Advances in Fluid Mechanics III, C.A. Brebbia & M. Rahman (Editors)
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Finite elements for shallow water equations: stabilized formulations and computational aspects

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Abstract

In this paper we present a space-time finite element formulation for problems governed by the shallow water equations. A linear time-discontinuous approximation is adopted, and linear three node triangles are used for the spatial discretization. Computational aspects are also discussed. It is shown that an edge-based data structure turns out to be a very fast and efficient way to solve the non-linear system of equations. Numerical examples are shown, illustrating the quality of the solution and the performance of the method.

1 Introduction

In this paper we are interested in the numerical solution of the so-called vertically averaged (2DH) model of the shallow water equations (SWE) which satisfactory describes the hydrodynamics (circulation of water) in a class of well-mixed estuaries and coastal embayments. One of the major numerical difficulties associated with the SWE is its convection-dominated character. Many numerical procedures rely on characteristic or semi-lagrangian based methods in order to circumvent this drawback. Another way to deal with this problem, in its full Eulerian description, is to use stabilized finite element methods [2,6,7,8]. In this work we adopt this methodology, using a space-time variational formulation for the non-conservative form of the SWE, written in terms of velocity-celerity

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variables. Following this approach, the CAU finite element method [1,4] is constructed.

Computational aspects are also addressed in this work. We present some results using the GMRES iterative method, implemented in element-by-element and in edge-by-edge basis. The space-time formulation with linear approximations in time, despite the fact of presenting accurate solutions, results in a system with a number of equations which is twice the number of equations of the correspondent semi-discrete version. However, as it will be shown at the end of this paper, the CPU time necessary to solve the system obtained in the space-time formulation can be significantly reduced by splitting the system in two "semi-discrete" systems and using an edge based data structure.

2 Problem Statement

The vertically averaged 2DH model for shallow water problems is described by the equations:

$$\begin{aligned}&\frac{\partial}{\partial t}(HU)+\frac{\partial}{\partial x}(HU^2+\frac{1}{2}gH^2)+\frac{\partial}{\partial y}(HUV)=\\&=\frac{1}{\rho}\tau_x^s-\gamma U+fHV+gH\frac{\partial h}{\partial x}+\frac{\partial}{\partial x}\left(\mu H\frac{\partial U}{\partial x}\right)+\frac{\partial}{\partial y}\left(\mu H\frac{\partial U}{\partial y}\right)\end{aligned} \tag{1}$$

$$\begin{aligned}&\frac{\partial}{\partial t}(HV)+\frac{\partial}{\partial x}(HUV)+\frac{\partial}{\partial y}(HV^2+\frac{1}{2}gH^2)=\\&=\frac{1}{\rho}\tau_y^s-\gamma V-fHU+gH\frac{\partial h}{\partial y}+\frac{\partial}{\partial x}\left(\mu H\frac{\partial V}{\partial x}\right)+\frac{\partial}{\partial y}\left(\mu H\frac{\partial V}{\partial y}\right)\end{aligned} \tag{2}$$

$$\frac{\partial H}{\partial t}+\frac{\partial}{\partial x}(HU)+\frac{\partial}{\partial y}(HV)=0 \tag{3}$$

In these expressions, $H=h+\eta$ is the shallow layer width and (U, V) are the averaged horizontal velocity components. The water depth and the water surface elevation, both measured from the undisturbed water surface, are respectively denoted by h and η. The gravity acceleration is denoted by g, f is the Coriolis parameter, ρ is the constant density (well mixed layers), τ_x^s and τ_y^s are the wind stress components at the free surface and μ is the eddy viscosity coefficient, which takes into account the horizontal diffusion effects for an "appropriate" turbulence model. Bottom friction forces are modeled by the Chezy formula $\gamma=g\left(U^2+V^2\right)^{1/2}/C^2$, where C is the Chezy coefficient.

3 Divergence × Advective forms of the SWE

Equations (1-3) can be written in a more compact form as

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$$\mathbf{U}_{,t} + \text{div}\,\mathscr{F} = \mathbf{F} \tag{4}$$

where $\mathbf{U}^t = [HU \quad HV \quad H]$ denotes the vector of conservation variables. The notation $\text{div}\,\mathscr{F}$ stands for

$$\text{div}\,\mathscr{F} = \frac{\partial}{\partial x}\mathscr{F}_x + \frac{\partial}{\partial y}\mathscr{F}_y \tag{5}$$

with $\mathscr{F}_x$ and $\mathscr{F}_y$ representing the hydraulic fluxes

$$\mathscr{F}_x^t = \left[HU^2 + \frac{1}{2}gH^2 \quad HUV \quad HU \right] \tag{6}$$

$$\mathscr{F}_y^t = \left[HUV \quad HV^2 + \frac{1}{2}gH^2 \quad HV \right] \tag{7}$$

The components of **F** are the terms at the right hand side of (1-3). The system (4) is written in the so-called conservation form and represents an incomplete parabolic system of equations. On the other hand, the reduced equation

$$\mathbf{U}_{,t} + \text{div}\,\mathscr{F} = \mathbf{0} \tag{8}$$

represents an hyperbolic system for which the convective term is expressed in its divergence form. Hyperbolic system of equations admit discontinuous solutions (shocks). In this case, the classical derivatives appearing in (8) do not make any sense and should be understood in the sense of distributions. This fact leads us to the use of a variational space-time formulation to conveniently represent the shallow water problem. This occurs for example in the well known "dam break" problem, for which an efficient shock/discontinuity-capturing technique should be used as long as a realistic shock discontinuous approximate solution is desired.

For the incomplete parabolic problem described by (4), discontinuous solutions are disregarded due to the dissipative character of the terms included in **F** (viscous effects). Nevertheless, the convection-dominated nature of the modeled physical problem requires its "correct" representation as long as approximate solutions are concerned. This fact guides us to the use of space-time advective form description of (4), over which a consistent stabilized space-time finite element approximation is straightforward constructed. This will be done in the next sections.

4 The Velocity-Celerity Symmetric Form of the SWE

For the SWE a symmetric advective form can be achieved using velocity-celerity variables. To this end let us define the change of variables $\mathbf{U} \rightarrow \mathbf{V}^*$ given by $H = \frac{\theta^2}{4g}$, where $\theta = 2c$ and $c = \sqrt{gH}$ is the gravitational wave propagation velocity. Using these definitions, equation (4) can be rewritten as

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$$\mathbf{A}_0^*\mathbf{V}_{,t}^* + \sum_{i=1}^{2}\mathbf{A}_i^*\mathbf{V}_{,i}^* = \mathbf{F} \tag{9}$$

where

$$(\mathbf{V}^*)^t = [U \quad V \quad \theta]; \quad \mathbf{A}_0^* = \mathbf{U}_{,\mathbf{V}^*} = \frac{\theta}{4g}\begin{bmatrix} \theta & 0 & 2U \\ 0 & \theta & 2V \\ 0 & 0 & 2 \end{bmatrix} \tag{10}$$

$$\mathbf{A}_1^* = \mathbf{A}_x\mathbf{A}_0^* = \frac{\theta}{4g}\begin{bmatrix} 2U\theta & 0 & \frac{\theta^2}{2}+2U^2 \\ V\theta & U\theta & 2UV \\ \theta & 0 & 2U \end{bmatrix} \tag{11}$$

$$\mathbf{A}_2^* = \mathbf{A}_y\mathbf{A}_0^* = \frac{\theta}{4g}\begin{bmatrix} V\theta & U\theta & 2UV \\ 0 & 2V\theta & \frac{\theta^2}{2}+2V^2 \\ 0 & \theta & 2V \end{bmatrix} \tag{12}$$

Pre-multiplying (9) by $(\mathbf{A}_0^*)^{-1}$ we finally arrive to the symmetric form:

$$\mathbf{V}_{,t}^* + \widetilde{\mathbf{A}}^*.\nabla\mathbf{V}^* - \nabla.(\boldsymbol{K}\nabla\mathbf{V}^*) = \mathbf{F}^* \tag{13}$$

where

$$\widetilde{\mathbf{A}}_1^* = \begin{bmatrix} U & 0 & c \\ 0 & U & 0 \\ c & 0 & U \end{bmatrix}; \qquad \widetilde{\mathbf{A}}_2^* = \begin{bmatrix} V & 0 & 0 \\ 0 & V & c \\ 0 & c & V \end{bmatrix} \tag{14}$$

$$\boldsymbol{K} = \begin{bmatrix} \boldsymbol{K}_{11} & \boldsymbol{K}_{12} \\ \boldsymbol{K}_{21} & \boldsymbol{K}_{22} \end{bmatrix}; \qquad \boldsymbol{K}_{ij} = \delta_{ij}\mu\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{15}$$

$$\mathbf{F}^* = \begin{bmatrix} g\frac{\partial h}{\partial x} - \frac{\gamma}{H}U + fV + \frac{1}{\rho H}\tau_x^s \\ g\frac{\partial h}{\partial y} - \frac{\gamma}{H}V - fU + \frac{1}{\rho H}\tau_y^s \\ 0 \end{bmatrix} \tag{16}$$

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5 Space-time Finite Element Approximations

In this section we will present some finite element approximations for the celerity-velocity variables of the shallow water problem described by (13). For convenience we will use the notation $\mathbf{U} = \mathbf{V}^*$, $\mathbf{A} = \widetilde{\mathbf{A}}^*$, $\mathbf{F} = \mathbf{F}^*$. We will assume that the spatial region of interest is mathematically represented by an open bounded set Ω of $\Re^2$, with boundary Γ; and the time period of observation is the interval $[0, T] \in \Re^+$.

We now proceed defining the space-time finite element discretization. To this end, for $n = 0,1,2,\ldots,N$ we will consider a partition of $[0,T]$, given by $t_0 = 0 < t_1 \ldots < t_n < t_{n+1} \ldots < t_N = T$. Denoting by $I_n = (t_n, t_{n+1})$ the n^{th} time interval, we will say that for each n, the space-time domain of interest is the "slab" $S_n = \Omega \times I_n$, with boundary $\overline{\Gamma} = \Gamma \times I_n$. Then for $n = 0,1,2,\ldots$ we will define a finite element triangulation of Ω such that:

$$\Omega = \bigcup_{e=1}^{(nel)_n} \Omega_n^e; \qquad \Omega_n^i \cap \Omega_n^j = \Phi \ \text{ for } \ i \neq j \tag{17}$$

To each Ω_n^e we associate the space-time finite element $S_n^e = \Omega_n^e \times I_n$, leading to the space-time partition $S_n = \bigcup_{e=1}^{(nel)_n} S_n^e$.

Under the above definitions we will assume that the finite element subspace $\hat{\mathcal{U}}_n^h$ is the set of continuous piecewise polynomials in S_n, which may be discontinuous across the slab interface at t_n, i.e:

$$\hat{\mathcal{U}}_n^h \equiv \left\{ \hat{\mathbf{U}}^h;\ \hat{\mathbf{U}}^h \in \left(C^0(S_n)\right)^3;\ \hat{\mathbf{U}}^h\Big|_{S_n^e} \in \left(P^k\left(S_n^e\right)\right)^3;\ \hat{\mathbf{U}}^h\Big|_{\overline{\Gamma}_n} = \mathbf{0} \right\} \tag{18}$$

where P^k is the set of polynomials of degree less than or equal k. For a prescribed boundary condition $\mathbf{g}$ on $\overline{\Gamma}_n$ a general trial function is then an element of $\mathcal{U}_n^h$, where:

$$\mathcal{U}_n^h \equiv \left\{ \mathbf{U}^h;\ \mathbf{U}^h \in \left(C^0(S_n)\right)^3;\ \mathbf{U}^h\Big|_{S_n^e} \in \left(P^k\left(S_n^e\right)\right)^3;\ \mathbf{U}^h\Big|_{\overline{\Gamma}_n} = \mathbf{g} \right\} \tag{19}$$

We present below some different finite element approximations.

5.1 Time-discontinuous Galerkin Method (TDG)

This method consists in: for each S_n, $n = 0,1,2,\ldots$ find $\mathbf{U}^h \in \mathcal{U}_n^h$ such that for all $\hat{\mathbf{U}}^h \in \hat{\mathcal{U}}_n^h$ the following variational equation is satisfied:

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$$\int_{S_n} \mathbf{R}.\hat{\mathbf{U}}^h \, d\Omega \, dt + \int_{\Omega} \left[\mathbf{U}^h(t_n^+)\right].\hat{\mathbf{U}}^h(t_n^+) \, d\Omega = 0 \tag{20}$$

where $\left[\mathbf{U}^h(t_n^+)\right] = \mathbf{U}^h(t_n^+) - \mathbf{U}^h(t_n^-)$ defines the jump of $\mathbf{U}^h$ in time t_n, and

$$\mathbf{R}(\mathbf{U}^h) = \mathbf{U}^h_{,t} + \mathbf{A}(\mathbf{U}^h).\nabla\mathbf{U}^h - \nabla.(\boldsymbol{K}\nabla\mathbf{U}^h) - \mathbf{F}(\mathbf{U}^h) \tag{21}$$

is the residual associated with the approximate solution $\mathbf{U}^h$.

5.2 STPG Method

For convection-dominated problems, Galerkin solutions present spurious oscillations, due to the lack of stability of the TDG variational formulation. We can circumvent this problem using the Space-Time Petrov Galerkin (STPG) stabilized finite element formulation based on the SUPG method [3]. This stabilization procedure can be achieved adding to the TDG formulation a term defined as:

$$\mathrm{T}_{STPG} = \sum_{e=1}^{(nel)_n} \int_{S_n^e} \mathbf{R}.\left(\boldsymbol{\tau}\left(\hat{\mathbf{U}}^h_{,t} + \mathbf{A} \,.\nabla\hat{\mathbf{U}}^h\right)\right) d\Omega \, dt \tag{22}$$

where $\boldsymbol{\tau}$ is a (3x3) symmetric positive matrix of intrinsic time-scales (for a general definition of this matrix see [9]).

Remark: Note that for hyperbolic problems $\mathbf{R} = \mathbf{U}^h_{,t} + \mathbf{A}.\nabla\mathbf{U}^h$, which means that for $\hat{\mathbf{U}}^h \equiv \mathbf{U}^h$ we arrive at a positive weighted square residual term:

$$\mathrm{T}_{STPG} = \|\mathbf{R}\|_\tau^2 = \sum_{e=1}^{(nel)_n} \int_{S_n^e} \mathbf{R}^t \, \boldsymbol{\tau} \;\; \mathbf{R} \, d\Omega \, dt \tag{23}$$

Some advantages of the $\boldsymbol{\tau}$ matrix definition for the celerity-velocity formulation were presented in [6].

5.3 CAU Method

We have also mentioned that for discontinuous (shock) solutions of hyperbolic problems, or even for sharp layers solutions related to highly convective dominant incomplete parabolic problems, an additional stabilization term is required if an "accurate" numerical approximation is desired. In the context of consistent variational weighted Petrov-Galerkin residual methods, this can be achieved using the CAU method. For this method an additional shock-capturing term is added to the STPG method which has the general form:

$$\mathrm{T}_{CAU} = \sum_{e=1}^{(nel)_n} \int_{S_n^e} \mathbf{R}.\left(\boldsymbol{\tau}_c\left(\mathbf{A} - \hat{\mathbf{A}}\right).\nabla\hat{\mathbf{U}}^h\right) d\Omega \, dt \tag{24}$$

where once again $\boldsymbol{\tau}_c$ is a symmetric positive matrix.

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Remark: In the design of the CAU method, the auxiliary matrix $\hat{\mathbf{A}}(\mathbf{U}^h)$ is constructed such that:

$$\left(\mathbf{A}-\hat{\mathbf{A}}\right).\nabla\mathbf{U}^h = \mathbf{R}\,; \qquad \mathbf{U}^h \xrightarrow[h\to 0]{} \mathbf{U} \Leftrightarrow \hat{\mathbf{A}} \xrightarrow[h\to 0]{} \mathbf{A} \tag{25}$$

This construction guarantees that always a quadratic positive weighted residual term is added to the variational form, since for $\hat{\mathbf{U}}^h = \mathbf{U}^h$:

$$\mathrm{T}_{CAU} = \|\mathbf{R}\|^2_{\tau_c} = \sum_{e=1}^{(nel)_n} \int_{S_n^e} \mathbf{R}^t \boldsymbol{\tau}_c \mathbf{R}\; d\Omega\, dt \tag{26}$$

Definitions for the $\boldsymbol{\tau}_c$ and the $\left(\mathbf{A}-\hat{\mathbf{A}}\right)$ matrices can be found in [1].

6 Computational Aspects

Let $\varphi_i(x,y)$, $(i=1,\ldots,nnode)$ denote the global interpolation functions in space and $N_i(t)$, $(i=1,2)$ represent the linear interpolation in time. Then over each "slab" S_n the finite element approximation is given by,

$$\mathbf{U}^h = \sum_{j=1}^{nnode} \varphi_j(x,y)\left[N_1(t)\mathbf{U}^h_{j,n^+} + N_2(t)\mathbf{U}^h_{j,n+1^-}\right] \tag{27}$$

where $\mathbf{U}^h_{j,n^+}$ and $\mathbf{U}^h_{j,n+1^-}$ are the unknown nodal values of $\mathbf{U}^h$ at times $t = t_n$ and $t = t_{n+1}$ respectively. Introducing the above approximations in the variational formulation, the following non-linear system of algebraic equations is obtained:

$$\mathbf{KU} = \mathbf{F}\text{, where:} \tag{28}$$

$$\mathbf{U} = \begin{bmatrix} \mathbf{U}^h_{n^+} \\ \mathbf{U}^h_{n+1^-} \end{bmatrix}; \quad \mathbf{K} = \begin{bmatrix} \left(\mathbf{K}_{11}+\mathbf{M}^G_{11}+\mathbf{M}\right) & \left(\mathbf{K}_{12}+\mathbf{M}^G_{12}\right) \\ \left(\mathbf{K}_{21}+\mathbf{M}^G_{21}\right) & \left(\mathbf{K}_{22}+\mathbf{M}^G_{22}\right) \end{bmatrix}; \quad \mathbf{F} = \begin{bmatrix} \mathbf{F}_1 \\ \mathbf{F}_2 \end{bmatrix} \tag{29}$$

The matrices $\mathbf{K}_{ij}$ and the vector $\mathbf{F}_i$ contain the time interpolation functions N_i and N_j $(i,j = 1, 2)$,

$$\mathbf{K}_{ij} = \mathbf{K}^G_{ij} + \mathbf{K}^{PG}_{ij} + \mathbf{K}^{DC}_{ij}; \quad \mathbf{F}_1 = \mathbf{F}^G_1 + \mathbf{F}^{PG}_1 + \mathbf{M}\mathbf{U}^h_{n^-}; \quad \mathbf{F}_2 = \mathbf{F}^G_2 + \mathbf{F}^{PG}_2 \tag{30}$$

The matrices $\mathbf{M}^G$, $\mathbf{K}^G_{ij}$ and $\mathbf{M}$ come from the Galerkin formulation (20). The matrices $\mathbf{K}^{PG}_{ij}$ and $\mathbf{K}^{DC}_{ij}$ come respectively from the STPG term (22) and the

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CAU contribution (24). Similarly, $\mathbf{F}_i^G$ and $\mathbf{F}_i^{PG}$ are the Galerkin and Petrov-Galerkin contributions to the right hand side of the system of equations.

The Modified Newton-Raphson method, applied to the system above, results in the following algorithm,

$$\left.\begin{aligned} &\mathbf{R}^i = \mathbf{F} - \mathbf{K}\,\mathbf{U}^i \\ &\mathbf{K}\,\Delta\mathbf{U}^i = \mathbf{R}^i \\ &\mathbf{U}^{i+1} = \mathbf{U}^i + \Delta\mathbf{U}^i \end{aligned}\right\} (i = 0,1,2,\ldots),\quad \mathbf{U}^0 = \begin{bmatrix} \mathbf{U}^0_{n^+} \\ \mathbf{U}^0_{n+1^-} \end{bmatrix} = \begin{bmatrix} \mathbf{U}^h_{n^-} \\ \mathbf{U}^h_{n^-} \end{bmatrix} \tag{31}$$

Splitting the system in two, the above algorithm can be rewritten as:

$$\left.\begin{aligned} \mathbf{R}_1^i &= \mathbf{F}_1 - \mathbf{K}_{11}^*\mathbf{U}^i_{n^+} - \mathbf{K}_{12}^*\mathbf{U}^i_{n+1^-} \\ \mathbf{K}_{11}^*\Delta\mathbf{U}^i_{n^+} &= \mathbf{R}_1^i \\ \mathbf{U}^{i+1}_{n^+} &= \mathbf{U}^i_{n^+} + \Delta\mathbf{U}^i_{n^+} \\ \mathbf{R}_2^i &= \mathbf{F}_2 - \mathbf{K}_{21}^*\mathbf{U}^{i+1}_{n^+} - \mathbf{K}_{22}^*\mathbf{U}^i_{n+1^-} \\ \mathbf{K}_{22}^*\Delta\mathbf{U}^i_{n+1^-} &= \mathbf{R}_2^i \\ \mathbf{U}^{i+1}_{n+1^-} &= \mathbf{U}^i_{n+1^-} + \Delta\mathbf{U}^i_{n+1^-} \end{aligned}\right\} (i = 0,1,2,\ldots) \tag{32}$$

where:

$$\begin{aligned} &\mathbf{K}_{11}^* = \mathbf{K}_{11} + \mathbf{M}_{11}^G + \mathbf{M}; \qquad &&\mathbf{K}_{12}^* = \mathbf{K}_{12} + \mathbf{M}_{12}^G \\ &\mathbf{K}_{21}^* = \mathbf{K}_{21} + \mathbf{M}_{21}^G; &&\mathbf{K}_{22}^* = \mathbf{K}_{22} + \mathbf{M}_{22}^G \end{aligned} \tag{33}$$

The main computational task is to solve the resulting nonlinear system of equations. This is done by solving a sequence of linear systems as in (31) or in (32), using the GMRES solver. The performance of any iterative method can be improved by using some type of preconditioning. Another way to improve the performance is to optimize the code, reducing memory requirement, the number of floating point operations (*flop*) and the number of indirect addressing operations (*i/a*).

Element by element techniques are convenient, as the sparsity of the system is fully exploited by the element level storage. Also, the matrix-vector product operation can be split into a global product, involving the nodal block diagonal of the system, and into another which is performed at an element by element basis. The assemblage of a global nodal block diagonal reduces the amount of necessary memory, but the coefficients relating two different nodes still remain spread over the contributing elements. This spreading of element contributions can be avoided if an edge data structure is used. This can be done performing a loop on the elements and assembling the coefficients by edges. For the shallow problem, Table 1 shows the reduction in terms of memory and *flop* for the

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element-by-element technique with block diagonal. Table 2 shows the reduction in terms of memory, *flop* and *i/a* for the edge-by-edge technique. The reduction, in both tables, is in relation to the simplest element by element technique, where the full element matrices are stored (i.e, no diagonal assembled). Note that for the element-by-element technique the number of *i/a* is not changed, while for the edge-by-edge technique there is no reduction for triangles, and the corresponding space-time prism presents an increase in terms of *i/a*. To reduce the number of *i/a*, the loop over single edges must be substituted for a loop over groups of edges [5]. In this case, the gain in *i/a* must compensate the increase in *flop*.

Table 1 - Element by element technique with block diagonal: reduction in terms of memory and *flop*.

Element	Memory (%)	*flop* (%)
Triangle	27.78	28.70
Quadrilateral	18.75	19.79
Tetrahedron	23.86	24.05
Space-time prism with triangular base	13.89	14.35

Table 2 - Edge by edge technique: reduction in terms of memory, *flop* and *i/a*.

Element	Memory (%)	*flop* (%)	*i/a* (%)
Triangle	61.11	62.04	0
Quadrilateral	43.75	44.79	-100
Tetrahedron	80.11	80.30	25
Space-time prism with triangular base	61.11	61.57	-116.17

7 Numerical Examples

The two examples shown here were selected in order to illustrate the quality of the solution and the computational performance of the proposed method. The first test case is the one-dimensional dam break problem, for which the stabilizing operators, in semi-discrete and space-time versions, are compared. The second example is a two dimensional extension of the first, and tests the performance of the GMRES solver using element-by-element and edge-by-edge techniques.

7.1 One-dimensional Dam Break

The first example is the well known *dam break* problem, which consists of a wall separating two undisturbed water levels that is suddenly removed. Friction effects are neglected and the spatial discretization is given by a 4x100 triangular elements mesh. Figures 1-2 show the results for $t = 7.50$, with $\Delta t = 0.25$. In these figures, we show the approximate solutions for the water elevation and the

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velocity obtained with the Galerkin, the STPG and the CAU methods. The results plotted in these figures show a remarkable accuracy improvement when the CAU method is employed. It can also be noted that the discontinuity-capturing term provided by this method behaves better than the one proposed in [9], here simply referred as DC. In Figure 3 we also present the results corresponding to the semi-discrete version of these methods. As expected, the approximations provided by the CAU method present the best results.

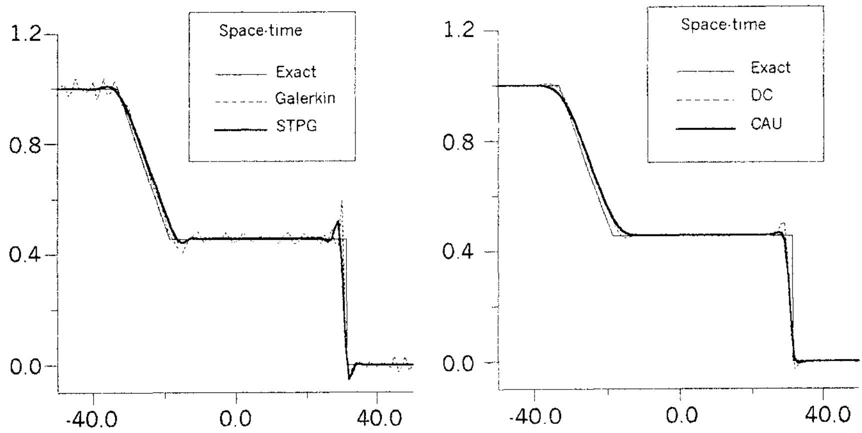

Figure 1 - Water elevation using space-time elements.

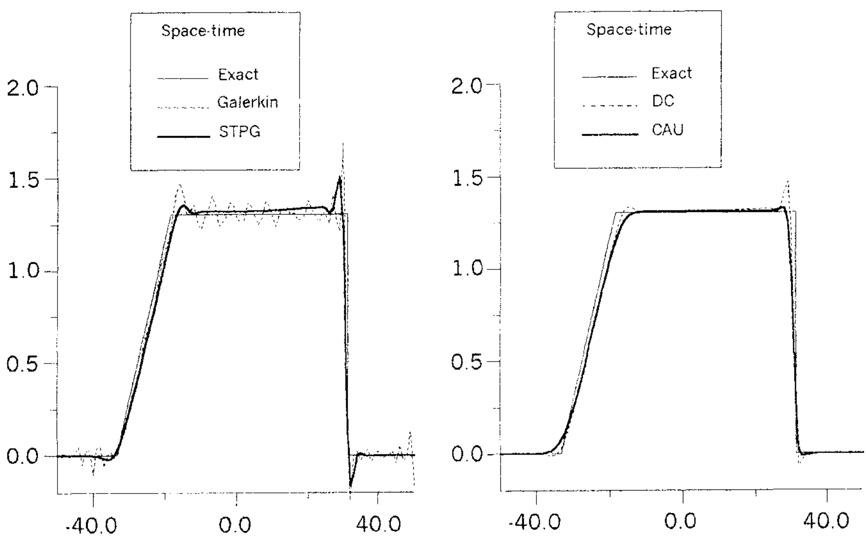

Figure 2 - Velocities using space-time elements.

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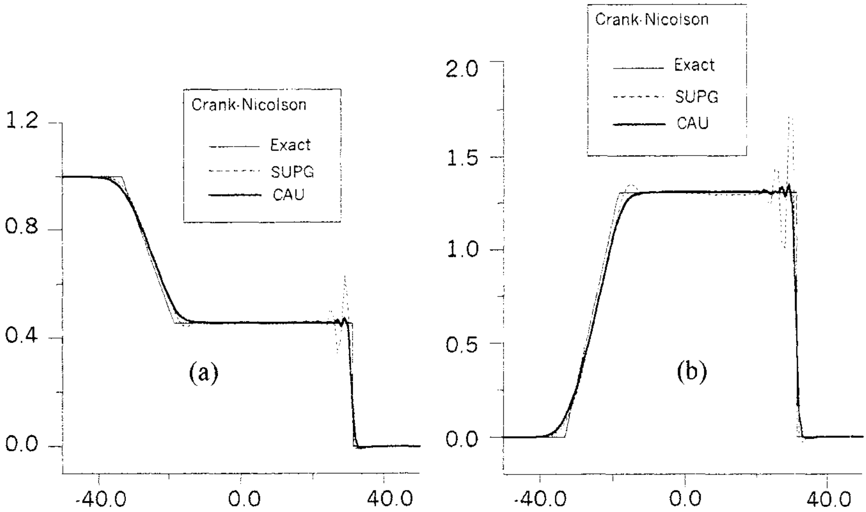

Figure 3 - Water elevation (a) and velocity (b) using semi-discrete formulation.

7.2 Two-dimensional Dam Break

This example is a two-dimensional extension of the first. The spatial discretization is given by a 2x100x100 triangular elements mesh, resulting in 59606 equations (29803 equations in the semi-discrete version).

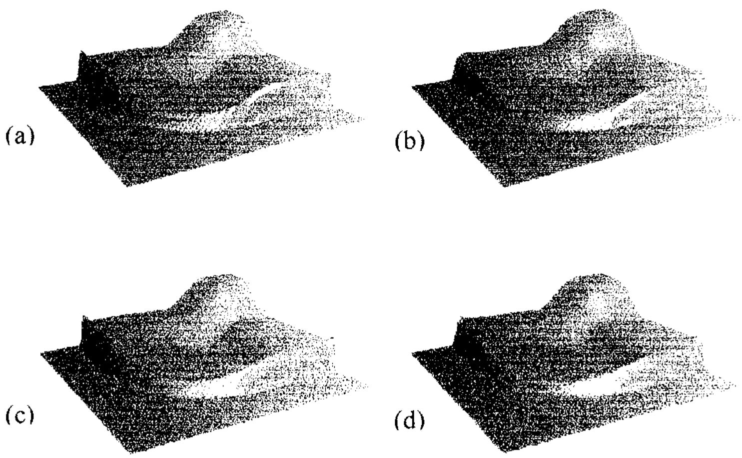

Figure 4 - Water elevation for $t = 7.5$: SUPG (a); semi-discrete CAU (b); STPG (c); space-time CAU (d).

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The semi-discrete and space-time solutions for $t = 7.5$ can be seen in Figure 4. The time step is the same as in the one-dimensional case ($\Delta t = 0.25$). The advantages of using edge data structure can be seen in Tables 3-4. These tables present the CPU time spent by the GMRES solver, to reach the solution at $t = 7.5$, or 30 time steps. The element-by-element techniques are denoted by EBE (full element matrices), EBE-D (simple main diagonal assembled) and EBE-BD (block diagonal assembled), while the edge-by-edge technique is denoted by EDGE. In space-time formulation with edge data structure, the results were obtained in both ways: solving the entire system or splitting the system in two (EDGE*). These figures also show the total number of non-linear iterations and the total number of GMRES iterations. A simple diagonal preconditioning has been used in all cases.

Table 3 - Semi-discrete (Crank-Nicolson): SUPG.

	CPU time (s)	Reduction (%)	Non-linear iterations	GMRES iterations
EBE	185.32	-	240	1961
EBE-D	161.90	12.64	240	1961
EBE-BD	147.35	20.49	240	1961
EDGE	105.92	42.84	240	1961

Table 4 - Space-time: STPG.

	CPU time (s)	Reduction (%)	Non-linear iterations	GMRES iterations
EBE	712.38	-	211	2342
EBE-D	679.23	4.65	211	2342
EBE-BD	626.93	12.00	211	2342
EDGE	439.82	38.26	211	2342
EDGE*	185.93	-	300	3399

8 Conclusions

In this paper we have presented a space-time finite element formulation for problems governed by the shallow water equations, using linear approximation in time and linear three node triangles for the spatial discretization. For the one-dimensional dam break problem, the numerical results were compared to the exact solution. Accurate results were obtained with the CAU method, showing its capability as a stabilizing operator. Although the number of equations in space-time formulation is twice the number of equations arising in the correspondent semi-discrete version, the CPU and memory requirements were significantly reduced by splitting the space-time system of equations in two "semi-discrete" systems, and using an edge based data structure.

Advances in Fluid Mechanics III, C.A. Brebbia & M. Rahman (Editors)

References

1. Almeida, R. C., Galeão, A. C., "An Adaptive Petrov-Galerkin Formulation for the Compressible Euler and Navier-Stokes Equations". *Comput. Meth. Appl. Mech. Engrg*, vol. 129, 157-176, 1996.

2. Bova, S. W., Carey, G. F., "An entropy Variable Formulation and Petrov-Galerkin Methods for the Shallow Water Equations", in: *Finite Element Modeling of Environmental Problems-Surface and Subsurface Flow and Transport*, ed. G. Carey, John Wiley, London, England, 1995.

3. Brooks, A. N., Hughes, T. J., "Streamline Upwind Petrov-Galerkin Formulation for Convection-Dominated Flows with Particular Emphasis on the Incompressible Navier-Stokes Equations", *Comput. Meth. Appl. Mech. Engrg*, vol. 32, 199-259, 1982.

4. Galeão, A. C., Do Carmo, E. G., "A Consistent Approximate Upwind Petrov-Galerkin Method for Convection-Dominated Problems", *Comput. Meth. Appl. Mech. Engrg*, vol. 32, 199-259, 1988.

5. Löhner, R., "Edges, Stars, Superedges and Chains", *Comput. Meth. App. Mech. Engrg.*, vol. 111, 255-263, 1994.

6. Ribeiro, F. L. B., Galeão, A. C. and Landau, L., "A Space-Time Finite Element Formulation for Shallow Water Equations", in: *Development and Application of Computer Techniques to Environmental Studies VI*, pp. 403-414, Computational Mechanics Publications, 1996.

7. Ribeiro, F. L. B., Castro, R. G. S., Galeão, A. C., Loula, A. F. D. and Landau, L., "A Space-Time Finite Element Formulation for Shallow Water Equations with Shock-capturing Operator", IV World Congress, Argentina, 1998.

8. Saleri, F., "Some Stabilization Techniques in Computational Fluid Dynamics", *Proceedings of the 9th International Conference on Finite Elements in Fluids*, Venezia, 1995.

9. Shakib, F., "Finite Element Analysis of the Compressible Euler and Navier-Stokes Equations", *Ph.D. Thesis*, Stanford University, 1988.