

Finite Elements and Fast Iterative  
Solvers: with Applications in  
Incompressible Fluid Dynamics

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