# FINITE ELEMENTS FOR DIV AND DIVDIV CONFORMING SYMMETRIC TENSORS IN ARBITRARY DIMENSION* 

LONG CHEN ${ }^{\dagger}$ AND XUEHAI HUANG ${ }^{\ddagger}$


#### Abstract

Several div-conforming and divdiv-conforming finite elements for symmetric tensors on simplexes in arbitrary dimension are constructed in this work. The shape function space is first split as the trace space and the bubble space. The later is further decomposed into the null space of the differential operator and its orthogonal complement. Instead of characterization of these subspaces of the shape function space, characterization of the dual spaces are provided. Vector divconforming finite elements are firstly constructed as an introductory example. Then new symmetric div-conforming finite elements are constructed. The dual subspaces are then used as build blocks to construct divdiv conforming finite elements.


Key words. symmetric tensor, div-conforming finite elements, divdiv-conforming finite elements, space decomposition, dual approach

AMS subject classifications. 65N30, 74S05

1. Introduction. In this paper we construct div-conforming finite elements and divdiv-conforming finite elements for symmetric tensors on simplexes in arbitrary dimension. A finite element on a geometric domain $K$ is defined as a triple ( $K, V, \mathrm{DoF}$ ) by Ciarlet in [17], where $V$ is the finite-dimensional space of shape functions and the set of degrees of freedom (DoFs) is a basis of the dual space $V^{\prime}$ of $V$. The shape functions are usually polynomials. The key is to identify an appropriate basis of $V^{\prime}$ to enforce the continuity of the functions across the boundary of the elements so that the global finite element space is a subspace of some Sobolev space $H(\mathrm{~d}, \Omega)$, where $\Omega \subset \mathbb{R}^{d}$ is a domain and d is a generic differential operator.


FIG. 1.1. Decomposition of a generic finite element space.
Denote by $\operatorname{tr}^{\mathrm{d}}$ the trace operator associated to d and the bubble function space $\mathbb{B}(\mathrm{d}):=\operatorname{ker}\left(\operatorname{tr}^{\mathrm{d}}\right) \cap V$. We shall decompose $V=\mathbb{B}(\mathrm{d}) \oplus \mathcal{E}\left(\operatorname{img}\left(\operatorname{tr}^{\mathrm{d}}\right)\right)$, where $\mathcal{E}$ is

[^0]an injective extension operator $\operatorname{img}\left(\operatorname{tr}^{\mathrm{d}}\right) \rightarrow V$, and find degrees of freedom of each subspace by:

1. characterization of $\left(\operatorname{img}\left(\operatorname{tr}^{\mathrm{d}}\right)\right)^{\prime}$ using the Green's formula;
2. characterization of $\mathbb{B}^{\prime}(\mathrm{d})$ through the polynomial complexes.

In the characterization of $\mathbb{B}^{\prime}(d)$, we will use the differential operator $d$ to further split $\mathbb{B}(\mathrm{d})$ into two subspaces $E_{0}:=\mathbb{B}(\mathrm{d}) \cap \operatorname{ker}(\mathrm{d})$ and $E_{0}^{\perp}:=\mathbb{B}(\mathrm{d}) / E_{0}$ (see Figure 1.1).

- A basis of $\left(E_{0}^{\perp}\right)^{\prime}$ is given by $\left\{(\mathrm{d} \cdot, p), p \in \mathrm{~d} \mathbb{B}(\mathrm{~d})=\mathrm{d} E_{0}^{\perp}\right\}$;
- On the other part $E_{0}^{\prime}$, we have two approaches:
- the primary approach: $E_{0}$ is the image of the previous bubble space,
- the dual approach: $E_{0}^{\prime}$ is the null space of a Koszul operator.

The dual approach is simpler and more general. For example, for the elasticity complex, the previous symmetric tensor space is related to the second order differential operator inc [3]. While in the dual approach, we prove that a basis of $E_{0}^{\prime}$ is given by $\mathcal{N}\left(\operatorname{ker}(\cdot \boldsymbol{x}) \cap \mathbb{P}_{k-2}(K ; \mathbb{S})\right)$. Here to simplify notation, we introduce operator $\mathcal{N}: U \rightarrow V^{\prime}$ as $\mathcal{N}(p):=(\cdot, p)$ with $U \subseteq V$ and $(\cdot, \cdot)$ is the inner product of space $V$ which is usually $L^{2}$-inner product. Generalization of inc and its bubble function space to $\mathbb{R}^{d}$ is unclear while $E_{0}^{\prime}=\mathcal{N}\left(\operatorname{ker}(\cdot \boldsymbol{x}) \cap \mathbb{P}_{k-2}(K ; \mathbb{S})\right)$ holds in arbitrary dimension. Such decomposition $V^{\prime} \cong E_{0}^{\prime} \oplus\left(E_{0}^{\perp}\right)^{\prime} \oplus\left(\mathrm{img}\left(\operatorname{tr}^{\mathrm{d}}\right)\right)^{\prime}$ also allows us to construct a new family of $H(\operatorname{div} ; \mathbb{S})$-conforming finite elements.

To show the main idea with easy examples, we first review the construction of the Brezzi-Douglas-Marini (BDM) element [8, 7] and Raviart-Thomas (RT) element $[27,25]$ for $H($ div $)$-conforming elements. The trace space $\operatorname{tr}^{\operatorname{div}}\left(\mathbb{P}_{k}\left(K ; \mathbb{R}^{d}\right)\right)=$ $\prod_{F \in \partial K} \mathbb{P}_{k}(F)$. By the aid of the space decomposition $\mathbb{P}_{k-1}\left(K ; \mathbb{R}^{d}\right)=\operatorname{grad} \mathbb{P}_{k}(K) \oplus$ $\operatorname{ker}(\cdot \boldsymbol{x}) \cap \mathbb{P}_{k-1}\left(K ; \mathbb{R}^{d}\right)$ derived from the dual complex, we can show $E_{0}^{\prime}=\mathcal{N}(\operatorname{ker}(\cdot \boldsymbol{x}) \cap$ $\left.\mathbb{P}_{k-1}\left(K ; \mathbb{R}^{d}\right)\right)$. For BDM element, the shape function space is $\mathbb{P}_{k}\left(K ; \mathbb{R}^{d}\right)$. The shape function space for RT element is $\mathbb{P}_{k+1}^{-}\left(K ; \mathbb{R}^{d}\right):=\mathbb{P}_{k}\left(K ; \mathbb{R}^{d}\right) \oplus \mathbb{H}_{k}(K) \boldsymbol{x}$. BDM and RT elements will share the same trace space and $E_{0}$, while

$$
\left(E_{0}^{\perp}\right)^{\prime}= \begin{cases}\mathcal{N}\left(\operatorname{grad} \mathbb{P}_{k-1}(K)\right) & \text { for BDM element } \\ \mathcal{N}\left(\operatorname{grad} \mathbb{P}_{k}(K)\right) & \text { for RT element }\end{cases}
$$

The dual space $\mathbb{B}_{k}^{\prime}(\operatorname{div}, K) \cong\left(E_{0}^{\perp}\right)^{\prime} \oplus E_{0}^{\prime}$ for BDM element can be further merged as

$$
\mathbb{B}_{k}^{\prime}(\operatorname{div}, K)=\mathcal{N}\left(\mathrm{ND}_{k-2}(K)\right):=\mathcal{N}\left(\mathbb{P}_{k-2}\left(K ; \mathbb{R}^{d}\right) \oplus \mathbb{H}_{k-2}(K ; \mathbb{K}) \boldsymbol{x}\right)
$$

We summarize DoFs for BDM element as follows

$$
\begin{align*}
(\boldsymbol{v} \cdot \boldsymbol{n}, q)_{F} & \forall q \in \mathbb{P}_{k}(F) \text { for each } F \in \partial K,  \tag{1.1}\\
(\boldsymbol{v}, \boldsymbol{q})_{K} & \forall \boldsymbol{q} \in \mathrm{ND}_{k-2}(K), \tag{1.2}
\end{align*}
$$

and the interior moments (1.2) can be further split as

$$
\begin{align*}
(\boldsymbol{v}, \boldsymbol{q})_{K} & \forall \boldsymbol{q} \in \operatorname{grad} \mathbb{P}_{k-1}(K)  \tag{1.3}\\
(\boldsymbol{v}, \boldsymbol{q})_{K} & \forall \boldsymbol{q} \in \operatorname{ker}(\cdot \boldsymbol{x}) \cap \mathbb{P}_{k-1}\left(K ; \mathbb{R}^{d}\right) \tag{1.4}
\end{align*}
$$

Enriching (1.3) to $\mathcal{N}\left(\operatorname{grad} \mathbb{P}_{k}(K)\right)$, we then get RT elements.
We then apply our approach for a more challenging problem: div-conforming finite elements for symmetric tensors, which is used in the mixed finite element methods for the stress-displacement formulation of the elasticity system. Several div-conforming finite elements for symmetric tensors were designed in $[6,1,3,23,20,22]$ on simplices,
but our elements are new and construction is more systematical. Let $\Pi_{F} \boldsymbol{\tau}$ be the projection of column vectors of $\boldsymbol{\tau}$ to the plane $F$. The space of shape functions is $\mathbb{P}_{k}(K ; \mathbb{S})$, and DoFs are

$$
\begin{align*}
\boldsymbol{\tau}(\delta) & \forall \delta \in \mathcal{V}(K)  \tag{1.5}\\
\left(\boldsymbol{n}_{i}^{\top} \boldsymbol{\tau} \boldsymbol{n}_{j}, q\right)_{F} & \forall q \in \mathbb{P}_{k+r-d-1}(F), F \in \mathcal{F}^{r}(K),  \tag{1.6}\\
& i, j=1, \cdots, r, \text { and } r=1, \cdots, d-1, \\
\left(\Pi_{F} \boldsymbol{\tau} \boldsymbol{n}, \boldsymbol{q}\right)_{F} & \forall \boldsymbol{q} \in \mathrm{ND}_{k-2}(F), F \in \mathcal{F}^{1}(K),  \tag{1.7}\\
(\boldsymbol{\tau}, \boldsymbol{q})_{K} & \forall \boldsymbol{q} \in \mathbb{P}_{k-2}(K ; \mathbb{S}) . \tag{1.8}
\end{align*}
$$

The symmetry of the shape function and the trace $\boldsymbol{\tau} \boldsymbol{n}$ on $(d-1)$-dimensional faces leads to the degrees of freedom (1.5)-(1.6), which will determine the normal-normal component $\boldsymbol{n}^{\top} \boldsymbol{\tau} \boldsymbol{n}$. The set of degrees of freedom (1.7) is for the face bubble part of the tangential-normal component $\Pi_{F} \boldsymbol{\tau} \boldsymbol{n}$, cf. (1.2), which differs from that of Hu's element in [20] for $d \geq 3$. The bubble function space $\mathbb{B}_{k}(\operatorname{div}, K ; \mathbb{S})$ can be decomposed into two parts $E_{0, k}(\mathbb{S}):=\mathbb{B}_{k}(\operatorname{div}, K ; \mathbb{S}) \cap \operatorname{ker}(\operatorname{div})$ and $E_{0, k}^{\perp}(\mathbb{S}):=\mathbb{B}_{k}(\operatorname{div}, K ; \mathbb{S}) / E_{0, k}(\mathbb{S})$. We show that

$$
\begin{equation*}
\left(E_{0, k}^{\perp}(\mathbb{S})\right)^{\prime}=\mathcal{N}\left(\operatorname{def} \mathbb{P}_{k-1}\left(K, \mathbb{R}^{d}\right)\right), \quad E_{0, k}^{\prime}(\mathbb{S})=\mathcal{N}\left(\operatorname{ker}(\cdot \boldsymbol{x}) \cap \mathbb{P}_{k-2}(K ; \mathbb{S})\right) \tag{1.9}
\end{equation*}
$$

A new family of $H(\operatorname{div} ; \mathbb{S})$-conforming elements is devised with the shape function space $\mathbb{P}_{k+1}^{-}(K ; \mathbb{S}):=\mathbb{P}_{k}(K ; \mathbb{S})+E_{0, k+1}^{\perp}(\mathbb{S})$, and enrich $\operatorname{DoF}\left(E_{0, k}^{\perp}(\mathbb{S})\right)^{\prime}$ to $\left(E_{0, k+1}^{\perp}(\mathbb{S})\right)^{\prime}$ so that $\operatorname{div} \mathbb{P}_{k+1}^{-}(K ; \mathbb{S})=\mathbb{P}_{k}\left(K ; \mathbb{R}^{d}\right)$.

Motivated by the recent construction [21] in two and three dimensions, the previous div-conforming finite elements for symmetric tensors are then revised to acquire $H$ (div div) $\cap H$ (div)-conforming finite elements for symmetric tensors in arbitrary dimension. Using the building blocks in the BDM element and div-conforming $\mathbb{P}_{k}(K ; \mathbb{S})$ element, we construct the following DoFs

$$
\begin{align*}
\boldsymbol{\tau}(\delta) & \forall \delta \in \mathcal{V}(K),  \tag{1.10}\\
\left(\boldsymbol{n}_{i}^{\top} \boldsymbol{\tau} \boldsymbol{n}_{j}, q\right)_{F} & \forall q \in \mathbb{P}_{k+r-d-1}(F), F \in \mathcal{F}^{r}(K),  \tag{1.11}\\
& i, j=1, \cdots, r, \text { and } r=1, \cdots, d-1, \\
\left(\Pi_{F} \boldsymbol{\tau} \boldsymbol{n}, \boldsymbol{q}\right)_{F} & \forall \boldsymbol{q} \in \mathrm{ND}_{k-2}(F), F \in \mathcal{F}^{1}(K),  \tag{1.12}\\
\left(\boldsymbol{n}^{\top} \operatorname{div} \boldsymbol{\tau}, p\right)_{F} & \forall p \in \mathbb{P}_{k-1}(F), F \in \mathcal{F}^{1}(K),  \tag{1.13}\\
(\boldsymbol{\tau}, \operatorname{def} \boldsymbol{q})_{K} & \forall \boldsymbol{q} \in \mathrm{ND}_{k-3}(K),  \tag{1.14}\\
(\boldsymbol{\tau}, \boldsymbol{q})_{K} & \forall \boldsymbol{q} \in \operatorname{ker}(\cdot \boldsymbol{x}) \cap \mathbb{P}_{k-2}(K ; \mathbb{S}) . \tag{1.15}
\end{align*}
$$

The degree of freedom (1.13) is to enforce $\operatorname{div} \boldsymbol{\tau}$ is $H$ (div)-conforming and thus $\boldsymbol{\tau} \in$ $H($ div div $) \cap H($ div $) . \operatorname{DoF}(1.15)$ is $E_{0, k}^{\prime}(\mathbb{S})$ shown in (1.9) and (1.13)-(1.14) is a further decomposition of $\operatorname{div} E_{0, k}^{\perp}(\mathbb{S})$ by the trace-bubble decomposition cf. (1.1)-(1.2).

We then modify this element slightly to get $H$ (div div)-conforming symmetric finite elements generalizing the divdiv-conforming element in two and three dimen-
sions $[13,11]$. The degrees of freedom are given by

$$
\begin{align*}
\boldsymbol{\tau}(\delta) & \forall \delta \in \mathcal{V}(K),  \tag{1.16}\\
\left(\boldsymbol{n}_{i}^{\top} \boldsymbol{\tau} \boldsymbol{n}_{j}, q\right)_{F} & \forall q \in \mathbb{P}_{k+r-d-1}(F), F \in \mathcal{F}^{r}(K),  \tag{1.17}\\
& i, j=1, \cdots, r, \text { and } r=1, \cdots, d-1, \\
\left(\Pi_{F} \boldsymbol{\tau} \boldsymbol{n}, \boldsymbol{q}\right)_{F} & \forall \boldsymbol{q} \in \mathrm{ND}_{k-2}(F), F \in \mathcal{F}^{1}(K),  \tag{1.18}\\
\left(\boldsymbol{n}^{\top} \operatorname{div} \boldsymbol{\tau}+\operatorname{div}_{F}(\boldsymbol{\tau} \boldsymbol{n}), p\right)_{F} & \forall p \in \mathbb{P}_{k-1}(F), F \in \mathcal{F}^{1}(K),  \tag{1.19}\\
(\boldsymbol{\tau}, \operatorname{def} \boldsymbol{q})_{K} & \forall \boldsymbol{q} \in \mathrm{ND}_{k-3}(K),  \tag{1.20}\\
(\boldsymbol{\tau}, \boldsymbol{q})_{K} & \forall \boldsymbol{q} \in \operatorname{ker}(\cdot \boldsymbol{x}) \cap \mathbb{P}_{k-2}(K ; \mathbb{S}) . \tag{1.21}
\end{align*}
$$

As we mentioned before, (1.16)-(1.18) will determine the trace $\boldsymbol{\tau} \boldsymbol{n}$, and consequently $\operatorname{div}_{F}(\boldsymbol{\tau} \boldsymbol{n})$. The only difference is that (1.13) is replaced by (1.19), which agrees with a trace operator of div div-operator. Such modification is from the requirement of $H$ (div div)-conformity: $\boldsymbol{n}^{\boldsymbol{\top}} \boldsymbol{\tau} \boldsymbol{n}$ and $\boldsymbol{n}^{\boldsymbol{\top}} \operatorname{div} \boldsymbol{\tau}+\operatorname{div}_{F}(\boldsymbol{\tau} \boldsymbol{n})$ are continuous. Therefore (1.18) for $\Pi_{F} \boldsymbol{\tau} \boldsymbol{n}$ is considered as interior to $K$, i.e., it is not single-valued across simplices.

In our recent work [11, 13], we have constructed $H$ (div div)-conforming symmetric finite elements for $d=2,3$. The dual space $\left(\operatorname{tr}^{\operatorname{div} \operatorname{div}}\left(\mathbb{P}_{k}(K ; \mathbb{S})\right)\right)^{\prime}$ is given by DoFs (1.16)-(1.19) but without (1.18) as $\Pi_{F} \boldsymbol{\tau} \boldsymbol{n}$ is not part of the trace of div div operator. Let $E_{0}(\operatorname{div} \operatorname{div}, \mathbb{S}):=\mathbb{B}_{k}(\operatorname{div} \operatorname{div}, K ; \mathbb{S}) \cap \operatorname{ker}(\operatorname{div} \operatorname{div})$ and $E_{0}^{\perp}(\operatorname{div} \operatorname{div}, \mathbb{S}):=$ $\mathbb{B}_{k}(\operatorname{div} \operatorname{div}, K ; \mathbb{S}) / E_{0}(\operatorname{div} \operatorname{div}, \mathbb{S})$. Then $\left(E_{0}^{\perp}(\operatorname{div} \operatorname{div}, \mathbb{S})\right)^{\prime}=\mathcal{N}\left(\nabla^{2} \mathbb{P}_{k-2}(K)\right)$ but the identification of $E_{0}^{\prime}(\operatorname{div} \operatorname{div}, \mathbb{S})$ is very tricky in three dimensions, We use the primary approach to get $E_{0}^{\prime}(\operatorname{div} \operatorname{div}, \mathbb{S})=\mathcal{N}\left(\operatorname{sym} \operatorname{curl} \mathbb{B}_{k+1}(\operatorname{sym} \operatorname{curl}, K ; \mathbb{T})\right)$. Characterization of $\mathbb{B}_{k+1}(\operatorname{sym}$ curl, $K ; \mathbb{T})$ is hard to generalize to arbitrary dimension. When using the dual approach, it turns out $\mathcal{N}\left(\operatorname{ker}\left(\boldsymbol{x}^{\boldsymbol{\top}} \cdot \boldsymbol{x}\right) \cap \mathbb{P}_{k-1}(K ; \mathbb{S})\right)$ is a strict subspace of $E_{0}^{\prime}(\operatorname{div} \operatorname{div}, \mathbb{S})$ as the dimension cannot match. An extra DoF on one face $(\boldsymbol{\tau} \boldsymbol{n}, \boldsymbol{n} \times \boldsymbol{x} q)_{F_{1}}, q \in \mathbb{P}_{k-2}\left(F_{1}\right)$ is introduced to fill the gap. Such fix in three dimensions seems not easy to be generalized to arbitrary dimension.

In (1.20), if we further decompose $\mathrm{ND}_{k-3}(K)=\operatorname{grad} \mathbb{P}_{k-2}(K) \oplus \mathbb{P}_{k-3}(K ; \mathbb{K}) \boldsymbol{x}$, based on our new element, we can obtain another characterization of $E_{0}^{\prime}(\operatorname{div} \operatorname{div}, \mathbb{S})$ as the sum of $\operatorname{DoFs}(1.18),(1.21)$, and $\mathcal{N}\left(\operatorname{def} \mathbb{P}_{k-3}(K ; \mathbb{K}) \boldsymbol{x}\right)$.

Furthermore, a new family of $\mathbb{P}_{k+1}^{-}(K ; \mathbb{S})$ type $H$ (div div) $\cap H$ (div)-conforming and $H$ (div div)-conforming finite elements are developed. The shape function space is enriched to $\mathbb{P}_{k+1}^{-}(K ; \mathbb{S}):=\mathbb{P}_{k}(K ; \mathbb{S}) \oplus \boldsymbol{x} \boldsymbol{x}^{\top} \mathbb{H}_{k-1}(K)$. The range div $\operatorname{div} \mathbb{P}_{k+1}^{-}(K ; \mathbb{S})$ is enriched to $\mathbb{P}_{k-1}(K)$ and so is $\left(E_{0}^{\perp}(\operatorname{div} \operatorname{div}, \mathbb{S})\right)^{\prime}=\mathcal{N}\left(\nabla^{2} \mathbb{P}_{k-1}(K)\right)$. But the trace DoFs and $E_{0}^{\prime}(\operatorname{div} \operatorname{div}, \mathbb{S})$ are unchanged. Such $\mathbb{P}_{k+1}^{-}(K ; \mathbb{S})$ type div div-conforming elements for symmetric tensors are new and not easy to construct without exploring the decomposition of the dual spaces.

The rest of this paper is organized as follows. Preliminaries are given in Section 2. We review the construction of div-conforming elements in Section 3. In Section 4, new div-conforming elements for symmetric tensors are designed. And we consider the construction of divdiv-conforming elements in Section 5.

## 2. Preliminary.

2.1. Notation. Let $K \subset \mathbb{R}^{d}$ be a simplex. For $r=1,2, \cdots, d$, denote by $\mathcal{F}^{r}(K)$ the set of all $(d-r)$-dimensional faces of $K$. The superscript $r$ in $\mathcal{F}^{r}(K)$ represents the co-dimension of a $(d-r)$-dimensional face $F$. Set $\mathcal{V}(K):=\mathcal{F}^{d}(K)$ as the set of vertices. Similarly, for $F \in \mathcal{F}^{r}(K)$, define

$$
\mathcal{F}^{1}(F):=\left\{e \in \mathcal{F}^{r+1}(K): e \subset \partial F\right\}
$$

For any $F \in \mathcal{F}^{r}(K)$ with $1 \leq r \leq d-1$, let $\boldsymbol{n}_{F, 1}, \cdots, \boldsymbol{n}_{F, r}$ be its mutually perpendicular unit normal vectors, and $\boldsymbol{t}_{F, 1}, \cdots, \boldsymbol{t}_{F, d-r}$ be its mutually perpendicular unit tangential vectors. We abbreviate $\boldsymbol{n}_{F, 1}$ as $\boldsymbol{n}_{F}$ or $\boldsymbol{n}$ when $r=1$. We also abbreviate $\boldsymbol{n}_{F, i}$ and $\boldsymbol{t}_{F, i}$ as $\boldsymbol{n}_{i}$ and $\boldsymbol{t}_{i}$ respectively if not causing any confusion. For any $F \in \mathcal{F}^{1}(K)$ and $e \in \mathcal{F}^{1}(F)$, denote by $\boldsymbol{n}_{F, e}$ the unit outward normal to $\partial F$ being parallel to $F$.

Given a face $F \in \mathcal{F}^{1}(K)$, and a vector $\boldsymbol{v} \in \mathbb{R}^{d}$, define

$$
\Pi_{F} \boldsymbol{v}=\left(\boldsymbol{n}_{F} \times \boldsymbol{v}\right) \times \boldsymbol{n}_{F}=\left(\boldsymbol{I}-\boldsymbol{n}_{F} \boldsymbol{n}_{F}^{\top}\right) \boldsymbol{v}
$$

as the projection of $\boldsymbol{v}$ onto the face $F$. For a matrix $\boldsymbol{\tau} \in \mathbb{R}^{d \times d}, \Pi_{F} \boldsymbol{\tau}$ is applied to each column vector of $\boldsymbol{\tau}$. Given a scalar function $v$, define the surface gradient on $F \in \mathcal{F}^{r}(K)$ as

$$
\nabla_{F} v:=\Pi_{F} \nabla v=\nabla v-\sum_{i=1}^{r} \frac{\partial v}{\partial n_{F, i}} \boldsymbol{n}_{F, i}=\sum_{i=1}^{d-r} \frac{\partial v}{\partial t_{F, i}} \boldsymbol{t}_{F, i}
$$

namely the projection of $\nabla v$ to the face $F$, which is independent of the choice of the normal vectors. Denote by $\operatorname{div}_{F}$ the corresponding surface divergence.
2.2. Polynomial spaces. We recall some results about polynomial spaces on a bounded and topologically trivial domain $D \subset \mathbb{R}^{d}$. Without loss of generality, we assume $\mathbf{0} \in D$. Given a non-negative integer $k$, let $\mathbb{P}_{k}(D)$ stand for the set of all polynomials in $D$ with the total degree no more than $k$, and $\mathbb{P}_{k}(D ; \mathbb{X})$ denote the tensor or vector version. Let $\mathbb{H}_{k}(D):=\mathbb{P}_{k}(D) \backslash \mathbb{P}_{k-1}(D)$ be the space of homogeneous polynomials of degree $k$. Recall that

$$
\operatorname{dim} \mathbb{P}_{k}(D)=\binom{k+d}{d}=\binom{k+d}{k}, \quad \operatorname{dim} \mathbb{H}_{k}(D)=\binom{k+d-1}{d-1}=\binom{k+d-1}{k}
$$

for a $d$-dimensional domain $D$.
By Euler's formula, we have

$$
\begin{align*}
\boldsymbol{x} \cdot \nabla q & =k q \quad \forall q \in \mathbb{H}_{k}(D)  \tag{2.1}\\
\operatorname{div}(\boldsymbol{x} q) & =(k+d) q \quad \forall q \in \mathbb{H}_{k}(D) \tag{2.2}
\end{align*}
$$

for integer $k \geq 0$.
2.3. Dual spaces. Consider a Hilbert space $V$ with the inner product $(\cdot, \cdot)$. Let $U \subseteq V$, then define $\mathcal{N}: U \rightarrow V^{\prime}$ as follows: for any $p \in U, \mathcal{N}(p) \in V^{\prime}$ is given by

$$
\langle\mathcal{N}(p), \cdot\rangle=(\cdot, p)
$$

When $V$ is a subspace of an ambient Hilbert space $W$, we use the inclusion $\mathcal{I}$ : $V \hookrightarrow W$ to denote the embedding of $V$ into $W$. Then the dual operator $\mathcal{I}^{\prime}: W^{\prime} \rightarrow V^{\prime}$ is onto. That is for any $N \in W^{\prime}, \mathcal{I}^{\prime} N \in V^{\prime}$ is defined as $\left\langle\mathcal{I}^{\prime} N, v\right\rangle=\langle N, \mathcal{I} v\rangle$.

Consider the case the finite-dimensional sub-space $V \subseteq W$ and a subspace $P^{\prime} \subseteq$ $W^{\prime}$, then to prove $V^{\prime}=\mathcal{I}^{\prime} P^{\prime}$, it suffices to show

$$
\begin{equation*}
\text { for any } v \in V, \text { if } N(v)=0, \text { for all } N \in P^{\prime}, \text { then } v=0 \tag{2.3}
\end{equation*}
$$

Note that it means $\mathcal{I}^{\prime}$ is onto but may not be into. That is $\operatorname{dim} P^{\prime}$ might be larger than $\operatorname{dim} V^{\prime}$. A DoF is in general a functional with a domain larger than $V$. It is
less rigorous to write $V^{\prime} \subseteq P^{\prime}$ as those two subspaces consists of functionals with different domains. The mapping $\mathcal{I}^{\prime}$ is introduced as a bridge for comparison. When $\mathcal{I}^{\prime}: P^{\prime} \rightarrow V^{\prime}$ is a bijection, we shall skip $\mathcal{I}^{\prime}$ and simply write as $V^{\prime}=P^{\prime}$. To prove $V^{\prime}=P^{\prime}$, besides (2.3), dimension count is applied to verify $\operatorname{dim} V^{\prime}=\operatorname{dim} P^{\prime}$.

The art of designing conforming finite element spaces is indeed identifying appropriate DoFs to enforce the continuity of the function across the boundary of the elements. Take $V=\mathbb{P}_{k}(K)$ as an example. A naive choice is $\mathcal{N}\left(\mathbb{P}_{k}(K)\right)=V^{\prime}$ but such basis enforces no continuity on $\partial K$. To be $H^{1}$-conforming we need a basis for $\left(\operatorname{tr}\left(\mathbb{P}_{k}(K)\right)\right)^{\prime}$ to ensure the continuity of the trace on lower-dimensional faces of an element $K$. Note that as the shape function is a polynomial inside the element, the trace is usually smoother than its Sobolev version, which is known as super-smoothness [16, 24, 28]. Choice of dual bases is not unique. For example, for $H^{1}$-conforming finite elements, $V=\mathbb{P}_{k}(K)$, Lagrange element and Hermite element will have different bases for $V^{\prime}$.

When counting the dimensions, we often use the following simple fact: for a linear operator $T$ defined on a finite dimensional linear space $V$, it holds

$$
\operatorname{dim} V=\operatorname{dim} \operatorname{ker}(T)+\operatorname{dim} \operatorname{img}(T)
$$

2.4. Simplex and barycentric coordinates. For $i=1, \cdots, d$, denote by $\boldsymbol{e}_{i} \in$ $\mathbb{R}^{d}$ the $d$-dimensional vector whose $j$ th component is $\delta_{i j}$ for $j=1, \cdots, d$. Let $K \subset \mathbb{R}^{d}$ be a non-degenerated simplex with vertices $\boldsymbol{x}_{0}, \boldsymbol{x}_{1}, \cdots, \boldsymbol{x}_{d}$. Let $F_{i} \in \mathcal{F}^{1}(K)$ be the $(d-1)$-dimensional face opposite to vertex $\boldsymbol{x}_{i}$, and $\lambda_{i}$ be the barycentric coordinate of $\boldsymbol{x}$ corresponding to vertex $\boldsymbol{x}_{i}$, for $i=0,1, \cdots, d$. Then $\lambda_{i}(\boldsymbol{x})$ is a linear polynomial and $\left.\lambda_{i}\right|_{F_{i}}=0$. For any sub-simplex $S$ not containing $\boldsymbol{x}_{i}$ (and thus $S \subseteq F_{i}$ ), $\left.\lambda_{i}\right|_{S}=0$. On the other hand, for a polynomial $p \in \mathbb{P}_{k}(K)$, if $\left.p\right|_{F_{i}}=0$, then $p=\lambda_{i} q$ for some $q \in \mathbb{P}_{k-1}(K)$. As $F_{i}$ is contained in the zero level set of $\lambda_{i}, \nabla \lambda_{i}$ is orthogonal to $F_{i}$ and a simple scaling calculation shows the relation $\nabla \lambda_{i}=-\left|\nabla \lambda_{i}\right| \boldsymbol{n}_{i}$, where $\boldsymbol{n}_{i}$ is the unit outward normal to the face $F_{i}$ of the simplex $K$ for $i=0,1, \cdots, d$. Clearly $\left\{\boldsymbol{n}_{1}, \boldsymbol{n}_{2}, \cdots, \boldsymbol{n}_{d}\right\}$ spans $\mathbb{R}^{d}$. We will identify its dual basis $\left\{\boldsymbol{l}_{1}, \boldsymbol{l}_{2}, \cdots, \boldsymbol{l}_{d}\right\}$, i.e. $\left(\boldsymbol{l}_{i}, \boldsymbol{n}_{j}\right)=\delta_{i j}$ for $i, j=1,2, \cdots, d$. Here the index 0 is single out for the ease of notation. We can set an arbitrary vertex as the origin.

Set $\boldsymbol{t}_{i, j}:=\boldsymbol{x}_{j}-\boldsymbol{x}_{i}$ for $0 \leq i \neq j \leq d$. By computing the constant directional derivative $\boldsymbol{t}_{i, j} \cdot \nabla \lambda_{\ell}$ by values on the two vertices, we have

$$
\boldsymbol{t}_{i, j} \cdot \nabla \lambda_{\ell}=\delta_{j \ell}-\delta_{i \ell}= \begin{cases}1, & \text { if } \ell=j  \tag{2.4}\\ -1, & \text { if } \ell=i \\ 0, & \text { if } \ell \neq i, j\end{cases}
$$

Then it is straightforward to verify $\left\{\boldsymbol{l}_{i}:=\left|\nabla \lambda_{i}\right| \boldsymbol{t}_{i, 0}\right\}$ is dual to $\left\{\boldsymbol{n}_{i}\right\}$. Note that in general neither $\left\{\boldsymbol{n}_{i}\right\}$ nor $\left\{\boldsymbol{l}_{i}\right\}$ is an orthonormal basis unless $K$ is a scaling of the reference simplex $\hat{K}$ with vertices $\mathbf{0}, \boldsymbol{e}_{1}, \cdots, \boldsymbol{e}_{d}$. By using the basis $\left\{\boldsymbol{n}_{i}, i=\right.$ $1,2, \cdots, d\}$, we avoid the pull back from the reference simplex.

Following notation in [5], denote by $\mathbb{N}^{d}$ the set of all multi-indices $\alpha=\left(\alpha_{1}, \cdots, \alpha_{d}\right)$ with integer $\alpha_{i} \geq 0$, and $\mathbb{N}_{0}^{d}$ the set of all multi-indices $\alpha=\left(\alpha_{0}, \alpha_{1}, \cdots, \alpha_{d}\right)$ with integer $\alpha_{i} \geq 0$. For $\boldsymbol{x}=\left(x_{1}, \cdots, x_{d}\right)$ and $\alpha \in \mathbb{N}^{d}$, define $\boldsymbol{x}^{\alpha}:=x_{1}^{\alpha_{1}} \cdots x_{d}^{\alpha_{d}}$ and $|\alpha|:=$ $\sum_{i=1}^{d} \alpha_{i}$. Similarly, for $\boldsymbol{\lambda}=\left(\lambda_{0}, \lambda_{1}, \cdots, \lambda_{d}\right)$ and $\alpha \in \mathbb{N}_{0}^{d}$, define $\boldsymbol{\lambda}^{\alpha}:=\lambda_{0}^{\alpha_{0}} \lambda_{1}^{\alpha_{1}} \cdots \lambda_{d}^{\alpha_{d}}$ and $|\alpha|:=\sum_{i=0}^{d} \alpha_{i}$. The Bernstein basis for the space $\mathbb{P}_{k}(K)$ consists of all monomials
of degree $k$ in the variables $\lambda_{i}$, i.e., the basis functions are given by

$$
\left\{\boldsymbol{\lambda}^{\alpha}:=\lambda_{0}^{\alpha_{0}} \lambda_{1}^{\alpha_{1}} \cdots \lambda_{d}^{\alpha_{d}}: \alpha \in \mathbb{N}_{0}^{d},|\alpha|=k\right\}
$$

Then $\mathbb{P}_{k}(K)=\left\{\sum_{\alpha \in \mathbb{N}_{0}^{d},|\alpha|=k} c_{\alpha} \boldsymbol{\lambda}^{\alpha}: c_{\alpha} \in \mathbb{R}\right\}$.
2.5. Tensors. Denote by $\mathbb{S}$ and $\mathbb{K}$ the subspace of symmetric matrices and skewsymmetric matrices of $\mathbb{R}^{d \times d}$, respectively. The set of symmetric tensors $\left\{\boldsymbol{T}_{i, j}:=\right.$ $\left.\boldsymbol{t}_{i, j} \boldsymbol{t}_{i, j}^{\top}\right\}_{0 \leq i<j \leq d}$ is dual to $\left\{\boldsymbol{N}_{i, j}\right\}_{0 \leq i<j \leq d}$, where

$$
\boldsymbol{N}_{i, j}:=\frac{1}{2\left(\boldsymbol{n}_{i}^{\top} \boldsymbol{t}_{i, j}\right)\left(\boldsymbol{n}_{j}^{\top} \boldsymbol{t}_{i, j}\right)}\left(\boldsymbol{n}_{i} \boldsymbol{n}_{j}^{\top}+\boldsymbol{n}_{j} \boldsymbol{n}_{i}^{\top}\right)
$$

That is, by direct calculation [9, (3.2)],

$$
\boldsymbol{T}_{i, j}: \boldsymbol{N}_{k, \ell}=\delta_{i k} \delta_{j \ell}, \quad 0 \leq i<j \leq d, 0 \leq k<\ell \leq d
$$

where : is the Frobenius inner product of matrices. Assuming $\sum_{0 \leq i<j \leq d} c_{i j} \boldsymbol{T}_{i j}=\mathbf{0}$, then apply the Frobenius inner product with $\boldsymbol{N}_{k, \ell}$ to conclude $c_{k \ell}=0$ for all $0 \leq k<$ $\ell \leq d$. Therefore both $\left\{\boldsymbol{T}_{i, j}\right\}_{0 \leq i<j \leq d}$ and $\left\{\boldsymbol{N}_{i, j}\right\}_{0 \leq i<j \leq d}$ are bases of $\mathbb{S}$. The basis $\left\{\boldsymbol{T}_{i, j}\right\}_{0 \leq i<j \leq d}$ is introduced in [15, 20] and $\left\{\boldsymbol{N}_{i, j}\right\}_{0 \leq i<j \leq d}$ is in [15, 9].
2.6. Characterization of DoFs for bubble spaces. We give a characterization of DoFs for bubble spaces by the dual approach and the decomposition of the bubble spaces through the bubble complex.

Lemma 2.1. Assume finite-dimensional Hilbert spaces $\mathbb{B}_{1}, \mathbb{B}_{2}, \ldots, \mathbb{B}_{n}$ with the inner product $(\cdot, \cdot)$ form an exact Hilbert complex

$$
0 \xrightarrow{C} \mathbb{B}_{1} \xrightarrow{\mathrm{~d}_{1}} \ldots \mathbb{B}_{i} \xrightarrow{\mathrm{~d}_{i}} \ldots \mathbb{B}_{n} \rightarrow 0
$$

where $\mathbb{B}_{i} \subseteq \operatorname{ker}\left(\operatorname{tr}^{\mathrm{d}_{i}}\right)$ for $i=1,2, \cdots, n-1$. Then the bubble space $\mathbb{B}_{i}$, for $i=$ $1, \ldots, n-1$, is uniquely determined by the DoFs

$$
\begin{align*}
\left(v, \mathrm{~d}_{i}^{*} q\right) & \forall q \in \mathrm{~d}_{i} \mathbb{B}_{i},  \tag{2.5}\\
(v, q) & \forall q \in \mathbb{Q} \cong\left(\mathrm{~d}_{i-1} \mathbb{B}_{i-1}\right)^{\prime} \tag{2.6}
\end{align*}
$$

where $\mathrm{d}_{i}^{*}$ is the adjoint of $\mathrm{d}_{i}: \mathbb{B}_{i} \rightarrow \mathbb{B}_{i+1}$ with respect to the inner product $(\cdot, \cdot)$ and the isomorphism $\mathbb{Q} \rightarrow\left(\mathrm{d}_{i-1} \mathbb{B}_{i-1}\right)^{\prime}$ is given by $p \rightarrow(p, \cdot)$ for $p \in \mathbb{Q}$.

Proof. By the splitting lemma in [19] (see also Theorem 2.2 in [10]),

$$
\begin{equation*}
\mathbb{B}_{i}=\mathrm{d}_{i}^{*} \mathrm{~d}_{i} \mathbb{B}_{i} \oplus \mathrm{~d}_{i-1} \mathbb{B}_{i-1} \tag{2.7}
\end{equation*}
$$

Since $\mathrm{d}_{i}^{*}$ restricted to $\mathrm{d}_{i} \mathbb{B}_{i}$ is injective, the number of DoFs (2.5)-(2.6) is same as $\operatorname{dim} \mathbb{B}_{i}$. Assume $v \in \mathbb{B}_{i}$ and all the DoFs (2.5)-(2.6) vanish. By the decomposition (2.7), there exist $v_{1} \in \mathbb{B}_{i}$ and $v_{2} \in \mathbb{B}_{i-1}$ such that $v=\mathrm{d}_{i}^{*} \mathrm{~d}_{i} v_{1}+\mathrm{d}_{i-1} v_{2}$. The vanishing (2.5) yields $\mathrm{d}_{i} v=0$, that is $\mathrm{d}_{i} \mathrm{~d}_{i}^{*}\left(\mathrm{~d}_{i} v_{1}\right)=0$. Noting that $\mathrm{d}_{i} \mathrm{~d}_{i}^{*}: \mathrm{d}_{i} \mathbb{B}_{i} \rightarrow \mathrm{~d}_{i} \mathbb{B}_{i}$ is isomorphic, we get $\mathrm{d}_{i} v_{1}=0$ and thus $v=\mathrm{d}_{i-1} v_{2}$. Now apply the vanishing (2.6) to get $v=0$.

When the bubble function space $\mathbb{B}$ can be characterized precisely, we can simply use $\mathcal{N}(\mathbb{B})$, i.e., $(v, q), q \in \mathbb{B}$ as DoFs. Lemma 2.1 tells us it suffices to identify the dual space without knowing the explicit form of the bubble functions. In the following, we present a way to identify $\mathbb{B}^{\prime}$ by a decomposition of the dual space.

Lemma 2.2. Consider linear map $\mathrm{d}: V \rightarrow P$ between two finite dimensional Hilbert spaces sharing the same inner product $(\cdot, \cdot)$. Let $\mathbb{B}=\operatorname{ker}\left(\operatorname{tr}^{\mathrm{d}}\right) \cap V, E_{0}=$ $\operatorname{ker}(\mathrm{d}) \cap \mathbb{B}$, and $E_{0}^{\perp}=\mathbb{B} / E_{0}$. Assume
(B1) $\mathbb{B}^{\prime}=\mathcal{I}^{\prime} \mathcal{N}(U)$ for some subspace $U \subseteq V$;
(B2) there exists an operator $\kappa$ leading to the inclusion:

$$
\begin{equation*}
U \subseteq \mathrm{~d}^{*}\left(H\left(\mathrm{~d}^{*}\right)\right) \oplus(\operatorname{ker}(\kappa) \cap U) \tag{2.8}
\end{equation*}
$$

where $\mathrm{d}^{*}$ is the adjoint of $\mathrm{d}: \mathbb{B} \rightarrow \mathrm{dB}$ with respect to the inner product $(\cdot, \cdot)$ and can be continuously extended to the space $H\left(\mathrm{~d}^{*}\right)$.
Then

$$
\begin{align*}
\left(E_{0}^{\perp}\right)^{\prime} & =\mathcal{N}\left(\mathrm{d}^{*}(\mathrm{~dB})\right)  \tag{2.9}\\
E_{0}^{\prime} & =\mathcal{I}^{\prime} \mathcal{N}(\operatorname{ker}(\kappa) \cap U) . \tag{2.10}
\end{align*}
$$

Proof. The characterization (2.9) is straightforward as $\mathrm{d}: E_{0}^{\perp} \rightarrow \mathrm{d} \mathbb{B}$ is a bijection. To prove (2.10), it suffices to show that: for any $u \in E_{0}$, if $(u, p)=0$ for all $p \in \operatorname{ker}(\kappa) \cap U$, then $u=0$. First of all, as $u \in E_{0}, u \perp \mathrm{~d}^{*}\left(H\left(\mathrm{~d}^{*}\right)\right)$, i.e.,

$$
\left(u, \mathrm{~d}^{*} p\right)=(\mathrm{d} u, p)=0 \quad \forall p \in H\left(\mathrm{~d}^{*}\right)
$$

Combined with the assumption (2.8), we have $(u, p)=0$ for all $p \in U$ and conclude $u=0$ by assumption $\mathbb{B}^{\prime}=\mathcal{I}^{\prime} \mathcal{N}(U)$.

As we mentioned before, in (2.10), $\mathcal{I}^{\prime}$ could be onto. For example, one can choose $U=V$. We want to choose the smallest subspace $U$ to get $E_{0}^{\prime}=\mathcal{N}(\operatorname{ker}(\kappa) \cap U)$. One guideline is the dimension count. On one hand, we have the following identity

$$
\operatorname{dim} E_{0}=\operatorname{dim} \mathbb{B}-\operatorname{dim} E_{0}^{\perp}=\operatorname{dim} V-\operatorname{dim}\left(\operatorname{img}\left(\operatorname{tr}^{d}\right)\right)-\operatorname{dim} E_{0}^{\perp}
$$

On the other hand, we have

$$
\operatorname{dim}(\operatorname{ker}(\kappa) \cap U)=\operatorname{dim} U-\operatorname{dim}(\kappa U)
$$

For specific examples, we only need to figure out the dimension not exact identification of subspaces.

Next we enrich space $V$ to derive another finite element.
Lemma 2.3. Consider linear map $\mathrm{d}: V \rightarrow P$ between two finite dimensional Hilbert spaces sharing the same inner product $(\cdot, \cdot)$. Let $\mathbb{B}=\operatorname{ker}\left(\operatorname{tr}^{\mathrm{d}}\right) \cap V, E_{0}=$ $\operatorname{ker}(\mathrm{d}) \cap \mathbb{B}$ and $E_{0}^{\perp}=\mathbb{B} / E_{0}$. Now we enrich the space to $V+\mathbb{H}$ and let $\mathbb{B}^{+}=$ $\operatorname{ker}\left(\operatorname{tr}^{\mathrm{d}}\right) \cap(V+\mathbb{H})$. Assume
(H1) $V \cap \mathbb{H}=\{0\}$ and $\mathrm{d} V \cap \mathrm{~d} \mathbb{H}=\{0\}$;
(H2) $\operatorname{tr}^{\mathrm{d}}(\mathbb{H}) \subseteq \operatorname{tr}^{\mathrm{d}}(V)$;
(H3) $\mathrm{d}: \mathbb{H} \rightarrow \mathrm{d} \mathbb{H}$ is bijective;
(H4) $E_{0}=\mathcal{N}(\mathbb{Q}),\left(E_{0}^{\perp}\right)^{\prime}=\mathcal{N}\left(\mathrm{d}^{*} \mathbb{P}\right)$;
(H5) $\left(\mathrm{d} \mathbb{B}^{+}\right)^{\prime}=\mathcal{I}^{\prime} \mathcal{N}(\mathbb{P} \oplus \mathrm{d} \mathbb{H})$.
Then

$$
\begin{equation*}
\left(\mathbb{B}^{+}\right)^{\prime}=\mathcal{N}(\mathbb{Q})+\mathcal{N}\left(\mathrm{d}^{*}(\mathbb{P}+\mathrm{d} \mathbb{H})\right), \tag{2.11}
\end{equation*}
$$

i.e., a function $v \in \mathbb{B}^{+}$is uniquely determined by DoFs

$$
\begin{align*}
\left(v, \mathrm{~d}^{*} q\right) & \forall q \in \mathbb{P}  \tag{2.12}\\
\left(v, \mathrm{~d}^{*} q\right) & \forall q \in \mathrm{~d} \mathbb{H}  \tag{2.13}\\
(v, q) & \forall q \in \mathbb{Q} \tag{2.14}
\end{align*}
$$

Proof. As $\operatorname{tr}^{\mathrm{d}}(\mathbb{H}) \subseteq \operatorname{tr}^{\mathrm{d}}(V)$, $\operatorname{dim} \mathbb{B}^{+}-\operatorname{dim} \mathbb{B}=\operatorname{dim} \mathbb{H}$. On the other hand, since $d^{*} d: \mathbb{H} \rightarrow d^{*} d \mathbb{H}$ is bijective, the number of DoFs increased is also $\operatorname{dim} \mathbb{H}$. Thus the dimensions in (2.11) are equal. Take a $v \in \mathbb{B}^{+}$and assume all the DoFs (2.12)(2.14) vanish. Thanks to the vanishing DoFs (2.12) and (2.13), we get from (H5) that $\mathrm{d} v=0$, which together with $\mathrm{d} V \cap \mathrm{~d} \mathbb{H}=\{0\}$ implies $v \in V$. Finally $v=0$ follows from the vanishing DoFs (2.12) and (2.14).

Assumptions $(H 1)-(H 3)$ are built into the construction of $\mathbb{H}$. Assumption (H4) can be verified from the characterization of $\mathbb{B}^{\prime}$ in Lemma 2.2. Only ( $H 5$ ) requires some work. One can show the kernel of d in the bubble space remains unchanged as $E_{0}$ but its imagine is enriched. The dual space is enriched from $(d \mathbb{B})^{\prime}=\mathcal{N}(\mathbb{P})$ to $\left(d \mathbb{B}^{+}\right)^{\prime}=\mathcal{N}(\mathbb{P} \oplus d \mathbb{H})$. Note that the precise characterization of $\mathbb{B}^{+}$is not easy and $\mathbb{H}$ may not be in $\mathbb{B}^{+}$.
3. Div-Conforming Finite Elements. In this section we shall construct the well-known div-conforming finite elements: Brezzi-Douglas-Marini (BDM) [8, 7, 26] and Raviart-Thomas (RT) elements [27, 25]. We start with this simple example to illustrate our approach and build some elementary blocks.
3.1. Div operator. We begin with the following result on the div operator.

Lemma 3.1. Let integer $k \geq 0$. The mapping div : $\boldsymbol{x} \mathbb{H}_{k}(D) \rightarrow \mathbb{H}_{k}(D)$ is bijective. Consequently div : $\mathbb{P}_{k+1}\left(D ; \mathbb{R}^{d}\right) \rightarrow \mathbb{P}_{k}(D)$ is surjective.

Proof. It is a simple consequence of the Euler's formulae (2.1) and (2.2).
3.2. Trace space. The trace operator for $H(\operatorname{div}, K)$ space

$$
\operatorname{tr}^{\mathrm{div}}: H(\operatorname{div}, K) \rightarrow H^{-1 / 2}(\partial K)
$$

is a continuous extension of $\operatorname{tr}^{\mathrm{div}} \boldsymbol{v}=\left.\boldsymbol{n} \cdot \boldsymbol{v}\right|_{\partial K}$ defined on smooth functions. We then focus on the restriction of the trace operator to the polynomial space. Denote by $\mathbb{P}_{k}\left(\mathcal{F}^{1}(K)\right):=\left\{q \in L^{2}(\partial K):\left.q\right|_{F} \in \mathbb{P}_{k}(F)\right.$ for each $\left.F \in \mathcal{F}^{1}(K)\right\}$, which is a Hilbert space with inner product $\sum_{F \in \mathcal{F}^{1}(K)}(\cdot, \cdot)_{F}$. Obviously $\operatorname{tr}^{\text {div }}\left(\mathbb{P}_{k}\left(K ; \mathbb{R}^{d}\right)\right) \subseteq \mathbb{P}_{k}\left(\mathcal{F}^{1}(K)\right)$. We prove it is indeed surjective.

Lemma 3.2. For integer $k \geq 1$, the mapping $\operatorname{tr}^{\text {div }}: \mathbb{P}_{k}\left(K ; \mathbb{R}^{d}\right) \rightarrow \mathbb{P}_{k}\left(\mathcal{F}^{1}(K)\right)$ is onto. Consequently

$$
\operatorname{dim} \operatorname{tr}^{\operatorname{div}}\left(\mathbb{P}_{k}\left(K ; \mathbb{R}^{d}\right)\right)=\operatorname{dim} \mathbb{P}_{k}\left(\mathcal{F}^{1}(K)\right)=(d+1)\binom{k+d-1}{k}
$$

Proof. By the linearity of the trace operator, it suffices to prove that: for any $F_{i} \in \mathcal{F}^{1}(K)$ and any $p \in \mathbb{P}_{k}\left(F_{i}\right)$, we can find a $\boldsymbol{v} \in \mathbb{P}_{k}\left(K ; \mathbb{R}^{d}\right)$ s.t. $\left.\boldsymbol{v} \cdot \boldsymbol{n}\right|_{F_{i}}=p$ and $\left.\boldsymbol{v} \cdot \boldsymbol{n}\right|_{F_{j}}=0$ for other $F_{j} \in \mathcal{F}^{1}(K)$ with $j \neq i$. Without loss of generality, we can assume $i=0$.

For any $p \in \mathbb{P}_{k}\left(F_{0}\right)$, it can be expanded in Bernstein basis $p=\sum_{\alpha \in \mathbb{N}^{d},|\alpha|=k} c_{\alpha} \boldsymbol{\lambda}^{\alpha}$, which can be naturally extended to the whole simplex by the definition of barycentric coordinates. Again by the linearity, we only need to consider one generic term still denoted by $p=c_{\alpha} \boldsymbol{\lambda}^{\alpha}$ for a multi-index $\alpha \in \mathbb{N}^{d},|\alpha|=k$. As $\sum_{i=1}^{d} \alpha_{i}=k>0$, there exists an index $1 \leq i \leq d$ s.t. $\alpha_{i} \neq 0$. Then we can write $p=\lambda_{i} q$ with $q \in \mathbb{P}_{k-1}(K)$.

Now we let $\boldsymbol{v}=\lambda_{i} q \boldsymbol{l}_{i} /\left(\boldsymbol{l}_{i}, \boldsymbol{n}_{0}\right)$. By construction,

$$
\begin{aligned}
\boldsymbol{v} \cdot \boldsymbol{n}_{0} & =\lambda_{i} q=p \\
\left.\left(\boldsymbol{v} \cdot \boldsymbol{n}_{j}\right)\right|_{F_{j}} & =\left.\lambda_{i} q\right|_{F_{j}}\left(\boldsymbol{l}_{i}, \boldsymbol{n}_{j}\right) /\left(\boldsymbol{l}_{i}, \boldsymbol{n}_{0}\right)=0, \quad j=1,2, \cdots, d .
\end{aligned}
$$

That is we find $\boldsymbol{v} \in \mathbb{P}_{k}\left(K ; \mathbb{R}^{d}\right)$ s.t. $\left.\left(\operatorname{tr}^{\operatorname{div}} \boldsymbol{v}\right)\right|_{F_{0}}=p$ and $\left.\left(\operatorname{tr}^{\operatorname{div}} \boldsymbol{v}\right)\right|_{F_{j}}=0$ for $j=1, \ldots, d . \square$
With this identification of the trace space, we clearly have $\mathcal{N}\left(\mathbb{P}_{k}\left(\mathcal{F}^{1}(K)\right)\right)=$ $\left(\operatorname{tr}^{\text {div }}\left(\mathbb{P}_{k}\left(K ; \mathbb{R}^{d}\right)\right)\right)^{\prime}$, and through $\left(\operatorname{tr}^{\text {div }}\right)^{\prime}$, we embed $\mathcal{N}\left(\mathbb{P}_{k}\left(\mathcal{F}^{1}(K)\right)\right.$ into $\mathbb{P}_{k}^{\prime}\left(K ; \mathbb{R}^{d}\right)$.

Lemma 3.3. Let integer $k \geq 1$. For any $\boldsymbol{v} \in \mathbb{P}_{k}\left(K ; \mathbb{R}^{d}\right)$, if the following degrees of freedom vanish, i.e.,

$$
(\boldsymbol{v} \cdot \boldsymbol{n}, p)_{F}=0 \quad \forall p \in \mathbb{P}_{k}(F), F \in \mathcal{F}^{1}(K)
$$

then $\operatorname{tr}^{\mathrm{div}} \boldsymbol{v}=0$.
Proof. Due to Lemma 3.2, the dual operator $\left(\operatorname{tr}^{\text {div }}\right)^{\prime}: \mathbb{P}_{k}^{\prime}\left(\mathcal{F}^{1}(K)\right) \rightarrow \mathbb{P}_{k}^{\prime}\left(K ; \mathbb{R}^{d}\right)$ is injective. Taking $N=(p, \cdot)_{F} \in \mathbb{P}_{k}^{\prime}\left(\mathcal{F}^{1}(K)\right)$ for any $F \in \mathcal{F}^{1}(K)$ and $p \in \mathbb{P}_{k}(F)$, we have

$$
\left(\left(\operatorname{tr}^{\mathrm{div}}\right)^{\prime} N\right)(\boldsymbol{v})=N\left(\operatorname{tr}^{\mathrm{div}} \boldsymbol{v}\right)=\left(p, \operatorname{tr}^{\mathrm{div}} \boldsymbol{v}\right)_{F}=(p, \boldsymbol{n} \cdot \boldsymbol{v})_{F} .
$$

By the assumption, we have $\boldsymbol{v} \perp \operatorname{img}\left(\left(\operatorname{tr}^{\text {div }}\right)^{\prime}\right)$, which indicates $\boldsymbol{v} \in \operatorname{ker}\left(\operatorname{tr}^{\mathrm{div}}\right)$.
Another basis of $\left(\operatorname{tr}^{\operatorname{div}}\left(\mathbb{P}_{k}\left(K ; \mathbb{R}^{d}\right)\right)\right)^{\prime}$ can be obtained by a geometric decomposition of vector Lagrange elements; see [14] for details.
3.3. Bubble space. After we characterize the range of the trace operator, we focus on its null space. Define the polynomial bubble space

$$
\mathbb{B}_{k}(\operatorname{div}, K)=\operatorname{ker}\left(\operatorname{tr}^{\text {div }}\right) \cap \mathbb{P}_{k}\left(K ; \mathbb{R}^{d}\right)
$$

As $\left\{\boldsymbol{n}_{i}, i=1,2, \cdots, d\right\}$ is a basis of $\mathbb{R}^{d}$, it is obvious that for $k=0, \mathbb{B}_{0}(\operatorname{div}, K)=\{\mathbf{0}\}$. As a direct consequence of dimension count, see Lemma 3.4 below, $\mathbb{B}_{1}(\operatorname{div}, K)$ is also the zero space.

Lemma 3.4. Let integer $k \geq 1$. It holds

$$
\operatorname{dim} \mathbb{B}_{k}(\operatorname{div}, K)=d\binom{k+d}{k}-(d+1)\binom{k+d-1}{k}=(k-1)\binom{k+d-1}{k}
$$

Proof. By the characterization of the trace space, we can count the dimension

$$
\operatorname{dim} \mathbb{B}_{k}(\operatorname{div}, K)=\operatorname{dim} \mathbb{P}_{k}\left(K ; \mathbb{R}^{d}\right)-\operatorname{dim} \operatorname{tr}^{\operatorname{div}}\left(\mathbb{P}_{k}\left(K ; \mathbb{R}^{d}\right)\right)
$$

as required.
Next we find different bases of $\mathbb{B}_{k}^{\prime}(\operatorname{div}, K)$. The primary approach is to find a basis for $\mathbb{B}_{k}(\operatorname{div}, K)$, which induces a basis of $\mathcal{N}\left(\mathbb{B}_{k}(\operatorname{div}, K)\right)$. For example, one can show

$$
\mathbb{B}_{k}(\operatorname{div}, K)=\sum_{0 \leq i<j \leq d} \lambda_{i} \lambda_{j} \mathbb{P}_{k-2}(K) \boldsymbol{t}_{i, j} \quad \text { for } k \geq 2
$$

Verification $\lambda_{i} \lambda_{j} \mathbb{P}_{k-2}(K) \boldsymbol{t}_{i, j} \subseteq \mathbb{B}_{k}(\operatorname{div}, K)$ is from the fact

$$
\left.\lambda_{i} \lambda_{j} \boldsymbol{t}_{i, j} \cdot \boldsymbol{n}_{\ell}\right|_{F_{\ell}}=0, \quad \ell=0,1, \cdots, d
$$

Indeed if $\ell=i$ or $\ell=j$, then $\left.\lambda_{i} \lambda_{j}\right|_{F_{\ell}}=0$. Otherwise $\boldsymbol{t}_{i, j} \cdot \boldsymbol{n}_{\ell}=0$ by (2.4). To show every function in $\mathbb{B}_{k}(\operatorname{div}, K)$ can be written as a linear combination of $\lambda_{i} \lambda_{j} \boldsymbol{t}_{i, j}$ is tedious and will be skipped. Obviously $\operatorname{dim} \mathbb{B}_{k}(\operatorname{div}, K) \neq d(d+1) / 2 \operatorname{dim} \mathbb{P}_{k-2}(K)$ as $\left\{\lambda_{i} \lambda_{j} \mathbb{P}_{k-2}(K) \boldsymbol{t}_{i, j}, 0 \leq i<j \leq d\right\}$ is linearly dependent. One can further expand the polynomials in $\mathbb{P}_{k-2}(K)$ in the Bernstein basis and $\boldsymbol{t}_{i j}$ in terms of $\mathrm{d} \lambda$ and add constraint on the multi-index to find a basis from this generating set; see [5]. Another systematical way to identify $\mathbb{B}_{k}(\operatorname{div}, K)$ is through a geometric decomposition of vector Lagrange elements and a $t-n$ basis decomposition at each sub-simplex; see [14] for details.

Fortunately we are interested in the dual space, which can let us find a basis of $\mathbb{B}_{k}^{\prime}(\operatorname{div}, K)$ without knowing one for $\mathbb{B}_{k}(\operatorname{div}, K)$. Following Lemma 2.2, we first find a larger space containing $\mathbb{B}_{k}^{\prime}(\operatorname{div}, K)$.

Lemma 3.5. Let integer $k \geq 1$. We have

$$
\mathbb{B}_{k}^{\prime}(\operatorname{div}, K)=\mathcal{I}^{\prime} \mathcal{N}\left(\mathbb{P}_{k-1}\left(K ; \mathbb{R}^{d}\right)\right)
$$

where $\mathcal{I}: \mathbb{B}_{k}(\operatorname{div}, K) \rightarrow \mathbb{P}_{k}\left(K ; \mathbb{R}^{d}\right)$ is the inclusion map.
Proof. It suffices to show that: for any $\boldsymbol{v} \in \mathbb{B}_{k}(\operatorname{div}, K)$ satisfying

$$
\begin{equation*}
(\boldsymbol{v}, \boldsymbol{q})_{K}=0 \quad \forall \boldsymbol{q} \in \mathbb{P}_{k-1}\left(K ; \mathbb{R}^{d}\right) \tag{3.1}
\end{equation*}
$$

then $\boldsymbol{v}=\mathbf{0}$.
Expand $\boldsymbol{v}$ in terms of $\left\{\boldsymbol{l}_{i}\right\}$ as $\boldsymbol{v}=\sum_{i=1}^{d} v_{i} \boldsymbol{l}_{i}$. Then $\left.\boldsymbol{v} \cdot \boldsymbol{n}_{i}\right|_{F_{i}}=0$ implies $\left.v_{i}\right|_{F_{i}}=0$ for $i=1, \cdots, d$, i.e., $v_{i}=\lambda_{i} p_{i}$ for some $p_{i} \in \mathbb{P}_{k-1}(K)$. Choose $\boldsymbol{q}=\sum_{i=1}^{d} p_{i} \boldsymbol{n}_{i} \in$ $\mathbb{P}_{k-1}\left(K ; \mathbb{R}^{d}\right)$ in (3.1) to get $\int_{K} \boldsymbol{v} \cdot\left(\sum_{i=1}^{d} p_{i} \boldsymbol{n}_{i}\right) \mathrm{d} x=\int_{K} \sum_{i=1}^{d} \lambda_{i} p_{i}^{2} \mathrm{~d} x=0$, which implies $p_{i}=0$, i.e. $v_{i}=0$ for all $i=1,2, \cdots, d$. Thus $\boldsymbol{v}=\mathbf{0}$.

Again the dimension count $\operatorname{dim} \mathbb{B}_{k}(\operatorname{div}, K) \neq \operatorname{dim} \mathbb{P}_{k-1}\left(K ; \mathbb{R}^{d}\right)$ implies $\mathcal{I}^{\prime}$ is not injective. We need to further refine our characterization. We use div operator to decompose $\mathbb{B}_{k}(\operatorname{div}, K)$ into

$$
E_{0}:=\mathbb{B}_{k}(\operatorname{div}, K) \cap \operatorname{ker}(\operatorname{div}), \quad E_{0}^{\perp}:=\mathbb{B}_{k}(\operatorname{div}, K) / E_{0}
$$

We can characterize the dual space of $E_{0}^{\perp}$ through div*, which is - grad restricting to the bubble function space and can be continuously extended to $H^{1}$.

Lemma 3.6. Let integer $k \geq 1$. The mapping

$$
\operatorname{div}: E_{0}^{\perp} \rightarrow \mathbb{P}_{k-1}(K) / \mathbb{R}
$$

is a bijection and consequently

$$
\operatorname{dim} E_{0}^{\perp}=\operatorname{dim} \mathbb{P}_{k-1}(K)-1=\binom{k-1+d}{d}-1
$$

Proof. The inclusion $\operatorname{div}\left(\mathbb{B}_{k}(\operatorname{div}, K)\right) \subseteq \mathbb{P}_{k-1}(K) / \mathbb{R}$ is proved through integration by parts

$$
(\operatorname{div} \boldsymbol{v}, p)_{K}=-(\boldsymbol{v}, \operatorname{grad} p)_{K}=0 \quad \forall p \in \mathbb{R}=\operatorname{ker}(\operatorname{grad})
$$

If $\operatorname{div}\left(\mathbb{B}_{k}(\operatorname{div}, K)\right) \neq \mathbb{P}_{k-1}(K) / \mathbb{R}$, then there exists a $p \in \mathbb{P}_{k-1}(K) / \mathbb{R}$ and $p \perp$ $\operatorname{div}\left(\mathbb{B}_{k}(\operatorname{div}, K)\right)$, which is equivalent to $\nabla p \perp \mathbb{B}_{k}(\operatorname{div}, K)$. Expand the vector $\nabla p$ in the basis $\left\{\boldsymbol{n}_{i}, i=1, \cdots, d\right\}$ as $\nabla p=\sum_{i=1}^{d} q_{i} \boldsymbol{n}_{i}$ with $q_{i} \in \mathbb{P}_{k-2}(K)$. Then set $\boldsymbol{v}_{p}=$ $\sum_{i=1}^{d} q_{i} \lambda_{0} \lambda_{i} \boldsymbol{l}_{i}=\sum_{i=1}^{d}\left|\nabla \lambda_{i}\right| q_{i} \lambda_{0} \lambda_{i} \boldsymbol{t}_{i, 0} \in \mathbb{B}_{k}(\operatorname{div}, K)$. We have

$$
\left(\operatorname{grad} p, \boldsymbol{v}_{p}\right)_{K}=\sum_{i=1}^{d} \int_{K} q_{i}^{2} \lambda_{0} \lambda_{i} \mathrm{~d} x=0
$$

which implies $q_{i}=0$ for all $i=1,2, \cdots, d$, i.e., $\operatorname{grad} p=0$ and $p=0$ as $p \in$ $\mathbb{P}_{k-1}(K) / \mathbb{R}$.

We have proved $\operatorname{div}\left(\mathbb{B}_{k}(\operatorname{div}, K)\right)=\mathbb{P}_{k-1}(K) / \mathbb{R}$, and thus div $: E_{0}^{\perp} \rightarrow \mathbb{P}_{k-1}(K) / \mathbb{R}$ is a bijection as $E_{0}^{\perp}=\mathbb{B}_{k}(\operatorname{div}, K) / \operatorname{ker}(\operatorname{div})$.

As an example of (2.9), we have the following characterization of $\left(E_{0}^{\perp}\right)^{\prime}$.
Corollary 3.7. Let integer $k \geq 1$. We have

$$
\left(E_{0}^{\perp}\right)^{\prime}=\mathcal{N}\left(\operatorname{grad} \mathbb{P}_{k-1}(K)\right)
$$

That is a function $\boldsymbol{v} \in E_{0}^{\perp}$ is uniquely determined by DoFs

$$
(\boldsymbol{v}, \boldsymbol{q})_{K} \quad \forall \boldsymbol{q} \in \operatorname{grad} \mathbb{P}_{k-1}(K)
$$

After we know the dimensions of $\mathbb{B}_{k}(\operatorname{div}, K)$ and $E_{0}^{\perp}$, we can calculate the dimension of $E_{0}$.

Lemma 3.8. Let integer $k \geq 1$. It holds

$$
\begin{equation*}
\operatorname{dim} E_{0}=\operatorname{dim} \mathbb{B}_{k}(\operatorname{div}, K)-\operatorname{dim} E_{0}^{\perp}=d\binom{k+d-1}{d}-\binom{k+d}{d}+1 \tag{3.2}
\end{equation*}
$$

The most difficult part is to characterize $E_{0}$. Using the de Rham complex, we can identify the null space of div : $\mathbb{B}_{k}(\operatorname{div}, K) \rightarrow \mathbb{P}_{k-1}(K) / \mathbb{R}$ as the image of another polynomial bubble spaces. For example,

$$
E_{0}= \begin{cases}\operatorname{curl}\left(\mathbb{P}_{k+1}(K) \cap H_{0}^{1}(K)\right) & \text { for } d=2 \\ \operatorname{curl}\left(\mathbb{P}_{k+1}\left(K ; \mathbb{R}^{3}\right) \cap H_{0}(\operatorname{curl}, K)\right) & \text { for } d=3\end{cases}
$$

Generalization of the curl operator can be done for de Rham complex, but it will be hard for elasticity complex and divdiv complex.

Instead, we take the dual approach. To identify the dual space of $E_{0}$, we resort to a polynomial decomposition of $\mathbb{P}_{k-1}\left(K ; \mathbb{R}^{d}\right)$.

Lemma 3.9. Let integer $k \geq 1$. We have the polynomial space decomposition

$$
\begin{equation*}
\mathbb{P}_{k-1}\left(K ; \mathbb{R}^{d}\right)=\operatorname{grad} \mathbb{P}_{k}(K) \oplus\left(\operatorname{ker}(\cdot \boldsymbol{x}) \cap \mathbb{P}_{k-1}\left(K ; \mathbb{R}^{d}\right)\right) \tag{3.3}
\end{equation*}
$$

Proof. Clearly it holds

$$
\operatorname{grad} \mathbb{P}_{k}(K) \oplus\left(\operatorname{ker}(\cdot \boldsymbol{x}) \cap \mathbb{P}_{k-1}\left(K ; \mathbb{R}^{d}\right)\right) \subseteq \mathbb{P}_{k-1}\left(K ; \mathbb{R}^{d}\right)
$$

And the sum is direct as by Euler's formulae (2.1),

$$
\operatorname{grad} \mathbb{P}_{k}(K) \cap\left(\operatorname{ker}(\cdot \boldsymbol{x}) \cap \mathbb{P}_{k-1}\left(K ; \mathbb{R}^{d}\right)\right)=\{\mathbf{0}\}
$$

The mapping $\cdot \boldsymbol{x}: \mathbb{P}_{k-1}\left(K ; \mathbb{R}^{d}\right) \rightarrow \mathbb{P}_{k-1}(K) \backslash \mathbb{R}$ is surjective, thus

$$
\operatorname{dim} \mathbb{P}_{k-1}\left(K ; \mathbb{R}^{d}\right)=\operatorname{dim}\left(\mathbb{P}_{k}(K) \backslash \mathbb{R}\right)+\operatorname{dim}\left(\operatorname{ker}(\cdot \boldsymbol{x}) \cap \mathbb{P}_{k-1}\left(K ; \mathbb{R}^{d}\right)\right)
$$

As $\operatorname{dim}\left(\mathbb{P}_{k}(K) \backslash \mathbb{R}\right)=\operatorname{dim} \operatorname{grad} \mathbb{P}_{k}(K)$, we obtain the decomposition (3.3) by dimension count.

Corollary 3.10. Let integer $k \geq 1$. We have

$$
E_{0}^{\prime}=\mathcal{N}\left(\operatorname{ker}(\cdot \boldsymbol{x}) \cap \mathbb{P}_{k-1}\left(K ; \mathbb{R}^{d}\right)\right)
$$

That is a function $\boldsymbol{v} \in E_{0}$ is uniquely determined by

$$
(\boldsymbol{v}, \boldsymbol{q})_{K} \quad \forall \boldsymbol{q} \in \operatorname{ker}(\cdot \boldsymbol{x}) \cap \mathbb{P}_{k-1}\left(K ; \mathbb{R}^{d}\right)
$$

Proof. We apply Lemma 2.2 with $V=\mathbb{P}_{k}\left(K ; \mathbb{R}^{d}\right), U=\mathbb{P}_{k-1}\left(K ; \mathbb{R}^{d}\right)$, and $\kappa=\cdot \boldsymbol{x}$. Lemmas 3.5 and 3.9 verify the assumptions (B1)-(B2), and we only need to count the dimension

$$
\begin{aligned}
\operatorname{dim}\left(\operatorname{ker}(\cdot \boldsymbol{x}) \cap \mathbb{P}_{k-1}\left(K ; \mathbb{R}^{d}\right)\right) & =\operatorname{dim} \mathbb{P}_{k-1}\left(K ; \mathbb{R}^{d}\right)-\operatorname{dim} \mathbb{P}_{k}(K)+1 \\
& =d\binom{k+d-1}{d}-\binom{k+d}{d}+1 \\
& =\operatorname{dim} E_{0},
\end{aligned}
$$

where the last step is based on (3.2). The desired result then follows.
Remark 3.11. A computational approach to find an explicit basis of $\operatorname{ker}(\cdot \boldsymbol{x}) \cap$ $\mathbb{P}_{k-1}\left(K ; \mathbb{R}^{d}\right)$ is as follows. First find a basis for $\mathbb{P}_{k-1}\left(K ; \mathbb{R}^{d}\right)$ and one for $\mathbb{P}_{k}(K)$. Then form the matrix representation $X$ of the operator $\cdot \boldsymbol{x}$. Afterwards the null space $\operatorname{ker}(X)$ can be found algebraically.

An explicit characterization of $\operatorname{ker}(\cdot \boldsymbol{x}) \cap \mathbb{P}_{k-1}\left(K ; \mathbb{R}^{d}\right)$ is shown in $[25$, Proposition 1], that is $\boldsymbol{v} \in \operatorname{ker}(\cdot \boldsymbol{x}) \cap \mathbb{H}_{k-1}\left(K ; \mathbb{R}^{d}\right)$ is equivalent to $\boldsymbol{v} \in \mathbb{H}_{k-1}\left(K ; \mathbb{R}^{d}\right)$ such that the symmetric part of $\nabla^{k-1} \boldsymbol{v}$ vanishes. We shall give another characterization of $\operatorname{ker}(\cdot \boldsymbol{x}) \cap \mathbb{P}_{k-1}\left(K ; \mathbb{R}^{d}\right)$.

Lemma 3.12. It holds

$$
\begin{equation*}
\operatorname{ker}(\cdot \boldsymbol{x}) \cap \mathbb{P}_{k-1}\left(K ; \mathbb{R}^{d}\right)=\mathbb{P}_{k-2}(K ; \mathbb{K}) \boldsymbol{x} \tag{3.4}
\end{equation*}
$$

where recall $\mathbb{K}$ is the subspace of skew-symmetric matrices of $\mathbb{R}^{d \times d}$, and

$$
\begin{equation*}
\mathbb{P}_{k-1}\left(K ; \mathbb{R}^{d}\right)=\operatorname{grad} \mathbb{P}_{k}(K) \oplus \mathbb{P}_{k-2}(K ; \mathbb{K}) \boldsymbol{x} \tag{3.5}
\end{equation*}
$$

Proof. Clearly we have $\mathbb{P}_{k-2}(K ; \mathbb{K}) \boldsymbol{x} \subseteq \operatorname{ker}(\cdot \boldsymbol{x}) \cap \mathbb{P}_{k-1}\left(K ; \mathbb{R}^{d}\right)$. By (3.3), it suffices to prove (3.5). Take $\boldsymbol{q} \in \mathbb{P}_{k-1}\left(K ; \mathbb{R}^{d}\right)$. Without loss of generality, by linearity, it is sufficient to assume $\boldsymbol{q}=\boldsymbol{x}^{\alpha} \boldsymbol{e}_{\ell}$ with $|\alpha|=k-1$ and $1 \leq \ell \leq d$. Let

$$
p=\frac{1}{k} \boldsymbol{x} \cdot \boldsymbol{q}=\frac{1}{k} \boldsymbol{x}^{\alpha+\boldsymbol{e}_{\ell}} \in \mathbb{P}_{k}(K), \quad \boldsymbol{\tau}=\frac{1}{k} \sum_{i=1}^{d} \alpha_{i} \boldsymbol{x}^{\alpha-\boldsymbol{e}_{i}}\left(\boldsymbol{e}_{\ell} \boldsymbol{e}_{i}^{\boldsymbol{\top}}-\boldsymbol{e}_{i} \boldsymbol{e}_{\ell}^{\boldsymbol{\top}}\right) \in \mathbb{P}_{k-2}(K ; \mathbb{K})
$$

Then

$$
\begin{aligned}
\operatorname{grad} p+\boldsymbol{\tau} \boldsymbol{x} & =\frac{1}{k} \sum_{i=1}^{d}\left(\alpha_{i}+\delta_{i \ell}\right) \boldsymbol{x}^{\alpha+\boldsymbol{e}_{\ell}-\boldsymbol{e}_{i}} \boldsymbol{e}_{i}+\frac{1}{k} \sum_{i=1}^{d} \alpha_{i} \boldsymbol{x}^{\alpha-\boldsymbol{e}_{i}}\left(\boldsymbol{e}_{\ell} x_{i}-\boldsymbol{e}_{i} x_{\ell}\right) \\
& =\frac{1}{k} \sum_{i=1}^{d}\left(\alpha_{i}+\delta_{i \ell}\right) \boldsymbol{x}^{\alpha+\boldsymbol{e}_{\ell}-\boldsymbol{e}_{i}} \boldsymbol{e}_{i}+\frac{1}{k} \sum_{i=1}^{d} \alpha_{i} \boldsymbol{x}^{\alpha} \boldsymbol{e}_{\ell}-\frac{1}{k} \sum_{i=1}^{d} \alpha_{i} \boldsymbol{x}^{\alpha+\boldsymbol{e}_{\ell}-\boldsymbol{e}_{i}} \boldsymbol{e}_{i} \\
& =\frac{1}{k} \boldsymbol{x}^{\alpha} \boldsymbol{e}_{\ell}+\frac{|\alpha|}{k} \boldsymbol{x}^{\alpha} \boldsymbol{e}_{\ell}=\boldsymbol{x}^{\alpha} \boldsymbol{e}_{\ell}
\end{aligned}
$$

Therefore it follows $\boldsymbol{q}=\operatorname{grad} p+\boldsymbol{\tau} \boldsymbol{x}$, i.e.

$$
\mathbb{P}_{k-1}\left(K ; \mathbb{R}^{d}\right) \subseteq \operatorname{grad} \mathbb{P}_{k}(K) \oplus \mathbb{P}_{k-2}(K ; \mathbb{K}) \boldsymbol{x}
$$

This combined with the fact $\operatorname{grad} \mathbb{P}_{k}(K) \oplus \mathbb{P}_{k-2}(K ; \mathbb{K}) \boldsymbol{x} \subseteq \mathbb{P}_{k-1}\left(K ; \mathbb{R}^{d}\right)$ gives (3.5).प The decomposition (3.3) and characterization (3.4) can be summarized as the following double-directional complex

$$
\mathbb{R} \underset{\pi_{0}}{\stackrel{C}{\rightleftarrows}} \mathbb{P}_{k}(K) \underset{\boldsymbol{v} \cdot \boldsymbol{x}}{\stackrel{\nabla}{\rightleftharpoons}} \mathbb{P}_{k-1}\left(K ; \mathbb{R}^{d}\right) \underset{\tau \boldsymbol{x}}{\stackrel{\text { skw }}{\underset{\tau}{\rightleftharpoons}}} \mathbb{P}_{k-2}(K ; \mathbb{K})
$$

Define, for an integer $k \geq 0$,

$$
\mathrm{ND}_{k}(K):=\mathbb{P}_{k}\left(K ; \mathbb{R}^{d}\right) \oplus \mathbb{H}_{k}(K ; \mathbb{K}) \boldsymbol{x}
$$

which is the shape function space of the first kind Nedéléc edge element in arbitrary dimension [25, 4].

Corollary 3.13. Let integer $k \geq 2$. We have

$$
\begin{equation*}
\mathbb{B}_{k}^{\prime}(\operatorname{div}, K)=\mathcal{N}\left(\operatorname{grad} \mathbb{P}_{k-1}(K) \oplus \mathbb{P}_{k-2}(K ; \mathbb{K}) \boldsymbol{x}\right)=\mathcal{N}\left(\mathrm{ND}_{k-2}(K)\right) \tag{3.6}
\end{equation*}
$$

Proof. The first identity is a direct consequence of Corollaries 3.7 and 3.10 and Lemma 3.12. We then write $\mathbb{P}_{k-2}(K ; \mathbb{K})=\mathbb{P}_{k-3}(K ; \mathbb{K}) \oplus \mathbb{H}_{k-2}(K ; \mathbb{K})$ and use the decomposition (3.3) to conclude the second identity.

Remark 3.14. The space $\mathrm{ND}_{k-2}(K)$ can be abbreviate as $\mathbb{P}_{k-1}^{-} \Lambda^{1}$ in the terminology of FEEC [4, 2]. The characterization of $\mathbb{B}_{k}^{\prime}(\operatorname{div}, K)$ in (3.6) can be written as

$$
\left(\mathbb{P}_{k} \Lambda^{n-1}(K)\right)^{*}=\mathbb{P}_{k-1}^{-} \Lambda^{1}(K)
$$

which is well documented for de Rham complex [5] but not easy for general complexes. Therefore we still stick to the vector/matrix calculus notation.

Hence we acquire the uni-solvence of BDM element from Lemma 3.3 and Corollary 3.13.

Theorem 3.15 (BDM element). Let integer $k \geq 1$. Choose the shape function space $V=\mathbb{P}_{k}\left(K ; \mathbb{R}^{d}\right)$. We have the following set of degrees of freedom for $V$

$$
\begin{align*}
(\boldsymbol{v} \cdot \boldsymbol{n}, q)_{F} & \forall q \in \mathbb{P}_{k}(F), F \in \mathcal{F}^{1}(K)  \tag{3.7}\\
(\boldsymbol{v}, \boldsymbol{q})_{K} & \forall \boldsymbol{q} \in \mathrm{ND}_{k-2}(K)=\operatorname{grad} \mathbb{P}_{k-1}(K) \oplus \mathbb{P}_{k-2}(K ; \mathbb{K}) \boldsymbol{x} \tag{3.8}
\end{align*}
$$

Define the global BDM element space

$$
\begin{aligned}
\boldsymbol{V}_{h}:=\left\{\boldsymbol{v} \in \boldsymbol{L}^{2}\left(\Omega ; \mathbb{R}^{d}\right):\right. & \left.\boldsymbol{v}\right|_{K} \in \mathbb{P}_{k}\left(K ; \mathbb{R}^{d}\right) \text { for each } K \in \mathcal{T}_{h} \\
& \text { the degree of freedom (3.7) is single-valued }\} .
\end{aligned}
$$

Since $\left.\boldsymbol{v} \cdot \boldsymbol{n}\right|_{F} \in \mathbb{P}_{k}(F)$, the single-valued degree of freedom (3.7) implies $\boldsymbol{v} \cdot \boldsymbol{n}$ is continuous across the boundary of elements, hence $\boldsymbol{V}_{h} \subset \boldsymbol{H}(\operatorname{div}, \Omega)$.
3.4. RT space. In this subsection, we assume integer $k \geq 0$. The space of shape functions for the Raviart-Thomas (RT) element is enriched to

$$
V^{\mathrm{RT}}:=\mathbb{P}_{k}\left(K ; \mathbb{R}^{d}\right) \oplus \mathbb{H}_{k}(K) \boldsymbol{x}
$$

The degrees of freedom are

$$
\begin{align*}
(\boldsymbol{v} \cdot \boldsymbol{n}, q)_{F} & \forall q \in \mathbb{P}_{k}(F), F \in \mathcal{F}^{1}(K)  \tag{3.9}\\
(\boldsymbol{v}, \boldsymbol{q})_{K} & \forall \boldsymbol{q} \in \mathbb{P}_{k-1}\left(K, \mathbb{R}^{d}\right)=\operatorname{grad} \mathbb{P}_{k}(K) \oplus \mathbb{P}_{k-2}(K ; \mathbb{K}) \boldsymbol{x} \tag{3.10}
\end{align*}
$$

Note that the DoFs to determine the trace remain the same and only the interior moments are increased from $\mathrm{ND}_{k-2}(K)$ to $\mathbb{P}_{k-1}\left(K ; \mathbb{R}^{d}\right)$. The range space is also increased, i.e., $\operatorname{div} V^{\mathrm{RT}}=\mathbb{P}_{k}(K)$ and therefore the approximation of $\operatorname{div} \boldsymbol{u}$ will be one order higher.

We follow our construction procedure to identify the dual spaces of each block.
Lemma 3.16. Let integer $k \geq 0$. It holds

$$
\begin{equation*}
\operatorname{tr}^{\operatorname{div}}\left(V^{\mathrm{RT}}\right)=\mathbb{P}_{k}\left(\mathcal{F}^{1}(K)\right) \tag{3.11}
\end{equation*}
$$

Proof. When $k \geq 1$, by Lemma 3.2 and the fact that $\left.\boldsymbol{n} \cdot \boldsymbol{x}\right|_{F_{i}}$ is a constant, it follows

$$
\operatorname{tr}^{\mathrm{div}}\left(V^{\mathrm{RT}}\right)=\operatorname{tr}^{\mathrm{div}}\left(\mathbb{P}_{k}\left(K ; \mathbb{R}^{d}\right)\right)=\mathbb{P}_{k}\left(\mathcal{F}^{1}(K)\right)
$$

Consider the case $k=0$. It is clear that $\operatorname{tr}^{\operatorname{div}}\left(V^{\mathrm{RT}}\right) \subseteq \mathbb{P}_{0}\left(\mathcal{F}^{1}(K)\right)$. To prove the other side, by the linearity, assume $q \in \mathbb{P}_{0}\left(\mathcal{F}^{1}(K)\right)$ such that $\left.q\right|_{F_{0}}=c \in \mathbb{R}$ and $\left.q\right|_{F_{i}}=0$ for $i=1, \cdots, d$. Set $\boldsymbol{v}=\frac{c}{\left(\boldsymbol{x}_{1}-\boldsymbol{x}_{0}\right) \cdot \boldsymbol{n}_{0}}\left(\boldsymbol{x}-\boldsymbol{x}_{0}\right) \in V^{\mathrm{RT}}$, then $\operatorname{tr}^{\mathrm{div}} \boldsymbol{v}=q$.

Define the bubble space

$$
\mathbb{B}_{k+1}^{-}(\operatorname{div}, K):=\operatorname{ker}\left(\operatorname{tr}^{\mathrm{div}}\right) \cap V^{\mathrm{RT}}
$$

By (3.11), $\operatorname{dim} \mathbb{B}_{k+1}^{-}(\operatorname{div}, K)=\operatorname{dim} V^{\mathrm{RT}}-\operatorname{dim} \operatorname{tr}^{\operatorname{div}}\left(V^{\mathrm{RT}}\right)=d\binom{k+d-1}{d}$ for $k \geq 1$ and $\operatorname{dim} \mathbb{B}_{1}^{-}(\operatorname{div}, K)=0$. We show the intersection of the null space of div operator and $\mathbb{B}_{k+1}^{-}(\operatorname{div}, K)$ remains unchanged.

Lemma 3.17. Let integer $k \geq 0$. It holds

$$
\begin{equation*}
\mathbb{B}_{k+1}^{-}(\operatorname{div}, K) \cap \operatorname{ker}(\operatorname{div})=E_{0} \tag{3.12}
\end{equation*}
$$

where $E_{0}:=\{\mathbf{0}\}$ for $k=0$.
Proof. For $\boldsymbol{v} \in V^{\mathrm{RT}}$, if $\operatorname{div} \boldsymbol{v}=0$, then $\boldsymbol{v} \in \mathbb{P}_{k}\left(K ; \mathbb{R}^{d}\right)$ as div $: \mathbb{H}_{k}(K) \boldsymbol{x} \rightarrow \mathbb{H}_{k}(K)$ is bijective. Then the desired result follows.

Define $E_{0}^{\perp,-}:=\mathbb{B}_{k+1}^{-}(\operatorname{div}, K) / E_{0}$. We give a characterization of $\left(E_{0}^{\perp,-}\right)^{\prime}$.

Lemma 3.18. Let integer $k \geq 0$. It holds

$$
\begin{equation*}
\left(E_{0}^{\perp,-}\right)^{\prime}=\mathcal{N}\left(\operatorname{grad} \mathbb{P}_{k}(K)\right) \tag{3.13}
\end{equation*}
$$

Proof. We first prove: given a $\boldsymbol{v} \in E_{0}^{\perp,-}$, i.e. $\operatorname{tr}^{\text {div }} \boldsymbol{v}=0$ and $\boldsymbol{v} \perp E_{0}$, if

$$
\begin{equation*}
(\boldsymbol{v}, \operatorname{grad} p)=0 \quad \forall p \in \mathbb{P}_{k}(K) \tag{3.14}
\end{equation*}
$$

then $\boldsymbol{v}=\mathbf{0}$. Indeed integration by parts of (3.14) and the fact $\operatorname{div} \boldsymbol{v} \in \mathbb{P}_{k}(K)$ imply $\operatorname{div} \boldsymbol{v}=0$, i.e. $\boldsymbol{v} \in E_{0}$. Then the only possibility to have $\boldsymbol{v} \perp E_{0}$ is $\boldsymbol{v}=\mathbf{0}$.

Then the dimension count gives

$$
\operatorname{dim} E_{0}^{\perp,-}=\operatorname{dim} \mathbb{B}_{k+1}^{-}(\operatorname{div}, K)-\operatorname{dim} E_{0}=\binom{k+d}{d}-1=\operatorname{dim} \operatorname{grad} \mathbb{P}_{k}(K)
$$

which indicates (3.13).
Hence we acquire the uni-solvence of RT element from (3.11), (3.12), (3.13) and Corollary 3.10. Global version of finite element space can be defined similarly.

Theorem 3.19 (Uni-solvence of RT element). Let integer $k \geq 0$. The degrees of freedom (3.9)-(3.10) are uni-solvent for $V^{\mathrm{RT}}$.

When $k \geq 1$, RT element can be enriched from BDM element by applying Lemma 2.3 with $\mathrm{d}=\operatorname{div}, V=\mathbb{P}_{k}\left(K ; \mathbb{R}^{d}\right), \mathbb{H}=\mathbb{H}_{k}(K) \boldsymbol{x}, \mathbb{P}=\mathbb{P}_{k-1}(K) / \mathbb{R}$ and $\mathbb{Q}=\mathbb{P}_{k-2}(K ; \mathbb{K}) \boldsymbol{x}$.
4. Symmetric Div-Conforming Finite Elements. In this section we shall construct div-conforming finite elements for symmetric matrices. For space $V=$ $\mathbb{P}_{k}(K, \mathbb{S})$, our element is slightly different from Hu's element constructed in [20]. A new family of $\mathbb{P}_{k+1}^{-}(K, \mathbb{S})$ type finite elements is also constructed. The trace space for symmetric div-conforming element seems hard to characterize, instead we identify the bubble function space and then only need to work on the dual of the trace space.

### 4.1. Div operator.

Lemma 4.1. Let $k \geq 0$. The operator div : $\operatorname{sym}\left(\mathbb{H}_{k}\left(D ; \mathbb{R}^{d}\right) \boldsymbol{x}^{\boldsymbol{\top}}\right) \rightarrow \mathbb{H}_{k}\left(D ; \mathbb{R}^{d}\right)$ is bijective and consequently div : $\mathbb{P}_{k+1}(D ; \mathbb{S}) \rightarrow \mathbb{P}_{k}\left(D ; \mathbb{R}^{d}\right)$ is surjective.

Proof. Noting that

$$
\begin{array}{r}
\operatorname{div}\left(\operatorname{sym}\left(\mathbb{H}_{k}\left(D ; \mathbb{R}^{d}\right) \boldsymbol{x}^{\boldsymbol{\top}}\right)\right) \subseteq \mathbb{H}_{k}\left(K ; \mathbb{R}^{d}\right), \\
\operatorname{dim}\left(\operatorname{sym}\left(\mathbb{H}_{k}\left(D ; \mathbb{R}^{d}\right) \boldsymbol{x}^{\boldsymbol{\top}}\right)\right)=\operatorname{dim} \mathbb{H}_{k}\left(K ; \mathbb{R}^{d}\right),
\end{array}
$$

it is sufficient to prove $\operatorname{sym}\left(\mathbb{H}_{k}\left(D ; \mathbb{R}^{\boldsymbol{d}}\right) \boldsymbol{x}^{\boldsymbol{\top}}\right) \cap \operatorname{ker}(\operatorname{div})=\{\mathbf{0}\}$. That is: for any $\boldsymbol{q} \in$ $\mathbb{H}_{k}\left(D ; \mathbb{R}^{d}\right)$ satisfying $\operatorname{div} \operatorname{sym}\left(\boldsymbol{q} \boldsymbol{x}^{\boldsymbol{\top}}\right)=\mathbf{0}$, we are going to prove $\boldsymbol{q}=\mathbf{0}$.

By (2.2), we have

$$
\begin{aligned}
2 \operatorname{div} \operatorname{sym}\left(\boldsymbol{q} \boldsymbol{x}^{\boldsymbol{\top}}\right) & =\operatorname{div}\left(\boldsymbol{q} \boldsymbol{x}^{\boldsymbol{\top}}\right)+\operatorname{div}\left(\boldsymbol{x} \boldsymbol{q}^{\boldsymbol{\top}}\right)=(k+d) \boldsymbol{q}+(\operatorname{grad} \boldsymbol{x}) \boldsymbol{q}+(\operatorname{div} \boldsymbol{q}) \boldsymbol{x} \\
& =(k+d+1) \boldsymbol{q}+(\operatorname{div} \boldsymbol{q}) \boldsymbol{x}
\end{aligned}
$$

It follows from $\operatorname{div} \operatorname{sym}\left(\boldsymbol{q} \boldsymbol{x}^{\boldsymbol{\top}}\right)=\mathbf{0}$ that

$$
\begin{equation*}
(k+d+1) \boldsymbol{q}+(\operatorname{div} \boldsymbol{q}) \boldsymbol{x}=\mathbf{0} \tag{4.1}
\end{equation*}
$$

Applying the divergence operator div on both side of (4.1), we get from (2.2) that

$$
2(k+d) \operatorname{div} \boldsymbol{q}=0
$$

Hence $\operatorname{div} \boldsymbol{q}=0$, which together with (4.1) gives $\boldsymbol{q}=\mathbf{0}$.
4.2. Bubble space. Define an $\boldsymbol{H}(\operatorname{div}, K ; \mathbb{S})$ bubble function space of polynomials of degree $k$ as

$$
\mathbb{B}_{k}(\operatorname{div}, K ; \mathbb{S}):=\left\{\boldsymbol{\tau} \in \mathbb{P}_{k}(K ; \mathbb{S}):\left.\boldsymbol{\tau} \boldsymbol{n}\right|_{\partial K}=\mathbf{0}\right\}
$$

It is easy to check that $\mathbb{B}_{1}(\operatorname{div}, K ; \mathbb{S})$ is merely the zero space. The following characterization of $\mathbb{B}_{k}(\operatorname{div}, K ; \mathbb{S})$ is given in [20, Lemma 2.2].

Lemma 4.2. For $k \geq 2$, it holds

$$
\begin{equation*}
\mathbb{B}_{k}(\operatorname{div}, K ; \mathbb{S})=\sum_{0 \leq i<j \leq d} \lambda_{i} \lambda_{j} \mathbb{P}_{k-2}(K) \boldsymbol{T}_{i, j} \tag{4.2}
\end{equation*}
$$

Consequently

$$
\operatorname{dim} \mathbb{B}_{k}(\operatorname{div}, K ; \mathbb{S})=\operatorname{dim} \mathbb{P}_{k-2}(K ; \mathbb{S})=\frac{d(d+1)}{2}\binom{d+k-2}{d}
$$

Lemma 4.3. For $k \geq 2$, it holds

$$
\mathbb{B}_{k}^{\prime}(\operatorname{div}, K ; \mathbb{S})=\mathcal{N}\left(\mathbb{P}_{k-2}(K ; \mathbb{S})\right)
$$

That is $\boldsymbol{\tau} \in \mathbb{B}_{k}(\operatorname{div}, K ; \mathbb{S})$ is uniquely determined by

$$
(\boldsymbol{\tau}, \boldsymbol{\varsigma})_{K} \quad \forall \boldsymbol{\varsigma} \in \mathbb{P}_{k-2}(K ; \mathbb{S})
$$

Proof. Given $\boldsymbol{\tau} \in \mathbb{B}_{k}(\operatorname{div}, K ; \mathbb{S})$, by (4.2), there exist $q_{i j} \in \mathbb{P}_{k-2}(K)$ with $0 \leq i<$ $j \leq d$ such that

$$
\boldsymbol{\tau}=\sum_{0 \leq i<j \leq d} \lambda_{i} \lambda_{j} q_{i j} \boldsymbol{T}_{i, j}
$$

Note that symmetric tensors $\left\{\boldsymbol{N}_{i, j}\right\}_{0 \leq i<j \leq d}$ are dual to $\left\{\boldsymbol{T}_{i, j}\right\}_{0 \leq i<j \leq d}$ with respect to the Frobenius inner product (cf. [9, Section 3.1] and also §2.5). Choosing $\varsigma=$ $\sum_{0 \leq i<j \leq d} q_{i j} \boldsymbol{N}_{i, j} \in \mathbb{P}_{k-2}(K ; \mathbb{S})$, we get

$$
(\boldsymbol{\tau}, \boldsymbol{\varsigma})_{K}=\sum_{0 \leq i<j \leq d}\left(\lambda_{i} \lambda_{j}, q_{i j}^{2}\right)_{K}=0 .
$$

Hence $q_{i j}=0$ for all $i, j$, and then $\boldsymbol{\tau}=\mathbf{0}$. As the dimensions match, we conclude the result.

Another characterization of $\mathbb{B}_{k}(\operatorname{div}, K ; \mathbb{S})$ and $\mathbb{B}_{k}^{\prime}(\operatorname{div}, K ; \mathbb{S})$ is given in [14].
4.3. Trace spaces. The mapping $\operatorname{tr}^{\text {div }}: \mathbb{P}_{k}(K ; \mathbb{S}) \rightarrow \mathbb{P}_{k}\left(\mathcal{F}^{1}\left(K ; \mathbb{R}^{d-1}\right)\right)$ is not onto due to the symmetry. Some compatible conditions should be imposed on lower dimensional simplexes. Fortunately, we only need its dimension.

Lemma 4.4. Let integer $k \geq 1$. It holds

$$
\begin{aligned}
\operatorname{dim} \operatorname{tr}^{\operatorname{div}}\left(\mathbb{P}_{k}(K ; \mathbb{S})\right) & =\operatorname{dim} \mathbb{P}_{k}(K ; \mathbb{S})-\operatorname{dim} \mathbb{B}_{k}(\operatorname{div}, K ; \mathbb{S}) \\
& =\operatorname{dim} \mathbb{H}_{k}(K ; \mathbb{S})+\operatorname{dim} \mathbb{H}_{k-1}(K ; \mathbb{S}) \\
& =\frac{1}{2} d(d+1)\left[\binom{d+k-1}{d-1}+\binom{d+k-2}{d-1}\right]
\end{aligned}
$$

We show the super-smoothness induced by the symmetry for $H(\operatorname{div} ; \mathbb{S})$ element. For a $(d-r)$-dimensional face $e \in \mathcal{F}^{r}(K)$ with $r=2, \cdots, d$ shared by two ( $d-1$ )dimensional faces $F, F^{\prime} \in \mathcal{F}^{1}(K)$, by the symmetry of $\boldsymbol{\tau},\left.\left(\boldsymbol{n}_{F}^{\top} \boldsymbol{\tau} \boldsymbol{n}_{F^{\prime}}\right)\right|_{e}$ is concurrently determined by $\left.\left(\boldsymbol{\tau} \boldsymbol{n}_{F}\right)\right|_{F}$ and $\left.\left(\boldsymbol{\tau} \boldsymbol{n}_{F^{\prime}}\right)\right|_{F^{\prime}}$. This implies the degrees of freedom $\boldsymbol{n}_{i}^{\top} \boldsymbol{\tau} \boldsymbol{n}_{j}$ on $e$ for all $i, j=1, \cdots, r$. In particular, for a 0 -dimensional vertex $\delta,\left(\boldsymbol{\tau}_{i j}(\delta)\right)_{d \times d}$ is taken as a degree of freedom.

The trace $\boldsymbol{\tau} \boldsymbol{n}$ restricted to a face $F \in \mathcal{F}^{1}(K)$ can be further split into two components: 1) the normal-normal component $\boldsymbol{n}^{\top} \boldsymbol{\tau} \boldsymbol{n}$ will be determined by $\boldsymbol{n}_{i}^{\top} \boldsymbol{\tau} \boldsymbol{n}_{j} ; 2$ ) the tangential-normal component $\Pi_{F} \boldsymbol{\tau} \boldsymbol{n}$ will be determined by the interior moments relative to $F$ after the trace $\operatorname{tr}^{\operatorname{div}_{F}}\left(\Pi_{F} \boldsymbol{\tau} \boldsymbol{n}\right)=\boldsymbol{n}_{F, e}^{\top} \boldsymbol{\tau} \boldsymbol{n}$ has been determined.

Lemma 4.5. A basis of

$$
\left(\operatorname{tr}^{\operatorname{div}}\left(\mathbb{P}_{k}(K ; \mathbb{S})\right)\right)^{\prime}
$$

is given by the degrees of freedom

$$
\begin{align*}
\boldsymbol{\tau}(\delta) & \forall \delta \in \mathcal{V}(K),  \tag{4.3}\\
\left(\boldsymbol{n}_{i}^{\top} \boldsymbol{\tau} \boldsymbol{n}_{j}, q\right)_{F} & \forall q \in \mathbb{P}_{k+r-d-1}(F), F \in \mathcal{F}^{r}(K),  \tag{4.4}\\
& i, j=1, \cdots, r, \text { and } r=1, \cdots, d-1, \\
\left(\Pi_{F} \boldsymbol{\tau} \boldsymbol{n}, \boldsymbol{q}\right)_{F} & \forall \boldsymbol{q} \in \mathrm{ND}_{k-2}(F), F \in \mathcal{F}^{1}(K) . \tag{4.5}
\end{align*}
$$

Proof. We first prove that if all the degrees of freedom (4.3)-(4.5) vanish, then $\boldsymbol{\tau}=\mathbf{0}$. As $\left.\boldsymbol{n}_{i}^{\top} \boldsymbol{\tau} \boldsymbol{n}_{j}\right|_{F} \in \mathbb{P}_{k}(F)$, by the vanishing degrees of freedom (4.3)-(4.4) and the uni-solvence of the Lagrange element, we get

$$
\left.\boldsymbol{n}_{i}^{\top} \boldsymbol{\tau} \boldsymbol{n}_{j}\right|_{F}=0 \quad \forall F \in \mathcal{F}^{r}(K), i, j=1, \cdots, d-r, \text { and } r=1, \cdots, d-1
$$

This implies

$$
\begin{equation*}
\left.\boldsymbol{n}^{\top} \boldsymbol{\tau} \boldsymbol{n}\right|_{F}=0,\left.\boldsymbol{n}_{F, e}^{\top} \boldsymbol{\tau} \boldsymbol{n}\right|_{e}=0 \quad \forall F \in \mathcal{F}^{1}(K), e \in \mathcal{F}^{1}(F) \tag{4.6}
\end{equation*}
$$

Notice that $\left.\Pi_{F} \boldsymbol{\tau} \boldsymbol{n}\right|_{F} \in \mathbb{P}_{k}\left(F ; \mathbb{R}^{d-1}\right)$. Due to the uni-solvence of the BDM element on $F$, cf. Theorem 3.15, we acquire from the second identity in (4.6) and the vanishing degrees of freedom (4.5) that $\left.\Pi_{F} \boldsymbol{\tau} \boldsymbol{n}\right|_{F}=\mathbf{0}$, which together with the first identity in (4.6) yields $\left.\boldsymbol{\tau} \boldsymbol{n}\right|_{F}=\mathbf{0}$.

We then count the dimension to finish the proof. By comparing DoFs of Hu element (cf. Remark 4.6) and DoFs (4.3)-(4.5), it follows from the DoFs of the first kind Nédélec element, cf. [25, 4], that the number of DoFs (4.3)-(4.5) is equal to the number of DoFs of Hu element, thus equals to $\operatorname{dim} \operatorname{tr}^{\operatorname{div}}\left(\mathbb{P}_{k}(K ; \mathbb{S})\right)$.

Remark 4.6. As a comparison, the degrees of freedom of Hu element on boundary in [20] are

$$
\begin{aligned}
\boldsymbol{\tau}(\delta) & \forall \delta \in \mathcal{V}(K), \\
\left(\boldsymbol{n}_{i}^{\top} \boldsymbol{\tau} \boldsymbol{n}_{j}, q\right)_{F} & \forall q \in \mathbb{P}_{k+r-d-1}(F), F \in \mathcal{F}^{r}(K), \\
& i, j=1, \cdots, r, \text { and } r=1, \cdots, d-1, \\
\left(\boldsymbol{t}_{i}^{\top} \boldsymbol{\tau} \boldsymbol{n}_{j}, q\right)_{F} & \forall q \in \mathbb{P}_{k+r-d-1}(F), F \in \mathcal{F}^{r}(K), \\
& i=1, \cdots, d-r, j=1, \cdots, r, \text { and } r=1, \cdots, d-1 .
\end{aligned}
$$

The difference is the way to impose the tangential-normal component.
4.4. Split of the bubble space. To construct $\boldsymbol{H}(\operatorname{div}, K ; \mathbb{S})$ elements, the interior degrees of freedom given by $\mathcal{N}\left(\mathbb{P}_{k-2}(K ; \mathbb{S})\right)$ are enough. For the construction of $\boldsymbol{H}(\operatorname{div} \operatorname{div}, K ; \mathbb{S})$ element, we use div operator to decompose $\mathbb{B}_{k}(\operatorname{div}, K ; \mathbb{S})$ into

$$
E_{0, k}(\mathbb{S}):=\mathbb{B}_{k}(\operatorname{div}, K ; \mathbb{S}) \cap \operatorname{ker}(\operatorname{div}), \quad E_{0, k}^{\perp}(\mathbb{S}):=\mathbb{B}_{k}(\operatorname{div}, K ; \mathbb{S}) / E_{0, k}(\mathbb{S})
$$

We will abbreviate $E_{0, k}(\mathbb{S})$ and $E_{0, k}^{\perp}(\mathbb{S})$ as $E_{0}(\mathbb{S})$ and $E_{0}^{\perp}(\mathbb{S})$ respectively if not causing any confusion. As before we can characterize the dual space of $E_{0, k}^{\perp}(\mathbb{S})$ through div*, which is - def $:=-$ sym grad restricting to the bubble space and can be extended to $H^{1}\left(K ; \mathbb{R}^{d}\right)$.

Lemma 4.7. Let integer $k \geq 2$. The mapping

$$
\operatorname{div}: E_{0, k}^{\perp}(\mathbb{S}) \rightarrow \mathbb{P}_{k-1, \mathrm{RM}}^{\perp}:=\mathbb{P}_{k-1}\left(K, \mathbb{R}^{d}\right) / \operatorname{ker}(\operatorname{def})
$$

is a bijection and consequently

$$
\begin{aligned}
\left(E_{0, k}^{\perp}(\mathbb{S})\right)^{\prime} & =\mathcal{N}\left(\operatorname{def} \mathbb{P}_{k-1}\left(K, \mathbb{R}^{d}\right)\right) \\
\operatorname{dim} E_{0, k}^{\perp}(\mathbb{S}) & =d\binom{k+d-1}{k-1}-\frac{1}{2}\left(d^{2}+d\right)
\end{aligned}
$$

Proof. The fact $\operatorname{div} \mathbb{B}_{k}(\operatorname{div}, K ; \mathbb{S})=\mathbb{P}_{k-1, \mathrm{RM}}^{\perp}$ was proved in [20, Theorem 2.2]. Here we recall it for the completeness.

The inclusion $\operatorname{div}\left(\mathbb{B}_{k}(\operatorname{div}, K ; \mathbb{S})\right) \subseteq \mathbb{P}_{k-1, \mathrm{RM}}^{\perp}$ can be proved through integration by parts

$$
(\operatorname{div} \boldsymbol{\tau}, \boldsymbol{v})_{K}=-(\boldsymbol{\tau}, \operatorname{def} \boldsymbol{v})_{K}=0 \quad \forall \boldsymbol{v} \in \operatorname{ker}(\operatorname{def}) .
$$

If $\operatorname{div}\left(\mathbb{B}_{k}(\operatorname{div}, K ; \mathbb{S})\right) \neq \mathbb{P}_{k-1, \mathrm{RM}}^{\perp}$, then there exists a function $\boldsymbol{v} \in \mathbb{P}_{k-1, \mathrm{RM}}^{\perp}$ satisfying $\boldsymbol{v} \perp \operatorname{div}\left(\mathbb{B}_{k}(\operatorname{div}, K ; \mathbb{S})\right)$, which is equivalent to $\operatorname{def} \boldsymbol{v} \perp \mathbb{B}_{k}(\operatorname{div}, K ; \mathbb{S})$. Expand the symmetric matrix def $\boldsymbol{v}$ in the basis $\left\{\boldsymbol{N}_{i j}, 0 \leq i<j \leq d\right\}$ as def $\boldsymbol{v}=\sum_{0 \leq i<j \leq d} q_{i j} \boldsymbol{N}_{i j}$ with $q_{i j} \in \mathbb{P}_{k-2}(K)$. Then set $\boldsymbol{\tau}_{v}=\sum_{0 \leq i<j \leq d} q_{i j} \lambda_{i} \lambda_{j} \boldsymbol{T}_{i j} \in \mathbb{B}_{k}(\operatorname{div}, K ; \mathbb{S})$. We have

$$
\left(\operatorname{def} \boldsymbol{v}, \boldsymbol{\tau}_{v}\right)_{K}=\sum_{0 \leq i<j \leq d} \int_{K} q_{i j}^{2} \lambda_{i} \lambda_{j} \mathrm{~d} x=0
$$

which implies $q_{i j}=0$ for all $0 \leq i<j \leq d$, i.e., $\operatorname{def} \boldsymbol{v}=0$ and $\boldsymbol{v}=0$ as $\boldsymbol{v} \in \mathbb{P}_{k-1, \mathrm{RM}}^{\perp}$. Since $\operatorname{div} E_{0, k}^{\perp}(\mathbb{S})=\operatorname{div} \mathbb{B}_{k}(\operatorname{div}, K ; \mathbb{S})$, the mapping div $: E_{0, k}^{\perp}(\mathbb{S}) \rightarrow \mathbb{P}_{k-1, \mathrm{RM}}^{\perp}$ is a bijection.

For $\boldsymbol{v} \in E_{0, k}^{\perp}(\mathbb{S}),(\boldsymbol{v}, \operatorname{def} \boldsymbol{q})_{K}=0$ for all $\boldsymbol{q} \in \mathbb{P}_{k-1}\left(K, \mathbb{R}^{d}\right)$ implies $\operatorname{div} \boldsymbol{v}=$ 0, i.e., $\boldsymbol{v} \in E_{0, k}(\mathbb{S})$. Then $\boldsymbol{v} \in E_{0, k}(\mathbb{S}) \cap E_{0, k}^{\perp}(\mathbb{S})=\{\mathbf{0}\}$. Hence $\left(E_{0, k}^{\perp}(\mathbb{S})\right)^{\prime}=$ $\mathcal{I}^{\prime} \mathcal{N}\left(\operatorname{def} \mathbb{P}_{k-1}\left(K, \mathbb{R}^{d}\right)\right)$. As the dimensions match, $\mathcal{I}^{\prime}$ is a bijection.

We then move to the space $E_{0, k}(\mathbb{S})$. Using the primary approach, we need the bubble space in the previous space and the differential operator. For example, we have $E_{0, k}(\mathbb{S})=\operatorname{curl} \operatorname{curl}\left(\mathbb{P}_{k+2}(K) \cap H_{0}^{2}(K)\right)$ in two dimensions [6], and in three dimensions [3, 12]

$$
E_{0, k}(\mathbb{S})=\operatorname{inc} \mathbb{B}_{k+2}(\mathrm{inc}, K ; \mathbb{S})
$$

with

$$
\begin{aligned}
\mathbb{B}_{k+2}(\text { inc, } K ; \mathbb{S}):= & \left\{\boldsymbol{\tau} \in \mathbb{P}_{k+2}(K ; \mathbb{S}): \boldsymbol{n} \times \boldsymbol{\tau} \times \boldsymbol{n}=\mathbf{0}\right. \\
& \left.2 \operatorname{def}_{F}\left(\boldsymbol{n} \cdot \boldsymbol{\tau} \Pi_{F}\right)-\Pi_{F} \partial_{n} \boldsymbol{\tau} \Pi_{F}=\mathbf{0} \forall F \in \mathcal{F}^{1}(K)\right\} .
\end{aligned}
$$

Such characterization is hard to be generalized to arbitrary dimension.
Instead we use the dual approach to identify $E_{0, k}^{\prime}(\mathbb{S})$. To this end, denote the space of rigid motions by

$$
\boldsymbol{R} \boldsymbol{M}:=\mathrm{ND}_{0}(K)=\left\{\boldsymbol{c}+\boldsymbol{N} \boldsymbol{x}: \boldsymbol{c} \in \mathbb{R}^{d}, \boldsymbol{N} \in \mathbb{K}\right\}
$$

Define operator $\boldsymbol{\pi}_{R M}: \mathcal{C}^{1}\left(D ; \mathbb{R}^{d}\right) \rightarrow \boldsymbol{R} \boldsymbol{M}$ as

$$
\boldsymbol{\pi}_{R M} \boldsymbol{v}:=\boldsymbol{v}(\mathbf{0})+(\operatorname{skw}(\nabla \boldsymbol{v}))(\mathbf{0}) \boldsymbol{x}
$$

Clearly it holds

$$
\boldsymbol{\pi}_{R M} \boldsymbol{v}=\boldsymbol{v} \quad \forall \boldsymbol{v} \in \boldsymbol{R} \boldsymbol{M}
$$

We shall establish the following short exact sequence

$$
\boldsymbol{R} \boldsymbol{M} \underset{\boldsymbol{\pi}_{R M}}{\subset} \mathbb{P}_{k+1}\left(D ; \mathbb{R}^{d}\right) \underset{\boldsymbol{\tau x}}{\stackrel{\text { def }}{\rightleftarrows}} \operatorname{def} \mathbb{P}_{k+1}(D ; \mathbb{S}) \rightleftarrows \mathbf{0}
$$

and derive a space decomposition from it.
Lemma 4.8. Let integer $k \geq 0$. If $\boldsymbol{q} \in \mathbb{P}_{k+1}\left(D ; \mathbb{R}^{d}\right)$ satisfying $(\operatorname{def} \boldsymbol{q}) \boldsymbol{x}=\mathbf{0}$, then $\boldsymbol{q} \in \boldsymbol{R M}$.

Proof. Since $\boldsymbol{x}^{\boldsymbol{\top}}(\boldsymbol{x} \cdot \nabla) \boldsymbol{q}=\boldsymbol{x}^{\boldsymbol{\top}}(\nabla \boldsymbol{q}) \boldsymbol{x}=\boldsymbol{x}^{\boldsymbol{\top}}(\operatorname{def} \boldsymbol{q}) \boldsymbol{x}=0$, we get

$$
(\boldsymbol{x} \cdot \nabla)\left(\boldsymbol{x}^{\top} \boldsymbol{q}\right)=\boldsymbol{x}^{\boldsymbol{\top}}(\boldsymbol{x} \cdot \nabla) \boldsymbol{q}+\boldsymbol{x}^{\top} \boldsymbol{q}=\boldsymbol{x}^{\top} \boldsymbol{q}
$$

By (2.1), this indicates $\boldsymbol{x}^{\top} \boldsymbol{q} \in \mathbb{P}_{1}(D)$. Noting that $(\nabla \boldsymbol{q}) \boldsymbol{x}=\nabla\left(\boldsymbol{x}^{\top} \boldsymbol{q}\right)-\boldsymbol{q}$, we obtain

$$
(\boldsymbol{x} \cdot \nabla) \boldsymbol{q}+\left(\nabla\left(\boldsymbol{x}^{\top} \boldsymbol{q}\right)-\boldsymbol{q}\right)=(\nabla \boldsymbol{q})^{\top} \boldsymbol{x}+(\nabla \boldsymbol{q}) \boldsymbol{x}=2(\operatorname{def} \boldsymbol{q}) \boldsymbol{x}=\mathbf{0}
$$

which implies $(\boldsymbol{x} \cdot \nabla) \boldsymbol{q}-\boldsymbol{q}=-\nabla\left(\boldsymbol{x}^{\boldsymbol{\top}} \boldsymbol{q}\right) \in \mathbb{P}_{0}\left(D ; \mathbb{R}^{d}\right)$. Hence $\boldsymbol{q} \in \mathbb{P}_{1}\left(D ; \mathbb{R}^{d}\right)$. Assume $\boldsymbol{q}=\boldsymbol{N} \boldsymbol{x}+\boldsymbol{C}$ with $\boldsymbol{N} \in \mathbb{M}$ and $\boldsymbol{C} \in \mathbb{R}^{d}$. Then

$$
\boldsymbol{x}^{\boldsymbol{\top}}(\operatorname{sym} \boldsymbol{N}) \boldsymbol{x}+\boldsymbol{x}^{\top} \boldsymbol{C}=\boldsymbol{x}^{\top} \boldsymbol{N} \boldsymbol{x}+\boldsymbol{x}^{\top} \boldsymbol{C}=\boldsymbol{x}^{\top} \boldsymbol{q} \in \mathbb{P}_{1}(D),
$$

which implies $\operatorname{sym} \boldsymbol{N}=\mathbf{0}$. Therefore $\boldsymbol{N} \in \mathbb{K}$ and $\boldsymbol{q} \in \boldsymbol{R} \boldsymbol{M}$.
Lemma 4.9. Let integer $k \geq 0$. We have

$$
\begin{equation*}
\left(\operatorname{def} \mathbb{P}_{k+1}\left(D ; \mathbb{R}^{d}\right)\right) \boldsymbol{x}=\mathbb{P}_{k}(D ; \mathbb{S}) \boldsymbol{x}=\mathbb{P}_{k+1}\left(D ; \mathbb{R}^{d}\right) \cap \operatorname{ker}\left(\boldsymbol{\pi}_{R M}\right) \tag{4.7}
\end{equation*}
$$

Proof. For any $\boldsymbol{\tau} \in \mathbb{P}_{k}(D ; \mathbb{S})$, it follows

$$
\boldsymbol{\pi}_{R M}(\boldsymbol{\tau} \boldsymbol{x})=(\operatorname{skw}(\nabla(\boldsymbol{\tau} \boldsymbol{x})))(\mathbf{0}) \boldsymbol{x}=\operatorname{skw}(\boldsymbol{\tau}(\mathbf{0})) \boldsymbol{x}=\mathbf{0}
$$

Thus $\mathbb{P}_{k}(D ; \mathbb{S}) \boldsymbol{x} \subseteq \mathbb{P}_{k+1}\left(D ; \mathbb{R}^{d}\right) \cap \operatorname{ker}\left(\boldsymbol{\pi}_{R M}\right)$. On the other hand, we obtain from Lemma 4.8 that

$$
\operatorname{dim}\left(\left(\operatorname{def} \mathbb{P}_{k+1}\left(D ; \mathbb{R}^{d}\right)\right) \boldsymbol{x}\right)=\operatorname{dim} \mathbb{P}_{k+1}\left(D ; \mathbb{R}^{d}\right)-\operatorname{dim} \boldsymbol{R} \boldsymbol{M}
$$

which equals to the dimension of $\mathbb{P}_{k+1}\left(D ; \mathbb{R}^{d}\right) \cap \operatorname{ker}\left(\boldsymbol{\pi}_{R M}\right)$. Thus (4.7) follows.
We denote by $\boldsymbol{x}: \mathbb{P}_{k}(D ; \mathbb{S}) \rightarrow \mathbb{P}_{k+1}\left(D ; \mathbb{R}^{d}\right)$ the mapping $\boldsymbol{\tau} \rightarrow \boldsymbol{\tau} \boldsymbol{x}$ as the matrixvector product $\boldsymbol{\tau} \boldsymbol{x}$ is applying row-wise inner product with vector $\boldsymbol{x}$.

Corollary 4.10. Let integer $k \geq 0$. We have the space decomposition

$$
\begin{equation*}
\mathbb{P}_{k}(D ; \mathbb{S})=\operatorname{def} \mathbb{P}_{k+1}\left(D ; \mathbb{R}^{d}\right) \oplus\left(\operatorname{ker}(\cdot \boldsymbol{x}) \cap \mathbb{P}_{k}(D ; \mathbb{S})\right) \tag{4.8}
\end{equation*}
$$

Proof. It follows from Lemma 4.8 that $\operatorname{def} \mathbb{P}_{k+1}\left(D ; \mathbb{R}^{d}\right) \cap\left(\operatorname{ker}(\cdot \boldsymbol{x}) \cap \mathbb{P}_{k}(D ; \mathbb{S})\right)=$ $\{\mathbf{0}\}$. Due to (4.7),

$$
\begin{aligned}
& \operatorname{dim} \operatorname{def} \mathbb{P}_{k+1}\left(D ; \mathbb{R}^{d}\right)+\operatorname{dim}\left(\operatorname{ker}(\cdot \boldsymbol{x}) \cap \mathbb{P}_{k}(D ; \mathbb{S})\right) \\
= & \operatorname{dim} \operatorname{def} \mathbb{P}_{k+1}\left(D ; \mathbb{R}^{d}\right)+\operatorname{dim} \mathbb{P}_{k}(D ; \mathbb{S})-\operatorname{dim}\left(\mathbb{P}_{k}(D ; \mathbb{S}) \boldsymbol{x}\right) \\
= & \operatorname{dim} \mathbb{P}_{k+1}\left(D ; \mathbb{R}^{d}\right)-\operatorname{dim} \boldsymbol{R} \boldsymbol{M}+\operatorname{dim} \mathbb{P}_{k}(D ; \mathbb{S})-\operatorname{dim}\left(\mathbb{P}_{k}(D ; \mathbb{S}) \boldsymbol{x}\right) \\
= & \operatorname{dim} \mathbb{P}_{k}(D ; \mathbb{S})
\end{aligned}
$$

which means (4.8).
Remark 4.11. In two and three dimensions, we have (cf. [11, 12])

$$
\operatorname{ker}(\cdot \boldsymbol{x}) \cap \mathbb{P}_{k}(D ; \mathbb{S})= \begin{cases}\boldsymbol{x}^{\perp}\left(\boldsymbol{x}^{\perp}\right)^{\top} \mathbb{P}_{k-2}(D), & \text { for } d=2 \\ \boldsymbol{x} \times \mathbb{P}_{k-2}(D ; \mathbb{S}) \times \boldsymbol{x}, & \text { for } d=3\end{cases}
$$

where $\boldsymbol{x}^{\perp}:=\binom{x_{2}}{-x_{1}}$, but generalization to arbitrary dimension is not easy and not necessary. A computation approach to find an explicit basis of $\operatorname{ker}(\cdot \boldsymbol{x}) \cap \mathbb{P}_{k-1}(K ; \mathbb{S})$ is as follows. Find a basis for $\mathbb{P}_{k-1}(K ; \mathbb{S})$ and one for $\mathbb{P}_{k}\left(K ; \mathbb{R}^{d}\right)$. Then form the matrix representation $X$ of the operator $\cdot \boldsymbol{x}$. Afterwards the null space $\operatorname{ker}(X)$ can be found algebraically.

Lemma 4.12. Let integer $k \geq 2$. We have

$$
\begin{equation*}
E_{0, k}^{\prime}(\mathbb{S})=\mathcal{N}\left(\operatorname{ker}(\cdot \boldsymbol{x}) \cap \mathbb{P}_{k-2}(K ; \mathbb{S})\right) \tag{4.9}
\end{equation*}
$$

That is a function $\boldsymbol{\tau} \in E_{0, k}(\mathbb{S})$ is uniquely determined by

$$
(\boldsymbol{\tau}, \boldsymbol{q})_{K} \quad \forall \boldsymbol{q} \in \operatorname{ker}(\cdot \boldsymbol{x}) \cap \mathbb{P}_{k-2}(K ; \mathbb{S})
$$

And

$$
\operatorname{dim} E_{0, k}(\mathbb{S})=\frac{d(d+1)}{2}\binom{k-2+d}{d}-d\binom{d+k-1}{d}+\frac{d(d+1)}{2}
$$

Proof. We apply Lemma 2.2 with $V=\mathbb{P}_{k}(K ; \mathbb{S}), U=\mathbb{P}_{k-2}(K ; \mathbb{S})$, and $\kappa=\cdot \boldsymbol{x}$. Lemma 4.3 and (4.7)-(4.8) verify the assumptions (B1)-(B2), and we only need to count the dimension.

By the space decomposition (4.8), Lemma 4.3 and Lemma 4.7,

$$
\begin{aligned}
\operatorname{dim}\left(\operatorname{ker}(\cdot \boldsymbol{x}) \cap \mathbb{P}_{k-2}(K ; \mathbb{S})\right) & =\operatorname{dim} \mathbb{P}_{k-2}(K ; \mathbb{S})-\operatorname{dim} \operatorname{def} \mathbb{P}_{k-1}\left(K ; \mathbb{R}^{d}\right) \\
& =\operatorname{dim} \mathbb{B}_{k}(\operatorname{div}, K ; \mathbb{S})-\operatorname{dim} E_{0, k}^{\perp}(\mathbb{S})=\operatorname{dim} E_{0, k}(\mathbb{S})
\end{aligned}
$$

as required.
4.5. $H($ div; $\mathbb{S})$-conforming elements. Combining Lemmas 4.5, 4.7, 4.12 and space decomposition (4.8) yields the degrees of freedom of $H$ (div; $\mathbb{S}$ )-conforming elements.

Theorem $4.13\left(\mathbb{P}_{k}(K ; \mathbb{S})\right.$-type $H(\operatorname{div} ; \mathbb{S})$-conforming elements). Take the shape function space $V(\mathbb{S})=\mathbb{P}_{k}(K ; \mathbb{S})$ with $k \geq d+1$. The degrees of freedom

$$
\begin{align*}
\boldsymbol{\tau}(\delta) & \forall \delta \in \mathcal{V}(K),  \tag{4.10}\\
\left(\boldsymbol{n}_{i}^{\top} \boldsymbol{\tau} \boldsymbol{n}_{j}, q\right)_{F} & \forall q \in \mathbb{P}_{k+r-d-1}(F), F \in \mathcal{F}^{r}(K),  \tag{4.11}\\
& i, j=1, \cdots, r, \text { and } r=1, \cdots, d-1, \\
\left(\Pi_{F} \boldsymbol{\tau} \boldsymbol{n}, \boldsymbol{q}\right)_{F} & \forall \boldsymbol{q} \in \mathrm{ND}_{k-2}(F), F \in \mathcal{F}^{1}(K),  \tag{4.12}\\
(\boldsymbol{\tau}, \boldsymbol{q})_{K} & \forall \boldsymbol{q} \in \mathbb{P}_{k-2}(K ; \mathbb{S}) \tag{4.13}
\end{align*}
$$

are uni-solvent for $\mathbb{P}_{k}(K ; \mathbb{S})$. The last degree of freedom (4.13) can be replaced by

$$
\begin{align*}
(\operatorname{div} \boldsymbol{\tau}, \boldsymbol{q})_{K} & \forall \boldsymbol{q} \in \mathbb{P}_{k-1}\left(K ; \mathbb{R}^{d}\right) / \boldsymbol{R} \boldsymbol{M}  \tag{4.14}\\
(\boldsymbol{\tau}, \boldsymbol{q})_{K} & \forall \boldsymbol{q} \in \operatorname{ker}(\cdot \boldsymbol{x}) \cap \mathbb{P}_{k-2}(K ; \mathbb{S}) \tag{4.15}
\end{align*}
$$

The global finite element space $\boldsymbol{V}_{h}(\operatorname{div} ; \mathbb{S}) \subset H(\operatorname{div}, \Omega ; \mathbb{S})$, where
$\boldsymbol{V}_{h}(\operatorname{div} ; \mathbb{S}):=\left\{\boldsymbol{\tau} \in \boldsymbol{L}^{2}(\Omega ; \mathbb{S}):\left.\boldsymbol{\tau}\right|_{K} \in \mathbb{P}_{k}(K ; \mathbb{S})\right.$ for each $K \in \mathcal{T}_{h}$,
the degrees of freedom (4.10)-(4.12) are single-valued $\}$.
Clearly $\boldsymbol{V}_{h}(\operatorname{div} ; \mathbb{S}) \subset H(\operatorname{div}, \Omega ; \mathbb{S})$ follows from the proof of Lemma 4.5.
For the most important three dimensional case, the degrees of freedom (4.10)(4.13) become

$$
\begin{aligned}
\boldsymbol{\tau}(\delta) & \forall \delta \in \mathcal{V}(K), \\
\left(\boldsymbol{n}_{i}^{\top} \boldsymbol{\tau} \boldsymbol{n}_{j}, q\right)_{e} & \forall q \in \mathbb{P}_{k-2}(e), e \in \mathcal{F}^{2}(K), i, j=1,2, \\
\left(\boldsymbol{n}^{\top} \boldsymbol{\tau} \boldsymbol{n}, q\right)_{F} & \forall q \in \mathbb{P}_{k-3}(F), F \in \mathcal{F}^{1}(K), \\
\left(\Pi_{F} \boldsymbol{\tau} \boldsymbol{n}, \boldsymbol{q}\right)_{F} & \forall \boldsymbol{q} \in \mathrm{ND}_{k-2}(F), F \in \mathcal{F}^{1}(K), \\
(\boldsymbol{\tau}, \boldsymbol{q})_{K} & \forall \boldsymbol{q} \in \mathbb{P}_{k-2}(K ; \mathbb{S}),
\end{aligned}
$$

which are slightly different from Hu-Zhang element in three dimensions [22].
Uni-solvence holds for $k \geq 1$. The requirement $k \geq d+1$ contains the degrees of freedom $(\boldsymbol{\tau} \boldsymbol{n}, \boldsymbol{q})_{F}$ for all $\boldsymbol{q} \in \mathbb{P}_{1}\left(F ; \mathbb{R}^{d-1}\right)$ on each face $F \in \mathcal{F}^{1}(K)$, by which the divergence of the global $H$ (div; $\mathbb{S}$ )-conforming element space will include the piecewise $\boldsymbol{R} \boldsymbol{M}$ space and combining with $\operatorname{div} \mathbb{B}_{k}(\operatorname{div}, K ; \mathbb{S})=\mathbb{P}_{k-1, \mathrm{RM}}^{\perp}$ will imply the following discrete inf-sup condition.

Lemma 4.14. Let $k \geq d+1$. It holds the inf-sup condition

$$
\left\|\boldsymbol{p}_{h}\right\|_{0} \lesssim \sup _{\boldsymbol{\tau}_{h} \in \boldsymbol{V}_{h}(\mathrm{div} ; \mathbb{S})} \frac{\left(\operatorname{div} \boldsymbol{\tau}_{h}, \boldsymbol{p}_{h}\right)}{\left\|\boldsymbol{\tau}_{h}\right\|_{H(\operatorname{div})}} \quad \forall \boldsymbol{p}_{h} \in \mathbb{P}_{k-1}\left(\mathcal{T}_{h} ; \mathbb{R}^{d}\right)
$$

where $\mathbb{P}_{k-1}\left(\mathcal{T}_{h} ; \mathbb{R}^{d}\right):=\left\{\boldsymbol{p}_{h} \in \boldsymbol{L}^{2}\left(\Omega ; \mathbb{R}^{d}\right):\left.\boldsymbol{p}_{h}\right|_{K} \in \mathbb{P}_{k-1}\left(K ; \mathbb{R}^{d}\right)\right.$ for each $\left.K \in \mathcal{T}_{h}\right\}$.

Proof. For $\boldsymbol{p}_{h} \in \mathbb{P}_{k-1}\left(\mathcal{T}_{h} ; \mathbb{R}^{d}\right)$, there exists $\boldsymbol{\tau} \in \boldsymbol{H}^{1}(\Omega ; \mathbb{S})$ such that [18]

$$
\operatorname{div} \boldsymbol{\tau}=\boldsymbol{p}_{h}, \quad\|\boldsymbol{\tau}\|_{1} \lesssim\left\|\boldsymbol{p}_{h}\right\|_{0}
$$

Let $\boldsymbol{\tau}_{h} \in \boldsymbol{V}_{h}(\operatorname{div} ; \mathbb{S})$ such that all the DoFs (4.10)-(4.12) and (4.14)-(4.15) vanish except

$$
\begin{aligned}
\left(\boldsymbol{\tau}_{h} \boldsymbol{n}, \boldsymbol{q}\right)_{F} & =(\boldsymbol{\tau} \boldsymbol{n}, \boldsymbol{q})_{F} & & \forall \boldsymbol{q} \in \mathbb{P}_{1}\left(F ; \mathbb{R}^{d}\right), F \in \mathcal{F}^{1}(K), \\
\left(\operatorname{div} \boldsymbol{\tau}_{h}, \boldsymbol{q}\right)_{K} & =(\operatorname{div} \boldsymbol{\tau}, \boldsymbol{q})_{K}=\left(\boldsymbol{p}_{h}, \boldsymbol{q}\right)_{K} & & \forall \boldsymbol{q} \in \mathbb{P}_{k-1}\left(K ; \mathbb{R}^{d}\right) / \boldsymbol{R} \boldsymbol{M}
\end{aligned}
$$

for all $K \in \mathcal{T}_{h}$. By the scaling argument, we have

$$
\begin{equation*}
\left\|\boldsymbol{\tau}_{h}\right\|_{0} \lesssim\|\boldsymbol{\tau}\|_{1} \lesssim\left\|\boldsymbol{p}_{h}\right\|_{0} \tag{4.16}
\end{equation*}
$$

Applying the integration by parts,

$$
\left(\operatorname{div} \boldsymbol{\tau}_{h}, \boldsymbol{q}\right)_{K}=(\operatorname{div} \boldsymbol{\tau}, \boldsymbol{q})_{K}=\left(\boldsymbol{p}_{h}, \boldsymbol{q}\right)_{K} \quad \forall \boldsymbol{q} \in \boldsymbol{R} \boldsymbol{M} .
$$

Hence

$$
\left(\operatorname{div} \boldsymbol{\tau}_{h}, \boldsymbol{q}\right)_{K}=(\operatorname{div} \boldsymbol{\tau}, \boldsymbol{q})_{K}=\left(\boldsymbol{p}_{h}, \boldsymbol{q}\right)_{K} \quad \forall \boldsymbol{q} \in \mathbb{P}_{k-1}\left(K ; \mathbb{R}^{d}\right)
$$

which implies $\operatorname{div} \boldsymbol{\tau}_{h}=\boldsymbol{p}_{h}$. Therefore we derive the inf-sup condition from (4.16).
4.6. $\mathbb{P}_{k+1}^{-}(K ; \mathbb{S})$-type elements. Let $k \geq d+1$. The space of shape functions is taken as

$$
\mathbb{P}_{k+1}^{-}(K ; \mathbb{S}):=\mathbb{P}_{k}(K ; \mathbb{S})+E_{0, k+1}^{\perp}(\mathbb{S})
$$

Since $E_{0, k+1}^{\perp}(\mathbb{S}) \subseteq \mathbb{B}_{k+1}(\operatorname{div}, K ; \mathbb{S})$ and $\operatorname{div} E_{0, k+1}^{\perp}(\mathbb{S})=\mathbb{P}_{k, \mathrm{RM}}^{\perp}$, we have

$$
\operatorname{tr}^{\operatorname{div}} \mathbb{P}_{k+1}^{-}(K ; \mathbb{S})=\operatorname{tr}^{\operatorname{div}}\left(\mathbb{P}_{k}(K ; \mathbb{S})\right), \quad \operatorname{div} \mathbb{P}_{k+1}^{-}(K ; \mathbb{S})=\mathbb{P}_{k}\left(K ; \mathbb{R}^{d}\right)
$$

By applying Lemma 2.3 with $\mathrm{d}=\operatorname{div}, V=\mathbb{P}_{k}(K ; \mathbb{S}), \mathbb{H}=E_{0, k+1}^{\perp}(\mathbb{S}) \backslash E_{0, k}^{\perp}(\mathbb{S}), \mathbb{P}=$ $\mathbb{P}_{k-1}\left(K, \mathbb{R}^{d}\right)$ and $\mathbb{Q}=\operatorname{ker}(\cdot \boldsymbol{x}) \cap \mathbb{P}_{k-2}(K ; \mathbb{S})$, we get the uni-solvent DoFs

$$
\begin{align*}
\boldsymbol{\tau}(\delta) & \forall \delta \in \mathcal{V}(K),  \tag{4.17}\\
\left(\boldsymbol{n}_{i}^{\top} \boldsymbol{\tau} \boldsymbol{n}_{j}, q\right)_{F} & \forall q \in \mathbb{P}_{k+r-d-1}(F), F \in \mathcal{F}^{r}(K),  \tag{4.18}\\
& i, j=1, \cdots, r, \text { and } r=1, \cdots, d-1, \\
\left(\Pi_{F} \boldsymbol{\tau} \boldsymbol{n}, \boldsymbol{q}\right)_{F} & \forall \boldsymbol{q} \in \mathrm{ND}_{k-2}(F), F \in \mathcal{F}^{1}(K),  \tag{4.19}\\
(\boldsymbol{\tau}, \boldsymbol{q})_{K} & \forall \boldsymbol{q} \in \operatorname{ker}(\cdot \boldsymbol{x}) \cap \mathbb{P}_{k-2}(K ; \mathbb{S}),  \tag{4.20}\\
(\operatorname{div} \boldsymbol{\tau}, \boldsymbol{q})_{K} & \forall \boldsymbol{q} \in \mathbb{P}_{k}\left(K ; \mathbb{R}^{d}\right) / \boldsymbol{R M} . \tag{4.21}
\end{align*}
$$

Since $\operatorname{div} \mathbb{P}_{k+1}^{-}(K ; \mathbb{S})=\mathbb{P}_{k}\left(K ; \mathbb{R}^{d}\right)$ and $\operatorname{div} \mathbb{P}_{k}(K ; \mathbb{S})=\mathbb{P}_{k-1}\left(K ; \mathbb{R}^{d}\right)$, it is expected that using $\mathbb{P}_{k+1}^{-}(K ; \mathbb{S})$ to discretize the mixed elasticity problem will possess one-order higher convergence rate of the divergence of the discrete stress than that of $\mathbb{P}_{k}(K ; \mathbb{S})$ symmetric element.

Remark 4.15. By the DoFs (4.10)-(4.13), we can find a basis $\left\{\phi_{i}\right\}_{i=1}^{N_{1}}$ of the bubble function space $\mathbb{B}_{k}(\operatorname{div}, K ; \mathbb{S})$. Let $\left\{\psi_{i}\right\}_{i=1}^{N_{2}}$ be a basis of $\mathbb{P}_{k-1}\left(K ; \mathbb{R}^{d}\right) \backslash \boldsymbol{R} \boldsymbol{M}$. Then form the matrix $\left(\left(\operatorname{div} \phi_{i}, \psi_{j}\right)_{K}\right)_{N_{1} \times N_{2}}$, whose kernel space combined with $\left\{\phi_{i}\right\}_{i=1}^{N_{1}}$ yields the basis of $E_{0, k}(\mathbb{S})$. Finally, a basis of $E_{0, k}^{\perp}(\mathbb{S})$ is achieved by finding the orthogonal complement of the basis of $E_{0, k}(\mathbb{S})$ under the inner product $(\cdot, \cdot)_{K}$.

The global finite element space $\boldsymbol{V}_{h}^{-}(\operatorname{div} ; \mathbb{S}) \subset H(\operatorname{div}, \Omega ; \mathbb{S})$, where
$\boldsymbol{V}_{h}^{-}(\operatorname{div} ; \mathbb{S}):=\left\{\boldsymbol{\tau} \in \boldsymbol{L}^{2}(\Omega ; \mathbb{S}):\left.\boldsymbol{\tau}\right|_{K} \in \mathbb{P}_{k+1}^{-}(K ; \mathbb{S})\right.$ for each $K \in \mathcal{T}_{h}$,
the degrees of freedom (4.17)-(4.19) are single-valued\}.
Similarly as Lemma 4.14, we have the following inf-sup condition.

Lemma 4.16. Let $k \geq d+1$. It holds the inf-sup condition

$$
\left\|\boldsymbol{p}_{h}\right\|_{0} \lesssim \sup _{\boldsymbol{\tau}_{h} \in \boldsymbol{V}_{h}^{-}(\mathrm{div} ; \mathbb{S})} \frac{\left(\operatorname{div} \boldsymbol{\tau}_{h}, \boldsymbol{p}_{h}\right)}{\left\|\boldsymbol{\tau}_{h}\right\|_{H(\text { div })}} \quad \forall \boldsymbol{p}_{h} \in \mathbb{P}_{k}\left(\mathcal{T}_{h} ; \mathbb{R}^{d}\right)
$$

As RT element, it is natural to enrich $\mathbb{P}_{k}(K ; \mathbb{S})$ to $\mathbb{P}_{k}(K ; \mathbb{S}) \oplus \operatorname{sym}\left(\mathbb{H}_{k}\left(K ; \mathbb{R}^{d}\right) \boldsymbol{x}^{\boldsymbol{\top}}\right)$. Unfortunately, $\operatorname{tr}^{\operatorname{div}}\left(\operatorname{sym}\left(\mathbb{H}_{k}\left(K ; \mathbb{R}^{d}\right) \boldsymbol{x}^{\boldsymbol{\top}}\right)\right) \nsubseteq \operatorname{tr}^{\text {div }}\left(\mathbb{P}_{k}(K ; \mathbb{S})\right.$ ), i.e. assumption (H2) in Lemma 2.3 does not hold, which ruins the discrete inf-sup condition.
5. Symmetric DivDiv-Conforming Finite Elements. We use the previous building blocks to construct divdiv-conforming finite elements in arbitrary dimension. Motivated by the recent construction [21] in two and three dimensions, we first construct $H$ (div div) $\cap H$ (div)-conforming finite elements for symmetric tensors and then apply a simple modification to construct $H$ (div div)-conforming finite elements. We also extend the construction to obtain a new family of $\mathbb{P}_{k+1}^{-}(\mathbb{S})$-type elements.

### 5.1. Divdiv operator and Green's identity.

Lemma 5.1. For integer $k \geq 1$, the operator

$$
\text { div div }: \boldsymbol{x}_{\boldsymbol{x}} \mathbb{H}_{k-1}(D) \rightarrow \mathbb{H}_{k-1}(D)
$$

is bijective. Consequently div div : $\mathbb{P}_{k+1}(D ; \mathbb{S}) \rightarrow \mathbb{P}_{k-1}(D)$ is surjective.
Proof. By (2.2), it follows

$$
\operatorname{div} \operatorname{div}\left(\boldsymbol{x} \boldsymbol{x}^{\boldsymbol{\top}} q\right)=\operatorname{div}((k+d) \boldsymbol{x} q)=(k+d)(k+d-1) q \quad \forall q \in \mathbb{H}_{k-1}(D)
$$

which ends the proof.
Next recall the Green's identity for operator divdiv in [13].
Lemma 5.2. We have for any $\boldsymbol{\tau} \in \mathcal{C}^{2}(K ; \mathbb{S})$ and $v \in H^{2}(K)$ that

$$
\begin{align*}
(\operatorname{div} \operatorname{div} \boldsymbol{\tau}, v)_{K}= & \left(\boldsymbol{\tau}, \nabla^{2} v\right)_{K}-\sum_{F \in \mathcal{F}^{1}(K)} \sum_{e \in \mathcal{F}^{1}(F)}\left(\boldsymbol{n}_{F, e}^{\top} \boldsymbol{\tau} \boldsymbol{n}, v\right)_{e} \\
& -\sum_{F \in \mathcal{F}^{1}(K)}\left[\left(\boldsymbol{n}^{\top} \boldsymbol{\tau} \boldsymbol{n}, \partial_{n} v\right)_{F}-\left(\boldsymbol{n}^{\boldsymbol{\top}} \operatorname{div} \boldsymbol{\tau}+\operatorname{div}_{F}(\boldsymbol{\tau} \boldsymbol{n}), v\right)_{F}\right] . \tag{5.1}
\end{align*}
$$

Proof. We start from the standard integration by parts

$$
\begin{aligned}
(\operatorname{div} \operatorname{div} \boldsymbol{\tau}, v)_{K} & =-(\operatorname{div} \boldsymbol{\tau}, \nabla v)_{K}+\sum_{F \in \mathcal{F}^{1}(K)}\left(\boldsymbol{n}^{\boldsymbol{\top}} \operatorname{div} \boldsymbol{\tau}, v\right)_{F} \\
& =\left(\boldsymbol{\tau}, \nabla^{2} v\right)_{K}-\sum_{F \in \mathcal{F}^{1}(K)}(\boldsymbol{\tau} \boldsymbol{n}, \nabla v)_{F}+\sum_{F \in \mathcal{F}^{1}(K)}\left(\boldsymbol{n}^{\boldsymbol{\top}} \operatorname{div} \boldsymbol{\tau}, v\right)_{F} .
\end{aligned}
$$

We then decompose $\nabla v=\partial_{n} v \boldsymbol{n}+\nabla_{F} v$ and apply the Stokes theorem to get

$$
\begin{aligned}
(\boldsymbol{\tau} \boldsymbol{n}, \nabla v)_{F} & =\left(\boldsymbol{\tau} \boldsymbol{n}, \partial_{n} v \boldsymbol{n}+\nabla_{F} v\right)_{F} \\
& =\left(\boldsymbol{n}^{\top} \boldsymbol{\tau} \boldsymbol{n}, \partial_{n} v\right)_{F}-\left(\operatorname{div}_{F}(\boldsymbol{\tau} \boldsymbol{n}), v\right)_{F}+\sum_{e \in \mathcal{F}^{1}(F)}\left(\boldsymbol{n}_{F, e}^{\top} \boldsymbol{\tau} \boldsymbol{n}, v\right)_{e} .
\end{aligned}
$$

Thus the Green's identity (5.1) follows from the last two identities.


Fig. 5.1. Decomposition of $\mathbb{P}_{k}(K, \mathbb{S})$ for an $H(\operatorname{div} \operatorname{div}) \cap H(\operatorname{div})$ conforming finite element.
5.2. $H(\operatorname{div} \operatorname{div} ; \mathbb{S}) \cap H(\operatorname{div} ; \mathbb{S})$-conforming elements. Based on (5.1), it suffices to enforce the continuity of both $\boldsymbol{\tau} \boldsymbol{n}$ and $\boldsymbol{n}^{\boldsymbol{\top}} \operatorname{div} \boldsymbol{\tau}$ so that the constructed finite element space is $H(\operatorname{div}, \mathbb{S}) \cap H$ (div div, $\mathbb{S}$ )-conforming. Such an approach is recently proposed in [21] to construct two and three dimensional finite elements. The readers are referred to Fig. 5.1 for an illustration of the space decomposition.

The subspaces $\operatorname{tr}^{\operatorname{div}}\left(\mathbb{P}_{k}(K, \mathbb{S})\right)$ and $E_{0}(\mathbb{S})$ are unchanged. The space $\operatorname{div} E_{0}^{\perp}(\mathbb{S})=$ $\mathbb{P}_{k-1, \mathrm{RM}}^{\perp}$ will be further split by the trace operator. Define

$$
F_{0}(\mathbb{S}) \subseteq E_{0}^{\perp}(\mathbb{S}), \text { satisfying } \operatorname{div} F_{0}(\mathbb{S})=\mathbb{B}_{k-1}(\operatorname{div}, K) \cap \boldsymbol{R} \boldsymbol{M}^{\perp}
$$

and $F_{\operatorname{tr}}(\mathbb{S}) \subseteq E_{0}^{\perp}(\mathbb{S})$ with $\operatorname{tr}^{\text {div }}\left(\operatorname{div} F_{\operatorname{tr}}(\mathbb{S})\right)=\operatorname{tr}{ }^{\operatorname{div}}\left(\operatorname{div} E_{0}^{\perp}\right)=\operatorname{tr}^{\text {div }}\left(\mathbb{P}_{k-1, \mathrm{RM}}^{\perp}\right)$, which is well defined as div restricted to $E_{0}^{\perp}(\mathbb{S})$ is a bijection. Here
$\mathbb{B}_{k-1}(\operatorname{div}, K) \cap \boldsymbol{R} \boldsymbol{M}^{\perp}:=\left\{\boldsymbol{v} \in \mathbb{B}_{k-1}(\operatorname{div}, K):(\boldsymbol{v}, \boldsymbol{q})_{K}=0\right.$ for all $\left.\boldsymbol{q} \in \boldsymbol{R} \boldsymbol{M}\right\}$.
Lemma 5.3. For integer $k \geq 3$, it holds

$$
\operatorname{tr}^{\operatorname{div}}\left(\operatorname{div} F_{\mathrm{tr}}(\mathbb{S})\right)=\operatorname{tr}^{\operatorname{div}}\left(\mathbb{P}_{k-1}\left(K ; \mathbb{R}^{d}\right)\right)
$$

Consequently $\left(\operatorname{tr}^{\text {div }}\left(\operatorname{div} F_{\mathrm{tr}}(\mathbb{S})\right)\right)^{\prime}=\mathcal{N}\left(\mathbb{P}_{k-1}\left(\mathcal{F}^{1}(K)\right)\right)$.
Proof. By definition, $\operatorname{tr}^{\operatorname{div}}\left(\operatorname{div} F_{\operatorname{tr}}(\mathbb{S})\right)=\operatorname{tr}^{\operatorname{div}}\left(\mathbb{P}_{k-1, \mathrm{RM}}^{\perp}\right) \subseteq \operatorname{tr}^{\operatorname{div}}\left(\mathbb{P}_{k-1}\left(K ; \mathbb{R}^{d}\right)\right)$. On the other hand, given a trace $p \in \operatorname{tr}^{\operatorname{div}}\left(\mathbb{P}_{k-1}\left(K ; \mathbb{R}^{d}\right)\right)$, by the uni-solvence of BDM element, cf. Theorem 3.15, we can find a $\boldsymbol{v} \in \mathbb{P}_{k-1}\left(K ; \mathbb{R}^{d}\right)$ such that $\boldsymbol{v} \cdot \boldsymbol{n}=p$ on $\partial K$ and $\boldsymbol{v} \perp \boldsymbol{R} \boldsymbol{M}$ as $\boldsymbol{R} \boldsymbol{M}=\mathrm{ND}_{0}(K) \subseteq \mathrm{ND}_{k-3}(K)$ when $k \geq 3$.

Lemma 5.4. For integer $k \geq 3$, we have

$$
F_{0}^{\prime}(\mathbb{S})=\mathcal{N}\left(\operatorname{def}\left(\mathrm{ND}_{k-3}(K)\right)\right)
$$

Proof. We pick a $\boldsymbol{\tau} \in F_{0}(\mathbb{S})$, i.e., $\boldsymbol{\tau}$ satisfies

$$
\left.(\boldsymbol{\tau} \boldsymbol{n})\right|_{\partial K}=0,\left.\quad \boldsymbol{n}^{\boldsymbol{\top}} \operatorname{div} \boldsymbol{\tau}\right|_{\partial K}=0, \quad \boldsymbol{\tau} \perp E_{0}(\mathbb{S})
$$

Assume

$$
(\boldsymbol{\tau}, \operatorname{def} \boldsymbol{q})_{K}=0, \quad \forall \boldsymbol{q} \in \mathrm{ND}_{k-3}(K)
$$

Note that $\boldsymbol{v}=\operatorname{div} \boldsymbol{\tau} \in \mathbb{B}_{k-1}(\operatorname{div}, K)$, and $(\boldsymbol{v}, \boldsymbol{q})_{K}=0$ for all $\boldsymbol{q} \in \mathrm{ND}_{k-3}(K)$, then $\boldsymbol{v}=\mathbf{0}$ by Theorem 3.15. Therefore $\operatorname{div} \boldsymbol{\tau}=\mathbf{0}$, i.e., $\boldsymbol{\tau} \in E_{0}(\mathbb{S})$. As $\boldsymbol{\tau} \perp E_{0}(\mathbb{S})$, the only possibility is $\boldsymbol{\tau}=\mathbf{0}$.

Then the dimension count

$$
\operatorname{dim} F_{0}(\mathbb{S})=\operatorname{dim} \mathbb{B}_{k-1}(\operatorname{div}, K)-\operatorname{dim} \boldsymbol{R} \boldsymbol{M}=\operatorname{dim} \mathrm{ND}_{k-3}(K)-\operatorname{dim} \operatorname{ker}(\operatorname{def})
$$

will finish the proof.
We summarize the construction in the following theorem.
Theorem 5.5. Let $V\left(\operatorname{div} \operatorname{div}^{+} ; \mathbb{S}\right):=\mathbb{P}_{k}(K, \mathbb{S})$ with $k \geq \max \{d, 3\}$. Then the following set of degrees of freedom determines an $H(\operatorname{div} \operatorname{div} ; \mathbb{S}) \cap H(\operatorname{div} ; \mathbb{S})$-conforming finite element

$$
\begin{align*}
\boldsymbol{\tau}(\delta) & \forall \delta \in \mathcal{V}(K),  \tag{5.2}\\
\left(\boldsymbol{n}_{i}^{\top} \boldsymbol{\tau} \boldsymbol{n}_{j}, q\right)_{F} & \forall q \in \mathbb{P}_{k+r-d-1}(F), F \in \mathcal{F}^{r}(K),  \tag{5.3}\\
& i, j=1, \cdots, r, \text { and } r=1, \cdots, d-1, \\
\left(\Pi_{F} \boldsymbol{\tau} \boldsymbol{n}, \boldsymbol{q}\right)_{F} & \forall \boldsymbol{q} \in \mathrm{ND}_{k-2}(F), F \in \mathcal{F}^{1}(K),  \tag{5.4}\\
\left(\boldsymbol{n}^{\boldsymbol{\top}} \operatorname{div} \boldsymbol{\tau}, q\right)_{F} & \forall q \in \mathbb{P}_{k-1}(F), F \in \mathcal{F}^{1}(K),  \tag{5.5}\\
(\operatorname{div} \operatorname{div} \boldsymbol{\tau}, q)_{K} & \forall q \in \mathbb{P}_{k-2}(K) / \mathbb{P}_{1}(K),  \tag{5.6}\\
(\operatorname{div} \boldsymbol{\tau}, \boldsymbol{q})_{K} & \forall \boldsymbol{q} \in\left(\mathbb{P}_{k-3}(K ; \mathbb{K}) / \mathbb{P}_{0}(K ; \mathbb{K})\right) \boldsymbol{x},  \tag{5.7}\\
(\boldsymbol{\tau}, \boldsymbol{q})_{K} & \forall \boldsymbol{q} \in \operatorname{ker}(\cdot \boldsymbol{x}) \cap \mathbb{P}_{k-2}(K ; \mathbb{S}) . \tag{5.8}
\end{align*}
$$

Proof. By Lemma 4.5, the vanishing degrees of freedom (5.2)-(5.4) imply $\left.\boldsymbol{\tau} \boldsymbol{n}\right|_{\partial K}=$ 0. Then applying Lemmas 5.3-5.4, we get from the vanishing degrees of freedom (5.5)(5.7) that $\boldsymbol{\tau} \in E_{0}(\mathbb{S})$. Finally combining (4.9) and (5.8) implies $\boldsymbol{\tau}=\mathbf{0}$.

We then count the dimensions. Compared to the degrees of freedom of BDM-type $H(\operatorname{div}, \mathbb{S})$ element, cf. Theorem 4.13 , the difference is (4.14) v.s. (5.5)-(5.7). Then from the uni-solvence of BDM div-conforming element, cf. Theorem 3.15, we have

$$
\operatorname{dim} \mathbb{P}_{k-1}\left(K ; \mathbb{R}^{d}\right)=\operatorname{dim} \mathrm{ND}_{k-3}(K)+\sum_{F \in \mathcal{F}^{1}(K)} \operatorname{dim} \mathbb{P}_{k-1}(F)
$$

and consequently the number of degrees of freedom (5.2)-(5.8) is $\operatorname{dim} \mathbb{P}_{k}(K ; \mathbb{S})$.
The global finite element space $\boldsymbol{V}_{h}\left(\operatorname{div} \operatorname{div}^{+} ; \mathbb{S}\right) \subset H(\operatorname{div} \operatorname{div}, \Omega ; \mathbb{S}) \cap H(\operatorname{div}, \Omega ; \mathbb{S})$ is defined as follows

$$
\boldsymbol{V}_{h}\left(\operatorname{div} \operatorname{div}^{+} ; \mathbb{S}\right):=\left\{\boldsymbol{\tau} \in \boldsymbol{L}^{2}(\Omega ; \mathbb{S}):\left.\boldsymbol{\tau}\right|_{K} \in \mathbb{P}_{k}(K, \mathbb{S}) \text { for each } K \in \mathcal{T}_{h}\right.
$$

the degrees of freedom (5.2)-(5.5) are single-valued\}.
The requirement $k \geq d$ ensures the degrees of freedom $\left(\boldsymbol{n}^{\top} \boldsymbol{\tau} \boldsymbol{n}, q\right)_{F}$ for all $q \in$ $\mathbb{P}_{0}(F)$ on each face $F \in \mathcal{F}^{1}(K)$, by which space $\operatorname{div} \operatorname{div} \boldsymbol{V}_{h}\left(\operatorname{div} \operatorname{div}{ }^{+} ; \mathbb{S}\right)$ will include all the piecewise linear functions.

Lemma 5.6. Let $k \geq \max \{d, 3\}$. It holds the inf-sup condition

$$
\left\|p_{h}\right\|_{0} \lesssim \sup _{\boldsymbol{\tau}_{h} \in \boldsymbol{V}_{h}\left(\operatorname{div} \operatorname{div}^{+} ; \mathbb{S}\right)} \frac{\left(\operatorname{div} \operatorname{div} \boldsymbol{\tau}_{h}, p_{h}\right)}{\left\|\boldsymbol{\tau}_{h}\right\|_{H(\operatorname{div})}+\left\|\operatorname{div} \operatorname{div} \boldsymbol{\tau}_{h}\right\|_{0}} \quad \forall p_{h} \in \mathbb{P}_{k-2}\left(\mathcal{T}_{h}\right)
$$

where $\mathbb{P}_{k-2}\left(\mathcal{T}_{h}\right):=\left\{p_{h} \in L^{2}(\Omega):\left.p_{h}\right|_{K} \in \mathbb{P}_{k-2}(K)\right.$ for each $\left.K \in \mathcal{T}_{h}\right\}$.

Proof. For $p_{h} \in \mathbb{P}_{k-2}\left(\mathcal{T}_{h}\right)$, there exists $\boldsymbol{\tau} \in \boldsymbol{H}^{2}(\Omega ; \mathbb{S})$ such that [18]

$$
\operatorname{div} \boldsymbol{\tau}=p_{h}, \quad\|\boldsymbol{\tau}\|_{2} \lesssim\left\|p_{h}\right\|_{0}
$$

Let $\boldsymbol{\tau}_{h} \in \boldsymbol{V}_{h}\left(\operatorname{div} \operatorname{div}^{+} ; \mathbb{S}\right)$ such that all the DoFs (5.2)-(5.8) vanish except

$$
\begin{aligned}
\left(\boldsymbol{\tau}_{h} \boldsymbol{n}, \boldsymbol{q}\right)_{F} & =(\boldsymbol{\tau} \boldsymbol{n}, \boldsymbol{q})_{F} & & \forall \boldsymbol{q} \in \mathbb{P}_{0}\left(F ; \mathbb{R}^{d}\right), F \in \mathcal{F}^{1}(K), \\
\left(\boldsymbol{n}^{\boldsymbol{\top}} \operatorname{div} \boldsymbol{\tau}_{h}, q\right)_{F} & =\left(\boldsymbol{n}^{\boldsymbol{\top}} \operatorname{div} \boldsymbol{\tau}, q\right)_{F} & & \forall q \in \mathbb{P}_{1}\left(F ; \mathbb{R}^{d}\right), F \in \mathcal{F}^{1}(K), \\
\left(\operatorname{div} \operatorname{div} \boldsymbol{\tau}_{h}, q\right)_{K} & =(\operatorname{div} \operatorname{div} \boldsymbol{\tau}, q)_{K}=\left(p_{h}, q\right)_{K} & & \forall q \in \mathbb{P}_{k-2}(K) / \mathbb{P}_{1}(K)
\end{aligned}
$$

for all $K \in \mathcal{T}_{h}$. By the scaling argument, we have

$$
\begin{equation*}
\left\|\boldsymbol{\tau}_{h}\right\|_{H(\text { div })} \lesssim\|\boldsymbol{\tau}\|_{2} \lesssim\left\|p_{h}\right\|_{0} \tag{5.9}
\end{equation*}
$$

Applying the integration by parts,

$$
\left(\operatorname{div} \operatorname{div} \boldsymbol{\tau}_{h}, q\right)_{K}=(\operatorname{div} \operatorname{div} \boldsymbol{\tau}, q)_{K}=\left(p_{h}, q\right)_{K} \quad \forall q \in \mathbb{P}_{1}(K)
$$

Hence

$$
\left(\operatorname{div} \operatorname{div} \boldsymbol{\tau}_{h}, q\right)_{K}=(\operatorname{div} \operatorname{div} \boldsymbol{\tau}, q)_{K}=\left(p_{h}, q\right)_{K} \quad \forall q \in \mathbb{P}_{k-2}(K)
$$

which implies $\operatorname{div} \operatorname{div} \boldsymbol{\tau}_{h}=p_{h}$. Therefore we derive the inf-sup condition from (5.9).प
5.3. $\mathbb{P}_{k+1}^{-}(\mathbb{S})$-type $H(\operatorname{div} \operatorname{div} ; \mathbb{S}) \cap H(\operatorname{div} ; \mathbb{S})$-conforming elements. The space of shape functions is taken as

$$
V^{-}\left(\operatorname{div} \operatorname{div}^{+} ; \mathbb{S}\right):=\mathbb{P}_{k}(K ; \mathbb{S}) \oplus \boldsymbol{x} \boldsymbol{x}^{\boldsymbol{\top}} \mathbb{H}_{k-1}(K)
$$

with $k \geq \max \{d, 3\}$. The degrees of freedom are

$$
\begin{align*}
\boldsymbol{\tau}(\delta) & \forall \delta \in \mathcal{V}(K),  \tag{5.10}\\
\left(\boldsymbol{n}_{i}^{\top} \boldsymbol{\tau} \boldsymbol{n}_{j}, q\right)_{F} & \forall q \in \mathbb{P}_{k+r-d-1}(F), F \in \mathcal{F}^{r}(K),  \tag{5.11}\\
& i, j=1, \cdots, r, \text { and } r=1, \cdots, d-1, \\
\left(\Pi_{F} \boldsymbol{\tau} \boldsymbol{n}, \boldsymbol{q}\right)_{F} & \forall \boldsymbol{q} \in \mathrm{ND}_{k-2}(F), F \in \mathcal{F}^{1}(K),  \tag{5.12}\\
\left(\boldsymbol{n}^{\boldsymbol{\top}} \operatorname{div} \boldsymbol{\tau}, p\right)_{F} & \forall p \in \mathbb{P}_{k-1}(F), F \in \mathcal{F}^{1}(K),  \tag{5.13}\\
(\operatorname{div} \operatorname{div} \boldsymbol{\tau}, q)_{K} & \forall q \in \mathbb{P}_{k-1}(K) / \mathbb{P}_{1}(K),  \tag{5.14}\\
(\operatorname{div} \boldsymbol{\tau}, \boldsymbol{q})_{K} & \forall \boldsymbol{q} \in\left(\mathbb{P}_{k-3}(K ; \mathbb{K}) / \mathbb{P}_{0}(K ; \mathbb{K})\right) \boldsymbol{x},  \tag{5.15}\\
(\boldsymbol{\tau}, \boldsymbol{q})_{K} & \forall \boldsymbol{q} \in \operatorname{ker}(\cdot \boldsymbol{x}) \cap \mathbb{P}_{k-2}(K ; \mathbb{S}) . \tag{5.16}
\end{align*}
$$

We can see that $\mathbb{P}_{k+1}^{-}(\mathbb{S})$-type $H(\operatorname{div} \operatorname{div} ; \mathbb{S}) \cap H(\operatorname{div} ; \mathbb{S})$-conforming elements follows from Lemma 2.3 with $\mathrm{d}=\operatorname{div} \operatorname{div}, V=\mathbb{P}_{k}(K ; \mathbb{S}), \mathbb{H}=\boldsymbol{x} \boldsymbol{x}^{\top} \mathbb{H}_{k-1}(K), \mathbb{P}=$ $\mathbb{P}_{k-1}(K) / \mathbb{P}_{1}(K)$ and $\mathbb{Q}=\operatorname{ker}(\cdot \boldsymbol{x}) \cap \mathbb{P}_{k-2}(K ; \mathbb{S})$. The assumption (H5) holds from the fact div $\operatorname{div} \mathbb{B}^{+}=\mathbb{P}_{k-1}(K) / \mathbb{P}_{1}(K)$ and $\nabla^{2}(\mathbb{P}+\mathrm{d} \mathbb{H})=\nabla^{2} \mathbb{P}_{k-1}(K)$.

Due to the added component $\boldsymbol{x} \boldsymbol{x}^{\boldsymbol{\top}} \mathbb{H}_{k-1}(K)$, the range of div div operator is increased to $\mathbb{P}_{k-1}(K)$ instead of $\mathbb{P}_{k-2}(K)$. The degree of freedom $(\operatorname{div} \boldsymbol{\tau}, \boldsymbol{q})_{K}$ is increased from $\boldsymbol{q} \in \mathrm{ND}_{k-3}(K)=\operatorname{grad} \mathbb{P}_{k-2}(K) \oplus \mathbb{P}_{k-3}(K ; \mathbb{K}) \boldsymbol{x}$ to $\mathbb{P}_{k-2}\left(K ; \mathbb{R}^{d}\right)=$ $\operatorname{grad} \mathbb{P}_{k-1}(K) \oplus \mathbb{P}_{k-3}(K ; \mathbb{K}) \boldsymbol{x}$. Hence the number of degrees of freedom (5.10)-(5.16) equals to $\operatorname{dim} V^{-}\left(\operatorname{div} \operatorname{div}^{+} ; \mathbb{S}\right)$. The boundary DoFs, however, remains the same as $\left.\left(\boldsymbol{x} \boldsymbol{x}^{\top} \mathbb{H}_{k-1}(K)\right) \boldsymbol{n}\right|_{F} \in \mathbb{P}_{k}\left(F ; \mathbb{R}^{d}\right)$. It is expected that using the $\mathbb{P}_{k+1}^{-}(\mathbb{S})$-type symmetric element to discretize the biharmonic problem will possess one-order higher convergence rate of the div div of the discrete bending moment than that of the $\mathbb{P}_{k}(\mathbb{S})$-type symmetric element while the computational cost is not increased significantly; see [11, Section 4]. When solving the linear algebraic equation, all interior degrees of freedom can be eliminated element-wisely.

Lemma 5.7. Let $\boldsymbol{\tau} \in V^{-}\left(\operatorname{div}^{\operatorname{div}}{ }^{+} ; \mathbb{S}\right)$. If the degrees of freedom (5.10)-(5.15) vanish, then $\boldsymbol{\tau} \in E_{0}(\mathbb{S})$.

Proof. Since $\boldsymbol{x} \cdot \boldsymbol{n}$ is constant on each $(d-1)$-dimensional face, the trace $\left.\boldsymbol{\tau} \boldsymbol{n}\right|_{F} \in$ $\mathbb{P}_{k}\left(F ; \mathbb{R}^{d}\right)$ and $\left.\left(\boldsymbol{n}^{\boldsymbol{\top}} \operatorname{div} \boldsymbol{\tau}\right)\right|_{F} \in \mathbb{P}_{k-1}(F)$ remain unchanged. Then we conclude $\operatorname{tr}^{\text {div }} \boldsymbol{\tau}=$ 0 and $\operatorname{tr}^{\operatorname{div}}(\operatorname{div} \boldsymbol{\tau})=0$ from Theorem 5.5.

Applying the Green's identity (5.1), we get

$$
(\operatorname{div} \operatorname{div} \boldsymbol{\tau}, v)_{K}=\left(\boldsymbol{\tau}, \nabla^{2} v\right)_{K}=0 \quad \forall v \in \mathbb{P}_{1}(K)
$$

Hence it follows from the vanishing degrees of freedom (5.14) that $\operatorname{div} \operatorname{div} \boldsymbol{\tau}=0$, which combined with Lemma 5.1 implies $\boldsymbol{\tau} \in \mathbb{P}_{k}(K ; \mathbb{S})$. Finally we achieve from Lemma 5.4 and the vanishing degrees of freedom (5.15) that $\boldsymbol{\tau} \in E_{0}(\mathbb{S})$.

Combining Lemma 5.7, (4.9) and the degree of freedom (5.16) shows the unisolvence of the $\mathbb{P}_{k+1}^{-}(\mathbb{S})$-type $H(\operatorname{div} \operatorname{div} ; \mathbb{S}) \cap H(\operatorname{div} ; \mathbb{S})$-conforming elements.

THEOREM 5.8. The degrees of freedom (5.10)-(5.16) are uni-solvent for the space $V^{-}\left(\operatorname{div} \operatorname{div}^{+} ; \mathbb{S}\right)=\mathbb{P}_{k}(K ; \mathbb{S}) \oplus \boldsymbol{x} \boldsymbol{x}^{\top} \mathbb{H}_{k-1}(K)$.

The finite element space $\boldsymbol{V}_{h}^{-}\left(\operatorname{div} \operatorname{div}^{+} ; \mathbb{S}\right) \subset H(\operatorname{div} \operatorname{div}, \Omega ; \mathbb{S}) \cap H(\operatorname{div}, \Omega ; \mathbb{S})$ is then defined as follows

$$
\begin{aligned}
\boldsymbol{V}_{h}^{-}\left(\operatorname{div} \operatorname{div}^{+} ; \mathbb{S}\right) & :=\left\{\boldsymbol{\tau} \in \boldsymbol{L}^{2}(\Omega ; \mathbb{S}):\left.\boldsymbol{\tau}\right|_{K} \in \mathbb{P}_{k}(K, \mathbb{S}) \oplus \boldsymbol{x} \boldsymbol{x}^{\boldsymbol{\top}} \mathbb{H}_{k-1}(K)\right. \text { for each } \\
& \left.K \in \mathcal{T}_{h}, \text { the degrees of freedom }(5.10) \text {-(5.13) are single-valued }\right\} .
\end{aligned}
$$

Similarly as Lemma 5.6, we have the following inf-sup condition.
Lemma 5.9. Let $k \geq \max \{d, 3\}$. It holds

$$
\left\|p_{h}\right\|_{0} \lesssim \sup _{\boldsymbol{\tau}_{h} \in \boldsymbol{V}_{h}^{-}\left(\operatorname{div} \operatorname{div}^{+} ; \mathbb{S}\right)} \frac{\left(\operatorname{div} \operatorname{div} \boldsymbol{\tau}_{h}, p_{h}\right)}{\left\|\boldsymbol{\tau}_{h}\right\|_{H(\operatorname{div})}+\left\|\operatorname{div} \operatorname{div} \boldsymbol{\tau}_{h}\right\|_{0}} \quad \forall p_{h} \in \mathbb{P}_{k-1}\left(\mathcal{T}_{h}\right)
$$

5.4. divdiv conforming elements. The requirement both $\boldsymbol{\tau} \boldsymbol{n}$ and $\boldsymbol{n}^{\boldsymbol{\top}} \operatorname{div} \boldsymbol{\tau}$ are continuous is sufficient but not necessary for a function to be in $H$ (div div). In addition to $\boldsymbol{n}^{\boldsymbol{\top}} \boldsymbol{\tau} \boldsymbol{n}$, the combination $\boldsymbol{n}^{\boldsymbol{\top}} \operatorname{div} \boldsymbol{\tau}+\operatorname{div}_{F}(\boldsymbol{\tau} \boldsymbol{n})$ to be continuous is enough due to the Green's identity (5.1).

Theorem 5.10. Take $V(\operatorname{div} \operatorname{div} ; \mathbb{S}):=\mathbb{P}_{k}(K ; \mathbb{S})$ with $k \geq \max \{d, 3\}$, as the space of shape functions. The degrees of freedom are given by

$$
\begin{align*}
\boldsymbol{\tau}(\delta) & \forall \delta \in \mathcal{V}(K),  \tag{5.17}\\
\left(\boldsymbol{n}_{i}^{\top} \boldsymbol{\tau} \boldsymbol{n}_{j}, q\right)_{F} & \forall q \in \mathbb{P}_{k+r-d-1}(F), F \in \mathcal{F}^{r}(K),  \tag{5.18}\\
& i, j=1, \cdots, r, \text { and } r=1, \cdots, d-1, \\
\left(\Pi_{F} \boldsymbol{\tau} \boldsymbol{n}, \boldsymbol{q}\right)_{F} & \forall \boldsymbol{q} \in \mathrm{ND}_{k-2}(F), F \in \mathcal{F}^{1}(K),  \tag{5.19}\\
\left(\boldsymbol{n}^{\top} \operatorname{div} \boldsymbol{\tau}+\operatorname{div}_{F}(\boldsymbol{\tau} \boldsymbol{n}), p\right)_{F} & \forall p \in \mathbb{P}_{k-1}(F), F \in \mathcal{F}^{1}(K),  \tag{5.20}\\
(\boldsymbol{\tau}, \operatorname{def} \boldsymbol{q})_{K} & \forall \boldsymbol{q} \in \mathrm{ND}_{k-3}(K),  \tag{5.21}\\
(\boldsymbol{\tau}, \boldsymbol{q})_{K} & \forall \boldsymbol{q} \in \operatorname{ker}(\cdot \boldsymbol{x}) \cap \mathbb{P}_{k-2}(K ; \mathbb{S}) . \tag{5.22}
\end{align*}
$$

The degree of freedom (5.19) is considered as interior to $K$, i.e., it is not single-valued across elements.

Proof. By Lemma 4.5, the $\operatorname{div}_{F}(\boldsymbol{\tau} \boldsymbol{n})$ can be determined by (5.17), (5.18), and (5.19). A linear combination with (5.20), the trace $\boldsymbol{n}^{\boldsymbol{\top}} \operatorname{div} \boldsymbol{\tau}$ can be determined. Then the uni-solvence is obtained from Theorem 5.5.

The finite element space $\boldsymbol{V}_{h}($ div div $)$ is defined as follows
$\boldsymbol{V}_{h}(\operatorname{div} \operatorname{div}, \Omega ; \mathbb{S}):=\left\{\boldsymbol{\tau} \in \boldsymbol{L}^{2}(\Omega ; \mathbb{S}):\left.\boldsymbol{\tau}\right|_{K} \in \mathbb{P}_{k}(K ; \mathbb{S})\right.$ for each $K \in \mathcal{T}_{h}$, the degrees of freedom (5.17)-(5.18) and (5.20) are single-valued\}.

As $\boldsymbol{n}^{\boldsymbol{\top}} \boldsymbol{\tau} \boldsymbol{n}$ and $\boldsymbol{n}^{\boldsymbol{\top}} \operatorname{div} \boldsymbol{\tau}+\operatorname{div}_{F}(\boldsymbol{\tau} \boldsymbol{n})$ are continuous, $\boldsymbol{V}_{h}(\operatorname{div} \operatorname{div}) \subset H(\operatorname{div} \operatorname{div}, \Omega ; \mathbb{S})$; see [13, Lemma 4.4].

Finally we present a $\mathbb{P}_{k+1}^{-}(\mathbb{S})$-type $H(\operatorname{div} \operatorname{div} ; \mathbb{S})$-conforming element.
Theorem 5.11. Let integer $k \geq \max \{d, 3\}$. Take the space of shape functions as

$$
V^{-}(\operatorname{div} \operatorname{div} ; \mathbb{S}):=\mathbb{P}_{k}(K ; \mathbb{S}) \oplus \boldsymbol{x} \boldsymbol{x}^{\top} \mathbb{H}_{k-1}(K)
$$

The degrees of freedom are

$$
\begin{align*}
\boldsymbol{\tau}(\delta) & \forall \delta \in \mathcal{V}(K),  \tag{5.23}\\
\left(\boldsymbol{n}_{i}^{\top} \boldsymbol{\tau} \boldsymbol{n}_{j}, q\right)_{F} & \forall q \in \mathbb{P}_{k+r-d-1}(F), F \in \mathcal{F}^{r}(K),  \tag{5.24}\\
& i, j=1, \cdots, r, \text { and } r=1, \cdots, d-1, \\
\left(\Pi_{F} \boldsymbol{\tau} \boldsymbol{n}, \boldsymbol{q}\right)_{F} & \forall \boldsymbol{q} \in \mathrm{ND}_{k-2}(F), F \in \mathcal{F}^{1}(K),  \tag{5.25}\\
\left(\boldsymbol{n}^{\boldsymbol{\top}} \operatorname{div} \boldsymbol{\tau}+\operatorname{div}_{F}(\boldsymbol{\tau} \boldsymbol{n}), p\right)_{F} & \forall p \in \mathbb{P}_{k-1}(F), F \in \mathcal{F}^{1}(K),  \tag{5.26}\\
(\boldsymbol{\tau}, \operatorname{def} \boldsymbol{q})_{K} & \forall \boldsymbol{q} \in \mathbb{P}_{k-2}\left(K ; \mathbb{R}^{d}\right),  \tag{5.27}\\
(\boldsymbol{\tau}, \boldsymbol{q})_{K} & \forall \boldsymbol{q} \in \operatorname{ker}(\cdot \boldsymbol{x}) \cap \mathbb{P}_{k-2}(K ; \mathbb{S}) . \tag{5.28}
\end{align*}
$$

Again the degree of freedom (5.25) is considered as interior to $K$, i.e., it is not singlevalued across elements.

Proof. By Lemma 4.5, the $\operatorname{div}_{F}(\boldsymbol{\tau} \boldsymbol{n})$ can be determined by (5.23), (5.24), and (5.25). A linear combination with (5.26), the trace $\boldsymbol{n}^{\boldsymbol{\top}} \operatorname{div} \boldsymbol{\tau}$ can be determined. Then the uni-solvence is obtained from Theorem 5.8.

The global finite element space $\boldsymbol{V}_{h}^{-}(\operatorname{div} \operatorname{div}) \subset H(\operatorname{div} \operatorname{div}, \Omega ; \mathbb{S})$, where
$\boldsymbol{V}_{h}^{-}(\operatorname{div} \operatorname{div}, \Omega ; \mathbb{S}):=\left\{\boldsymbol{\tau} \in \boldsymbol{L}^{2}(\Omega ; \mathbb{S}):\left.\boldsymbol{\tau}\right|_{K} \in V^{-}(\operatorname{div} \operatorname{div} ; \mathbb{S})\right.$ for each $K \in \mathcal{T}_{h}$, the
degrees of freedom (5.23)-(5.24) and (5.26) are single-valued\}.
Finally we list inf-sup conditions for divdiv conforming elements.
Lemma 5.12. Let $k \geq \max \{d, 3\}$. We have

$$
\begin{align*}
& \left\|p_{h}\right\|_{0} \lesssim \sup _{\boldsymbol{\tau}_{h} \in \boldsymbol{V}_{h}(\operatorname{div} \operatorname{div}, \Omega ; \mathbb{S})} \frac{\left(\operatorname{div} \operatorname{div} \boldsymbol{\tau}_{h}, p_{h}\right)}{\left\|\boldsymbol{\tau}_{h}\right\|_{0}+\left\|\operatorname{div} \operatorname{div} \boldsymbol{\tau}_{h}\right\|_{0}} \quad \forall p_{h} \in \mathbb{P}_{k-2}\left(\mathcal{T}_{h}\right),  \tag{5.29}\\
& \left\|p_{h}\right\|_{0} \lesssim \sup _{\boldsymbol{\tau}_{h} \in \boldsymbol{V}_{h}^{-}(\operatorname{div} \operatorname{div}, \Omega ; \mathbb{S})} \frac{\left(\operatorname{div} \operatorname{div} \boldsymbol{\tau}_{h}, p_{h}\right)}{\left\|\boldsymbol{\tau}_{h}\right\|_{0}+\left\|\operatorname{div} \operatorname{div} \boldsymbol{\tau}_{h}\right\|_{0}} \quad \forall p_{h} \in \mathbb{P}_{k-1}\left(\mathcal{T}_{h}\right) .
\end{align*}
$$

Proof. Since $\left\|\boldsymbol{\tau}_{h}\right\|_{0} \leq\left\|\boldsymbol{\tau}_{h}\right\|_{H(\operatorname{div})}$ and $\boldsymbol{V}_{h}\left(\operatorname{div} \operatorname{div}^{+} ; \mathbb{S}\right) \subseteq \boldsymbol{V}_{h}(\operatorname{div} \operatorname{div}, \Omega ; \mathbb{S})$, the inf-sup condition (5.29) follows from Lemma 5.6. Similarly, the inf-sup condition (5.30) follows from Lemma 5.9 and $\boldsymbol{V}_{h}^{-}\left(\operatorname{div} \operatorname{div}^{+} ; \mathbb{S}\right) \subseteq \boldsymbol{V}_{h}^{-}(\operatorname{div} \operatorname{div}, \Omega ; \mathbb{S})$.
[1] S. Adams and B. Cockburn, A mixed finite element method for elasticity in three dimensions, Journal of Scientific Computing, 3 (2005), pp. 515-521. 2
[2] D. N. Arnold, Finite element exterior calculus, vol. 93 of CBMS-NSF Regional Conference Series in Applied Mathematics, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2018. 14
[3] D. N. Arnold, G. Awanou, and R. Winther, Finite elements for symmetric tensors in three dimensions, Math. Comp., 77 (2008), pp. 1229-1251. 2, 19
[4] D. N. Arnold, R. S. Falk, and R. Winther, Finite element exterior calculus, homological techniques, and applications, Acta Numer., 15 (2006), pp. 1-155. 14, 18
[5] D. N. Arnold, R. S. Falk, and R. Winther, Geometric decompositions and local bases for spaces of finite element differential forms, Computer Methods in Applied Mechanics and Engineering, 198 (2009), pp. 1660-1672. 6, 11, 14
[6] D. N. Arnold and R. Winther, Mixed finite elements for elasticity, Numer. Math., 92 (2002), pp. 401-419. 2, 19
[7] F. Brezzi, J. Douglas, Jr., R. Durán, and M. Fortin, Mixed finite elements for second order elliptic problems in three variables, Numer. Math., 51 (1987), pp. 237-250. 2, 9
[8] F. Brezzi, J. Douglas, Jr., and L. D. Marini, Recent results on mixed finite element methods for second order elliptic problems, in Vistas in applied mathematics, Transl. Ser. Math. Engrg., Optimization Software, New York, 1986, pp. 25-43. 2, 9
[9] L. Chen, J. Hu, and X. Huang, Fast auxiliary space preconditioners for linear elasticity in mixed form, Math. Comp., 87 (2018), pp. 1601-1633. 7, 17
[10] L. Chen and X. Huang, Decoupling of mixed methods based on generalized Helmholtz decompositions, SIAM J. Numer. Anal., 56 (2018), pp. 2796-2825. 7
[11] L. Chen and X. Huang, Finite elements for divdiv-conforming symmetric tensors, arXiv preprint arXiv:2005.01271, (2020). 4, 21, 27
[12] L. Chen and X. Huang, A finite element elasticity complex in three dimensions, arXiv preprint arXiv:2106.12786, (2021). 19, 21
[13] L. Chen and X. Huang, Finite elements for divdiv-conforming symmetric tensors in three dimensions, Math. Comp., (2021). 4, 24, 29
[14] L. Chen and X. Huang, Geometric decompositions of div-conforming finite element tensors, arXiv preprint arXiv:2112.14351, (2021). 10, 11, 17
[15] S. H. Christiansen, On the linearization of Regge calculus, Numer. Math., 119 (2011), pp. 613640. 7
[16] C. K. Chui and M.-J. Lai, Multivariate vertex splines and finite elements, Journal of approximation theory, 60 (1990), pp. 245-343. 6
[17] P. G. Ciarlet, The finite element method for elliptic problems, North-Holland Publishing Co., Amsterdam, 1978. 1
[18] M. Costabel and A. McIntosh, On Bogovski乞 and regularized Poincaré integral operators for de Rham complexes on Lipschitz domains, Math. Z., 265 (2010), pp. 297-320. 22, 27
[19] A. Hatcher, Algebraic topology, Cambridge University Press, Cambridge, 2002. 7
[20] J. Hu, Finite element approximations of symmetric tensors on simplicial grids in $\mathbb{R}^{n}$ : the higher order case, J. Comput. Math., 33 (2015), pp. 283-296. 2, 3, 7, 16, 17, 18, 19
[21] J. Hu, R. Ma, and M. Zhang, A family of mixed finite elements for the biharmonic equations on triangular and tetrahedral grids, Sci. China Math., (2021). 3, 24, 25
[22] J. Hu and S. Zhang, A family of symmetric mixed finite elements for linear elasticity on tetrahedral grids, Sci. China Math., 58 (2015), pp. 297-307. 2, 22
[23] J. Hu and S. Zhang, Finite element approximations of symmetric tensors on simplicial grids in $\mathbb{R}^{n}$ : the lower order case, Math. Models Methods Appl. Sci., 26 (2016), pp. 1649-1669. 2
[24] M.-J. Lai and L. L. Schumaker, Spline functions on triangulations, vol. 110, Cambridge University Press, 2007. 6
[25] J.-C. NÉdÉLEC, Mixed finite elements in $\mathbf{R}^{3}$, Numer. Math., 35 (1980), pp. 315-341. 2, 9, 13, 14, 18
[26] J.-C. NÉdélec, A new family of mixed finite elements in $\mathbf{R}^{3}$, Numer. Math., 50 (1986), pp. 5781. 9
[27] P.-A. Raviart and J. M. Thomas, A mixed finite element method for 2nd order elliptic problems, in Mathematical aspects of finite element methods (Proc. Conf., Consiglio Naz. delle Ricerche (C.N.R.), Rome, 1975), Springer, Berlin, 1977, pp. 292-315. Lecture Notes in Math., Vol. 606. 2, 9
[28] T. Sorokina, Intrinsic supersmoothness of multivariate splines, Numerische Mathematik, 116 (2010), pp. 421-434. 6


[^0]:    *Submitted to the editors January 31, 2022.
    Funding: The first author was supported by NSF DMS-1913080 and DMS-2012465. The second author was supported by the National Natural Science Foundation of China Projects 12171300 and 11771338, the Natural Science Foundation of Shanghai 21ZR1480500, and the Fundamental Research Funds for the Central Universities 2019110066.
    ${ }^{\dagger}$ Department of Mathematics, University of California at Irvine, Irvine, CA 92697, USA (chenlong@math.uci.edu).
    $\ddagger$ Corresponding author. School of Mathematics, Shanghai University of Finance and Economics, Shanghai 200433, China (huang.xuehai@sufe.edu.cn).

