldsymbol{V} \supset P_{d-1}(\boldsymbol{T}), \quad \tilde{\boldsymbol{V}} \supset P_{\tilde{d}-1}(\boldsymbol{T}), \tag{2.5}$$

being the spaces of all polynomials over $\boldsymbol{T}$ of degree $d-1$ and $\tilde{d}-1$ respectively. We put

$$\gamma = \sup\{s : \boldsymbol{V} \subset H^s(\boldsymbol{T})\}, \quad \tilde{\gamma} = \sup\{s : \tilde{\boldsymbol{V}} \subset H^s(\boldsymbol{T})\}.$$

We define sequences of 'global' primal and dual finite element type spaces $(V_j)_{j\geq 0}$ and $(\tilde{V}_j)_{j\geq 0}$ on Γ by

$$V_j = \{u \in C(\Gamma) : u \circ \kappa_i \circ \lambda_T^{-1} \in \boldsymbol{V},\, T \in \tau_j,\, 1 \leq i \leq p\},$$

and analogous definition of $\tilde{V}_j$. Both sequences are nested by assumption $(\mathcal{R})$.

We assume that some bases $\boldsymbol{\Phi} = \{\boldsymbol{\phi}_\lambda : \lambda \in \boldsymbol{I}\}$, $\tilde{\boldsymbol{\Phi}} = \{\tilde{\boldsymbol{\phi}}_\lambda : \lambda \in \boldsymbol{I}\}$ for $\boldsymbol{V}$, $\tilde{\boldsymbol{V}}$ are available with index set $\boldsymbol{I} \subset \boldsymbol{T}$. To be able to use these bases as building blocks for constructing bases for V_j and $\tilde{V}_j$, we assume that

($\mathcal{V}$) $\boldsymbol{\phi}_\lambda$ vanishes on any face that does not include λ,

($\mathcal{S}$) $\pi(\boldsymbol{I} \cap \partial \boldsymbol{T}) = \boldsymbol{I} \cap \partial \boldsymbol{T}$ and $\boldsymbol{\phi}_\lambda|_{\partial \boldsymbol{T}} = (\boldsymbol{\phi}_{\pi(\lambda)} \circ \pi)|_{\partial \boldsymbol{T}}$ for any permutation $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$,

($\mathcal{I}$) For $\boldsymbol{e} = \boldsymbol{T}$, or for $\boldsymbol{e}$ being any face of $\boldsymbol{T}$, $\{\boldsymbol{\phi}_\lambda|_{\boldsymbol{e}} : \lambda \in \boldsymbol{I} \cap \boldsymbol{e}\}$ is independent,

and analogous conditions on $\tilde{\boldsymbol{\Phi}}$.

A connection between (V_j) and $(\tilde{V}_j)$ will be established by assuming that

$$\mathrm{Re}\, \langle \boldsymbol{\Phi}, \tilde{\boldsymbol{\Phi}} \rangle_{\boldsymbol{\mu}} > 0, \tag{2.6}$$

where $\langle \boldsymbol{u}, \boldsymbol{v} \rangle_{\boldsymbol{\mu}} = \int_{\boldsymbol{T}} \boldsymbol{u}\overline{\boldsymbol{v}} d\boldsymbol{\mu}$ with $\boldsymbol{\mu}$ being the induced Lebesgue measure on $\boldsymbol{T}$.

With $I_j := \cup_{i=1}^p \cup_{T \in \tau_j} \kappa_i(\lambda_T^{-1}(\boldsymbol{I}))$, we define collections $\Phi_j = \{\phi_{j,x} : x \in I_j\}$ of functions on Γ by

(2.7)
$$\phi_{j,x}(y) = \begin{cases} 2^{jn/2} \boldsymbol{\phi}_{\lambda_T(\kappa_i^{-1}(x))}(\lambda_T(\kappa_i^{-1}(y))) & \text{if } x, y \in \kappa_i(T) \text{ for some } 1 \le i \le p, T \in \tau_j, \\ 0 & \text{elsewhere.} \end{cases}$$

So these global functions result from connecting the local basis functions over the interfaces between the 'elements' $\kappa_i(T)$. Note that because of our assumption that $n \le 3$, we have an automatic matching of triangulations at interfaces. That is, if $y \in \kappa_i(T) \cap \kappa_{\breve{i}}(\breve{T})$ with $1 \le i \ne \breve{i} \le p$ or $T \ne \breve{T} \in \tau_j$, then $\lambda_T(\kappa_i^{-1}(y))$ is equal to $\lambda_{\breve{T}}(\kappa_{\breve{i}}^{-1}(y))$ modulo some permutation. Using ($\mathcal{V}$), ($\mathcal{S}$), one therefore concludes that the $\phi_{j,x}$ are well-defined, continuous functions on Γ, and that the Φ_j are *uniformly local* in the sense that

$$\mathrm{diam}(\mathrm{supp}(\phi_{j,x})) \eqsim 2^{-j}.$$

Together with ($\mathcal{I}$) it even follows that Φ_j is a basis for V_j.

Obviously, similar observations hold for the dual collections $\tilde{\Phi}_j$ defined analogously using $\tilde{\boldsymbol{\Phi}}$.

Remark 2.2. Although in this paper we focus on a construction of wavelets on compact manifolds Γ, clearly it also applies to domains Ω. Possible essential homogeneous boundary conditions can easily be incorporated just by removing the points on $\partial\Omega$ from the index sets I_j. In that case, for $s \ge 0$, $H^s(\Gamma)$ should read as $H^s(\Omega) \cap H_0^1(\Omega)$. In case Ω is a polygon, we may assume that the κ_i are affine mappings, which implies that $\mu = \mu_{j_0}$ for all $j_0 \in \mathbb{N}$.

We constructed $\langle\, , \,\rangle_{\mu_{j_0}}$ from $\langle\, , \,\rangle_\mu$ by 'freezing' the Jacobian determinant on the pull-back of each $\kappa_i(\hat{T})$ for $1 \le i \le p$ and $\hat{T} \in \tau_{j_0}$. As a consequence, for $j \ge j_0 \ge 0$ and $x, y \in I_j$, we have

$$
\begin{aligned}
(2.8)\quad & \langle \phi_{j,x}, \tilde{\phi}_{j,y}\rangle_{\mu_{j_0}} \\
&= \sum_{i=1}^{p} \sum_{\{\hat{T}\in\tau_{j_0}, T\in\tau_j : T\subset\hat{T},\, \kappa_i(T)\ni x,y\}} |\partial\kappa_i(\zeta(\hat{T}))| 2^{jn}\frac{\mu(T)}{\mu(\boldsymbol{T})} \langle \boldsymbol{\phi}_{\lambda_T(\kappa_i^{-1}(x))}, \tilde{\boldsymbol{\phi}}_{\lambda_T(\kappa_i^{-1}(y))}\rangle_{\boldsymbol{\mu}} \\
&\eqsim \sum_{i=1}^{p} \sum_{\{T\in\tau_j : \kappa_i(T)\ni x,y\}} \langle \boldsymbol{\phi}_{\lambda_T(\kappa_i^{-1}(x))}, \tilde{\boldsymbol{\phi}}_{\lambda_T(\kappa_i^{-1}(y))}\rangle_{\boldsymbol{\mu}}.
\end{aligned}
$$

Here and below, whenever it is relevant, the $\lesssim$, $\gtrsim$ and $\eqsim$ symbols will not only refer to uniformity in j (and here in $x, y \in I_j$), but also in $j_0 \in I\!N$. By replacing $\tilde{\phi}_{j,y}$ by $\phi_{j,y}$ in (2.8), one easily infers that the Φ_j, and analogously the $\tilde{\Phi}_j$, are *uniform* $L^2(\Gamma)$*-Riesz systems*, with which we mean that $\|\mathbf{c}^T\Phi_j\|_{L^2(\Gamma)} \eqsim \|\mathbf{c}\|_{\ell^2}$ holds also uniformly in j.

Furthermore, using (2.8) one deduces that (2.6) implies that for $j \geq j_0 \geq 0$, $\mathrm{Re}\langle\Phi_j, \tilde{\Phi}_j\rangle_{\mu_{j_0}} \gtrsim 1$. Since Φ_j and $\tilde{\Phi}_j$ are uniform $L^2(\Gamma)$-Riesz bases, the latter result shows that for $j \geq j_0 \geq 0$,

$$
\inf_{0\neq\tilde{u}_j\in\tilde{V}_j} \sup_{0\neq u_j\in V_j} \frac{|\langle u_j, \tilde{u}_j\rangle_{\mu_{j_0}}|}{\|u_j\|_{L^2(\Gamma)}\|\tilde{u}_j\|_{L^2(\Gamma)}} \gtrsim 1. \tag{2.9}
$$

From (2.4) we conclude that in any case when j_0 is sufficiently large, for $j \geq j_0$,

$$
\inf_{0\neq\tilde{u}_j\in\tilde{V}_j} \sup_{0\neq u_j\in V_j} \frac{|\langle u_j, \tilde{u}_j\rangle_{\mu}|}{\|u_j\|_{L^2(\Gamma)}\|\tilde{u}_j\|_{L^2(\Gamma)}} \gtrsim 1, \tag{$\mathcal{A}$}
$$

meaning that the $\langle\, ,\,\rangle_\mu$-angle between V_j and $\tilde{V}_j$ stays away from $\frac{\pi}{2}$ uniformly in $j \geq j_0$.

As shown in [DSt99, Theorem 2.1], ($\mathcal{A}$) implies that there exists a unique sequence $(Q_j)_{j\geq j_0}$ of uniformly bounded projectors $Q_j : L^2(\Gamma) \to L^2(\Gamma)$ such that

$$
\mathrm{Im}(Q_j) = V_j, \quad \mathrm{Im}(I - Q_j) = \tilde{V}_j^{\perp_{\langle,\rangle_\mu}},
$$

and so for the adjoints,

$$
\mathrm{Im}(Q_j^*) = \tilde{V}_j, \quad \mathrm{Im}(I - Q_j^*) = V_j^{\perp_{\langle,\rangle_\mu}}.
$$

The existence of continuous, uniformly local, uniform $L^2(\Gamma)$-Riesz bases implies (cf. [Osw94]) the validity of the *Bernstein inequality*

$$
\|u_j\|_{H^s(\Gamma)} \lesssim 2^{js}\|u_j\|_{L^2(\Gamma)} \qquad (u_j \in V_j,\ s \in [0, \min\{\tfrac{3}{2}, \gamma\}) \text{ with } s \leq m \text{ or } s < t), \tag{$\mathcal{B}$}
$$

and likewise for the dual sequence with γ replaced by $\tilde{\gamma}$. Assumption (2.5) implies the *Jackson estimate*

$$
\inf_{u_j\in V_j} \|u - u_j\|_{L^2(\Gamma)} \lesssim 2^{-js}\|u\|_{H^s(\Gamma)} \quad (u \in H^s(\Gamma),\ s \in [0, d] \text{ with } s \leq m \text{ or } s < t), \tag{$\mathcal{J}$}
$$

and likewise for the dual sequence with d replaced by $\tilde{d}$. By ($\mathcal{A}$), ($\mathcal{B}$), ($\mathcal{J}$), and the nestedness of both sequences, the general theory about stability of biorthogonal space decompositions (cf. [Dah96, DSt99]) shows that with $Q_{j_0-1} := 0$,

$$\|u\|^2_{H^s(\Gamma)} \eqsim \sum_{j=j_0}^{\infty} 4^{js}\|(Q_j - Q_{j-1})u\|^2_{L^2(\Gamma)} \qquad (u \in H^s(\Gamma), s \in (-\tfrac{3}{2}, \tfrac{3}{2}) \cap (-\tilde{\gamma}, \gamma) \text{ with } |s| \leq m \text{ or } |s| < t). \tag{2.10}$$

and

$$\|u\|^2_{H^s(\Gamma)} \eqsim \sum_{j=j_0}^{\infty} 4^{js}\|(Q_j^* - Q_{j-1}^*)u\|^2_{L^2(\Gamma)} \qquad (u \in H^s(\Gamma), s \in (-\tfrac{3}{2}, \tfrac{3}{2}) \cap (-\gamma, \tilde{\gamma}) \text{ with } |s| \leq m \text{ or } |s| < t). \tag{2.11}$$

Remark 2.3. In all examples constructed in [DSt99, Ste00], the functions in $\boldsymbol{V}$ and $\tilde{\boldsymbol{V}}$ are either polynomials or continuous piecewise polynomials. As a consequence, the values of γ and $\tilde{\gamma}$ are either ∞ or $\frac{3}{2}$, meaning that in ($\mathcal{B}$), (2.10) and (2.11), the conditions involving γ and $\tilde{\gamma}$ are superfluous. Therefore, for ease of presentation in the following we will drop these conditions. Yet, on the other hand one may think of interesting examples were in particular $\tilde{\boldsymbol{V}}$ contains functions that are implicitly defined as the solution of some refinement equation, which may have a lower regularity. For these cases, results derived in this paper based on the Bernstein inequalities should be restricted to the corresponding smaller ranges of Sobolev norms.

Remark 2.4. If one, at least formally, wants to include unbounded manifolds or domains, yielding infinite dimensional spaces V_j and $\tilde{V}_j$, the maximum angle condition ($\mathcal{A}$) should be appended with the analogous condition, also resulting from (2.6), in which the roles of V_j and $\tilde{V}_j$ are interchanged.

Below, possibly for a j_0 larger than in (2.10), we will construct uniform $L^2(\Gamma)$-Riesz bases Ψ_j for the spaces

$$\operatorname{Im}(Q_{j+1} - Q_j) = V_{j+1} \cap \tilde{V}_j^{\perp_{\langle,\rangle_\mu}} \qquad (j \geq j_0),$$

which elements are then called *wavelets*. Then (2.10) shows that

$$\Phi_{j_0} \cup \cup_{j=j_0}^{\infty} 2^{-js}\Psi_j \text{ is a Riesz basis for } H^s(\Gamma),$$

for the range of s as in (2.10).

For simplicity, let us assume that $\boldsymbol{I}$ is a subset of the 'refined index set'

$$\boldsymbol{I}^{(r)} := \bigcup_{k=1}^{2^n} \lambda^{-1}_{\boldsymbol{T}_k}(\boldsymbol{I}).$$

Suppose that collections $\boldsymbol{\Theta} = \{\boldsymbol{\theta}_\lambda : \lambda \in \boldsymbol{I}\}$ and $\boldsymbol{\Xi} = \{\boldsymbol{\xi}_\lambda : \lambda \in \boldsymbol{I}^{(r)} \backslash \boldsymbol{I}\}$ of functions on $\boldsymbol{T}$ are available, such that $\boldsymbol{\Theta} \cup \boldsymbol{\Xi}$ satisfies ($\mathcal{V}$), ($\mathcal{S}$) and ($\mathcal{I}$), $\boldsymbol{\Theta} \cup \boldsymbol{\Xi}$ is a basis for $\boldsymbol{V}^{(r)}$, and

$$\langle \boldsymbol{\Theta}, \tilde{\boldsymbol{\Phi}} \rangle_{\boldsymbol{\mu}} = I. \tag{2.12}$$

As Φ_j and $\tilde{\Phi}_j$ were defined from $\mathbf{\Phi}$ and $\tilde{\mathbf{\Phi}}$, above $\mathbf{\Theta}$ and $\mathbf{\Xi}$ give rise to collections $\Theta_j = \{\theta_{j,x} : x \in I_j\}$ and $\Xi_j = \{\xi_{j,y} : y \in I_{j+1}\backslash I_j\}$ of functions on Γ defined as in (2.7). The same arguments that were used earlier show that $\Theta_j \cup \Xi_j$ are uniform $L^2(\Gamma)$-Riesz bases for the spaces V_{j+1}.

Example 2.5. From [DSt99], we recall an example of such collections $\mathbf{\Phi}$, $\tilde{\mathbf{\Phi}}$, $\mathbf{\Theta}$ and $\mathbf{\Xi}$, which quadruple will determine the whole wavelet construction. Let $\boldsymbol{I}$ be the set of vertices of the n-simplex $\boldsymbol{T}$, so that $\boldsymbol{I}^{(r)}$ is the set of vertices and midpoints of edges of $\boldsymbol{T}$. The sets $\tilde{\mathbf{\Phi}} = \mathbf{\Phi}$ are defined by $\boldsymbol{\phi}_\lambda(\mu) = \delta_{\lambda\mu}$ ($\lambda, \mu \in \boldsymbol{I}$). It holds that $\boldsymbol{V} = \tilde{\boldsymbol{V}} = \operatorname{span}\mathbf{\Phi} = P_1(\boldsymbol{T})$, giving $d = \tilde{d} = 2$. Since $\boldsymbol{V} = \tilde{\boldsymbol{V}}$, in this case (2.10) refers to an *orthogonal* space decomposition. Note that in the domain case, the spaces $V_j = \tilde{V}_j$ are just the standard P_1 finite element spaces. With $\boldsymbol{\phi}_\lambda^{(r)} \in \boldsymbol{V}^{(r)}$ defined by $\boldsymbol{\phi}_\lambda^{(r)}(\mu) = \delta_{\lambda\mu}$ ($\lambda, \mu \in \boldsymbol{I}^{(r)}$), sets $\mathbf{\Theta}$ and $\mathbf{\Xi}$ satisfying above conditions are given by $\boldsymbol{\theta}_\lambda = \frac{2^{n+1}(n+1)!}{\sqrt{n+1}}(\boldsymbol{\phi}_\lambda^{(r)} - 2^{-(n+1)}\boldsymbol{\phi}_\lambda)$ ($\lambda \in \boldsymbol{I}$), and $\boldsymbol{\xi}_\lambda = \boldsymbol{\phi}_\lambda^{(r)}$ ($\lambda \in \boldsymbol{I}^{(r)}\backslash\boldsymbol{I}$), see Figure 1.

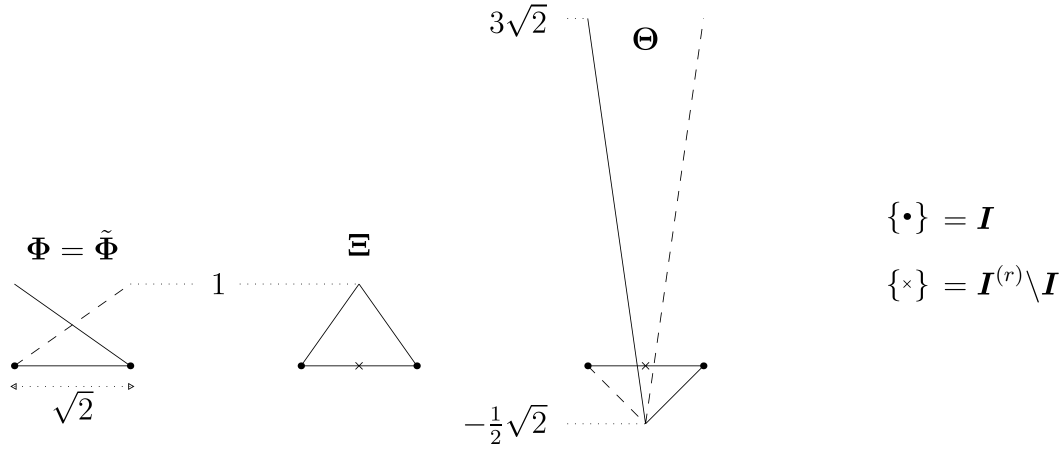


FIGURE 1. $\mathbf{\Phi}, \tilde{\mathbf{\Phi}}, \mathbf{\Theta}, \mathbf{\Xi}$ from Example 2.5 for $n = 1$

Anticipating to the discussion at the end of §3, to get wavelets with more vanishing moments, or more generally, a cancellation property of higher order, it makes sense to select $\tilde{\boldsymbol{V}} \neq \boldsymbol{V}$ such that $\tilde{\boldsymbol{V}}$ includes all polynomials of some higher degree. Examples are given in [DSt99].

From (2.6) we obtained (2.9). So comparing (2.12) with (2.6), we may conclude that for $j \geq j_0 \geq 0$,

$$\inf_{0 \neq \tilde{u}_j \in \tilde{V}_j} \sup_{0 \neq v_j \in \operatorname{span}\Theta_j} \frac{|\langle v_j, \tilde{u}_j \rangle_{\mu_{j_0}}|}{\|v_j\|_{L^2(\Gamma)} \|\tilde{u}_j\|_{L^2(\Gamma)}} \gtrsim 1, \tag{2.13}$$

and thus that for j_0 being sufficiently large and $j \geq j_0$,

$$\inf_{0 \neq \tilde{u}_j \in \tilde{V}_j} \sup_{0 \neq v_j \in \operatorname{span}\Theta_j} \frac{|\langle v_j, \tilde{u}_j \rangle_{\mu}|}{\|v_j\|_{L^2(\Gamma)} \|\tilde{u}_j\|_{L^2(\Gamma)}} \gtrsim 1. \tag{2.14}$$

In [Ste00] it was shown that (2.14), together with the fact that $\Theta_j \cup \Xi_j$ and $\tilde{\Phi}_j$ are uniform $L^2(\Gamma)$-Riesz bases for V_{j+1} and $\tilde{V}_j$ respectively, implies that for $j \geq j_0$,

$$\Psi_j := \Xi_j - \langle \Xi_j, \tilde{\Phi}_j \rangle_\mu \langle \Theta_j, \tilde{\Phi}_j \rangle_\mu^{-1} \Theta_j \tag{2.15}$$

are uniform $L^2(\Gamma)$-Riesz bases for the spaces $V_{j+1} \cap \tilde{V}_j^{\perp_{\langle,\rangle_\mu}}$. Note that Ψ_j is the result of projecting Ξ_j along spanΘ_j onto $\tilde{V}_j^{\perp_{\langle,\rangle_\mu}}$. In particular this means that Ψ_j is independent of the choice of the bases of spanΘ_j and $\tilde{V}_j$. In the terminology from [Dah97], Ξ_j and Ψ_j correspond to 'initial' and 'target' 'stable completions' of Θ_j in V_{j+1}.

Analogously, using (2.13), we conclude that for $j \geq j_0 \geq 0$, the 'auxiliary' collections

$$\Psi_j^{(j_0)} := \Xi_j - \langle \Xi_j, \tilde{\Phi}_j \rangle_{\mu_{j_0}} \langle \Theta_j, \tilde{\Phi}_j \rangle_{\mu_{j_0}}^{-1} \Theta_j \tag{2.16}$$

are uniform $L^2(\Gamma)$-Riesz bases for $V_{j+1} \cap \tilde{V}_j^{\perp_{\langle,\rangle_{\mu_{j_0}}}}$, where here 'uniform' also refers to j_0. The fact that the $\Psi_j^{(j_0)}$ are uniform $L^2(\Gamma)$-Riesz systems will be used in §4.

3. Constant Jacobian determinants

In general, $\langle \Theta_j, \tilde{\Phi}_j \rangle_\mu^{-1}$ will be a densely populated matrix, meaning that (2.15) yields wavelets with global supports, which is undesirable for practical computations. On the other hand, formula (2.8) shows that assumption (2.12), i.e. $\langle \boldsymbol{\Theta}, \tilde{\boldsymbol{\Phi}} \rangle_{\boldsymbol{\mu}} = I$, implies that $\langle \Theta_j, \tilde{\Phi}_j \rangle_{\mu_{j_0}}$ is diagonal for $j \geq j_0 \geq 0$. By also expanding $\langle \Xi_j, \tilde{\Phi}_j \rangle_{\mu_{j_0}}$ in terms of local scalar products using (2.8), we infer that $\Psi_j^{(j_0)} = \{\psi_{j,y}^{(j_0)} : y \in I_{j+1} \backslash I_j\}$ is given by

$$\psi_{j,y}^{(j_0)} = \xi_{j,y} - \sum_{x \in I_j} \frac{\sum_{\{i, \hat{T} \in \tau_{j_0}, T \in \tau_j : T \subset \hat{T}, \kappa_i(T) \ni x, y\}} |\partial \kappa_i(\zeta(\hat{T}))| \, \langle \boldsymbol{\xi}_{\lambda_T(\kappa_i^{-1}(y))}, \tilde{\boldsymbol{\phi}}_{\lambda_T(\kappa_i^{-1}(x))} \rangle_{\boldsymbol{\mu}}}{\sum_{\{i, \hat{T} \in \tau_{j_0}, T \in \tau_j : T \subset \hat{T}, \kappa_i(T) \ni x\}} |\partial \kappa_i(\zeta(\hat{T}))|} \, \theta_{j,x}, \tag{3.1}$$

and in particular,

$$\psi_{j,y}^{(0)} = \xi_{j,y} - \sum_{x \in I_j} \frac{\sum_{\{i, T \in \tau_j : \kappa_i(T) \ni x, y\}} |\partial \kappa_i(\zeta(T_0))| \, \langle \boldsymbol{\xi}_{\lambda_T(\kappa_i^{-1}(y))}, \tilde{\boldsymbol{\phi}}_{\lambda_T(\kappa_i^{-1}(x))} \rangle_{\boldsymbol{\mu}}}{\sum_{\{i, T \in \tau_j : \kappa_i(T) \ni x\}} |\partial \kappa_i(\zeta(T_0))|} \, \theta_{j,x}. \tag{3.2}$$

From the fact that Ξ_j, $\tilde{\Phi}_j$ and Θ_j are uniformly local, we conclude that the sum over $x \in I_j$ in (3.1) is uniformly finite, and thus that the $\Psi_j^{(j_0)}$ are uniformly local. In particular, with

$$\Lambda_{j,y}(i) = \{T \in \tau_j : \exists\, 1 \leq \breve{\imath} \leq p,\, \breve{T} \in \tau_j \text{ with } y \in \kappa_{\breve{\imath}}(\breve{T}) \text{ and } \kappa_i(T) \cap \kappa_{\breve{\imath}}(\breve{T}) \neq \emptyset\}, \tag{3.3}$$

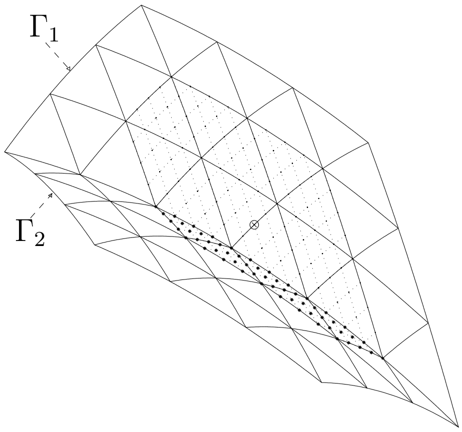
it holds that

$$\text{supp}\psi_{j,y}^{(j_0)} \subset \cup_{i=1}^p \kappa_i(\Lambda_{j,y}(i)), \tag{3.4}$$

see Figure 2.

In view of above observations, as in [DSt99], *throughout this section we will assume that*

$$\mu = \mu_0,$$



$\otimes$: $y \in I_{j+1} \backslash I_j \subset \overline{\Gamma_1}$

$\{\triangle\} = \kappa_1(\Lambda_{j,y}(1))$

$\{\triangle\} = \kappa_2(\Lambda_{j,y}(2))$

FIGURE 2. Illustration of sets $\kappa_i(\Lambda_{j,y}(i))$ for a 2-dimensional manifold

meaning that all Jacobian determinants $|\partial\kappa_i|$ are constant functions. Apart from the polygonal domain case discussed in Remark 2.2, manifolds consisting of patches that for example are parts of hyperplanes, spheres or cylinders, can be described by such parametrizations. Under this assumption, $(\mathcal{A})$ and thus (2.10) are valid for $j \geq 0$, and $\Psi_j = \Psi_j^{(0)}$. We conclude that $\Phi_0 \cup \cup_{j\geq 0} 2^{-js}\Psi_j$ is a Riesz basis for $H^s(\Gamma)$ for the range of s as in (2.10), where moreover now the collections Ψ_j are uniformly local.

In the following three remarks, we discuss some generalizations or extensions of the results we obtained so far.

Remark 3.1. Instead of $\mu = \mu_0$, we could also have assumed that $\mu = \mu_{j_0}$ for some $j_0 \in \mathbb{N}$. By breaking the $\overline{\Gamma_i}$ into the smaller patches $\kappa_i(T)$ $(T \in \tau_{j_0})$, it is easily seen that this generalization can be reduced to the previous situation.

Remark 3.2. As discussed in [Ste00], the condition (2.12), i.e. $\langle \mathbf{\Theta}, \tilde{\mathbf{\Phi}} \rangle_{\boldsymbol{\mu}} = I$, can be relaxed as follows: With respect to some partitioning $\boldsymbol{I} = \cup_{\ell=1}^{q} \boldsymbol{I}^{(\ell)}$, where $\pi(\boldsymbol{I}^{(\ell)} \cap \partial \boldsymbol{T}) = \boldsymbol{I}^{(\ell)} \cap \partial \boldsymbol{T}$ for all permutations π, let $\langle \mathbf{\Theta}, \tilde{\mathbf{\Phi}} \rangle_{\boldsymbol{\mu}}$ be a block triangular matrix with identity matrices as diagonal blocks. Then with respect to a corresponding partitioning of the sets I_j into q subsets, for $j \geq j_0$ the matrices $\langle \Theta_j, \tilde{\Phi}_j \rangle_{\mu_{j_0}}$ are block triangular with diagonal matrices as diagonal blocks. It follows that both the matrices $\langle \Theta_j, \tilde{\Phi}_j \rangle_{\mu_{j_0}}$ and $\langle \Theta_j, \tilde{\Phi}_j \rangle_{\mu_{j_0}}^{-1}$ are uniformly bounded, and uniformly local in the sense that entries corresponding to $x, y \in I_j$ with distance larger than some multiple of 2^{-j} are zero. The first property shows that $\tilde{\Phi}_j$ and $\langle \Theta_j, \tilde{\Phi}_j \rangle_{\mu_{j_0}}^{-1} \Theta_j$ are $\langle\,,\,\rangle_{\mu_{j_0}}$-biorthogonal uniformly $L^2(\Gamma)$-Riesz bases for $\tilde{V}_j$ and $\mathrm{span}\Theta_j$ respectively. The existence of such bases implies (2.13). As we have seen, (2.13) in turn shows that the collections $\Psi_j^{(j_0)}$ from (2.16) are uniform $L^2(\Gamma)$-Riesz bases for $V_{j+1} \cap \tilde{V}_j^{\perp_{\langle,\rangle_{\mu_{j_0}}}}$. The uniform locality of $\langle \Theta_j, \tilde{\Phi}_j \rangle_{\mu_{j_0}}^{-1}$ shows that the $\Psi_j^{(j_0)}$ are uniformly local. Concluding, assuming that $\mu = \mu_0$, the wavelets $\Psi_j = \Psi_j^{(0)}$ are uniformly local, uniform $L^2(\Gamma)$-Riesz bases for $V_{j+1} \cap \tilde{V}_j^{\perp_{\langle,\rangle_\mu}}$. Note however that (3.1), and thus (3.2), and also (3.4) are no longer valid.

Also the wavelet construction presented in §4 can be carried out when (2.12) is replaced by above relaxed assumption. Yet, for ease of presentation, in the remainder of this paper we stick to assumption (2.12), i.e., $\langle \boldsymbol{\Theta}, \tilde{\boldsymbol{\Phi}} \rangle_{\boldsymbol{\mu}} = I$.

Remark 3.3. In [Ste00], examples of quadruples $(\boldsymbol{\Phi}, \tilde{\boldsymbol{\Phi}}, \boldsymbol{\Theta}, \boldsymbol{\Xi})$ are given with $\boldsymbol{\Theta} = \boldsymbol{\Phi}$, that is, $\langle \boldsymbol{\Phi}, \tilde{\boldsymbol{\Phi}} \rangle_{\boldsymbol{\mu}} = I$, or more generally, $\langle \boldsymbol{\Phi}, \tilde{\boldsymbol{\Phi}} \rangle_{\boldsymbol{\mu}}$ is a block triangular matrix as in Remark 3.2. In these cases, and assuming that $\mu = \mu_0$, the sets Φ_j and $\langle \Phi_j, \tilde{\Phi}_j \rangle_\mu^{-1} \tilde{\Phi}_j$ are uniformly local, $\langle , \rangle_\mu$-biorthogonal scaling functions. It can be shown that as a consequence also uniformly local *dual wavelets* become available. Note that for $\boldsymbol{\Theta} = \boldsymbol{\Phi}$, it follows that $\mathrm{span}\Theta_j = V_j$, and so (2.9) and (2.13), and also $(\mathcal{A})$ and (2.14) are equal.

Apart from generating Riesz bases, the other essential property that makes wavelets suitable for solving operator equations is that they have vanishing moments, or more generally, to cover cases where piecewise polynomials are not included in the dual spaces, that they have cancellation properties. Still assuming that $\mu = \mu_0$, the wavelets $\Psi_j = \Psi_j^{(0)}$ satisfy a *cancellation property of order* $\tilde{d}$, with with we mean that following estimate is valid:

Proposition 3.4. *For v being a continuous function on Γ, which is patchwise smooth, one has*

$$|\langle v, \psi_{j,y} \rangle_\mu| \lesssim 2^{-j(\tilde{d}+n/2)} \max_{1 \leq i \leq p,\, T \in \Lambda_{j,y}(i)} |v \circ \kappa_i|_{W^{\tilde{d},\infty}(T)} \tag{3.5}$$

Proof. For $q \in \mathbb{N}$, let $\boldsymbol{N}_q : C(\boldsymbol{T}) \to P_q(\boldsymbol{T})$ be the interpolant defined by $(\boldsymbol{N}_q \boldsymbol{v})(\lambda) = \boldsymbol{v}(\lambda)$ for $\lambda \in (\mathbb{N}/q)^{n+1} \cap \boldsymbol{T}$. We define $N_{j,q} : C(\Gamma) \to \prod_{i=1}^p \prod_{T \in \tau_j} \kappa_i(P_q(T))$ by

$$(N_{j,q} v) \circ \kappa_i \circ \lambda_T^{-1} = \boldsymbol{N}_q (v \circ \kappa_i \circ \lambda_T^{-1}) \qquad (1 \leq i \leq p, T \in \tau_j).$$

Since $\boldsymbol{N}_q$ reproduces polynomials of order q, the Bramble-Hilbert lemma and a homogeneity argument show that for continuous, patchwise smooth v,

$$\|(I - N_{j,q}) v\|_{L^\infty(\kappa_i(T))} \lesssim 2^{-(q+1)j} |v \circ \kappa_i|_{W^{q+1,\infty}(T)}. \tag{3.6}$$

The choice of the interpolation points and the matching condition (2.1) ensure that $N_{j,q}$ maps into $C(\Gamma)$. As a consequence, from our assumption that $\tilde{\boldsymbol{V}} \supset P_{\tilde{d}-1}(\boldsymbol{T})$ we infer that $N_{j,\tilde{d}-1}$ maps into $\tilde{V}_j$. Finally, from $\psi_{j,y} \perp_{\langle,\rangle_\mu} \tilde{V}_j$ and $\mathrm{diam}(\mathrm{supp}(\psi_{j,y})) \eqsim 2^{-j}$, we obtain that

$$\begin{aligned} |\langle v, \psi_{j,y} \rangle_\mu| &= |\langle (I - N_{j,\tilde{d}-1}) v, \psi_{j,y} \rangle_\mu| \\ &\lesssim \|(I - N_{j,\tilde{d}-1}) v\|_{L^2(\mathrm{supp}(\psi_{j,y}))} \lesssim 2^{-jn/2} \|(I - N_{j,\tilde{d}-1}) v\|_{L^\infty(\mathrm{supp}(\psi_{j,y}))}. \end{aligned}$$

The proof is completed by (3.4) and (3.6). $\square$

With a cancellation property of sufficiently high order, a wavelet representation of an integral operator can be approximated by a sparse matrix without lowering the order of convergence of the resulting discretization. For details, we refer to [Sch98, Dah97].

Remark 3.5. For the domain case discussed in Remark 2.2, if the functions from $\tilde{V}_j$ satisfy essential homogeneous boundary conditions, then (3.5) restricts to those v that also satisfy these conditions.

4. General parametrizations

The assumption that $\mu = \mu_0$ made in §3 clearly restricts the field of applications. Therefore we now study the situation that this assumption is not valid. Then (2.15) will generally not result in uniformly local wavelets.

A potential solution is to replace throughout §2, the Lebesgue measure μ on Γ by the measure μ_0, that is, to consider space decompositions that are biorthogonal with respect to $\langle\, ,\,\rangle_{\mu_0}$ instead of with respect to $\langle\, ,\,\rangle_{\mu}$. Then (2.10) holds with $j_0 = 0$ and the wavelet collections yielded by (2.15) are just the collections $\Psi_j^{(0)}$.

A point however that deserves attention is the interpretation of (2.10) if $s < 0$. The operators Q_j should be interpreted as extensions of mappings $L^2(\Gamma) \to V_j$ to mappings $H^s(\Gamma) \to V_j$, by identifying $u \in L^2(\Gamma)$ with the functional $v \mapsto \langle v, u\rangle_\mu$, yielding a set that is dense in $H^s(\Gamma)$. Likewise, for the consequence that $\|\sum_j \mathbf{c}_j^T 2^{-js}\Psi_j\|^2_{H^s(\Gamma)} \eqsim \sum_j \|\mathbf{c}_j\|^2_{\ell^2}$, the $H^s(\Gamma)$-norm of the series of functions in $L^2(\Gamma)$ should be interpreted with respect to the same dense embedding of $L^2(\Gamma)$ into $H^s(\Gamma)$.

Replacing μ by μ_0 changes this embedding from

$$E : u \mapsto (v \mapsto \langle v, u\rangle_\mu)$$

into $E_0 : u \mapsto (v \mapsto \langle v, u\rangle_{\mu_0})$. For $s \leq -\frac{1}{2}$, and for an m_0, defined in (2.2), that has jumps over the interfaces between patches, both embeddings result in a non-equivalent $H^s(\Gamma)$-norms of $L^2(\Gamma)$-functions. Indeed, suppose that the norms would be equivalent, then for $v \in H^{-s}(\Gamma)$,

$$\begin{aligned}\|v\|_{H^{-s}(\Gamma)} &= \sup_{0\neq f\in H^s(\Gamma)} \frac{|f(v)|}{\|f\|_{H^s(\Gamma)}} = \sup_{0\neq u\in L^2(\Gamma)} \frac{|\langle v, u\rangle_\mu|}{\|E(u)\|_{H^s(\Gamma)}} \eqsim \sup_{0\neq u\in L^2(\Gamma)} \frac{|\langle v, u\rangle_\mu|}{\|E_0(u)\|_{H^s(\Gamma)}}\\ &= \sup_{0\neq u\in L^2(\Gamma)} \frac{|\langle v/m_0, u\rangle_\mu|}{\|E(u)\|_{H^s(\Gamma)}} = \|v/m_0\|_{H^{-s}(\Gamma)},\end{aligned}$$

which is known not to be valid for $s \leq -\frac{1}{2}$ and such m_0. We conclude that for m_0 having jumps and $s \leq -\frac{1}{2}$, a space decomposition that is biorthogonal with respect to $\langle\, ,\,\rangle_{\mu_0}$ results in a wavelet system that cannot be a Riesz basis for $H^s(\Gamma)$ with respect to the embedding of $L^2(\Gamma)$ into $H^s(\Gamma)$ using $\langle\, ,\,\rangle_\mu$, and vice versa.

The application of wavelets that we focus on is that of Galerkin discretizations of operator equations. In applications the variational formulations of these equations are formed using the duality pairing with respect to $\langle\, ,\,\rangle_\mu$. This implies that the relevant embedding of

$L^2(\Gamma)$ into $H^s(\Gamma)$ for $s < 0$ is the embedding E based on $\langle\,,\,\rangle_\mu$. Another consequence is that cancellation properties should indeed be measured with respect to $\langle\,,\,\rangle_\mu$.

Instead of μ_0, more generally one may consider the option to replace μ by μ_g defined by $d\mu_g = g d\mu$, where $g > 0$ with $g, 1/g \in L^\infty(\Gamma)$. Above analysis shows that the approach to construct space decompositions that are biorthogonal with respect to $\langle\,,\,\rangle_{\mu_g}$ give rise to 'stable splittings' of $H^s(\Gamma)$ for $s < 0$, in the sense of (2.10) and with respect to the embedding E, if and only if

$$f \mapsto fg \quad \text{is a homeomorphism in } H^{-s}(\Gamma). \tag{4.1}$$

On the other hand, our approach to construct uniformly local, uniform $L^2(\Gamma)$-Riesz bases for the subspaces $V_{j+1} \cap \tilde{V}_j^{\perp_{\langle,\rangle_{\mu_g}}}$ only applies when for each $1 \le i \le p$,

$$\Gamma_i \to \mathbb{C} : x \mapsto g(x)|\partial\kappa_i(\kappa_i^{-1}(x))| \quad \text{is constant.} \tag{4.2}$$

Before trying to circumvent these restrictive conditions, in the following simple one-dimensional example we illustrate above findings with numerical results, at the same time exemplifying the wavelet formula (3.2):

Example 4.1. Let $\Gamma = \cup_{i=1}^2 \overline{\Gamma_i}$ be the unit circle in $\mathbb{R}^2$, and $T_0 = [0,1]$. We use $(\mathbf{\Phi}, \tilde{\mathbf{\Phi}}, \mathbf{\Theta}, \mathbf{\Xi})$ from Example 2.5 (with $n = 1$). We take

$$\begin{aligned} \kappa_1 &: z \mapsto (\cos(\tfrac{2}{3}\pi z), \sin(\tfrac{2}{3}\pi z)),\\ \kappa_2 &: z \mapsto (\cos(\tfrac{4}{3}\pi(z+\tfrac{1}{2})), \sin(\tfrac{4}{3}\pi(z+\tfrac{1}{2}))). \end{aligned}$$

Both Jacobian determinants are constants, with values $\frac{2}{3}\pi$ and $\frac{4}{3}\pi$, and so

$$\langle u, v\rangle_\mu = \sum_{i=1}^2 |\partial\kappa_i| \int_0^1 u(\kappa_i(z))\overline{v(\kappa_i(z))}dz.$$

Since $\mu = \mu_0$, formula (3.2) yields locally supported wavelets $\psi_{j,y} = \psi_{j,y}^{(0)}$. Yet, to illustrate the preceding analysis, here we also consider wavelets, denoted by $\breve{\psi}_{j,y}$, that result from ignoring the jump in the Jacobian determinant, which approach has been followed in the literature. These wavelets $\breve{\psi}_{j,y}$ arise from replacing μ by μ_g throughout §2, where $g(x) = |\partial\kappa_i(\kappa_i^{-1}(x))|^{-1}$ if $x \in \Gamma_i$, or

$$\langle u, v\rangle_{\mu_g} = \sum_{i=1}^2 \int_0^1 u(\kappa_i(z))\overline{v(\kappa_i(z))}dz.$$

Note that this g does not satisfy (4.1) for $s \le -\frac{1}{2}$.

For $y \in I_{j+1}$, let us denote with y_L and y_R both its direct neighbours in I_{j+1}. Using that $\langle \mathbf{\Xi}, \tilde{\mathbf{\Phi}}\rangle_{\boldsymbol{\mu}} = \begin{bmatrix}\frac{1}{4}\sqrt{2} & \frac{1}{4}\sqrt{2}\end{bmatrix}$ and $\xi_{j,y} = \phi_{j+1,y}$, formula (3.2) yields

$$\psi_{j,y} = \phi_{j+1,y} - \tfrac{1}{4}\sqrt{2} \sum_{x \in \{y_L, y_R\}} \frac{w(y)}{\tilde{w}(x)}\,\theta_{j,x},$$

where

$$w(y)=|\partial\kappa_i| \quad \text{if } y\in\Gamma_i, \qquad \tilde{w}(x)=\begin{cases} 2|\partial\kappa_i| & \text{if } x\in\Gamma_i,\\ |\partial\kappa_1|+|\partial\kappa_2| & \text{if } x\in\overline{\Gamma_1}\cap\overline{\Gamma_2}.\end{cases}$$

By substituting

$$\theta_{j,x}=3\sqrt{2}\,\phi_{j+1,x}-\tfrac{1}{2}\sqrt{2}\,(\phi_{j+1,x_L}+\phi_{j+1,x_R}),$$

we find $\psi_{j,y}$ given as a linear combination of 5 nodal basis functions, generalizing the well-known 'prewavelet' construction on uniform partitions of the line, which can for example be found in [CW92]. Replacing μ_0 by μ_g yields

$$\breve{\psi}_{j,y}=\phi_{j+1,y}-\tfrac{1}{8}\sqrt{2}\sum_{x\in\{y_L,y_R\}}\theta_{j,x}.$$

Both $\psi_{j,y}$ and $\breve{\psi}_{j,y}$ are illustrated in Figure 3. Note that $\psi_{j,y}$ is equal to $\breve{\psi}_{j,y}$ except when

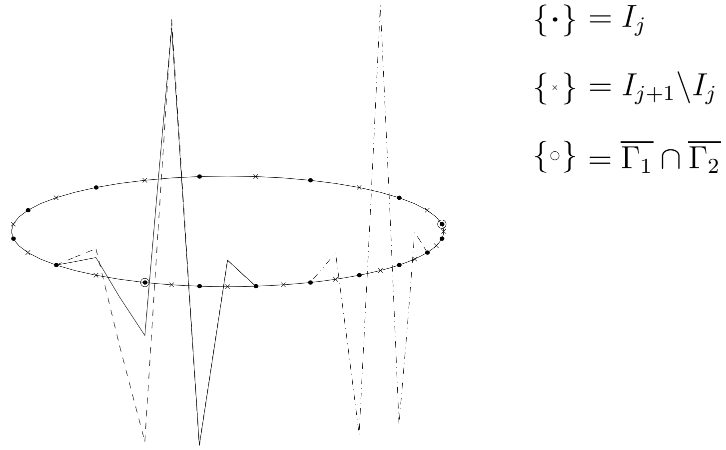


FIGURE 3. Wavelets $\psi_{j,y}$ ('$-$') and $\breve{\psi}_{j,y}$ ('$--$') with supports that intersect an interface, and wavelets $\psi_{j,y}=\breve{\psi}_{j,y}$ ('$-\cdot$') with support inside one patch

their supports intersect an interface between the two patches $\overline{\Gamma_1}$ and $\overline{\Gamma_2}$, in which case $\breve{\psi}_{j,y}$ has no cancellation properties.

Let us define $\Psi_s^{(j)}=\Phi_0\cup\cup_{\ell=0}^{j-1}2^{-\ell s}\Psi_\ell$ and $\breve{\Psi}_s^{(j)}=\Phi_0\cup\cup_{\ell=0}^{j-1}2^{-\ell s}\breve{\Psi}_\ell$. We are interested in $\kappa_{H^s(\Gamma)}(\Psi_s^{(j)})$ and $\kappa_{H^s(\Gamma)}(\breve{\Psi}_s^{(j)})$, where for a countable collection of functions $\Upsilon\subset H^s(\Gamma)\cap L^2(\Gamma)$,

$$\kappa_{H^s(\Gamma)}(\Upsilon):=\sup_{0\neq\mathbf{c}=(c_v)_{v\in\Upsilon}}\frac{\|\mathbf{c}^T\Upsilon\|^2_{H^s(\Gamma)}}{\|\mathbf{c}\|^2}\Bigg/\inf_{0\neq\mathbf{c}=(c_v)_{v\in\Upsilon}}\frac{\|\mathbf{c}^T\Upsilon\|^2_{H^s(\Gamma)}}{\|\mathbf{c}\|^2},$$

where thus for $s<0$ we use the embedding $E:L^2(\Gamma)\to H^s(\Gamma)$. We start with searching for equivalent quantities that are computable for general $|s|\leq 1$.

As norm on $H^1(\Gamma)$, we may use $\|u\|_{H^1(\Gamma)}:=\sqrt{\sum_{i=1}^2\|u\circ\kappa_i\|^2_{H^1(T_0)}}$. We have

$$\|\mathbf{u}_j^T\Phi_j\|^2_{H^1(\Gamma)}=\langle\breve{\mathbf{A}}_j\mathbf{u}_j,\mathbf{u}_j\rangle_{\ell^2}+\langle\breve{\mathbf{M}}_j\mathbf{u}_j,\mathbf{u}_j\rangle_{\ell^2},$$

where

$$\breve{\mathbf{A}}_j = \left(\sum_{i=1}^{2} \int_0^1 (\phi_{j,x} \circ \kappa_i)'(z) \overline{(\phi_{j,y} \circ \kappa_i)'(z)} dz \right)_{x,y \in I_j}$$

and

$$\breve{\mathbf{M}}_j \left(= \langle \Phi_j, \Phi_j \rangle_{\mu_g}\right) = \left(\sum_{i=1}^{2} \int_0^1 \phi_{j,x}(\kappa_i(z)) \overline{\phi_{j,y}(\kappa_i(z))} dz \right)_{x,y \in I_j}$$

are $2^{j+1} \times 2^{j+1}$ Toeplitz matrices with 'stencils' $4^j[-1 \quad 2 \quad -1]$ and $[\frac{1}{6} \quad \frac{2}{3} \quad \frac{1}{6}]$ respectively. Using $\|\mathbf{u}_j^T \Phi_j\|_{L^2(\Gamma)} \eqsim \|\mathbf{u}_j\|_{\ell^2}$, and by applying interpolation, we find that

$$\|\mathbf{u}_j^T \Phi_j\|_{H^s(\Gamma)} \eqsim \|(\breve{\mathbf{A}}_j + \breve{\mathbf{M}}_j)^{\frac{s}{2}} \mathbf{u}_j\|_{\ell^2} \qquad (s \in [0,1]). \tag{4.3}$$

As follows from (2.10), the $\langle , \rangle_\mu$-orthogonal projector $Q_j : L^2(\Gamma) \to V_j$ satisfies $\|Q_j\|_{H^s(\Gamma) \leftarrow H^s(\Gamma)} \lesssim 1$ ($|s| < \frac{3}{2}$). As a consequence, for $u_j \in V_j$ and $s \in (-\frac{3}{2}, 0]$, we have

$$\sup_{0 \neq v_j \in V_j} \frac{|\langle u_j, v_j \rangle_\mu|}{\|v_j\|_{H^{-s}(\Gamma)}} \leq \|u_j\|_{H^s(\Gamma)} = \sup_{0 \neq v \in H^{-s}(\Gamma)} \frac{|\langle u_j, Q_j v \rangle_\mu|}{\|v\|_{H^{-s}(\Gamma)}} \lesssim \sup_{0 \neq v_j = Q_j v \in V_j} \frac{|\langle u_j, v_j \rangle_\mu|}{\|v_j\|_{H^{-s}(\Gamma)}},$$

and so for $s \in [-1, 0]$,

$$\|\mathbf{u}_j^T \Phi_j\|_{H^s(\Gamma)} \eqsim \sup_{0 \neq v_j = \mathbf{v}_j^T \Phi_j \in V_j} \frac{|\langle \mathbf{M}_j \mathbf{u}_j, \mathbf{v}_j \rangle|}{\|(\breve{\mathbf{A}}_j + \breve{\mathbf{M}}_j)^{-\frac{s}{2}} \mathbf{v}_j\|_{\ell^2}} = \|(\breve{\mathbf{A}}_j + \breve{\mathbf{M}}_j)^{\frac{s}{2}} \mathbf{M}_j \mathbf{u}_j\|_{\ell^2}, \tag{4.4}$$

where $\mathbf{M}_j = \langle \Phi_j, \Phi_j \rangle_\mu$.

From (4.3), (4.4), one infers that for Υ_j being a basis for V_j, and $\mathbf{T}_{\Upsilon_j}^{\Phi_j}$ the matrix such that $\Upsilon_j^T = \Phi_j^T \mathbf{T}_{\Upsilon_j}^{\Phi_j}$, and $(\mathbf{T}_{\Upsilon_j}^{\Phi_j})^*$ its matrix adjoint,

$$\kappa_{H^s(\Gamma)}(\Upsilon_j) \eqsim \kappa_{s,j}(\Upsilon_j) := \begin{cases} \kappa((\mathbf{T}_{\Upsilon_j}^{\Phi_j})^* (\breve{\mathbf{A}}_j + \breve{\mathbf{M}}_j)^s \mathbf{T}_{\Upsilon_j}^{\Phi_j}) & \text{if } s \in (0,1], \\ \kappa((\mathbf{T}_{\Upsilon_j}^{\Phi_j})^* \mathbf{M}_j (\breve{\mathbf{A}}_j + \breve{\mathbf{M}}_j)^s \mathbf{M}_j \mathbf{T}_{\Upsilon_j}^{\Phi_j}) & \text{if } s \in [-1,0]. \end{cases} \tag{4.5}$$

We have computed numerical values of $\kappa_{s,j}(\Psi_s^{(j)})$ and $\kappa_{s,j}(\breve{\Psi}_s^{(j)})$ using the Lanczos method. By evaluating the application of $(\breve{\mathbf{A}}_j + \breve{\mathbf{M}}_j)^s$ using the FFT, each iteration can be performed in $\mathcal{O}(\mathrm{dim} V_j \log(\mathrm{dim} V_j))$ operations. As expected, the results given in Figures 4 and 5 show that in contrast to $\kappa_{s,j}(\Psi_s^{(j)})$, for $s \leq -\frac{1}{2}$, $\kappa_{s,j}(\breve{\Psi}_s^{(j)})$ is not bounded as function of j. In the limit case $s = -\frac{1}{2}$, the growth is approximately linear in j. For $s < -\frac{1}{2}$, $\kappa_{s,j}(\breve{\Psi}_s^{(j)})$ turns out to be exponentially increasing as function of j.

For general parametrizations, often a g satisfying both (4.1) for $s \leq -\frac{1}{2}$ and (4.2) does not exist. Therefore, below we will give up biorthogonality of the space decompositions. That is, we will construct collections

$$\Psi_j = \{\psi_{j,y} : y \in I_{j+1} \backslash I_j\} \subset V_{j+1}, \tag{4.6}$$

that will not (exactly) span spaces $V_{j+1} \cap \tilde{V}_j^{\perp_{\langle , \rangle_{\mu_g}}}$. Nevertheless, as it will turn out, they will give rise to Riesz bases for a range of Sobolev spaces, including $H^s(\Gamma)$ for $s \leq -\frac{1}{2}$, and

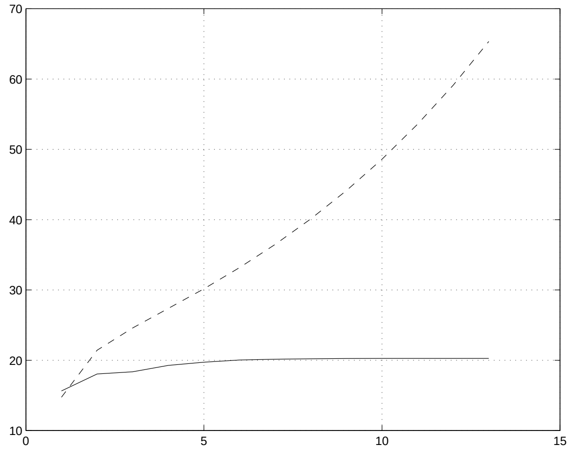


FIGURE 4. $\kappa_{s,j}(\Psi_s^{(j)})$ ('$-$') and $\kappa_{s,j}(\breve{\Psi}_s^{(j)})$ ('$--$') for $s = -\frac{1}{2}$ and $j = 2, \ldots 13$

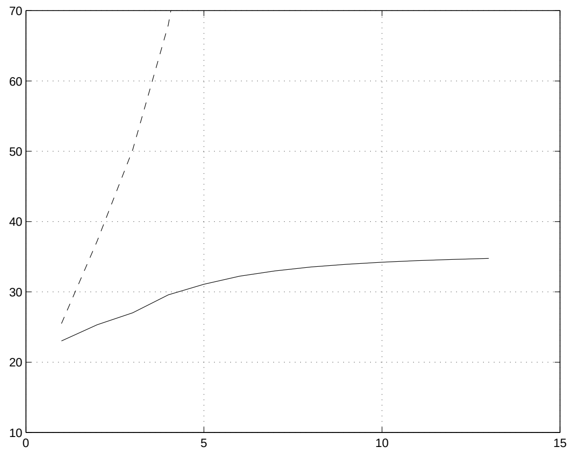


FIGURE 5. $\kappa_{s,j}(\Psi_s^{(j)})$ ('$-$') and $\kappa_{s,j}(\breve{\Psi}_s^{(j)})$ ('$--$') for $s = -\frac{3}{4}$ and $j = 2, \ldots 13$. For $s = -\frac{3}{4}$ and $j = 13$, we found $\kappa_{s,j}(\breve{\Psi}_s^{(j)}) = 8.3 \times 10^3$

their elements $\psi_{j,y}$ will satisfy cancellation properties, which means that it is appropriate to call them wavelets. Note that the notations $\psi_{j,y}$ and Ψ_j that up to now were reserved for wavelets that span $V_{j+1} \cap \tilde{V}_j^{\perp_{\langle,\rangle_\mu}}$ are now used for the new collections.

Given $j \in I\!N$ and $y \in I_{j+1} \backslash I_j$, for all $1 \leq i \leq p$ for which $\Lambda_{j,y}(i)$, defined in (3.3) and illustrated in Figure 2, is non-empty, select some

$$z_{j,y}(i) \in \Lambda_{j,y}(i). \tag{4.7}$$

Now define

$$\psi_{j,y} = \xi_{j,y} - \sum_{x \in I_j} \frac{\sum_{\{1 \leq i \leq p, T \in \tau_j : \kappa_i(T) \ni x,y\}} |\partial \kappa_i(z_{j,y}(i))| \langle \boldsymbol{\xi}_{\lambda_T(\kappa_i^{-1}(y))}, \tilde{\boldsymbol{\phi}}_{\lambda_T(\kappa_i^{-1}(x))} \rangle_{\boldsymbol{\mu}}}{\sum_{\{1 \leq i \leq p, T \in \tau_j : \kappa_i(T) \ni x\}} |\partial \kappa_i(z_{j,y}(i))|} \, \theta_{j,x}. \tag{4.8}$$

Note that as supp$\psi^{(j_0)}_{j,y}$, supp$\psi_{j,y}$ is contained in $\cup_{i=1}^p \kappa_i(\Lambda_{j,y}(i))$. Furthermore, if all but one sets $\Lambda_{j,y}(i)$ are empty, i.e. supp$\psi_{j,y}$ is contained inside one patch $\overline{\Gamma_i}$, then $\psi_{j,y} = \psi^{(0)}_{j,y}$, and the choice of $z_{j,y}(i)$ is irrelevant. So in this case the non-constant Jacobian determinant is ignored, which however is assumed to be smooth on supp$\psi_{j,y}$. In the other case that supp$\psi_{j,y}$ extends to different patches, the non-constant Jacobian determinant is taken into account, in the sense that it is replaced by a piecewise constant. Generally $\psi_{j,y}$ and $\psi^{(0)}_{j,y}$ are now different.

We start by showing that these new wavelets induce a 'stable two-level splitting'. By using (2.3), comparison of (4.8) and (3.1) shows that for $0 \leq j_0 \leq j$,

$$\|\psi_{j,y} - \psi^{(j_0)}_{j,y}\|_{L^2(\Gamma)} \lesssim 2^{-j_0}. \tag{4.9}$$

By the uniform locality of both Ψ_j and $\Psi^{(j_0)}_j$, it follows that

$$\|\mathbf{c}_j^T(\Psi_j - \Psi^{(j_0)}_j)\|_{L^2(\Gamma)} \lesssim 2^{-j_0}\|\mathbf{c}_j\|_{\ell^2}.$$

Since, as was shown in §2, for $j \geq j_0 \geq 0$ the $\Psi^{(j_0)}_j$ are uniform $L^2(\Gamma)$-Riesz systems, we conclude that for j_0 being sufficiently large and $j \geq j_0$, the Ψ_j are uniform $L^2(\Gamma)$-Riesz systems.

For $j \geq j_0$, let $\hat{W}_j := \operatorname{span}\Psi_j$. By (4.9) and (2.4) it holds that for $x \in I_j$, $y \in I_{j+1}\backslash I_j$,

$$|\langle\tilde{\phi}_{j,x}, \psi_{j,y}\rangle_\mu| = |\langle\tilde{\phi}_{j,x}, \psi_{j,y} - \psi^{(j)}_{j,y}\rangle_\mu + \langle\tilde{\phi}_{j,x}, \psi^{(j)}_{j,y}\rangle_\mu - \langle\tilde{\phi}_{j,x}, \psi^{(j)}_{j,y}\rangle_{\mu_j}| \lesssim 2^{-j}.$$

Since $\tilde{\Phi}_j$, Ψ_j are uniformly local, uniform $L^2(\Gamma)$-Riesz bases for $\tilde{V}_j$, $\hat{W}_j$, we conclude that

$$|\langle\tilde{v}_j, \hat{w}_j\rangle_\mu| \lesssim 2^{-j}\|\tilde{v}_j\|_{L^2(\Gamma)}\|\hat{w}_j\|_{L^2(\Gamma)} \qquad (\tilde{v}_j \in \tilde{V}_j,\ \hat{w}_j \in \hat{W}_j), \tag{4.10}$$

meaning that Ψ_j spans a subspace of V_{j+1} which is nearly orthogonal to $\tilde{V}_j$.

Possibly for a larger j_0, for $j \geq j_0$ let $Q_j : L^2(\Gamma) \to L^2(\Gamma)$ be the uniformly bounded projectors from §2, satisfying $\operatorname{Im}(Q_j) = V_j$ and $\operatorname{Im}(I - Q_j) = \tilde{V}_j^{\perp_{\langle,\rangle_\mu}}$, and so for the adjoints, $\operatorname{Im}(Q_j^*) = \tilde{V}_j$ and $\operatorname{Im}(I - Q_j^*) = V_j^{\perp_{\langle,\rangle_\mu}}$. From (4.10), for $\hat{w}_j \in \hat{W}_j$ we have

$$\begin{aligned}\|Q_j\hat{w}_j\|_{L^2(\Gamma)} &\eqsim \sup_{0\neq v_j\in V_j}\frac{|\langle v_j, Q_j\hat{w}_j\rangle_\mu|}{\|v_j\|_{L^2(\Gamma)}} = \sup_{0\neq v_j\in V_j}\frac{|\langle Q_j^* v_j, \hat{w}_j\rangle_\mu|}{\|v_j\|_{L^2(\Gamma)}}\\ &\lesssim 2^{-j}\|Q_j^*\|_{L^2(\Gamma)\leftarrow L^2(\Gamma)}\|\hat{w}_j\|_{L^2(\Gamma)} \lesssim 2^{-j}\|\hat{w}_j\|_{L^2(\Gamma)}.\end{aligned} \tag{4.11}$$

With $W_j := \operatorname{Im}(Q_{j+1} - Q_j) = \operatorname{Im}((I - Q_j)|_{V_{j+1}})$, the uniform boundedness of the projectors Q_j shows that the pairs (V_j, W_j) satisfy the following uniform strengthened Cauchy-Schwarz inequality,

$$\sup_{j\geq j_0}\ \sup_{0\neq v_j\in V_j, 0\neq w_j\in W_j}\frac{|\langle v_j, w_j\rangle_\mu|}{\|v_j\|_{L^2(\Gamma)}\|w_j\|_{L^2(\Gamma)}} < 1. \tag{4.12}$$

Writing for $v_j \in V_j$ and $\hat{w}_j \in \hat{W}_j$,

$$\langle v_j, \hat{w}_j\rangle_\mu = \langle v_j, Q_j\hat{w}_j\rangle_\mu + \langle v_j, (I - Q_j)\hat{w}_j\rangle_\mu,$$

from (4.11) and (4.12), we infer that for j_0 being sufficiently large and $j \geq j_0$, also the $(V_j, \hat{W}_j)$ satisfy a uniform strengthened Cauchy-Schwarz inequality. Since furthermore $V_j, \hat{W}_j \subset V_{j+1}$ and $\dim V_{j+1} = \dim V_j + \dim \hat{W}_j$, we may conclude that for $j \geq j_0$ there exist uniformly bounded projectors

$$\hat{Q}_j : L^2(\Gamma) \supset V_{j+1} \to V_j \subset L^2(\Gamma),$$

such that $\mathrm{Im}\hat{Q}_j = V_j$ and $\mathrm{Im}(I - \hat{Q}_j) = \hat{W}_j$, which result we meant by stability of the two-level splitting. Note that $\Phi_{j_0} \cup \cup_{j=j_0}^{\ell} \Psi_j$ is a basis for $V_{\ell+1}$.

An immediate consequence of (4.11) and the uniform boundedness of $\hat{Q}_j$ is that for $j \geq j_0$,

$$\|Q_j - \hat{Q}_j\|_{L^2(\Gamma) \leftarrow L^2(\Gamma)} = \|Q_j(I - \hat{Q}_j)\|_{L^2(\Gamma) \leftarrow L^2(\Gamma)} \lesssim 2^{-j}. \tag{4.13}$$

Theorem 4.2. *Consider the wavelet collections Ψ_j defined by* (4.6), (4.8). *From* (4.13), *and the fact that these Ψ_j are uniform $L^2(\Gamma)$-Riesz bases for $\hat{W}_j = \mathrm{Im}(I - \hat{Q}_j)$, it follows that $\Phi_{j_0} \cup \cup_{j \geq j_0} 2^{-js}\Psi_j$ is a Riesz basis for $H^s(\Gamma)$ when $s \in (-1, \frac{3}{2})$ with $|s| \leq m$ or $|s| < t$.*

Proof. We define the auxiliary spaces $H_s(\Gamma)$ for $s \geq 0$ as the closure of

$$U_s := \{u \in C(\Gamma) : u \circ \kappa_i \in H^s(T_0),\ 1 \leq i \leq p\}$$

with respect to the norm $\|u\|_{H_s(\Gamma)} = \sqrt{\sum_{i=1}^p \|u \circ \kappa_i\|^2_{H^s(T_0)}}$, and for $s < 0$ as $H_{-s}(\Gamma)'$. For $s \in [0, \frac{3}{2})$ with $s \leq m$ or $s < t$, U_s is also a dense subset of $H^s(\Gamma)$. Since furthermore $\|u \circ \kappa_i\|_{H^s(T_0)} \eqsim \|u\|_{H^s(\Gamma_i)}$, we infer that $H^s(\Gamma)$ and $H_s(\Gamma)$ agree as sets and have equivalent norms. By duality, these results extend to $s \in (-\frac{3}{2}, 0)$ with $s \geq -m$ or $s > -t$. We conclude that it is sufficient to prove that

$$\Phi_{j_0} \cup \cup_{j \geq j_0} 2^{-js}\Psi_j \text{ is a Riesz basis for } H_s(\Gamma) \text{ when } s \in (-1, \tfrac{3}{2}). \tag{4.14}$$

The point of introducing the spaces $H_s(\Gamma)$ is that it is now sufficient to prove the Riesz basis property for s in an interval that is always open.

The spaces $H_s(\Gamma)$ were also used in [DSt99] to prove the stability (2.10) of biorthogonal space decompositions. With respect to the $H_s(\Gamma)$ spaces, the Bernstein inequalities ($\mathcal{B}$), and the Jackson estimates ($\mathcal{J}$) hold for the 'full' ranges $s \in [0, \frac{3}{2})$, and $s \in [0, d]$ or $s \in [0, \tilde{d}]$ respectively, yielding for $|s| < \frac{3}{2}$,

$$\|u\|_{H_s(\Gamma)} \eqsim \sum_{j=j_0}^{\infty} 4^{js} \|(Q_j - Q_{j-1})u\|^2_{L^2(\Gamma)} \qquad (u \in H_s(\Gamma)), \tag{4.15}$$

and

$$\|u\|_{H_s(\Gamma)} \eqsim \sum_{j=j_0}^{\infty} 4^{js} \|(Q_j^* - Q_{j-1}^*)u\|^2_{L^2(\Gamma)} \qquad (u \in H_s(\Gamma)). \tag{4.16}$$

We will show that for any $s \in (-1, \frac{3}{2})$, there exists an $\omega < 1$ such that

$$|\langle \hat{w}_j, \hat{w}_\ell \rangle_{H^s(\Gamma)}| \lesssim \omega^{\ell - j} 2^{js} \|\hat{w}_j\|_{L^2(\Gamma)} 2^{\ell s} \|\hat{w}_\ell\|_{L^2(\Gamma)} \qquad (j_0 \leq j \leq \ell), \tag{4.17}$$

and that for any $s \in (-1, 0]$,

$$\sup_{\ell \geq j \geq j_0} \|\hat{Q}_j \hat{Q}_{j+1} \cdots \hat{Q}_\ell\|_{H_s(\Gamma) \leftarrow H_s(\Gamma)} < \infty. \tag{4.18}$$

Then, using (4.15), for $s \in (-1, \frac{3}{2})$ an application of [Ste98, Theorem 3.1] (with 'r'$= 0$ and 'q'$\in (-1, \min\{s, 0\}]$) shows that

$$\|v_{j_0} + \sum_{j=j_0}^{\ell} \hat{w}_j\|^2_{H_s(\Gamma)} \eqsim \|v_{j_0}\|^2_{L^2(\Gamma)} + \sum_{j=j_0}^{\ell} 4^{js} \|\hat{w}_j\|^2_{L^2(\Gamma)} \qquad (v_{j_0} \in V_{j_0},\ \hat{w}_j \in \hat{W}_j).$$

Since Φ_{j_0}, Ψ_j are uniform $L^2(\Gamma)$-Riesz bases for V_{j_0}, $\hat{W}_j$ respectively, it follows that $\Phi_{j_0} \cup \cup_{j=j_0}^{\ell} \Psi_j$ are uniform (in ℓ) $H_s(\Gamma)$-Riesz bases for $V_{\ell+1}$, and thus that $\Phi_{j_0} \cup \cup_{j=j_0}^{\infty} \Psi_j$ is a Riesz system in $H_s(\Gamma)$. Since its span includes $\cup_j V_j$, we conclude (4.14).

First we prove (4.17). It is sufficient to show that for $s \in (-1, \frac{3}{2})$,

$$\|\hat{w}_j\|_{H_s(\Gamma)} \lesssim 2^{js} \|\hat{w}_j\|_{L^2(\Gamma)} \qquad (\hat{w}_j \in \hat{W}_j), \tag{4.19}$$

since this implies that for $s \in (-1, \frac{3}{2})$, and with $\epsilon > 0$ such that $s \pm \epsilon \in (-1, \frac{3}{2})$,

$$|\langle \hat{w}_j, \hat{w}_\ell \rangle_{H_s(\Gamma)}| \lesssim \|\hat{w}_j\|_{H_{s+\epsilon}(\Gamma)} \|\hat{w}_\ell\|_{H_{s-\epsilon}(\Gamma)} \lesssim (2^{-\epsilon})^{(\ell-j)} (2^{js} \|\hat{w}_j\|_{L^2(\Gamma)}) (2^{\ell s} \|\hat{w}_\ell\|_{L^2(\Gamma)}).$$

For $s \geq 0$, (4.19) follows from the Bernstein inequality. Now let $s < 0$. Then the uniform boundedness of $\|Q^*_{j+1}\|_{H_{-s}(\Gamma) \leftarrow H_{-s}(\Gamma)}$, which is an easy consequence of (4.15) or (4.16), shows that

$$\begin{aligned} \|\hat{w}_j\|_{H_s(\Gamma)} &= \sup_{0 \neq v \in H_{-s}(\Gamma)} \frac{|\langle \hat{w}_j, v \rangle_\mu|}{\|v\|_{H_{-s}(\Gamma)}} = \sup_{0 \neq v \in H_{-s}(\Gamma)} \frac{|\langle \hat{w}_j, Q^*_{j+1} v \rangle_\mu|}{\|v\|_{H_{-s}(\Gamma)}} \\ &\lesssim \sup_{0 \neq v \in H_{-s}(\Gamma)} \frac{|\langle \hat{w}_j, Q^*_{j+1} v \rangle_\mu|}{\|Q^*_{j+1} v\|_{H_{-s}(\Gamma)}} = \sup_{0 \neq \tilde{v}_{j+1} \in \tilde{V}_{j+1}} \frac{|\langle (I - \hat{Q}_j) \hat{w}_j, \tilde{v}_{j+1} \rangle_\mu|}{\|\tilde{v}_{j+1}\|_{H_{-s}(\Gamma)}}. \end{aligned}$$

Now by

$$\begin{aligned} |\langle (I - \hat{Q}_j) \hat{w}_j, \tilde{v}_{j+1} \rangle_\mu| &= |\langle (Q_j - \hat{Q}_j) \hat{w}_j, \tilde{v}_{j+1} \rangle_\mu + \langle \hat{w}_j, (Q^*_j - Q^*_{j+1}) \tilde{v}_{j+1} \rangle_\mu| \\ &\lesssim 2^{-j} \|\hat{w}\|_{L^2(\Gamma)} \|\tilde{v}_{j+1}\|_{L^2(\Gamma)} + \|\hat{w}\|_{L^2(\Gamma)} 2^{js} \|\tilde{v}_{j+1}\|_{H_{-s}(\Gamma)} \end{aligned}$$

which follows from (4.13) and (4.16), we conclude (4.19) and thus (4.17).

Now we will show (4.18), which is the crucial part of this proof. Given some $s \in (-1, 0]$, for $j_0 \leq j \leq \ell + 1$, let

$$\begin{aligned} \rho_j^{(\ell)} &:= \max_{j_0 \leq k \leq j} ||Q_k \hat{Q}_j \hat{Q}_{j+1} \cdots \hat{Q}_\ell||_{H_s(\Gamma) \leftarrow H_s(\Gamma)}, \\ \epsilon_j &:= \max_{j_0 \leq k \leq j} ||Q_k (\hat{Q}_j - Q_j)||_{H_s(\Gamma) \leftarrow H_s(\Gamma)}. \end{aligned}$$

Then from $Q_k Q_j = Q_k$, and thus

$$Q_k \hat{Q}_j \hat{Q}_{j+1} \cdots \hat{Q}_\ell = Q_k (\hat{Q}_j - Q_j) \hat{Q}_{j+1} \cdots \hat{Q}_\ell + Q_k \hat{Q}_{j+1} \cdots \hat{Q}_\ell,$$

we find that $\rho_j^{(\ell)} \leq (\epsilon_j + 1)\rho_{j+1}^{(\ell)}$. By the uniform boundedness of $\|Q_k\|_{H_s(\Gamma)\leftarrow H_s(\Gamma)}$, we have $\rho_{\ell+1}^{(\ell)} \lesssim 1$ and $\epsilon_j \lesssim \|\hat{Q}_j - Q_j\|_{H_s(\Gamma)\leftarrow H_s(\Gamma)} \lesssim 2^{-js}\|\hat{Q}_j - Q_j\|_{L^2(\Gamma)\leftarrow L^2(\Gamma)} \lesssim 2^{j(-1-s)}$ by (4.13). We infer that

$$\sup_{\ell\geq j\geq j_0} \|\hat{Q}_j\hat{Q}_{j+1}\cdots\hat{Q}_\ell\|_{H_s(\Gamma)\leftarrow H_s(\Gamma)} \leq \sup_{\ell\geq j\geq j_0} \rho_j^{(\ell)} \lesssim \sup_{\ell\geq j\geq j_0} \sum_{m=j}^{\ell} \epsilon_m \lesssim \sum_{m=0}^{\infty} 2^{m(-1-s)} < \infty,$$

which completes the proof of the theorem. □

We now discuss the cancellation properties of the wavelets defined in (4.8). Let $y \in I_{j+1}\backslash I_j$, and let $\Lambda_{j,y}(i)$ and $z_{j,y}(i)$ be as in (3.3) and (4.7). Define g on Γ by

$$g(x) = |\partial\kappa_i(z_{j,y}(i))||\partial\kappa_i(x)|^{-1} \quad \text{if } x \in \Gamma_i \text{ with } i \text{ such that } \Lambda_{j,y}(i) \neq \emptyset, \tag{4.20}$$

and say $g(x) = 1$ otherwise. Then by construction, $\psi_{j,y} \perp_{\langle,\rangle_{\mu_g}} \tilde{V}_j$. Proposition 3.4 with $\langle\,,\,\rangle_\mu$ replaced by $\langle\,,\,\rangle_{\mu_g}$ shows that for v being a continuous function on Γ, which is patchwise smooth, and $I\!N \ni k \leq \tilde{d}$ it holds that

$$|\langle v, \psi_{j,y}\rangle_{\mu_g}| \lesssim 2^{-j(k+n/2)} \max_{i,T\in\Lambda_{j,y}(i)} |v\circ\kappa_i|_{W^{k,\infty}(T)}. \tag{4.21}$$

In fact, it is sufficient when v restricted to $\cup_i\kappa_i(\Lambda_{j,y}(i)) \supset \operatorname{supp}\psi_{j,y}$ is continuous, and smooth on each $\kappa_i(\Lambda_{j,y}(i))$.

Obviously, one has

$$\langle v, \psi_{j,y}\rangle_\mu = \langle v/g, \psi_{j,y}\rangle_{\mu_g}. \tag{4.22}$$

So in case all but one sets $\Lambda_{j,y}(i)$ are empty, and so $\operatorname{supp}\psi_{j,y}$ is contained in one patch $\overline{\Gamma_i}$, the smoothness of g on this patch shows that

$$|\langle v, \psi_{j,y}\rangle_\mu| \lesssim 2^{-j(\tilde{d}+n/2)} \max_{i,T\in\Lambda_{j,y}(i)} \|v\circ\kappa_i\|_{W^{\tilde{d},\infty}(T)},$$

i.e., $\psi_{j,y}$ has the cancellation property of the full order $\tilde{d}$.

Now consider $\psi_{j,y}$ with support that extends to more than one patches $\overline{\Gamma_i}$. Then, if the $z_{j,y}(i)$ can be selected such that the function g from (4.20) is continuous on $\cup_i\kappa_i(\Lambda_{j,y}(i))$, then above arguments show that again $\psi_{j,y}$ has the cancellation property of the full order $\tilde{d}$. For example, for a one-dimensional manifold this can always be realized by selecting $z_{j,y}(i)$ as the pull-back of the interface point inside $\operatorname{supp}\psi_{j,z}$.

Finally, if above requirement is not satisfied, then from $\sup_{x\in\operatorname{supp}(\psi_{j,y})} |1/g(x) - 1| \lesssim 2^{-j}$, (4.22) and (4.21), one infers that

$$|\langle v, \psi_{j,y}\rangle_\mu| \lesssim 2^{-j(1+n/2)} \max_{i,T\in\Lambda_{j,y}(i)} \|v\circ\kappa_i\|_{W^{1,\infty}(T)},$$

or, in any case $\psi_{j,y}$ has the cancellation property of order 1.

Remark 4.3. Instead of applying the Ψ_j defined by (4.8), another option to handle the general case of non-constant Jacobian determinants would be to use the collections $\Psi_j^{(j)}$. As shown in §2, these $\Psi_j^{(j)}$ are uniform $L^2(\Gamma)$-Riesz bases for $V_{j+1} \cap \tilde{V}_j^{\perp_{\langle,\rangle_{\mu_j}}}$. Furthermore, the same arguments that were used to prove Theorem 4.2 show that $\Phi_0 \cup \cup_{j\geq 0} 2^{-js}\Psi_j^{(j)}$ is a Riesz basis for $H^s(\Gamma)$ when $s \in (-1, \frac{3}{2})$ with $|s| \leq m$ or $|s| < t$. The reason however not to propose this wavelet construction is that *each* $\psi_{j,y}^{(j)}$, thus also when its support is contained in one $\overline{\Gamma_i}$, generally has the cancellation property of only order 1.

Remark 4.4. Just as the wavelets corresponding to the case of constant Jacobian determinants, our new wavelets are given in the form $\Psi_j = \Xi_j - \mathbf{G}_j^T \Theta_j$, where the $\mathbf{G}_j$ are matrices that are uniformly local. This means that the discussion from [DSt99] about constructing an efficient implementation of the inverse wavelet transform, i.e., the transformation from wavelet to single-scale basis, here applies without modification. Instead of expressing an expansion $\mathbf{d}_j^T \Psi_j$ directly in the form $\mathbf{c}_{j+1}^T \Phi_{j+1}$, the idea is to express it first as $\mathbf{d}_j^T \Xi_j - (\mathbf{G}_j \mathbf{d}_j)^T \Theta_j$, and then to write $\mathbf{d}_j^T \Xi_j$ in the form $\tilde{\mathbf{c}}_{j+1}^T \Phi_{j+1}$, and $(\mathbf{G}_j \mathbf{d}_j)^T \Theta_j$ in the form $\sum_{k=0}^{\ell} \check{\mathbf{c}}_{j+1-k}^T \Phi_{j+1-k}$ for some fixed ℓ; the latter step by expressing each $\theta_{j,x}$ as a minimal linear combination of elements from $\Phi_{j+1}, \ldots, \Phi_{j+1-\ell}$. Often Ξ_j is just a subset of Φ_{j+1}, whereas the transformation involving Θ_j is cheap since $\mathrm{card}\Theta_j / \mathrm{card}\Psi_j \approx (2^n - 1)^{-1}$.

Remark 4.5. As was already noted in Remark 3.3, in [Ste00], examples of quadruples $(\mathbf{\Phi}, \tilde{\mathbf{\Phi}}, \mathbf{\Theta}, \mathbf{\Xi})$ are given with $\mathbf{\Theta} = \mathbf{\Phi}$, meaning that when $\mu = \mu_0$, the sets Φ_j and $\langle \Phi_j, \tilde{\Phi}_j \rangle_\mu^{-1} \tilde{\Phi}_j$ are uniformly local, $\langle , \rangle_\mu$-biorthogonal scaling functions. With non-constant Jacobian determinants, this biorthogonality on the global level is lost, and so we do not obtain formulas for the dual wavelets. On the other hand, since for $\mathbf{\Theta} = \mathbf{\Phi}$ each $\psi_{j,y}$ is given as $\xi_{j,y}$ minus a uniformly finite linear combination of coarse-grid scaling functions $\theta_{j,x} = \phi_{j,x}$, the wavelet transform, i.e., the transformation from single-scale to wavelet basis, is of optimal complexity also in case of non-constant Jacobian determinants.

Finally, we give some numerical results obtained with the newly introduced wavelets:

Example 4.6. As in Example 4.1, let $\Gamma = \cup_{i=1}^2 \overline{\Gamma_i}$ be the unit circle in $I\!R^2$, and $T_0 = [0, 1]$. Again we take $(\mathbf{\Phi}, \tilde{\mathbf{\Phi}}, \mathbf{\Theta}, \mathbf{\Xi})$ from Example 2.5 (with $n = 1$). This time, we take $\kappa_1(z) = \kappa(z)$, $\kappa_2(z) = \kappa(z + 1)$, where

$$\kappa(z) := (\cos(2\pi(2^{z/2} - 1)), \sin(2\pi(2^{z/2} - 1))),$$

yielding

$$\langle u, v \rangle_\mu = \pi \log(2) \int_0^2 u(\kappa(z)) \overline{v(\kappa(z))}\, 2^{z/2} dz.$$

Note that the Jacobian determinant is not equal to any piecewise constant function, and so $\mu \neq \mu_{j_0}$ for all $j_0 \in I\!N$.

The new wavelets defined by (4.8) read as

$$\psi_{j,y} = \phi_{j+1,y} - \tfrac{1}{4}\sqrt{2} \sum_{x \in \{y_L, y_R\}} \frac{w(y)}{\tilde{w}(x)}\, \theta_{j,x},$$

where now with $z_{j,y}(i)$ being some point in $\Lambda_{j,y}(i)$,

$$w(y) = |\partial\kappa_i(z_{j,y}(i))| \quad \text{if } y \in \Gamma_i, \quad \tilde{w}(x) = \begin{cases} 2|\partial\kappa_i(z_{j,y}(i))| & \text{if } x \in \Gamma_i, \\ |\partial\kappa_1(z_{j,y}(1))| + |\partial\kappa_2(z_{j,y}(2))| & \text{if } x \in \overline{\Gamma_1} \cap \overline{\Gamma_2}, \end{cases}$$

and

$$\theta_{j,x} = 3\sqrt{2}\,\phi_{j+1,x} - \tfrac{1}{2}\sqrt{2}\,(\phi_{j+1,x_L} + \phi_{j+1,x_R}).$$

If both $y_l, y_R \notin \overline{\Gamma_1} \cap \overline{\Gamma_2}$, the choice of $z_{j,y}(i)$ is irrelevant. In the other case, to ensure that for $j > 0$ all $\psi_{j,y}$ satisfy the cancellation property of the full order order 2, we take $z_{j,y}(i)$ being the pull-back of the interface point inside $\operatorname{supp}\psi_{j,z}$. That is, either $z_{j,y}(1) = 1$ and $z_{j,y}(2) = 0$ and so $|\partial\kappa_1(z_{j,y}(1))| = |\partial\kappa_2(z_{j,y}(2))|$ thus yielding an 'unmodified' wavelet, which is appropriate since the Jacobian determinant connects continuously over this interface, or $z_{j,y}(1) = 0$ and $z_{j,y}(2) = 1$ and so $|\partial\kappa_1(z_{j,y}(1))| = \frac{1}{2}|\partial\kappa_2(z_{j,y}(2))|$ yielding a wavelet adapted to the jump in the Jacobian determinant over the other interface, cf. Figure 6. The lowest

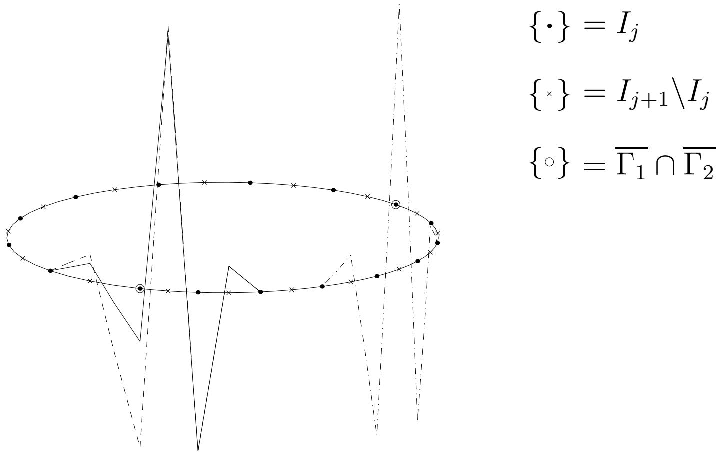


FIGURE 6. Wavelets $\psi_{j,y}$ ('$-$') and $\breve{\psi}_{j,y}$ ('$--$') with supports that intersect the interface where the Jacobian determinant has a jump, and wavelets $\psi_{j,y} = \breve{\psi}_{j,y}$ ('$-\cdot$') with support inside one patch

level corresponds to an exceptional case: Both wavelets $\psi_{0,y}$ for $y \in I_1 \backslash I_0$ have supports equal to Γ and therefore intersect both interfaces. We took $z_{0,y}(1) = 0$, $z_{0,y}(2) = 1$.

With Ψ_j being the resulting wavelet collections defined by (4.6) and (4.8), and $\Psi_s^{(j)} = \Phi_0 \cup \cup_{\ell=0}^{j-1} 2^{-\ell s}\Psi_\ell$, we computed $\kappa_{s,j}(\Psi_s^{(j)})$ defined as in (4.5), where obviously $\mathbf{M}_j = \langle \Phi_j, \Phi_j \rangle_\mu$ and $\mathbf{T}^{\Phi_j}_{\Psi_s^{(j)}}$ now refer to the current parametrizations and wavelet collections. As in Example 4.1, for comparison we also computed $\kappa_{s,j}(\breve{\Psi}_s^{(j)})$ where $\breve{\Psi}_s^{(j)} = \Phi_0 \cup \cup_{\ell=0}^{j-1} 2^{-\ell s}\breve{\Psi}_\ell$, and $\breve{\Psi}_j$ results from ignoring the non-constant Jacobian determinants, i.e.,

$$\breve{\psi}_{j,y} = \phi_{j+1,y} - \tfrac{1}{8}\sqrt{2} \sum_{x \in \{y_L, y_R\}} \theta_{j,x}.$$

Recall that $\breve{\Psi}_j$ spans $V_{j+1} \cap V_j^{\perp_{\mu_g}}$ where $g(x) = |\partial\kappa_i(\kappa_i^{-1}(x))|^{-1}$ if $x \in \Gamma_i$, or

$$\langle u, v \rangle_{\mu_g} = \int_0^2 u(\kappa(z))\overline{v(\kappa(z))}dz.$$

As in Example 4.1, the results given in Figures 7 and 8 show that in contrast to $\kappa_{s,j}(\Psi_s^{(j)})$,

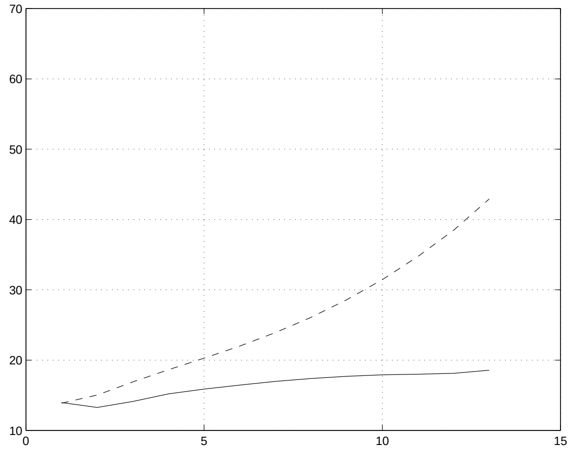


FIGURE 7. $\kappa_{s,j}(\Psi_s^{(j)})$ ('$-$') and $\kappa_{s,j}(\breve{\Psi}_s^{(j)})$ ('$--$') for $s = -\frac{1}{2}$ and $j = 2, \ldots 13$

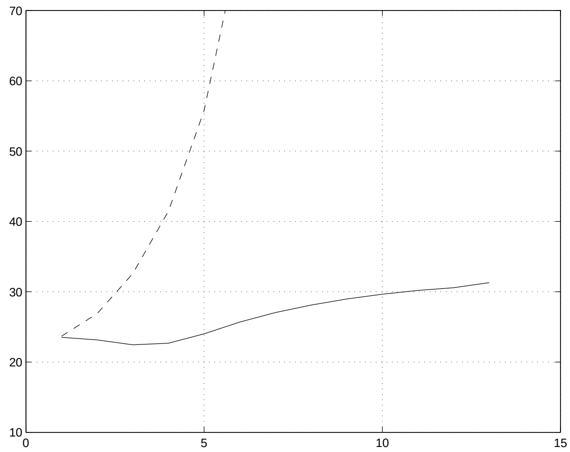


FIGURE 8. $\kappa_{s,j}(\Psi_s^{(j)})$ ('$-$') and $\kappa_{s,j}(\breve{\Psi}_s^{(j)})$ ('$--$') for $s = -\frac{3}{4}$ and $j = 2, \ldots 13$. For $s = -\frac{3}{4}$ and $j = 13$, we found $\kappa_{s,j}(\breve{\Psi}_s^{(j)}) = 4.2 \times 10^3$

for $s \leq -\frac{1}{2}$, $\kappa_{s,j}(\breve{\Psi}_s^{(j)})$ is not bounded as function of j. In the limit case $s = -\frac{1}{2}$, the growth is approximately linear in j. For $s < -\frac{1}{2}$, $\kappa_{s,j}(\breve{\Psi}_s^{(j)})$ turns out to be exponentially increasing as function of j. Unfortunately, although the $\Psi_s^{(j)}$ are uniform $H^s(\Gamma)$-Riesz systems, our computation of $\kappa_{s,j}(\Psi_s^{(j)})$ as the spectral condition number of a product of a number of matrices which are not all uniformly well-conditioned starts to become numerically unstable around level $j = 13$, which slightly shows up in the figures.

An alternative would have been to compare with the wavelets that span $V_{j+1} \cap V_j^{\perp_{\mu_0}}$, that is, the wavelets yielded by (3.2). Since $|\partial\kappa_1(\frac{1}{2})|/|\partial\kappa_2(\frac{1}{2})| = \frac{1}{2}\sqrt{2}$, this construction yields ‘wrong’ wavelets at both interfaces. We may expect similar results as obtained with $\breve{\Psi}_j$.

References

[CES00] A. Cohen, L.M. Echeverry, and Q. Sun. Finite element wavelets. Technical report, Laboratoire d’Analyse Numérique, Université Pierre et Marie Curie, 2000.

[Coh00] A. Cohen. Wavelet methods in numerical analysis. In P.G. Ciarlet and J. L. Lions, editors, *Handbook of numerical analysis. Vol. VII.*, pages 417–711. North-Holland, Amsterdam, 2000.

[CTU99] C. Canuto, A. Tabacco, and K. Urban. The wavelet element method part I: Construction and analysis. *Appl. Comput. Harmonic Anal*, 6:1–52, 1999.

[CW92] C.K. Chui and J.Z. Wang. On compactly supported spline wavelets and a duality principle. *Trans. Amer. Math. Soc.*, 330:903–916, 1992.

[Dah96] W. Dahmen. Stability of multiscale transformations. *J. Fourier Anal. Appl.*, 4:341–362, 1996.

[Dah97] W. Dahmen. Wavelet and multiscale methods for operator equations. *Acta Numerica*, 55:55–228, 1997.

[DSch99a] W. Dahmen and R. Schneider. Composite wavelet bases for operator equations. *Math. Comp.*, 68:1533–1567, 1999.

[DSch99b] W. Dahmen and R. Schneider. Wavelets on manifolds I: Construction and domain decomposition. *SIAM J. Math. Anal.*, 31:184–230, 1999.

[DSt99] W. Dahmen and R.P. Stevenson. Element-by-element construction of wavelets satisfying stability and moment conditions. *SIAM J. Numer. Anal.*, 37(1):319–352, 1999.

[FQ99] M. S. Floater and E. G. Quak. Piecewise linear prewavelets on arbitrary triangulations. *Numer. Math.*, 82(2):221–252, 1999.

[FQ00] M. S. Floater and E. G. Quak. Linear independence and stability of piecewise linear prewavelets on arbitrary triangulations. *SIAM J. Numer. Anal.*, 38(1):58–79, 2000.

[HM00] D. Hong and Y.A. Mu. Construction of prewavelets with minimum support over triangulations. In *Wavelet analysis and multiresolution methods,* proceedings Urbana-Champaign, IL, 1999, Lecture Notes in Pure and Appl. Math. 212, pages 145–165, Marcel Dekker, Inc., New York, 2000.

[KO95] U. Kotyczka and P. Oswald. Piecewise linear prewavelets of small support. In C.K. Chui and L.L. Schumaker, editors, *Approximation Theory VIII.* World Scientific Publishing Co. Inc., 1995.

[Osw94] P. Oswald. *Multilevel finite element approximation: Theory and applications*. B.G. Teubner, Stuttgart, 1994.

[Sch98] R. Schneider. *Multiskalen- und Wavelet-Matrixkompression: Analysisbasierte Methoden zur Lösung großer vollbesetzter Gleigungssysteme.* Habilitationsschrift, 1995. Advances in Numerical Mathematics. Teubner, Stuttgart, 1998.

[Ste98] R.P. Stevenson. Stable three-point wavelet bases on general meshes. *Numer. Math.*, 80:131–158, 1998.

[Ste00] R.P. Stevenson. Locally supported, piecewise polynomial biorthogonal wavelets on non-uniform meshes. Technical Report 1157, University of Utrecht, September 2000. Submitted.

Department of Mathematics, Utrecht University, P.O. Box 80.010, NL-3508 TA Utrecht, The Netherlands.

E-mail address: nguyen@math.uu.nl, stevenso@math.uu.nl