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FINITE ENERGY CYLINDERS OF SMALL AREA

H. HOFER [1], K. WYSOCKI [2], AND E. ZEHNDER [3]

ABSTRACT. Investigated are $\widetilde{J}$-holomorphic cylinders in $\mathbb{R} \times M$, where M is a contact manifold. It is shown that a sufficiently long cylinder having small area is close to a constant map, if its center action vanishes. If its center action is positive, it is close to a cylinder over a periodic orbit of the Reeb vector field, and has a well determined shape. A compactness result is deduced, which is useful in symplectic field theory.

CONTENTS

1. INTRODUCTION AND RESULTS

In the following we denote by M a compact (2n+1)-dimensional manifold and assume that M is equipped with the contact form λ. By definition, a contact form is a 1-form having the property that $\lambda \wedge (d\lambda)^n$ is a volume form. The contact form determines the hyperplane bundle

$$\xi = \ker(\lambda),$$

called the associated contact structure on M. The restriction $d\lambda|_\xi$ is a symplectic form on every fibre. The contact form λ determines also the associated Reeb vector field X by the conditions

$$i_X\lambda = 1 \quad \text{and} \quad i_X d\lambda = 0.$$

The Reeb vector field is transversal to the contact structure so that the tangent bundle splits

$$TM = \mathbb{R}X \oplus \xi.$$

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We shall denote by

$$\pi : TM \to \xi$$

the projection along X. We choose a compatible almost complex structure J on ξ. This is a smooth fiber preserving fiberwise linear map $J : \xi \to \xi$ satisfying $J^2 = -\operatorname{Id}$ so that, in addition,

$$g_J(m)\big(k', k\big) := d\lambda(m)\big(k', J(m)k\big), \quad m \in M,$$

defines a fiberwise metric on ξ. Associated with J we now choose a special almost complex structure $\widetilde{J}$ on the $(2n+2)$-manifold $\mathbb{R} \times M$. It is defined by

$$\widetilde{J}(h, k) = \big(-\lambda(k), J(\pi k) + hX(m)\big), \tag{1}$$

where $(h, k) \in T_m(\mathbb{R} \times M)$. The almost complex structure $\widetilde{J}$ has the distinguished property that it is $\mathbb{R}$-invariant. A $\widetilde{J}$-holomorphic curve or a pseudoholomorphic curve is a smooth map

$$\widetilde{u} = (a, u) : \dot{S} \to \mathbb{R} \times M \tag{2}$$

defined on a punctured Riemann surface $\dot{S}$ equipped with the complex structure i, and solving the Cauchy-Riemann equations

$$\widetilde{J}(\widetilde{u}) \circ T\widetilde{u} = T\widetilde{u} \circ i. \tag{3}$$

Recall that a punctured Riemann surface is a closed Riemann surface (S, i) with a finite set Γ of punctures removed, that is, $\dot{S} = S \setminus \Gamma$. We shall impose the energy condition

$$E\big(\widetilde{u}\big) < \infty, \tag{4}$$

where the energy E is defined as

$$E\big(\widetilde{u}\big) = \sup_{\varphi \in \Sigma} \int_S \widetilde{u}^* d[\varphi\lambda].$$

The class Σ of functions is defined by $\Sigma = \big\{f \in C^\infty\big(\mathbb{R}, [0,1]\big) | f' \geq 0\big\}$. We point out that the integrand for a solution $\widetilde{u} = (a, u)$ is always nonnegative. Indeed, for a solution one computes in holomorphic coordinates $s + it$,

$$\begin{aligned} 2\widetilde{u}^* d[\varphi\lambda] &= \varphi'(a)\big[a_s^2 + a_t^2 + \lambda(u_s)^2 + \lambda(u_t)^2\big]\ ds \wedge dt \\ &\quad + \varphi(a)\big[|\pi u_s|_J^2 + |\pi u_t|_J^2\big]\ ds \wedge dt, \end{aligned}$$

where we have used the norm $|h|_J = g_J(h, h)$ on ξ. We shall also call a map $\widetilde{u} : S \to \mathbb{R} \times M$ satisfying the equation (3) and the energy bound (4) a finite energy surface. If $S = \mathbb{C}$, it is called a finite energy plane.

To describe an example important later on we assume that the Reeb vector field X possesses a T-periodic solution $x(t)$. One verifies easily that the smooth map

$$\widetilde{u} = (a, u) : \mathbb{R} \times S^1 \to \mathbb{R} \times M,$$

defined by

$$\widetilde{u}(s, t) = \big(Ts + a_0, x(Tt + \vartheta_0)\big) \tag{5}$$

for two real numbers a_0, ϑ_0, solves the equation (2) with the standard complex structure i on the cylinder $\mathbb{R} \times S^1$, where $S^1 = \mathbb{R}/\mathbb{Z}$. The energy of $\widetilde{u}$ satisfies

$$E(\widetilde{u}) = T$$

while the $d\lambda$-energy vanishes,

$$\int_{\mathbb{R}\times S^1} u^* d\lambda = 0.$$

The special finite energy surface (5) will be called a cylinder over a periodic orbit.

In the following we study non constant finite $\widetilde{J}$-holomorphic cylinders

$$\widetilde{u} = (a, u) : [-R, R] \times S^1 \to \mathbb{R} \times M$$

having finite energy and small $d\lambda$-energy

$$\int_{[-R,R]\times S^1} u^* d\lambda \leqslant \gamma.$$

We shall show that for R large and γ small, the finite cylinder $\widetilde{u}$ is close to a cylinder over a periodic orbit. The main point of this paper will be to obtain precise estimates determining the shape of the cylinder. This requires, however, assumptions on M and λ.

We shall assume that the set of periodic orbits of the Reeb vector field X is not empty. In view of the results of Hofer [5], this is, for example, guaranteed for three-manifolds M in the following cases

(1) $M = S^3$
(2) $\pi_2(M) \neq 0$
(3) The contact structure ξ is overtwisted.

Denote by $\mathcal{P} \subset \mathbb{R}$ the set containing $0 \in \mathbb{R}$ and all periods T of periodic orbits of the Reeb vector field X. We call $\mathcal{P} = \mathcal{P}_\lambda$ the action spectrum of λ. We assume that there exists a constant $E_0 > 0$ having the property that all periodic orbits of the Reeb vector field X with periods $T \leqslant E_0$ are non-degenerate. Hence there are only finitely many of them. Recall that a T-periodic orbit is non-degenerate, if it has only one Floquet multiplier equal to 1. A non-degenerate T-periodic orbit is, in particular, isolated among periodic orbits having periods close to T. We define the real number $\gamma_0 > 0$ by

$$\gamma_0 = \min\bigl\{|T_1 - T_2|;\ \text{where } T_1, T_2 \in \mathcal{P} \text{ are} \leqslant E_0 \text{ and } T_1 \neq T_2\bigr\}.$$

One can associate with sufficiently long $\widetilde{J}$-holomorphic cylinders having sufficiently small area, a unique element $T \in \mathcal{P}$, as the first result shows.

Theorem 1.1. *Let λ be a contact form on M and $E_0 > 0$ be given, so that all periodic orbits with period $T \leqslant E_0$ are non-degenerate. Let γ_0 be the number defined above. Fix numbers γ and σ satisfying $0 < \gamma < \gamma_0$ and $0 < \sigma < \gamma_0 - \gamma$. Then there exists a constant $h_0 > 0$ so that the following holds. For every $R > h_0$ and every $\widetilde{J}$-holomorphic cylinder*

$$\widetilde{u} = (a, u) : [-R, R] \times S^1 \to \mathbb{R} \times M$$

satisfying

$$E(\widetilde{u}_0) \leqslant E_0$$

$$\int_{[-R,R]\times S^1} u^* d\lambda \leqslant \gamma, \tag{6}$$

there exists a unique element $T \in \mathcal{P}$ satisfying $T \leqslant E_0$ and

$$\left|\int_{S^1} u(0)^*\lambda - T\right| < \frac{\sigma}{2} \tag{7}$$

for the action of the loop $u(0)$ in the center of the cylinder. Here and in the following, $u(s)$ is the loop in M defined by $u(s)(t) := u(s,t)$, where $(s,t) \in [-R,R] \times S^1$.

The unique element $T \in \mathcal{P}$ associated with the cylinder $\widetilde{u}$ satisfying the hypotheses of Theorem 1.1 will be called the center action of $\widetilde{u}$ and abbreviated by

$$T = A(\widetilde{u}). \tag{8}$$

If the cylinder $\widetilde{u} = (a,u) : [-R,R] \times S^1 \to \mathbb{R} \times M$ meets the hypotheses of Theorem 1.1, the actions of all the loops are estimated by

$$\left|\int_{S^1} u(s)^*\lambda - T\right| < \frac{\sigma}{2} + \gamma < \gamma_0 \tag{9}$$

for all $s \in [-R,R]$, where $T = A\big(\widetilde{u}\big)$ is the center action. This follows by means of Stokes' theorem from (6) and (7).

Proof of Theorem 1.1. For every $\delta > 0$ there exists a constant $C > 0$ so that the gradients of all the $\widetilde{J}$-holomorphic cylinders $\widetilde{u} : [-R,R] \times S^1 \to \mathbb{R} \times M$ with $R > \delta$ which satisfy the energy estimates (6), are uniformly bounded on $[-R+\delta, R-\delta] \times S^1$ by the constant C. Indeed, if the gradients are not uniformly bounded, then a bubbling off analysis as in [5] or in [10] produces a finite energy plane $\widetilde{v} : \mathbb{C} \to \mathbb{R} \times M$ whose asymptotic limit in M is a periodic solution of the Reeb vector field whose period T satisfies $0 < T \leqslant E_0$ and

$$T = \int_{\mathbb{C}} v^* d\lambda \leqslant \gamma.$$

By assumption, $\gamma < \gamma_0$ and by the definition of γ_0 there exists no such period. This contradiction shows that the gradients are uniformly bounded. From gradient bounds one deduces uniform bounds for all higher derivatives by means of elliptic regularity theory. This is well known and we refer to [5].

In order to prove Theorem 1.1 we argue indirectly and find a sequence of numbers $R_n \to \infty$ and a sequence $\widetilde{u}_n = (a_n, u_n) : [-R_n, R_n] \times S^1 \to \mathbb{R} \times M$ of $\widetilde{J}$-holomorphic cylinders satisfying the energy estimates (6) of the theorem, and in addition,

$$\left|\int_{S^1} u_n(0)^*\lambda - T\right| \geq \frac{\sigma}{2} \tag{10}$$

for every $T \in \mathcal{P}$ satisfying $T \leqslant E_0$. In view of the previous observation we may assume after taking a subsequence and using Ascoli-Arzela's theorem, that $\widetilde{u}_n$ converges in $C^\infty_{\text{loc}}(\mathbb{R} \times S^1, \mathbb{R} \times M)$ to some $\widetilde{J}$-holomorphic cylinder $\widetilde{u} = (a,u) : \mathbb{R} \times S^1 \to \mathbb{R} \times M$ satisfying, in view of (6) and (10),

$$\begin{gathered} E(\widetilde{u}) \leqslant E_0 \\ \int u^* d\lambda \leqslant \gamma \\ \left|\int_{S^1} u(0)^*\lambda - T\right| \geq \frac{1}{2}\sigma \ \text{ for all } \ T \in \mathcal{P},\ T \leqslant E_0. \end{gathered} \tag{11}$$

The map $\widetilde{u}$ may be viewed as a finite energy map defined on a 2-punctured Riemann sphere. It is well known, see [8], that a puncture is either removable or has a periodic orbit of the Reeb vector field as asymptotic limit. In both cases the limits

$$\lim_{s\to\pm\infty}\int_{S^1} u(s)^*\lambda \in \mathbb{R}$$

do exist. Moreover, the limit is equal to 0 if the puncture is removable, and equal to the period of the asymptotic limit if the puncture is not removable. Consequently, by means of Stokes' theorem the $d\lambda$-energy of $\widetilde{u}$ has the form

$$\int_{\mathbb{R}\times S^1} u^* d\lambda = T_2 - T_1$$

with $T_2 \geq T_1$, where $T_1, T_2 \in \mathcal{P}$ and $T_1, T_2 \leqslant E_0$. Since by (11) the $d\lambda$-energy of $\widetilde{u}$ is $\leqslant \gamma$, and since $\gamma < \gamma_0$, we conclude from the definition of the constant γ_0 that $T_1 = T_2$. Set $T := T_1 = T_2$. If $T = 0$, then both punctures are removable and so $\widetilde{u}$ has an extension to a $\widetilde{J}$-holomorphic finite energy sphere $S^2 \to \mathbb{R}\times M$. Consequently, the map $\widetilde{u}$ must be constant and so $\int_{S^1} u(0)^*\lambda = 0 = T$, contradicting the third estimate in (11). If $T > 0$, then the finite energy cylinder $\widetilde{u}$ is non constant and has vanishing $d\lambda$-energy. Non constant finite energy surfaces having vanishing $d\lambda$-energies are classified in [8], and we conclude from Proposition 3.11 of [8] that $\widetilde{u}$ must be a cylinder over a periodic orbit $x(t)$ of the form $\widetilde{u}(s,t) = \big(Ts+c, x(Tt+d)\big)$ for constants c and d and with a period T satisfying $0 < T \leqslant E_0$. Hence $\int_{S^1} u(0)^*\lambda - T = 0 < \sigma/2$ again contradicting the third estimate in (11).

To sum up we have proved that there exists a constant $h_0 > 0$ so that every $\widetilde{J}$-holomorphic cylinder $\widetilde{u} = (a, u) : [-R, R] \times S^1 \to \mathbb{R}\times M$ with $R > h_0$ and satisfying the energy estimates (6), has a center loop $u(0)$ whose action is close to an element $T \in \mathcal{P}$ with $T \leqslant E_0$ in the sense that

$$\left|\int_{S^1} u(0)^*\lambda - T\right| < \frac{\sigma}{2}. \tag{12}$$

Assume now that two elements $T_1, T_2 \in \mathcal{P}$ with $T_i \leqslant E_0$ satisfy the above estimate. Then

$$|T_1 - T_2| < \frac{\sigma}{2} + \frac{\sigma}{2} = \sigma.$$

By assumption, $\sigma < \gamma_0 - \gamma < \gamma_0$ and it follows again from the definition of γ_0 that $T_1 = T_2$. Hence the element $T \in \mathcal{P}$ satisfying $T \leqslant E_0$ and the estimate (12) is unique and the proof of Theorem 1.1 is complete. □

From the definition of the constant γ_0 it follows for the center action $A(\widetilde{u})$ of a long cylinder $\widetilde{u}$ meeting the hypotheses of Theorem 1.1, that

$$\begin{array}{ll} \text{either} & A(\widetilde{u}) = 0 \\ \text{or} & A(\widetilde{u}) \geq \gamma_0. \end{array}$$

In the first case the $\widetilde{J}$-holomorphic cylinder is close to a constant map as the next result shows. Fix a Riemannian metric g on M and define an associated $\mathbb{R}$-invariant metric $\widetilde{g}$ on $\mathbb{R}\times M$ by

$$\widetilde{g}\big((h,k),(h',k')\big) = hh' + g(k,k').$$

We have

Theorem 1.2. *Let E_0, γ_0, γ and σ be as in the previous theorem and let $h_0 > 0$ be the number guaranteed by the Theorem 1.1. There exists for given $\varepsilon > 0$ a number $h_1 \geq h_0$, so that for every $R > h_1$ and every finite energy cylinder $\widetilde{u} = (a, u) : [-R, R] \times S^1 \to \mathbb{R} \times M$ satisfying*

$$
\begin{aligned}
A(\widetilde{u}) &= 0 \\
E(\widetilde{u}) &\leqslant E_0 \\
\int_{[-R,R]\times S^1} u^* d\lambda &\leqslant \gamma,
\end{aligned}
$$

the following holds,

$$\widetilde{u}\big([-R + h_1, R - h_1] \times S^1\big) \subset B_\varepsilon\big(\widetilde{u}(0, 0)\big).$$

Postponing the proof we would like to point out an immediate consequence. Since most of the $\widetilde{J}$-holomorphic cylinder $\widetilde{u}$ of Theorem 1.2 lies in a compact part of $\mathbb{R} \times M$ which, equipped with the 2-form $\omega = d(\varphi\lambda)$ where $\varphi' > 0$, is a compact symplectic manifold, the familiar theory in the compact case initiated by Gromov [4] is applicable.

In contrast to the case $A(\widetilde{u}) = 0$, a long $\widetilde{J}$-holomorphic cylinder having small area but positive center action is close to a cylinder over a periodic orbit of the Reeb vector field and has a well defined shape. This is our main result.

Theorem 1.3. *Let E_0 and γ_0 be the positive numbers introduced above and denote by h_0 the number guaranteed by Theorem 1.1 and associated with the constants $0 < \gamma < \gamma_0$ and $0 < \sigma < \gamma_0 - \gamma$. Then there exist positive constants δ_0, C_α, μ and $\nu < \min\{4\pi, 2\mu\}$ such that the following holds. Given $0 < \delta \leqslant \delta_0$, there exists a constant $h \geq h_0$ such that for every $R > h$ and every $\widetilde{J}$-holomorphic cylinder*

$$\widetilde{u} = (a, u) : [-R, R] \times S^1 \to \mathbb{R} \times M$$

satisfying

$$
\begin{aligned}
A\big(\widetilde{u}\big) &> 0 \\
E(\widetilde{u}) &\leqslant E_0 \\
\int_{[-R,R]\times S^1} u^* d\lambda &\leqslant \gamma,
\end{aligned}
$$

there exists a unique (up to phase shift) periodic solution $x(t)$ of the Reeb vector field X having period $T = A\big(\widetilde{u}\big) \leqslant E_0$ satisfying

$$\left| \int_{S^1} u(0)^*\lambda - T \right| < \frac{\sigma}{2} \quad \textit{and} \quad \left| \int_{S^1} u(s)^*\lambda - T \right| < \gamma_0, \textit{ for all } s \in [-R, R].$$

In addition, there exists a tubular neighborhood $U \cong S^1 \times \mathbb{R}^{2n}$ around the periodic orbit $x(\mathbb{R}) \cong S^1 \times \{0\}$ such that $u(s, t) \in U$ for all $(s, t) \in [-R + h, R - h] \times S^1$. Using the covering $\mathbb{R}$ of $S^1 = \mathbb{R}/\mathbb{Z}$, the map $\widetilde{u}$ is in the local coordinates $\mathbb{R} \times U$ represented as

$$
\begin{aligned}
\widetilde{u}(s, t) &= \big(a(s, t), \vartheta(s, t), z(s, t)\big) \\
&= \big(Ts + a_0 + \widetilde{a}(s, t), kt + \vartheta_0 + \widetilde{\vartheta}(s, t), z(s, t)\big),
\end{aligned}
$$

where $(a_0, \vartheta_0) \in \mathbb{R}^2$ are constants. The functions $\widetilde{a}$, $\widetilde{\vartheta}$ and z are 1-periodic in t and the positive integer k is the covering number of the T-periodic orbit represented by $x(Tt) = (kt, 0, 0)$. For all multi-indices α and for all $(s,t) \in [-R+h, R-h] \times S^1$ the following estimates hold

$$\left|\partial^\alpha z(s,t)\right|^2 \leqslant C_\alpha \cdot \delta^2 \cdot \frac{\cosh[\mu s]}{\cosh[\mu(R-h)]}$$

and

$$\left|\partial^\alpha \widetilde{a}(s,t)\right|^2, \ \left|\partial^\alpha \widetilde{\vartheta}(s,t)\right|^2 \ \leqslant C_\alpha \cdot \delta^2 \cdot \frac{\cosh[\nu s]}{\cosh[\nu(R-h)]}.$$

For the next theorem we view M as embedded in $\mathbb{R}^m$ for some large m. On the loop space $C^\infty(S^1, \mathbb{R}^m)$ there is a translation-invariant metric d which induces a metric on $C^\infty(S^1, M)$. As an immediate consequence of Theorem 1.3 we have

Theorem 1.4. *Let E_0, γ_0 be as introduced in Theorem 1.1 and let $0 < \gamma < \gamma_0$. Then for given $\varepsilon > 0$ there exists a constant $h > 0$ such that the following holds true. For every $R > h$ and every finite $\widetilde{J}$-holomorphic cylinder $\widetilde{u} = (a, u) : [-R, R] \times S^1 \to \mathbb{R} \times M$ such that*

$$A(\widetilde{u}) > 0$$
$$E(\widetilde{u}) \leqslant E_0$$
$$\int_{[-R,R]\times S^1} u^* d\lambda \leqslant \gamma,$$

the loops $t \mapsto u(s,t)$ satisfy

$$u(s, \cdot) \in \mathcal{U}_\varepsilon\big(u(0,\cdot)\big)$$

for all $s \in [-R+h, R-h]$. Here $\mathcal{U}_\varepsilon\big(u(0,\cdot)\big)$ is the ε-ball around the loop $t \mapsto u(0,t)$ in $C^\infty(S^1, M)$.

Corollary 1.5. *Under the assumptions of Theorem 1.4 there exists for given $\varepsilon > 0$ a constant $h > 0$ so that for every $R > h$ and every finite energy cylinder $\widetilde{u} = (a, u) : [-R, R] \times S^1 \to \mathbb{R} \times M$ satisfying*

$$A(\widetilde{u}) > 0$$
$$E(\widetilde{u}) \leqslant E_0$$
$$\int_{[-R,R]\times S^1} u^* d\lambda \leqslant \gamma,$$

the following holds. There exists a periodic orbit $x(t)$ of the Reeb vector field X having period $T \leqslant E_0$ such that the loops $t \mapsto u(s,t)$ satisfy

$$u(s,\cdot) \in \mathcal{U}_\varepsilon\big(x(T\cdot)\big) \quad \text{and} \quad u(0,\cdot) \in \mathcal{U}_{\varepsilon/2}\big(x(T\cdot)\big)$$

for all $s \in [-R+h, R-h]$.

We point out that the neighborhoods $\mathcal{U}_\varepsilon$ are not required to be S^1-invariant. Therefore, Corollary 1.5 implies, in particular, that the ϑ-variable in the local coordinates of Theorem 1.3 turns very little inside an arbitrary long cylinder as long as one stays away from the boundary at a distance larger than h.

In Section 4, Theorem 1.4 will be used in order to prove a compactness result for sequences of $\widetilde{J}$-holomorphic finite energy maps, which is important in the symplectic field theory of [2].

2. Proof of Theorem 1.2

The first ingredient of the proof of Theorem 1.2 is a well known a-priori estimate due to M. Gromov. Fix a smooth map $\varphi : \mathbb{R} \to [0,1]$ satisfying $\varphi' > 0$. Then the 2-form $\omega = d(\varphi\lambda)$ is a symplectic form on $\mathbb{R} \times M$ compatible with the almost complex structure $\widetilde{J}$. By a result of Gromov for which we refer to [14] (Chapter II, Theorem 1.3), there exist constants $r_0 > 0$ and $C > 0$ such that the following holds true. For a $\widetilde{J}$-holomorphic curve $\widetilde{u} : S \to \mathbb{R} \times M$ defined on a compact Riemann surface S with boundary, and an interior point $p \in S \backslash \partial S$, satisfying $\widetilde{u}(p) \in \{0\} \times M$ and $\widetilde{u}(\partial S) \cap B_r\big(\widetilde{u}(p)\big) = \emptyset$, we have

$$\int_{\widetilde{u}^{-1}\left(\bar{B}_r\left(\widetilde{u}(p)\right)\right)} \widetilde{u}^*\omega \geq C \cdot r^2 \text{ for all } r \in (0, r_0].$$

Using the $\mathbb{R}$-invariance of the almost complex structure $\widetilde{J}$ and the definition of the energy E one immediately extends the result to all of $\mathbb{R} \times M$ as follows. For a compact Riemann surface S with boundary, an interior point $p \in S \setminus \partial S$ and a $\widetilde{J}$-holomorphic map $\widetilde{u} : S \to \mathbb{R} \times M$ satisfying $\widetilde{u}(\partial S) \cap B_r\big(\widetilde{u}(p)\big) = \emptyset$, we have

$$E(\widetilde{u}|_{\widetilde{u}^{-1}\left(\bar{B}_r(\widetilde{u}(p))\right)}) \geq C \cdot r^2 \quad \text{for all } r \in (0, r_0]. \tag{13}$$

The second ingredient are the following properties of long $\widetilde{J}$-holomorphic cylinders having vanishing center actions.

Lemma 2.1. *Recall the constant E_0 and γ_0. Choose $0 < \gamma < \gamma_0$ and $0 < \sigma < \gamma_0 - \gamma$ and let $h_0 > 0$ be the associated constant guaranteed by Theorem 1.1. Given $\delta > 0$ there exists a constant $h \geq h_0$ such that for every $R > h$ and every $\widetilde{J}$-holomorphic cylinder $\widetilde{u} = (a, u); [-R, R] \times S^1 \to \mathbb{R} \times M$ satisfying the hypotheses of Theorem 1.1 and having center action $A(\widetilde{u}) = 0$, the loops $\widetilde{u}(s)$ have the following properties,*

(i) *diam* $\big(\widetilde{u}(s)\big) \leqslant \delta$
(ii) $\big|\lambda(u_t(s))\big| \leqslant \delta$

for all $s \in [-R + h, R - h]$.

Proof. We start with the proof of (i). Arguing indirectly we find a constant $\delta_0 > 0$, a sequence $R_n \geq n + h_0$ and a sequence $\widetilde{u}_n = (a_n, u_n) : [-R_n, R_n] \times S^1 \to \mathbb{R} \times M$ of $\widetilde{J}$-holomorphic cylinders satisfying

$$E(\widetilde{u}_n) \leqslant E_0$$
$$\int_{[-R_n, R_n] \times S^1} u_n^* d\lambda \leqslant \gamma$$
$$\left| \int_{S^1} u_n(0)^* \lambda \right| \leqslant \frac{\sigma}{2}$$
$$\text{diam } \widetilde{u}_n(s_n) \geq \delta_0$$

for a sequence $s_n \in [-R_n+n, R_n-n]$. Using the $\mathbb{R}$-invariance of $\widetilde{J}$ we may assume that $s_n = 0$ for all n. Define the sequence

$$\widetilde{v}_n(s,t) = \big(a_n(s,t) - a_n(0,0), u_n(s,t)\big)$$

of $\widetilde{J}$-holomorphic curves. Then, by the arguments of Theorem 1.1, a subsequence of $\widetilde{v}_n$ converges in C^∞_{loc} to a $\widetilde{J}$-holomorphic cylinder $\widetilde{v} = (b,v) : \mathbb{R}\times S^1 \to \mathbb{R}\times M$ having the following properties

$$\begin{aligned} E(\widetilde{v}) &\leqslant E_0 \\ \int_{\mathbb{R}\times S^1} v^* d\lambda &= 0 \\ \left|\int_{S^1} v(0)^*\lambda\right| &\leqslant \frac{\sigma}{2} \\ \operatorname{diam}\big(\widetilde{v}(0)\big) &\geq \delta_0. \end{aligned} \tag{14}$$

Hence $\widetilde{v}$ is, in particular, a non constant finite energy cylinder having vanishing $d\lambda$-energy. We conclude from Proposition 3.11 in [8] that $\widetilde{v}$ is a cylinder over a periodic orbit of period $0 < T \leqslant E_0$. Consequently,

$$\int_{S^1} v(0)^*\lambda = T \geq \gamma_0 > \gamma$$

in contradiction to (14). Hence $\operatorname{diam}\big(\widetilde{u}(s)\big) \leqslant \delta$ for all $s \in [-R+h, R-h]$. The second statement in Lemma 2.1 is proved the same way. The proof of the lemma is complete. $\square$

Let now $\varepsilon > 0$ be given as in Theorem 1.2. Then we choose $\delta > 0$ and $0 < r \leqslant r_0$ so small that

$$2\delta < Cr^2 \quad \text{and} \quad 4\delta + r \leqslant \varepsilon/2. \tag{15}$$

With h as in the lemma, the $\widetilde{J}$-holomorphic curve $\widetilde{u} : [-R,R]\times S^1 \to \mathbb{R}\times M$ with $R > h$, satisfying the hypotheses of Theorem 1.2 has the properties $\operatorname{diam}\big(\widetilde{u}(s)\big) \leqslant \delta$ and $\big|\lambda\big(u_t(s)\big)\big| \leqslant \delta$ on $[-R+h, R-h]$. This implies, using the definition of the energy E and Stokes' theorem

$$E\big(\widetilde{u}|_{[-R+h,R-h]\times S^1}\big) \leqslant 2\delta. \tag{16}$$

If the conclusion of Theorem 1.2 is not true for this h,we find a point $(s_0,t_0) \in [-R+h, R-h]\times S^1$ satisfying

$$d\big(\widetilde{u}(s_0,t_0), \widetilde{u}(0,0)\big) \geq \varepsilon.$$

From $\operatorname{diam}\big(\widetilde{u}(s)\big) \leqslant \delta$ we deduce

$$d\big(\widetilde{u}(s_0,t), \widetilde{u}(0,t')\big) \geq \varepsilon - 2\delta$$

for all $t, t' \in S^1$. Choose a point s_1 between 0 and s_0 so that

$$\begin{aligned} d\big(\widetilde{u}(s_1,t), \widetilde{u}(s_0,t')\big) &\geq \frac{\varepsilon}{2} - 4\delta \\ d\big(\widetilde{u}(s_1,t), \widetilde{u}(0,t')\big) &\geq \frac{\varepsilon}{2} - 4\delta \end{aligned}$$

for all $t, t' \in S^1$. Since $r \leqslant \varepsilon/2 - 4\delta$ we can apply the a-priori estimate (13) to the open ball $B_r(\widetilde{u}(s_1, t_1))$ and hence conclude

$$E(\widetilde{u}|_{[-R+h,R-h]\times S^1}) \geq C \cdot r^2.$$

This implies by (16) that $C \cdot r^2 \leqslant 2\delta$ in contradiction to the choice in (15). This contradiction shows that $\widetilde{u}(s,t) \in B_\varepsilon\big(\widetilde{u}(0,0)\big)$ for all $(s,t) \in [-R+h, R-h] \times S^1$ as claimed in Theorem 1.2.

3. Proof of the Main Results

The proof of Theorem 1.3 will follow from a sequence of lemmata. We first show that a finite energy cylinder having positive center action is close to a periodic solution of the Reeb vector field X if its area is sufficiently small.

There is a natural action of S^1 on $C^\infty(S^1, M)$ defined by $\big(e^{2\pi i\vartheta} * y\big)(t) = y(t+\vartheta)$ for $e^{2\pi i\vartheta} \in S^1$. We choose an S^1-invariant neighborhood $\mathcal{W}$ in the loop space $C^\infty\big(S^1, M\big)$ of the finitely many loops $t \mapsto x(Tt)$, $0 \leqslant t \leqslant 1$, defined by the periodic solutions $x(t)$ of X having periods $T \leqslant E_0$. Moreover, we choose the neighborhood $\mathcal{W}$ so small that it separates these distinguished loops from each other.

Lemma 3.1. *Let $E_0 > 0$ and $\gamma_0 > 0$ be as defined in the introduction. Given any S^1-invariant neighborhood $\mathcal{W} \subset C^\infty\big(S^1, M\big)$ in the loop space of the loops defined by the periodic solutions of X having periods $T \leqslant E_0$ and given $\gamma \in (0, \gamma_0)$, there exists a constant $h > h_0$ (the constant h_0 is guaranteed by Theorem 1.1) having the following property. For every $R > h$ and for every $\widetilde{J}$-holomorphic cylinder $\widetilde{u} = (a, u) : [-R, R] \times S^1 \to \mathbb{R} \times M$ satisfying*

$$\begin{aligned} A(\widetilde{u}) &> 0 \\ E(\widetilde{u}) &\leqslant E_0 \\ \int_{[-R,R]\times S^1} u^* d\lambda &\leqslant \gamma, \end{aligned} \tag{17}$$

the loops $t \mapsto u(s,t)$ satisfy

$$u\big(s, \cdot\big) \in \mathcal{W} \quad \text{for all } s \in [-R+h, R-h].$$

Moreover, with $T = A(\widetilde{u})$ being the center action, the loops $u(s)$ will be in the S^1-invariant neighborhood of a loop $t \mapsto x(Tt)$ associated with a T-periodic orbit $x(t)$ of the Reeb vector field.

Since $\mathcal{W}$ separates the loops of the periodic orbits having periods $T \leqslant E_0$, all these loops $u(s, \cdot)$ for $s \in [-R+h, R-h]$ are contained in the neighborhood component of $\mathcal{W}$ containing precisely one of the distinguished loops defined by a periodic orbit (x, T) having period $T \leqslant E_0$. From $\lambda(X) = 1$ we deduce immediately

$$T = \int_{S^1} x(T\cdot)^* \lambda. \tag{18}$$

Hence, given $\varepsilon > 0$ we can choose $\mathcal{W}$ so small that

$$\left| \int_{S^1} u(s, \cdot)^* \lambda - T \right| \leqslant \varepsilon. \tag{19}$$

for all $s \in [-R+h, R+h]$.

Proof. Arguing by contradiction we find a constant $\gamma \in (0, \gamma_0)$, a constant $\sigma < \gamma_0 - \gamma$, a sequence R_n with $R_n \geq n + h_0$, and a sequence of $\widetilde{J}$-holomorphic cylinders $\widetilde{u}_n = (a_n, u_n) : [-R_n, R_n] \times S^1 \to \mathbb{R} \times M$ having positive center actions and satisfying

$$\begin{aligned} E(\widetilde{u}_n) &\leqslant E_0 \\ \int_{[-R_n, R_n] \times S^1} u_n^* d\lambda &\leqslant \gamma \end{aligned} \tag{20}$$

and

$$u_n(s_n, \cdot) \notin \mathcal{W} \tag{21}$$

for some sequence $s_n \in [-R_n + n, R_n - n]$. By assumption, the center actions are positive and hence $A(\widetilde{u}_n) = T_n \geq \gamma_0$ so that we deduce from (9)

$$\int_{S^1} u_n(s)^* \lambda \geq \gamma_0 - \gamma - \sigma =: \varepsilon_0 > 0$$

for all n and all $s \in [-R_n, R_n]$.

In view of the $\mathbb{R}$-invariance of $\widetilde{J}$ we may assume that $s_n = 0$ for all n. In that case, of course, the finite energy map is defined on $[-R_n - s_n, R_n - s_n] \times S^1$. In any case the left hand boundary converges to $-\infty$ and the right hand boundary to $+\infty$. Define now the sequence of maps $\widetilde{v}_n = (b_n, v_n)$ by setting

$$\widetilde{v}_n(s,t) = \big(a_n(s,t) - a_n(0,0), u_n(s,t)\big).$$

The maps $\widetilde{v}_n$ are still $\widetilde{J}$-holomorphic maps since $\widetilde{J}$ is $\mathbb{R}$-invariant and satisfy (20) and (21). We claim that the gradients of $\widetilde{v}_n$ are bounded uniformly in n. Indeed, otherwise a bubbling off analysis produces as in [5] and [11] a non constant finite energy plane

$$\widetilde{u} := (a, u) : \mathbb{C} \to \mathbb{R} \times M.$$

According to the results in [5] and [6], the projection into M has a T-periodic solution $x(t)$ of X as asymptotic limit so that

$$u\big(Re^{2\pi i t}\big) \to x(Tt)$$

as $R \to \infty$, uniformly in t, and one shows using the estimates (20), that

$$T = \int_{S^1} x(T\cdot)^* \lambda = \lim_{R \to \infty} \int_{S^1} u\big(Re^{2\pi i \cdot}\big)^* \lambda = \int_{\mathbb{C}} u^* d\lambda \leqslant \gamma < \gamma_0.$$

However, according to the definition of γ_0, a periodic solution having period $T < \gamma_0$ does not exist. Therefore, the gradients of $\widetilde{v}_n$ must be uniformly bounded. From gradient bounds one concludes bounds for all derivatives using elliptic regularity theory, see [13]. Consequently, by Ascoli-Arzela's theorem, a subsequence of $\widetilde{v}_n$ converges in C^∞_{loc},

$$\widetilde{v}_n \to \widetilde{v} \quad \text{in } C^\infty_{\text{loc}}\big(\mathbb{R} \times S^1, \mathbb{R} \times M\big). \tag{22}$$

The limit $\widetilde{v} = (b, v) : \mathbb{R} \times S^1 \to \mathbb{R} \times M$ of the subsequence is a $\widetilde{J}$-holomorphic cylinder having the properties

$$
\begin{aligned}
&E(\widetilde{v}) \leqslant E_0 \\
&\int_{\mathbb{R}\times S^1} v^* d\lambda \leqslant \gamma \qquad (23)\\
&\int_{S^1} v(s,\cdot)^*\lambda \geq \varepsilon_0 \quad \text{for all } s \in \mathbb{R}.
\end{aligned}
$$

The last property shows that $\widetilde{v}$ is non constant. Since the energy $E(\widetilde{v})$ is finite, we know from the results in [6] that the projection v into M converges as $s \to \pm\infty$ to periodic orbits $x_\pm$ of X having periods $T_\pm \leqslant E_0$ so that $v(s,t) \to x_\pm(T_\pm t)$ as $s \to \pm\infty$, uniformly in t. So,

$$
T_\pm = \int_{S^1} x_\pm(T_\pm \cdot)^*\lambda = \lim_{s\to\pm\infty} \int_{S^1} v(s,\cdot)^*\lambda, \tag{24}
$$

and applying Stokes' theorem we conclude

$$
|T_+ - T_-| = \int_{\mathbb{R}\times S^1} v^* d\lambda \leqslant \gamma < \gamma_0.
$$

By the definition of γ_0, the constant γ is smaller than the difference between any two distinct periods in $(0, E_0]$, so that $T_- = T_+$ and hence

$$
\int_{\mathbb{R}\times S^1} v^* d\lambda = 0.
$$

Non constant finite energy cylinders having $d\lambda$-energy equal to 0 are classified in [8]. From Proposition 3.11 in [8] we deduce that $\widetilde{v}$ must be a cylinder over a T-periodic orbit z of the form

$$
\widetilde{v}(s,t) = (Ts + c, z(Tt + d))
$$

for some constants c and d. In view of (24), $T = T_- = T_+ \leqslant E_0$ so that the loop $z(T\cdot +d)$ belongs to $\mathcal{W}$. From (22) we obtain, setting $s = 0$,

$$
u_n(0,\cdot) = v_n(0,\cdot) \to v(0,\cdot) = z(T\cdot +d).
$$

Since the loop $z(T\cdot +d)$ lies in the interior of $\mathcal{W}$, this contradicts $u_n(0,\cdot) \notin \mathcal{W}$ required in (21). The proof of Lemma 3.1 is complete. □

In view of Lemma 3.1 we can fix a non-degenerate periodic solution $x(t)$ of period $T \leqslant E_0$ and study $\widetilde{J}$-holomorphic cylinders $\widetilde{u} = (a, u) : [-R, R] \times S^1 \to \mathbb{R} \times M$ whose projections into M satisfy

$$
u\big([-R, R] \times S^1\big) \subset \mathcal{U}
$$

for a small tubular neighborhood $\mathcal{U}$ of $x(\mathbb{R})$.

For our study of long cylinders with positive center action we need special coordinates as in [6]. In the lemma below we denote by λ_0 the standard contact form

$$
\lambda_0 = d\vartheta + \sum_{i=1}^n x_i dy_i
$$

on $S^1 \times \mathbb{R}^{2n}$ with coordinates $(\vartheta, x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n)$.

Lemma 3.2. *Let (M, λ) be a (2n+1)-dimensional manifold equipped with a contact form, and let $x(t)$ be a T-periodic solution of the corresponding Reeb vector field $\dot{x} = X_\lambda(x)$ on M. Let τ be the minimal period such that $T = k\tau$ for some positive integer k. Then there is an open neighborhood $U \subset S^1 \times \mathbb{R}^{2n}$ of $S^1 \times \{0\}$ and an open neighborhood $V \subset M$ of $P = \{x(t) \mid t \in \mathbb{R}\}$ and a diffeomorphism $\varphi \colon U \to V$ mapping $S^1 \times \{0\}$ onto P such that*

$$\varphi^* \lambda = f \cdot \lambda_0, \tag{25}$$

with a positive smooth function $f \colon U \to \mathbb{R}$ satisfying

$$f \equiv \tau \quad \text{and} \quad df \equiv 0 \tag{26}$$

on $S^1 \times \{0\}$.

The proof can be found in [6]. Since $S^1 = \mathbb{R}/\mathbb{Z}$ we work in the covering space and denote by $(\vartheta, x, y) = (\vartheta, x_1, x_2, \ldots x_n, y_1, y_2, \ldots, y_n) \in \mathbb{R}^{2n+1}$ the coordinates where ϑ is mod 1. The contact form λ in these coordinates is equal to

$$\lambda = f \cdot \lambda_0 \tag{27}$$

with a smooth function $f : \mathbb{R}^{2n+1} \to (0, \infty)$ defined near $S^1 \times \{0\}$ and periodic in ϑ, $f(\vartheta + 1, x, y) = f(\vartheta, x, y)$ and satisfying (26). The Reeb vector vector field

$$X = \big(X_0, X_1, \ldots, X_{2n}\big)$$

has the components

$$X_0 = \frac{1}{f^2}\big(f + \sum_i^n x_i \partial_{x_i} f\big), \quad X_i = \frac{1}{f^2}\big(\partial_{y_i} f - x_i \partial_\vartheta f\big), \quad X_{i+n} = -\frac{1}{f^2} \partial_{x_i} f$$

for $i = 1, \ldots, n$. The vector field X is periodic in ϑ of period 1 and constant along the periodic orbit $x(\mathbb{R})$,

$$X(\vartheta, 0) = \frac{1}{\tau}\big(1, 0\big). \tag{28}$$

The periodic solution is represented as

$$x(Tt) = \big(kt, 0\big), \tag{29}$$

where $T = k \cdot \tau$ is the period, τ the minimal period and k the covering number of the periodic solution.

Lemma 3.3. *Let $E_0 > 0$ and $\gamma_0 > 0$ be as in the Introduction. For every $N \in \mathbb{N}$, $\delta > 0$ and $0 < \gamma < \gamma_0$ there exists $h > 0$ having the following properties. For every $R > h$ and for every $\widetilde{J}$-holomorhic cylinder $\widetilde{u} = (a, u) : [-R, R] \times S^1 \to \mathbb{R} \times M$ satisfying the properties listed in* (17) *of Lemma 3.1, the following holds true for the representation*

$$\widetilde{u}(s, t) = \big(a(s, t), \vartheta(s, t), z(s, t)\big)$$

of the cylinder in the local coordinates above. For all $(s, t) \in [-R + h, R - h] \times S^1$,

$$\big|\partial^\alpha [a(s, t) - Ts]\big| \leqslant \delta$$
$$\big|\partial^\alpha [\vartheta(s, t) - kt]\big| \leqslant \delta$$

for the derivatives satisfying $1 \leqslant |\alpha| \leqslant N$, and

$$\big|\partial^\alpha z(s, t)\big| \leqslant \delta$$

for all $0 \leqslant |\alpha| \leqslant N$. *Here* T *is the period and* k *the covering number of the distinguished periodic solution sitting in the center of the tubular neighborhood.*

Proof. Arguing by contradiction we assume the existence of a sequence of numbers $R_n \geq 2n$ and and a sequence of $\widetilde{J}$-holomorphic maps $\widetilde{u}_n = (a_n, u_n) : [-R_n, R_n] \times S^1 \to \mathbb{R} \times M$ satisfying for all n,

$$E(\widetilde{u}_n) \leqslant E_0$$
$$\int_{[-R_n,R_n]\times S^1} u_n^* d\lambda \leqslant \gamma.$$

Moreover, representing the maps $\widetilde{u}_n$ in local coordinates by

$$\widetilde{u}_n(s,t) = (a_n(s,t), \vartheta_n(s,t), z_n(s,t)),$$

we assume the existence of a number $\delta_0 > 0$, a sequence $(s_n, t_n) \in [-R_n + n, R_n - n] \times S^1$ and a multi-index α in $1 \leqslant |\alpha| \leqslant N$, satisfying

$$\left|\partial^\alpha \big[(a_n - Ts, \vartheta_n - kt)\big](s_n, t_n)\right| \geq \delta_0. \tag{30}$$

We define the translated sequence $\widetilde{v}_n : [-n, n] \times S^1 \to \mathbb{R} \times M$ by

$$\widetilde{v}_n(s,t) = \big(b_n(s,t), v_n(s,t)\big) = \big(a_n(s + s_n, t) - a_n(s_n, t_n), u_n(s + s_n, t)\big).$$

Since $\widetilde{J}$ is $\mathbb{R}$-invariant, the maps $\widetilde{v}_n$ are $\widetilde{J}$-holomorphic and satisfy the energy estimates

$$E(\widetilde{v}_n) \leqslant E_0 \quad \text{and} \quad \int_{[-n,n]\times S^1} v_n^* d\lambda \leqslant \gamma.$$

Since $\gamma < \gamma_0$ one concludes as in Lemma 3.1 that the sequence $\widetilde{v}_n$ has uniformly bounded gradients on $[-n+1, n-1] \times S^1$ and hence possesses a C^∞_{loc} converging subsequence. Its limit $\widetilde{v} = (b, v) : \mathbb{R} \times S^1 \to \mathbb{R} \times M$ is a $\widetilde{J}$-holomorphic cylinder having the energy bounds

$$E(\widetilde{v}) \leqslant E_0 \quad \text{and} \quad \int_{\mathbb{R}\times S^1} v^* d\lambda \leqslant \gamma.$$

In addition, due to (30), the map $\widetilde{v}$ is non constant. Therefore, arguing as in Lemma 3.1, $\widetilde{v}$ is a cylinder over a periodic orbit $z(t)$ of period $T' \leqslant E_0$ and hence of the form $\widetilde{v}(s,t) = \big(T's + a_0, z(T't)\big)$. In view of the estimate (19), the period T' is close to the period T of the distinguished periodic orbit $x(t)$. Since this periodic orbit is non-degenerate, there exists a tubular neighborhood U of $x(\mathbb{R})$ which does not contain any other periodic orbit having period close to T. Hence choosing the tubular neighborhood sufficiently small we conclude that $T' = T$ and $z(Tt) = x(Tt)$, so that in the local coordinates of Lemma 3.2,

$$\widetilde{v}(s,t) = (Ts + a_0, kt + \vartheta_0, 0)$$

for two constants a_0 and ϑ_0. From $\widetilde{v}_n \to \widetilde{v}$ in C^∞_{loc} we conclude, setting $s = 0$, that

$$\partial^\alpha \big[(a_n - Ts, \vartheta_n - kt)\big](s_n, t_n) \to \big(0, 0\big)$$

for $|\alpha| \geq 1$ in contradiction to (30). Similarly, the last estimate in Lemma 3.3 is proved assuming that

$$\big|\partial^\alpha z_n(s_n, t_n)\big| \geq \delta_0 \tag{31}$$

for some α in $0 \leqslant |\alpha| \leqslant N$ and some $\delta_0 > 0$. Arguing as above we find, since the limit map $\widetilde{v}$ has its z-component equal to zero, that

$$\partial^\alpha z_n(s_n, t_n) \to 0$$

contradicting (31). The proof of Lemma 3.3 is complete. □

Corollary 3.4. *If $\widetilde{u} = (a, u) : [-R, R] \times S^1 \to \mathbb{R} \times M$ satisfies the assumptions of Lemma 3.3, then*

$$\int_{S^1} u(s, \cdot)^* \lambda = T + O(\delta)$$

for all $s \in [-R + h, R - h]$.

Proof. From the lemma we conclude

$$\frac{\partial}{\partial t} u(s, \cdot) = (k, 0) + O(\delta).$$

At the distance $O(\delta)$ from the periodic orbit, we have

$$\lambda(k, 0) = \tau\lambda_0(k, 0) + O(\delta) = \tau k \lambda_0(1, 0) + O(\delta) = T + O(\delta),$$

and the corollary follows. □

Next we briefly recall from [6] the Cauchy-Riemann equations for the representation

$$\begin{aligned}
\widetilde{u}(s, t) &= \big(a(s, t), u(s, t)\big) = \big(a(s, t), \vartheta(s, t), z(s, t)\big) \\
&= \big(a(s, t), \vartheta(s, t), x(s, t), y(s, t)\big) \\
&= \big(a(s, t), \vartheta(s, t), x^1(s, t), \ldots, x^n(s, t), y^1(s, t), \ldots, y^n(s, t)\big).
\end{aligned}$$

of a $\widetilde{J}$-holomorphic cylinder in the local coordinates $\mathbb{R} \times \mathbb{R}^{2n+1}$ of our tubular neighborhood given in Lemma 3.2. On $\mathbb{R}^{2n+1}$ we have the contact form $\lambda = f \cdot \lambda_0$. At the point $m = (t, x, y) \in \mathbb{R}^{2n+1}$, the contact structure $\xi_m = \ker\lambda_m$ is spanned by $2n$ vectors

$$E_i = e_{i+1}, \qquad E_{i+n} = -x_i e_1 + e_{1+i+n}, \quad i = 1, \ldots n$$

with $e_1, \ldots e_{2n+1}$ denoting the standard basis of $\mathbb{R}^{2n+1}$. We denote by $J(m)$ the $(2n) \times (2n)$ matrix representing the compatible almost complex structure on the plane ξ_m in the basis $\{E_1, \ldots E_{2n}\}$. The symplectic structure $d\lambda|_{\xi_m}$ is, in the basis $\{E_1, \ldots E_{2n}\}$, given by the skew symmetric matrix function $f(m)J_0$, where

$$J_0 = \begin{bmatrix} 0 & -\operatorname{Id} \\ \operatorname{Id} & 0 \end{bmatrix}.$$

Therefore, in view of the compatibility requirement, the complex multiplication $J(m)$ has the properties

$$J(m)^2 = -\operatorname{Id}, \quad J(m)^T J_0 J(m) = J_0, \quad -J_0 J(m) > 0. \tag{32}$$

In particular, $J_0 J(m)$ is a symmetric matrix. It follows that

$$\langle\!\langle x, y \rangle\!\rangle := \langle x, -J_0 J(m) y \rangle$$

is an inner product on $\mathbb{R}^{2n}$ which is left invariant under $J(m)$,

$$\langle\!\langle J(m)x, J(m)y \rangle\!\rangle = \langle\!\langle x, y \rangle\!\rangle.$$

Denoting by

$$X = \big(X_0, X_1, \dots, X_{2n}\big) \in \mathbb{R} \times \mathbb{R}^{2n}$$

the Reeb vector field associated with λ, we define, abbreviating $z = (x, y) \in \mathbb{R}^{2n}$,

$$Y(t, z) = \big(X_1(t, z), \dots, X_{2n}(t, z)\big) \in \mathbb{R}^{2n}.$$

Since $X(t, 0) = (1/\tau, 0)$ we have

$$Y(t, z) = D(t, z)z,$$

with the matrix function

$$D(t, z) = \int_0^1 dY(t, \rho z)d\rho, \tag{33}$$

where d is the derivative with respect to the z-variable. In particular, if $z = 0$,

(34)

$$D(t,0) = dY(t,0) = \frac{1}{\tau^2}\begin{pmatrix} \partial_{x_1y_1}f & \dots & \partial_{x_ny_1}f & \partial_{y_1y_1}f & \dots & \partial_{y_ny_1}f \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \partial_{x_1y_n}f & \dots & \partial_{x_ny_1}f & \partial_{y_1y_n}f & \dots & \partial_{y_ny_n}f \\ -\partial_{x_1x_1}f & \dots & -\partial_{x_nx_1}f & -\partial_{x_1y_1}f & \dots & -\partial_{x_1y_n}f \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -\partial_{x_1x_n}f & \dots & -\partial_{x_nx_n}f & -\partial_{x_ny_1}f & \dots & -\partial_{x_ny_n}f \end{pmatrix},$$

the last matrix being evaluated at $(t, 0)$. We introduce the $(2n) \times (2n)$ matrices depending on $\widetilde{u}(s, t)$,

$$\begin{aligned} J(s,t) &= J\big(u(s,t)\big) = J\big(\vartheta(s,t), z(s,t)\big) \\ S(s,t) &= [a_t - a_s J(s,t)]D\big(u(s,t)\big). \end{aligned} \tag{35}$$

Writing $\pi u_s + J(u)\pi u_t = 0$ in the above basis $\big(E_j\big)$ of the contact plane ξ_m at $m = u(s, t)$, one sees as in [6], that the Cauchy-Riemann equations (3) for the representation $\widetilde{u}(s, t) = \big(a(s, t), \vartheta(s, t), z(s, t)\big) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}^{2n}$ in the local coordinates of Lemma 3.2, become the following partial differential equations.

$$z_s + J(s, t)z_t + S(s, t)z = 0 \tag{36}$$

and, with $z(s, t) = \big(x(s, t), y(s, t)\big)$,

$$\begin{aligned} a_s &= \ \ \Big(\vartheta_t + \sum_{i=1}^n x_i \partial_t y_i\Big) f(u) \\ a_t &= -\Big(\vartheta_s + \sum_{i=1}^n x_i \partial_s y_i\Big) f(u). \end{aligned} \tag{37}$$

It will be convenient to decompose the matrix $S(s, t)$ into its symmetric and anti-symmetric parts with respect to the inner product $\big\langle \cdot\ , -J_0 J(s, t)\cdot\big\rangle$ on $\mathbb{R}^{2n}$ introducing

$$\begin{aligned} B(s,t) &= \frac{1}{2}\big[S(s,t) + S^*(s,t)\big] \\ C(s,t) &= \frac{1}{2}\big[S(s,t) - S^*(s,t)\big]. \end{aligned} \tag{38}$$

Here S^* denotes the transpose of S with respect to the inner product $\langle \cdot\,, -J_0J(s,t)\cdot\rangle$. Explicitly,

$$S^* = JJ_0S^TJ_0J,$$

where S^T is the transpose matrix of S with respect to the Euclidean inner product $\langle \cdot\,, \cdot\rangle$ in $\mathbb{R}^{2n}$. With this notation, the equation (36) for $z(s,t)$ becomes

$$z_s + J(s,t)z_t + B(s,t)z + C(s,t)z = 0. \tag{39}$$

The operator $A(s) : W^{1,2}(S^1, \mathbb{R}^{2n}) \subset L^2(S^1, \mathbb{R}^{2n}) \to L^2(S^1, \mathbb{R}^{2n})$ is defined by

$$A(s) = -J(s,t)\frac{d}{dt} - B(s,t).$$

The operator $A(s)$ is self-adjoint with respect to the inner product $\langle \cdot\,, \cdot\rangle_s$ in L^2 defined for $x, y \in L^2(S^1, \mathbb{R}^{2n})$ by

$$\langle x, y\rangle_s := \int_0^1 \langle x(t), -J_0J\big(u(s,t)\big)y(t)\rangle dt.$$

The norms

$$\|x\|_s^2 := \langle x, x\rangle_s$$

are equivalent to the standard $L^2(S^1, \mathbb{R}^{2n})$-norm denoted by $\|\cdot\|$. Indeed, there exists a constant c not depending on s such that

$$\frac{1}{c}\|x\|_s \leqslant \|x\| \leqslant c\|x\|_s \tag{40}$$

for all $x \in L^2(S^1, \mathbb{R}^{2n})$.

Lemma 3.5. *For every $0 < \gamma < \gamma_0$ there exists a constant $h > 0$ such that for every $R > h$ and every $\widetilde{J}$-holomorphic cylinder $\widetilde{u} : [-R, R] \times S^1 \to \mathbb{R} \times M$ satisfying the conditions* (17) *in Lemma 3.1, the following holds true.*

If $\widetilde{u} = \big(a(s,t), u(s,t)\big)$ is the representation in the local coordinates and $A(s)$ the associated operators, then there exists a constant $\eta > 0$ such that

$$\|A(s)\xi\|_s \geq \eta\|\xi\|_s$$

for all $s \in [-R+h, R-h]$ and all $\xi \in W^{1,2}(S^1, \mathbb{R}^{2n})$.

Proof. Arguing by contradiction we assume the existence of a sequence of numbers $R_n \geq 2n$ and a sequence $\widetilde{u}_n = (a_n, u_n) : [-R_n, R_n] \times S^1 \to \mathbb{R} \times M$ of $\widetilde{J}$-holomorphic cylinders satisfying

$$E(\widetilde{u}_n) \leqslant E_0$$

$$\int_{[-R_n,R_n]\times S^1} u_n^* d\lambda \leqslant \gamma$$

$$\int_{S^1} u_n^* d\lambda \geq \varepsilon_0 \quad \text{for all } s \in [-R_n, R_n] \text{ and some } \varepsilon_0 > 0.$$

Representing $\widetilde{u}_n$ in local coordinates as $\widetilde{u}_n(s,t) = \big(a_n(s,t), \vartheta_n(s,t), z_n(s,t)\big)$ we have the associated operators

$$A_n(s) = -J_n(s,t)\frac{d}{dt} - B_n(s,t), \tag{41}$$

where $J_n(s,t) = J\big(u_n(s,t)\big)$ and where $S_n(s,t)$ and hence $B_n(s,t)$ is defined by replacing in the definitions the map $\widetilde{u}$ by the map $\widetilde{u}_n$. We assume that there exists a sequence $s_n \in [-R_n - n, R_n + n]$, $\xi_n \in W^{1,2}(S^1, \mathbb{R}^{2n})$ such that

$$\big\|\xi_n\big\|_{s_n} = 1 \quad \text{and} \quad \big\|A_n(s_n)\xi_n\big\|_{s_n} \to 0. \tag{42}$$

Proceeding as above we introduce the translated maps $\widetilde{v}_n = (b_n, v_n)$ by defining

$$\widetilde{v}_n(s,t) = (b_n(s,t), v_n(s,t)) = \big(a_n(s+s_n,t) - a_n(s_n,0), u_n(s+s_n,t)\big) \tag{43}$$

for all n and $(s,t) \in [-n,n] \times S^1$. Again $\widetilde{v}_n$ is a $\widetilde{J}$-holomorphic cylinder having the energy bounds

$$E(\widetilde{v}_n) \leqslant E_0 \quad \text{and} \quad \int_{[-n,n]\times S^1} v_n^* d\lambda \leqslant \gamma.$$

It follows as in the previous lemma,

$$\widetilde{v}_n \to \widetilde{v} \quad \text{in } C^\infty_{\text{loc}}(\mathbb{R} \times S^1, \mathbb{R} \times M), \tag{44}$$

where $\widetilde{v}$ is a cylinder over a distinguished periodic orbit $x(t)$ sitting in the center of the tubular neighborhood, hence having in the local coordinates the representation

$$\widetilde{v}(s,t) = \big(Ts + a_0, kt + \vartheta_0, 0\big)$$

with two constants a_0 and ϑ_0. Setting $s = 0$ in (44) we deduce

$$\begin{gathered}
\frac{\partial}{\partial t} a_n(s_n,t) \to 0 \\
\frac{\partial}{\partial s} a_n(s_n,t) \to T \\
\vartheta_n(s_n,t) \to kt + \vartheta_0 \\
z_n(s_n,t) \to 0
\end{gathered}$$

as $s \to \infty$, uniformly in t. As a consequence, in view of (32)-(35) and (38), one finds

$$\begin{gathered}
B_n(s_n,t) \to TJ(kt+\vartheta_0,0)\cdot dY(kt+\vartheta_0,0) \\
J_n(s_n,t) \to J(kt+\vartheta_0,0)
\end{gathered} \tag{45}$$

as $n \to \infty$, uniformly in t. Since $\big\|J_n(s,\cdot)\xi\big\|_s = \big\|\xi\big\|_s$ for every $\xi \in L^2(S^1, \mathbb{R}^{2n})$ and since the s-norms are equivalent to the L^2-norms, there exists a constant $C > 0$ such that for all n and $\xi \in W^{1,2}(S^1, \mathbb{R}^{2n})$

$$\frac{1}{C}\big\|\dot{\xi}\big\| \leqslant \big\|A_n(s_n)\xi\big\| + \big\|B_n(s_n,\cdot)\xi\big\|. \tag{46}$$

Consequently, the sequence ξ_n in (43) is bounded in $W^{1,2}$. Since $W^{1,2}$ is compactly embedded in L^2, a subsequence of ξ_n converges in L^2. Therefore, by the assumptions (42), the limits (45) and the estimates (46), the subsequence (still denoted by ξ_n) is a Cauchy sequence in $W^{1,2}(S^1, \mathbb{R}^{2n})$ so that

$$\xi_n \to \xi \quad \text{in } W^{1,2}(S^1, \mathbb{R}^{2n}).$$

From

$$A_n(s_n)\xi_n = -J_n(s_n,\cdot)\dot{\xi}_n - B_n(s_n,\cdot)\xi_n \to 0 \quad \text{in } L^2(S^1,\mathbb{R}^{2n})$$

and (45) one concludes that ξ solves the equation

$$\frac{d}{dt}\xi(t) = TdY\big(x(Tt)\big)\xi(t),$$

where we have used that in our coordinates the periodic solution is represented as $x(Tt) = (kt+\vartheta_0, 0)$. Since the linearized flow of the Reeb vector field X leaves the splitting $TM = \mathbb{R}X \oplus \xi$ invariant, and since $\xi \neq 0$ we have found a second Floquet multiplier equal to 1 so that the T-periodic orbit $x(t)$ is degenerate. This contradicts our assumption that all the periodic solutions having periods $T \leqslant E_0$ are non degenerate. The proof of the lemma is complete. □

In the next lemma we estimate the L^2-norms for smooth loops $z(s)$ defined by

$$z(s)(t) := z(s,t)$$

and use the notation

$$\big\|z(s)\big\|^2 = \int_0^1 |z(s,t)|^2 dt$$

for the L^2-norm.

Lemma 3.6. *There exists $\delta_0 > 0$ such that for every $0 < \delta \leqslant \delta_0$ there exists constants $h>0$ and $\mu > 0$ with the following property. For every $R > h$ and every $\widetilde{J}$-holomorphic cylinder $\widetilde{u} = (a,u) : [-R,R]\times S^1 \to \mathbb{R}\times M$ satisfying the properties (17) we have for the representation $\widetilde{u} = \big(a(s,t),\vartheta(s,t),z(s,t)\big)$ in local cordinates the estimates*

$$\begin{aligned}\frac{1}{c^2}\big\|z(s)\big\|^2 &\leqslant \frac{1}{\sinh[2\mu r]}\Big(\big\|z(-r)\big\|^2\sinh[\mu(r-s)] + \big\|z(r)\big\|^2\sinh[\mu(r+s)]\Big)\\ &\leqslant \max\Big\{\big\|z(-r)\big\|^2, \big\|z(r)\big\|^2\Big\}\cdot\frac{\cosh[\mu s]}{\cosh[\mu r]}\\ &\leqslant \delta^2\frac{\cosh[\mu s]}{\cosh[\mu r]}\end{aligned}$$

for all $s\in[-r,r]$ and $r = R-h$. The constant c occurs in the equivalence relation (40) for the norms. The last estimate follows from lemma 3.3.

Proof. Let δ be as in Lemma 3.3 and Lemma 3.5 and choose $R>h$. Repeatedly we shall make use of Lemma 3.3 to make δ smaller and so h possibly larger. We consider in the local coordinates the $\widetilde{J}$-holomorphic cylinder $\widetilde{u} = \big(a(s,t),\vartheta(s,t),z(s,t)\big)$ on $[-r,r]\times S^1$ with $r = R-h$. We define the function $g(s)$ for $s\in[-r,r]$ by

$$g(s) = \frac{1}{2}\big\|z(s)\big\|_s^2 = \int_0^1 \big\langle z(s,t), -J_0J(s,t)z(s,t)\big\rangle dt. \tag{47}$$

To prove the result it suffices to show that

$$g''(s) \geq \mu^2 g(s) \tag{48}$$

for some constant $\mu>0$ and all $s\in[-r,r]$. Indeed, assume that g satisfies the estimate (48) and introduce the function

$$f(s) = Ae^{-\mu s} + Be^{\mu s},$$

where the constants A and B are chosen so that

$$f(-r)=g(-r) \quad \text{and} \quad f(r)=g(r).$$

Explicitly,

$$f(s)=\frac{1}{\sinh[2\mu r]}\Big(g(-r)\sinh[\mu(r-s)]+g(r)\sinh[\mu(r+s)]\Big).$$

We claim

$$g(s)\leqslant f(s), \quad s\in[-r,r]. \tag{49}$$

To prove the claim we observe that $f''(s)=\mu^2 f(s)$. Hence, the function $h(s)=g(s)-f(s)$ satisfies $h''(s)\geq \mu^2 h(s)$ and $h(-r)=h(r)=0$. We have to prove that $h\leqslant 0$ on $[-r,r]$. In order to do so we assume that $I\subset[-r,r]$ is an interval on which $h\geq 0$ and $h=0$ at the boundary points. On I the function h is convex and hence attains the maximum value at the boundary points where the values are equal to 0. This implies $h=0$ on I. Consequently, $h(s)\leqslant 0$ on $[-r,r]$ proving the desired estimate (49). It remains to prove (48).

In order to derive the inequality (48) we shall differentiate the function g. Using equation (39) for $z(s,t)$ and recalling that the matrix $J_0J(s,t)$ is symmetric with respect to the Euclidean inner product $\langle\cdot\,,\cdot\rangle$ while $C(s,t)$ is anti-symmetric with respect to $\langle\cdot\,,-J_0J(s,t)\cdot\rangle$ we obtain

$$\begin{aligned}
g'(s)&=\frac{1}{2}\int_0^1\langle z_s,-J_0Jz\rangle dt+\frac{1}{2}\int_0^1\langle z,-J_0Jz_s\rangle dt+\frac{1}{2}\int_0^1\langle z,-J_0J_sz\rangle dt\\
&=\langle z_s,z\rangle_s+\frac{1}{2}\int_0^1\langle z,-J_0J_sz\rangle dt\\
&=\langle A(s)z,z\rangle_s+\frac{1}{2}\int_0^1\langle z,-J_0J_sz\rangle dt.
\end{aligned}$$

We differentiate the function g once more and use the fact that the operator $A(s)$ is self-adjoint with respect to the inner product $\langle\cdot,\cdot\rangle_s$ on $L^2(S^1,\mathbb{R}^{2n})$,

$$\begin{aligned}
g''(s)&=2\|A(s)z\|_s^2-\langle A(s)z,Cz\rangle_s-2\langle A(s)z,JJ_sz\rangle_s+\langle Cz,JJ_sz\rangle_s\\
&\quad-\frac{1}{2}\langle z,JJ_{ss}z\rangle_s+\int_0^1\langle -J_sz_t-B_sz,-J_0Jz\rangle dt.
\end{aligned}\tag{50}$$

We reformulate the last term containing z_t. Recall (32). By taking the derivative of $J^TJ_0J=J_0$ in s,

$$J_s^TJ_0J+J^TJ_0J_s=0.$$

Using this identity and recalling the operator $A(s)z=-J(s,t)z_t-B(s,t)z$ we obtain, with the inner product $\langle\cdot\,,\cdot\rangle$ in $\mathbb{R}^{2n}$, the pointwise identity

$$\langle -J_sz_t,-J_0Jz\rangle=\langle A(s)z,J_0J_sz\rangle+\langle Bz,J_0J_sz\rangle.$$

Integrating and recalling $J^2=-\operatorname{Id}$,

$$\int_0^1\langle -J_sz_t-B_sz,-J_0Jz\rangle dt=\langle A(s)z,JJ_sz\rangle_s+\langle B_sz,z\rangle_s-\langle Bz,JJ_sz\rangle_s. \tag{51}$$

Recall that for the derivatives of the solution u, the estimates $\left|\partial_s^\alpha u(s,t)\right|\leqslant\delta$ for $\alpha\geq 1$ hold true by Lemma 3.3. Consequently, the derivatives in s of the matrices $B(s,t)$ and $J(s,t)$ contain factors estimated by δ, which we can choose as small as desired. Denoting by $O(\delta)$ a positive function satisfying $O(\delta) \to 0$ as $\delta \to 0$, we therefore have

$$\big|J_s(s,t)\big|,\ \big|J_{ss}(s,t)\big|,\ \big|B_s(s,t)\big| = O(\delta). \tag{52}$$

We must show that also

$$\big|C(s,t)\big| = O(\delta). \tag{53}$$

Recall that

$$\begin{aligned} C(s,t) &= \frac{1}{2}\big[S(s,t) - S(s,t)^*\big] \\ &= \frac{a_t}{2}\big[D - JJ_0D^TJ_0J\big] - \frac{a_s}{2}\big[JD - JJ_0D^TJ_0\big]. \end{aligned}$$

The first term contains a_t which, by Lemma 3.3, is estimated by $\big|a_t(s,t)\big| \leqslant \delta$. To estimate the second term we observe, using $J^2 = -\operatorname{Id}$ and $J_0^T = -J_0$, that

$$JD - JJ_0D^TJ_0 = JJ_0\big[(J_0D)^T - (J_0D)\big].$$

Define the anti-symmetric matrix $R := (J_0D)^T - (J_0D)$. Since $J_0D(t,0)$ is, in view of (32) a symmetric matrix we have for R the representation

$$R(\vartheta, z) = \left[\int_0^1 (\partial_z R)(\vartheta, \tau z)d\tau\right] \cdot z.$$

Here $\partial_z R$ stands for the derivative of R with respect to the second variable z. Since $|z(s,t)| \leqslant \delta$, by Lemma 3.3, we have verified the estimate (53) for $C(s,t)$. Finally, recalling form Lemma 3.6 that $\big\|A(s)\xi\big\|_s \geq \eta\big\|\xi\big\|_s$ we can estimate $g''(s)$ in (50) using (51), (52) and (53) for δ sufficiently small as follows

$$\begin{aligned} g''(s) &\geq 2\big\|A(s)z\big\|_s^2 - O(\delta)\big\|A(s)z\big\|_s \cdot \big\|z\big\|_s - O(\delta)\big\|z\big\|_s^2 \\ &\geq 2\big\|A(s)z\big\|_s \cdot \big(\big\|A(s)z\big\|_s - O(\delta)\big\|z\big\|_s\big) - O(\delta)\big\|z\big\|_s^2 \\ &\geq 4\eta \cdot \big[\eta - O(\delta)\big]\frac{\big\|z\big\|_s^2}{2} - O(\delta)\big\|z\big\|_s^2 \\ &\geq \mu^2 g(s), \end{aligned}$$

with $\mu^2 = \eta^2$, provided δ is sufficiently small. We have verified the desired inequality (48) and the proof of Lemma 3.6 is complete. ☐

For the L^2-norms of the higher derivatives we find the same estimates.

Lemma 3.7. *Given $N \in \mathbb{N}$ and $0 < \gamma \leqslant \gamma_0$. Then there exist constants $\delta_0 > 0$ and $C_\alpha > 0$ such that for every $0 < \delta \leqslant \delta_0$ there exist constants $h > 0$ and $\mu > 0$ having the following property. For every $R > h$ and every $\widetilde{J}$-holomorphic cylinder $\widetilde{u} = (a,u) : [-R,R] \times S^1 \to \mathbb{R} \times M$ satisfying the requirements* (17) *of Lemma 3.1, the representation $\widetilde{u}(s,t) = \big(a(s,t), \vartheta(s,t), z(s,t)\big)$ in local coordinates admits the following estimates for the partial derivatives in s,*

$$\big\|\partial_s^\alpha z(s,\cdot)\big\|^2 \leqslant C_\alpha \cdot \delta^2 \cdot \frac{\cosh[\mu s]}{\cosh[\mu r]}$$

for all $s \in [-r,r]$ with $r = R - h$ and $0 \leqslant \alpha \leqslant N$.

Proof. Fix $N \in \mathbb{N}$ and denote by W the column vector

$$W(s,t) = \begin{pmatrix} z(s,t) \\ \partial_s z(s,t) \\ \vdots \\ \partial_s^N z(s,t) \end{pmatrix}.$$

For a given k, the derivative $\partial_s^k z(s,t)$ satisfies the equation

$$\begin{aligned}
\partial_s \partial_s^k z &= -\sum_{l=0}^{k} \binom{k}{l} \big[\big(\partial_s^k J\big)\big(\partial_s^{k-l}\partial_t z\big) + \big(\partial_s^k B\big)\big(\partial_s^{k-l} z\big) + \big(\partial_s^k C\big)\big(\partial_s^{k-l} z\big) \big] \\
&= \big(J\partial_t \partial_s^k z + B\partial_s^k z\big) - \sum_{l=1}^{k} \binom{k}{l} \big(\partial_s^k J\big)\big(\partial_s^{k-l}\partial_t z\big) - \sum_{l=1}^{k} \binom{k}{l} \big(\partial_s^k B\big)\big(\partial_s^{k-l} z\big) \\
&\quad - \sum_{l=0}^{k} \binom{k}{l} \big(\partial_s^k C\big)\big(\partial_s^{k-l} z\big).
\end{aligned}$$

Hence the vector W solves the equation

$$\partial_s W + \widehat{J}\partial_t W + \widehat{B} W + \widehat{C} W + \Delta \partial_t W = 0, \tag{54}$$

where we abbreviated the $(N+1)\times(N+1)$ matrix functions

$$\widehat{J} = \operatorname{diag}\big[J, \cdots, J\big], \quad \widehat{B} = \operatorname{diag}\big[B, \cdots, B\big],$$

$$\widehat{C} = \begin{bmatrix} C & 0 & 0 & \cdots & 0 \\ C_{1,1} & C & 0 & \cdots & 0 \\ C_{2,2} & C_{2,1} & C & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \\ C_{N,N} & C_{N,N-1} & \cdots & C_{N,1} & C \end{bmatrix}, \quad \Delta = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ \Delta_{1,1} & 0 & 0 & \cdots & 0 \\ \Delta_{2,2} & \Delta_{2,1} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \\ \Delta_{N,N} & \Delta_{N,N-1} & \cdots & \Delta_{N,1} & 0 \end{bmatrix},$$

where

$$\Delta_{k,l} = \binom{k}{l} \partial_s^l J, \quad C_{k,l} = \binom{k}{l} \big[\partial_s^l B + \partial_s^l C\big]$$

for $1 \leqslant l \leqslant k \leqslant N$, with the matrices $B(s,t)$ and $C(s,t)$ defined in (38), and with $J(s,t)$ defined in (35). We introduce the operator $\widehat{A}(s)$ in $L^2(S^1, \mathbb{R}^{2n(N+1)})$ by

$$\widehat{A}(s)\xi = -\widehat{J}\frac{d}{dt}\xi - \widehat{B}\xi$$

for $\xi \in W^{1,2}(S^1, \mathbb{R}^{2n(N+1)})$. Denoting by $\widehat{J}_0$ the block-diagonal matrix

$$\widehat{J}_0 = \operatorname{diag}\big[J_0, J_0, \cdots, J_0\big],$$

we introduce the family of s-dependent inner products and corresponding norms

$$\begin{aligned}
\big\langle x, y \big\rangle_s &= \int_0^1 \big\langle x(t), -\widehat{J}_0 \widehat{J}(s,t) y(t) \big\rangle \, dt \\
\big\| x \big\|_s^2 &= \big\langle x, x \big\rangle_s
\end{aligned}$$

for $x, y \in L^2(S^1, \mathbb{R}^{2n(N+1)})$. The operator $\widehat{A}$ is self-adjoint with respect to the s-dependent inner product $\langle \cdot\,,\cdot \rangle_s$ in L^2. By Lemma 3.5,

$$\|\widehat{A}(s)\xi\|_s \geq \eta\|\xi\|_s \tag{55}$$

for $\xi \in W^{1,2}$ with a positive constant η.

Introducing the loops $W(s) = W(s, \cdot)$ we define the function

$$g(s) = \frac{1}{2}\|W(s)\|_s^2 = \frac{1}{2}\int_0^1 \langle W(s,t), -\widehat{J}_0\widehat{J}(s,t)W(s,t)\rangle dt.$$

Proceeding as in Lemma 3.6 it suffices to prove

$$g''(s) \geq \mu^2 g(s) \tag{56}$$

for all $s \in [-r, r]$ where $r = R - h$ with some positive constant μ.

Taking the derivative of g one obtains

$$g'(s) = \langle W_s, W\rangle_s + \frac{1}{2}\int_0^1 \langle W, -\widehat{J}_0\widehat{J}_sW\rangle dt.$$

For the second derivative one finds, using $\widehat{J}^2 = -\operatorname{Id}$,

$$g''(s) = \langle W_{ss}, W\rangle_s + \|W_s\|_s^2 - 2\langle W_s, \widehat{J}\widehat{J}_sW\rangle_s - \frac{1}{2}\langle W, \widehat{J}\widehat{J}_{ss}W\rangle_s. \tag{57}$$

Our solution $u(s,t)$ satisfies, in view of Lemma 3.3, the estimates $|\partial^\alpha u(s,t)| \leqslant \delta$ for $\alpha \geq 1$. Hence the matrices $\widehat{J}_s$ and $\widehat{J}_{ss}$ contain factors which are estimated by δ, in the following chosen sufficiently small. Denoting by $O(\delta)$ a positive function satisfying $O(\delta) \to 0$ as $\delta \to 0$, we have the estimates $|\langle W_s, \widehat{J}\widehat{J}_sW\rangle_s|$, $|\langle W, \widehat{J}\widehat{J}_sW\rangle_s| \leqslant O(\delta)\big(\|W\|_s^2 + \|W_s\|_s^2\big)$. Consequently,

$$g''(s) \geq \langle W_{ss}, W\rangle_s - O(\delta)\big(\|W\|_s^2 + \|W_s\|_s^2\big). \tag{58}$$

In order to estimate the first term we differentiate equation (54) in s and obtain

$$\begin{aligned} \langle W_{ss}, W\rangle_s &= \langle \widehat{A}(s)W_s, W\rangle_s - \langle \Delta W_{st}, W\rangle_s - \langle \widehat{J}_sW_t, W\rangle_s \\ &\quad - \langle \Delta_sW_t, W\rangle_s - \langle \widehat{B}_sW, W\rangle_s - \langle \widehat{C}W_s, W\rangle_s - \langle \widehat{C}_sW, W\rangle_s. \end{aligned} \tag{59}$$

In view of the proof of the previous lemma, $|\widehat{C}(s,t)| = O(\delta)$, so that the last three terms in (59) are estimated by $O(\delta)\big(\|W\|_s^2 + \|W_s\|_s^2\big)$. In order to estimate the two terms $\langle \widehat{J}_sW_t, W\rangle_s$ and $\langle \Delta_sW_t, W\rangle_s$ containing the first derivative of W in t we multiply equation (54) by $\widehat{J}$, and using $\widehat{J}^2 = -\operatorname{Id}$, obtain the equality

$$\big(\operatorname{Id} - \widehat{J}\Delta\big)W_t = \widehat{J}W_s + \widehat{J}\widehat{B}W + \widehat{J}\widehat{C}W. \tag{60}$$

Since the matrix entries of Δ are derivatives of $J(s,t)$, we have $|\widehat{J}\Delta| = O(\delta)$ so that the matrix $\big(\operatorname{Id} - \widehat{J}\Delta\big)$ is invertible if δ is sufficiently small. With $\big|\big(\operatorname{Id} - \widehat{J}\Delta\big)^{-1}\big| \leqslant (1 - |\widehat{J}\Delta|)^{-1}$ one concludes

$$\|W_t\|_s \leqslant C \cdot \big(\|W\|_s^2 + \|W_s\|_s^2\big). \tag{61}$$

Consequently,

$$\begin{aligned}\left|\left\langle \widehat{J}_s W_t, W\right\rangle_s\right| &\leqslant O(\delta)\left(\left\|W\right\|_s^2 + \left\|W_s\right\|_s^2\right)\\ \left|\left\langle \Delta_s W_t, W\right\rangle_s\right| &\leqslant O(\delta)\left(\left\|W\right\|_s^2 + \left\|W_s\right\|_s^2\right).\end{aligned}$$

In order to estimate the term $\left\langle \Delta W_{st}, W\right\rangle_s$ in (59) we differentiate the function $\left\langle \Delta(s,t)W_s(s,t), -\widehat{J}_0\widehat{J}(s,t)W(s,t)\right\rangle$ in t to obtain

$$\begin{aligned}\frac{d}{dt}\left\langle \Delta W_s, -\widehat{J}_0\widehat{J}W\right\rangle =&\left\langle \Delta_t W_s, -\widehat{J}_0\widehat{J}W\right\rangle + \left\langle \Delta W_{st}, -\widehat{J}_0\widehat{J}W\right\rangle\\ &+\left\langle \Delta W_s, -\widehat{J}_0\widehat{J}_t W\right\rangle + \left\langle \Delta W_s, -\widehat{J}_0\widehat{J}W_t\right\rangle.\end{aligned}$$

Integrating in t over $[0,1]$ the left hand side vanishes in view of the periodicity in t and, using $\widehat{J}^2 = -\operatorname{Id}$, one has

$$\left\langle \Delta W_{st}, W\right\rangle_s = -\left\langle \Delta_t W_s, W\right\rangle_s + \left\langle \Delta W_s, \widehat{J}\widehat{J}_t W\right\rangle_s - \left\langle \Delta W_s, W_t\right\rangle_s \tag{62}$$

so that, by (61),

$$\left|\left\langle \Delta W_{st}, W\right\rangle_s\right| \leqslant O(\delta)\left(\left\|W\right\|_s^2 + \left\|W_s\right\|_s^2\right).$$

Summarizing the estimates for the term $\left\langle W_{ss}, W\right\rangle_s$ in (59) so far we have, in view of the selfadjointness of the operator $\widehat{A}(s)$,

$$\left\langle W_{ss}, W\right\rangle_s \geq \left\langle W_s, \widehat{A}(s)W\right\rangle_s - O(\delta)\left(\left\|W\right\|_s^2 + \left\|W_s\right\|_s^2\right). \tag{63}$$

In order to estimate $\left\langle W_s, \widehat{A}(s)W\right\rangle_s$ we first solve the equation (54) for $\widehat{A}(s)W$ and find, using $\left|\Delta\right|$, $\left|\widehat{C}\right| = O(\delta)$ and using the estimates (61) for W_t, if δ is sufficiently small,

$$\begin{aligned}\left\langle W_s, \widehat{A}(s)W\right\rangle_s &= \left\|W_s\right\|_s^2 - \left\langle W_s, \widehat{C}W\right\rangle_s - \left\langle W_s, \Delta W_t\right\rangle_s\\ &\geq \left(1 - O(\delta)\right)\left\|W_s\right\|_s^2 - O(\delta)\left\|W\right\|_s^2\\ &\geq \frac{1}{2}\left\|W_s\right\|_s^2 - O(\delta)\left\|W\right\|_s^2.\end{aligned} \tag{64}$$

It remains to estimate $\left\|W_s\right\|_s$ from below. We first claim that for δ sufficiently small,

$$\left\|\widehat{A}(s)\xi - \Delta\dot{\xi}\right\|_s \geq \left(\eta/2\right)\left\|\xi\right\|_s \tag{65}$$

for all $s \in [-r,r]$ and all $\xi \in W^{1,2}(S^1, \mathbb{R}^{2n(N+1)})$, with the constant η occuring in (55). To prove the claim we note that, using $\widehat{J}^2 = -\operatorname{Id}$,

$$\widehat{A}(s)\xi - \Delta\dot{\xi} = \left(\operatorname{Id} - \Delta\widehat{J}\right)\widehat{A}(s)\xi - \Delta\widehat{J}\widehat{B}\xi.$$

Using again $\left|\Delta\widehat{J}\right| = O(\delta)$, the matrix $\left(\operatorname{Id} - \Delta\widehat{J}\right)$ is invertible and we find, recalling the estimate (55) for the operator $\widehat{A}(s)$, if δ is sufficiently small,

$$\begin{aligned}\left\|\widehat{A}(s)\xi - \Delta\xi_t\right\|_s &\geq \left\|\left(\operatorname{Id} - \Delta\widehat{J}\right)^{-1}\right\|^{-1} \cdot \left\|\widehat{A}(s)\xi\right\|_s - O(\delta)\left\|\xi\right\|_s\\ &\geq \left(1 + O(\delta)\right)\eta\left\|\xi\right\|_s - O(\delta)\left\|\xi\right\|_s\\ &\geq \left(\eta/2\right)\left\|\xi\right\|_s\end{aligned}$$

as claimed in (65). Using once more equation (54) for W_s we can estimate, using (65) and choosing δ sufficiently small,

$$\begin{aligned}\|W_s\| &\geq \|A(s)W - \Delta W_t\| - \|\widehat{C}W\|_s \\ &\geq (\eta/2 - O(\delta)) \cdot \|W\|_s \geq (\eta/4) \cdot \|W\|_s.\end{aligned} \tag{66}$$

From (57), (62), (63) and (65) we finally conclude

$$\begin{aligned}g''(s) &\geq \left[\frac{1}{2}(\eta/4)^2 - O(\delta)\right]\|W\|_s^2 \\ &\geq \mu^2 \cdot \frac{1}{2}\|W\|_s^2 = \mu^2 g(s)\end{aligned}$$

with $\mu^2 = (\eta/4)^2/2$, provided δ is sufficiently small. This is the desired estimate (56) and the proof of Lemma 3.7 is complete. □

In the following lemmata the constants C_α are independent of δ and h. However, h depends on the choice of δ in $0 < \delta \leqslant \delta_0$ and $h \to \infty$ as $\delta \to 0$. From Lemma 3.7 we deduce immediately L^2-estimates for all derivatives.

Lemma 3.8. *Under the same conditions as in Lemma 3.7,*

$$\|\partial^\alpha z(s,\cdot)\|^2 \leqslant C_\alpha \cdot \delta^2 \cdot \frac{\cosh[\mu s]}{\cosh[\mu r]}$$

for all derivatives $\partial^\alpha = \partial_s^{\alpha_1}\partial_t^{\alpha_2}$, $0 \leqslant |\alpha| \leqslant N$, *for all* $s \in [-r, r]$ *with* $r = R - h$. *The constants* C_α *do not depend on* δ *and* h.

Proof. Let $W(s,t)$ be a vector from Lemma 3.7. It solves the equation(54). Multiplying (54) by $\widehat{J}$ and using $\widehat{J}^2 = -\operatorname{Id}$ we find, by Lemma 3.3, and recalling $|\Delta| = O(\delta)$, the estimate

$$\|\partial_t W(s,\cdot)\|_s \leqslant C\left(\delta^2 \frac{\cosh[\mu s]}{\cosh[\mu r]}\right)^{1/2} + O(\delta)\|\partial_t W(s,\cdot)\|_s.$$

Consequently, for all α, we obtain for the first derivatives in t, if δ in Lemma 3.3 is sufficiently small,

$$\|\partial_t \partial_s^\alpha z(s,\cdot)\|_s^2 \leqslant C_{1,\alpha} \cdot \delta^2 \cdot \frac{\cosh[\mu s]}{\cosh[\mu r]}.$$

Taking more derivatives of equation (54) in t, the statement follows by induction from Lemma 3.7. □

Using the Sobolev estimates we deduce from Lemma 3.7 and Lemma 3.8 immediately the desired pointwise estimates.

Lemma 3.9. *Under the same conditions as in Lemma 3.7,*

$$|\partial^\alpha z(s,t)|^2 \leqslant C_\alpha \cdot \delta^2 \cdot \frac{\cosh[\mu s]}{\cosh[\mu r]}$$

for all $0 \leqslant |\alpha| \leqslant N$ *and for all* $s \in [-r, r]$ *and* $t \in S^1$ *with* $r = R - h$. *The constants* C_α *do not depend on* δ *and on* h.

We turn to the functions $a(s,t)$ and $\vartheta(s,t)$ in the coordinate representation of the $\widetilde{J}$-holomorphic cylinder $\widetilde{u}:[-R,R]\times S^1\to\mathbb{R}\times M$ in Lemma 3.7. They are smooth solutions of

$$\begin{aligned} a_s &= \Big(\vartheta_t+\sum_{i=1}^n x_i\partial_t y_i\Big)f(\vartheta,z)\\ a_t &= -\Big(\vartheta_s+\sum_{i=1}^n x_i\partial_s y_i\Big)f(\vartheta,z), \end{aligned}\tag{67}$$

where $z(s,t)=\big(x(s,t),y(s,t)\big)$. Moreover, $f(t,0)=\tau$ and $f_z(t,0)=0$. Consequently,

$$\begin{aligned} f(t,z)&=\tau+\left[\int_0^1 f_z(t,\sigma z)\ d\sigma\right]z=\tau+\left[\int_0^1\left[\int_0^1\sigma f_{zz}(t,\lambda\sigma z)d\lambda\right]z\ d\sigma\right]z\\ &=\tau+\big\langle b(t,z)z,z\big\rangle. \end{aligned}$$

Introduce the functions $\widetilde{a}$ and $\widetilde{\vartheta}$ by

$$\begin{aligned} \widetilde{a}(s,t)&=a(s,t)-Ts\\ \widetilde{\vartheta}(s,t)&=\vartheta(s,t)-kt. \end{aligned}\tag{68}$$

The functions $\widetilde{a}$ and $\widetilde{\vartheta}$ are 1-periodic in t. This follows from Lemma 3.3 and the representation $x(Tt)=(kt+c,0)$ of the distinguished periodic orbit. Recalling $T=\tau k$, equations (67) become

$$\begin{aligned} \widetilde{a}_s-\tau\widetilde{\vartheta}_t&=\tau\sum_{i=1}^n x_i\partial_t y_i+\Big(\vartheta_t+\sum_{i=1}^n x_i\partial_t y_i\Big)\big\langle b(t,z)z,z\big\rangle\\ \widetilde{\vartheta}_s+\frac{1}{\tau}\widetilde{a}_t&=-\sum_{i=1}^n x_i\partial_s y_i-\frac{1}{\tau}\Big(\vartheta_s+\sum_{i=1}^n x_i\partial_s y_i\Big)\big\langle b(t,z)z,z\big\rangle. \end{aligned}\tag{69}$$

Introducing $w(s,t)$ by

$$w=\begin{pmatrix}\widetilde{a}\\ \widetilde{\vartheta}\end{pmatrix},\tag{70}$$

the equations (69) take the form

$$w_s+\widehat{J}w_t=h,\quad \widehat{J}=\begin{pmatrix}0&-\tau\\ 1/\tau&0\end{pmatrix},\tag{71}$$

where $h(s,t)$ is the right hand side of (69). It is quadratic in z so that, in view of Lemma 3.9, the map $h(s,t)$ satisfies the following estimates for the derivatives $0\leqslant|\alpha|\leqslant N$,

$$|\partial^\alpha h(s,t)|\leqslant C_\alpha\cdot\delta^2\cdot\frac{\cosh[\mu s]}{\cosh[\mu r]}.\tag{72}$$

The next lemma is inspired by J. Robbin and D. Salomon [18].

Lemma 3.10. *Assume $w(s,t)$ and $h(s,t)$ are smooth, 1-periodic in t and solve the partial differential equation*

$$w_s+\widehat{J}w_t=h\quad\text{on } [-r,r]\times S^1\tag{73}$$

for some $r > 0$*. Assume, in addition, that*

$$(i) \qquad \int_0^1 w(s,t)dt = 0$$

$$(ii) \qquad \big\|\partial_s h(s,\cdot)\big\|^2 + \big\|\partial_t h(s,\cdot)\big\|^2 \leqslant f(s)$$

for a smooth function f *satisfying* $f''(s) = (2\mu)^2 f(s)$*, with some constant* $\mu > 0$*. Then, choosing* $0 < \nu < 2\mu$ *and* $\nu < 4\pi$ *we have for all* $s \in [-r,r]$

$$\frac{1}{2}\big\|w(s)\big\|^2 \leqslant \frac{c}{\sinh[2\nu r]}\Big(\beta(-r)\sinh[\nu(r-s)] + \beta(r)\sinh[\nu(r+s)]\Big),$$

where

$$\beta(-r) = \frac{1}{2}\big\|w(-r)\big\|^2 + bf(-r)$$

$$\beta(r) = \frac{1}{2}\big\|w(r)\big\|^2 + bf(r)$$

with a positive constants c *and* $b = b(\nu)$ *independent of* r *and* s*.*

Proof. In terms of the operator A, defined by

$$A = \widehat{J}\frac{d}{dt},$$

the equation (73) looks as follows,

$$w' + Aw = h, \tag{74}$$

where during the proof prime denotes ∂_s and dot denotes ∂_t. The operator A is self-adjoint with respect to the inner product $\langle\cdot,\cdot\rangle$ in L^2 defined by $\langle x,y\rangle = \int_0^1 \langle x(t), -J_0\widehat{J}y(t)\rangle\, dt$ for $x, y \in L^2(S^1, \mathbb{R}^{2n})$. The spectrum of A is equal to $2\pi\mathbb{Z}$ and A is non-degenerate when restricted to the closed subspace of functions having mean value equal to 0. Denoting by $|\cdot|^2 = \langle\cdot,\cdot\rangle$ the corresponding norm in L^2 we define

$$\alpha(s) = \frac{1}{2}|w(s)|^2.$$

Differentiation in s gives in view of the equation (74) for w,

$$\alpha' = \langle w, w'\rangle = \langle w, h - Aw\rangle.$$

Together with $|Ah| \leqslant C|\dot{h}|$, where $C \geq 1$ is a constant, we obtain for the second derivative of α the estimate

$$\begin{aligned}
\alpha''(s) &= 2|Aw|^2 - 3\langle Ah, w\rangle + \langle w, h'\rangle + |h|^2 \\
&\geq 2(2\pi)^2|\dot{w}|^2 - |w|\cdot|h'| - C|w|\cdot|\dot{h}| \\
&\geq 8\pi^2|w|^2 - \lambda|w|^2 - \frac{C^2}{2\lambda}\{\ |h'|^2 + |\dot{h}|^2\ \} \\
&\geq (16\pi^2 - 2\lambda)\frac{1}{2}|w|^2 - \frac{C^2}{2\lambda}\{\ |h'|^2 + |\dot{h}|^2\ \}
\end{aligned}$$

for every $\lambda > 0$. Choosing $\lambda > 0$ small one concludes, using assumption (ii), the estimate

$$\alpha''(s) \geq \nu^2\alpha(s) - af(s) \tag{75}$$

for $\nu < 4\pi$, with a constant $a > 0$. Choose $\nu < 2\mu$ and define

$$\beta(s) = \alpha(s) + bf(s), \quad b = \frac{a}{(2\mu)^2 - \nu^2}. \tag{76}$$

Then

$$\alpha(s) \leqslant \beta(s). \tag{77}$$

From (75), (76) and $f''(s) = (2\mu)^2 f(s)$ we deduce

$$\beta''(s) \geq \nu^2 \beta(s).$$

Consequently, arguing as in Lemma 3.6 we find the desired estimates and the proof of Lemma 3.10 is complete. □

As a consequence we have

Lemma 3.11. *Under the assumptions of Lemma 3.7 we fix $N \in \mathbb{N}$. Then there exist constants C_α such that for all derivatives*

$$\partial^\alpha = \partial_s^{\alpha_1} \partial_t^{\alpha_2}, \quad \alpha_2 \geq 1, |\alpha| \leqslant N$$

the following estimates hold for the functions $\widetilde{a}$ and $\widetilde{\vartheta}$ introduced in (68),

$$\left\|\partial^\alpha \widetilde{a}(s,t)\right\|^2, \ \left\|\partial^\alpha \widetilde{\vartheta}(s,t)\right\|^2 \leqslant C_\alpha \cdot \delta^2 \cdot \frac{\cosh[\nu s]}{\cosh[\nu r]}$$

for all $s \in [-r, r]$ where $r = R - h$, and where $0 < \nu < 2\mu$, $\nu < 4\pi$ is the constant introduced in Lemma 3.10 and where μ is as in Lemma 3.7. The constants C_α do not depend on δ and h.

Proof. The function w is a solution of (71). Abbreviate

$$v = \partial^\alpha w = \partial_s^{\alpha_1} \partial_t^{\alpha_2} w.$$

Since $\alpha_2 \geq 1$, the mean values over a period vanish and v solves the equation

$$v_s + \widehat{J} v_t = \partial^\alpha h =: g.$$

In view of (72),

$$\left\|\partial_s g(s,\cdot)\right\|^2 + \left\|\partial_t g(s,\cdot)\right\|^2 \leqslant C_\alpha \cdot \delta^2 \cdot \frac{\cosh[2\mu s]}{\cosh[2\mu r]}, \tag{78}$$

with different constants C_α. We have assumed that $\delta \leqslant 1$. Choosing $f(s)$ equal to the right hand side, the assumptions of Lemma 3.10 are met. By Lemma 3.3, $\left\|v(-r)\right\|^2$, $\left\|v(r)\right\|^2 \leqslant \delta^2$, and the desired estimates follow from Lemma 3.10. □

Lemma 3.12. *Under the assumptions of Lemma 3.7 and for fixed $N \in \mathbb{N}$ the following estimates hold true*

$$\left\|\partial_s^\alpha [\widetilde{\vartheta}(s,\cdot) - \vartheta_0]\right\|^2, \ \left\|\partial_s^\alpha [\widetilde{a}(s,\cdot) - a_0]\right\|^2 \leqslant C_\alpha \cdot \delta^2 \cdot \frac{\cosh[\nu s]}{\cosh[\nu r]}$$

for all $s \in [-r, r]$ and $0 \leqslant \alpha \leqslant N$, where $r = R - h$ and $0 < \nu < 2\mu$, $\nu < 4\pi$ is the constant introduced in Lemma 3.10. The constants (a_0, ϑ_0) are the mean values over a period at $s = 0$,

$$a_0 = \int_0^1 a(0,t)\ dt, \quad \vartheta_0 = \int_0^1 \big[\vartheta(0,t) - kt\big]\ dt,$$

and the constants C_α do not depend on δ and on h.

Proof. Recall from (70) the function $w(s,t) = \big(\widetilde{a}(s,t), \widetilde{\vartheta}(s,t)\big)$ and abbreviate $v = \partial_s^\alpha w$, $0 \leqslant \alpha \leqslant N$. It satisfies the equation

$$v_s + \widehat{J} v_t =: g$$

with $g(s,t) = \partial_s^\alpha h(s,t)$. Subtracting the mean values, the function

$$\widetilde{v}(s,t) = v(s,t) - \int_0^1 v(s,t)\ dt$$

solves the equation

$$\widetilde{v}_s + \widehat{J}\widetilde{v}_t = g - \int_0^1 g(s,t)\ dt.$$

and satisfies

$$\int_0^1 \widetilde{v}(s,t)\ dt = 0.$$

Therefore, by Lemma 3.11,

$$|\widetilde{v}(s,t)|^2 \leqslant \big\|\partial_t \widetilde{v}(s,\cdot)\big\|^2 = \big\|\partial_t v(s,\cdot)\big\|^2 \leqslant C_\alpha \cdot \delta^2 \cdot \frac{\cosh[\nu s]}{\cosh[\nu r]}. \tag{79}$$

Abbreviate the mean values

$$\alpha(s) = \int_0^1 v(s,t) dt.$$

Then $v(s,t) - \alpha(0) = \widetilde{v}(s,t) + [\alpha(s) - \alpha(0)]$ where $\widetilde{v}$ and $\alpha(s) - \alpha(0)$ are orthogonal in $L^2(S^1)$ so that

$$\big\|v(s,\cdot) - \alpha(0)\big\|^2 = \big\|\widetilde{v}(s,\cdot)\big\|^2 + |\alpha(s) - \alpha(0)|^2. \tag{80}$$

The term $\big\|\widetilde{v}(s,\cdot)\big\|^2$ is estimated in (79). In order to deal with the last term we integrate equation (71) for v over a period and find, in view of the periodicity in time,

$$\int_0^1 v_s(s,t)\ dt = \int_0^1 g(s,t)\ dt.$$

Consequently, recalling the estimate (72),

$$\begin{aligned} |\alpha(s) - \alpha(0)| = \left|\int_0^s \alpha'(\tau)\ d\tau\right| &\leqslant \frac{C_\alpha \cdot \delta^2}{\cosh[\mu r]} \cdot \left|\int_0^s \cosh[\mu\tau]\ d\tau\right| \\ &\leqslant \frac{C_\alpha \cdot \delta^2}{\mu \cosh[\mu r]} \cdot \big|\sinh[\mu s]\big|. \end{aligned}$$

Hence, if $s \in [-r, r]$,

$$|\alpha(s) - \alpha(0)|^2 \leqslant \frac{C_\alpha^2 \cdot \delta^4}{\mu^2} \cdot \frac{\sinh^2[\mu s]}{\cosh^2[\mu r]} \leqslant \frac{C_\alpha^2 \cdot \delta^4}{\mu^2} \cdot \frac{\cosh[2\mu s]}{\cosh[2\mu r]} \leqslant \frac{C_\alpha^2 \cdot \delta^4}{\mu^2} \cdot \frac{\cosh[\nu s]}{\cosh[\nu r]}.$$

We have used that the function $\cosh[2\mu s]/\cosh[\nu s]$ is increasing on $[0, r]$, since $\nu < 2\mu$. Assuming $\delta \leqslant 1$, the desired estimates now follow from the estimate (79) and from (80) above. □

By means of the Sobolev embedding theorem one concludes from Lemma 3.11 and Lemma 3.12 the following pointwise estimates.

Lemma 3.13. *Under the assumptions of Lemma 3.7 the following estimates hold true for all* $(s,t) \in [-r,r] \times S^1$ *with* $r = R - h$,

$$\left|\partial^\alpha\left[a(s,t) - Ts - a_0\right]\right|^2,\ \left|\partial^\alpha\left[\vartheta(s,t) - kt - \vartheta_0\right]\right|^2 \leqslant C_\alpha \cdot \delta^2 \frac{\cosh[\nu s]}{\cosh[\nu r]},$$

$0 \leqslant |\alpha| \leqslant N$, *with the constants* a_0 *and* ϑ_0 *defined in Lemma 3.12, and with* $0 < \nu < 2\mu$, $\nu < 4\pi$ *as introduced in Lemma 3.10. The constants* C_α *do not depend on* δ *and* h.

With Lemma 3.13, the proof of Theorem 1.3 in the introduction is complete.

4. Application

The following results will have applications in the compactness proof of sequences of $\widetilde{J}$-holomorphic finite energy maps, which will be important for the symplectic field theory in [2].

We begin by introducing several concepts. In the following S_+ and S_- are two compact disk-like Riemann surfaces with smooth boundaries. Their complex structures will be denoted by j_+ and j_-. Let $o_\pm$ be interior points of $S_\pm$. Then a noded surface is obtained by identifying in the disjoint union $S_- \cup S_+$ the two points o_- and o_+. The noded surface obtained this way is denoted by S and its node by o.

Definition 4.1 (Deformation). *A deformation of a compact Riemann surface* (A, j) *of annulus type is a continuous surjection map*

$$f : A \to S$$

onto the nodal surface, so that $f^{-1}(o)$ *is a smooth embedded circle, and*

$$f : A \setminus f^{-1}(o) \to S \setminus \{o\}$$

is an orientation preserving diffeomorphism. On $S \setminus \{o\}$ *we have the pushed forward complex structure* $f_* j$

We consider a sequence of compact Riemann surfaces (S_n, j_n) of annulus type whose moduli converge to ∞,

$$\operatorname{mod}(S_n, j_n) \to \infty.$$

On this sequence of surfaces we consider a sequence

$$\widetilde{u}_n = (a_n, u_n) : (S_n, j_n) \to \mathbb{R} \times M$$

of $\widetilde{J}$-holomorphic finite energy maps. Let $\widetilde{u} = (a, u) : S \setminus \{o\} \to \mathbb{R} \times M$ be a $\widetilde{J}$-holomorphic finite energy map having the negative puncture at $o_+ \in S_+$ and the positive puncture $o_- \in S_-$ and assume that their asymptotic limits are the same periodic orbit of the Reeb vector field.

Definition 4.2 (Convergence modulo $\mathbb{R}$). *The sequence* $\widetilde{u}_n : (S_n, j_n) \to \mathbb{R} \times M$ *of* $\widetilde{J}$*-holomorphic maps is said to converge to* $\widetilde{u} : S \setminus \{o\} \to \mathbb{R} \times M$ *if there exists a sequence of deformations* $f_n : (S_n, j_n) \to (S, j)$ *satisfying*

(i) $u_n \circ f_n^{-1} \to u$

(ii) $(f_n)_* j_n \to j_\pm$

in $C^\infty_{loc}(S_\pm \setminus \{o_\pm\})$.

We will next introduce a second notion of convergence related to the notion of asymptotically marked points. Let R be a Riemann surface and let $r \in R$ be an interior point. An asymptotic marker for r consists of a choice of an oriented real line $\widehat{r} \subset T_r R$ in the tangent space at r. The oriented line $\widehat{r}$ together with the underlying point r will be called an asymptotically marked point.

Given the Riemann surface R with an asymptotically marked point $\widehat{r}$ there is a distinguished class of holomorphic coordinate systems around r, called compatible with the asymptotic marker and defined as follows. We take any compact disk-like neighborhood $\mathcal{D}$ with smooth boundary around r. Then we take the unique biholomorphic map σ from the closed unit disk D onto $\mathcal{D}$ mapping 0 to r so that the tangent $T\sigma(0)$ maps $1 \in \mathbb{R}$ to an (oriented) basis vector in $\widehat{r}$. We note that for two such coordinate systems σ and τ compatible with the asymptotic marker, the linearized transition map at 0, $D\big(\sigma^{-1} \circ \tau\big)(0) : \mathbb{C} \to \mathbb{C}$, acts via multiplication by a positive real number.

Assume now that $\widetilde{v} = (a, v) : R \setminus \{r\} \to \mathbb{R} \times M$ is a $\widetilde{J}$-holomorphic finite energy map and assume that r is a non removable positive puncture. Moreover, assume that the associated asymptotic limit is a non degenerate periodic orbit of the Reeb vector field. Let $\widehat{r}$ be an asymptotic marker. Then there is a special class of holomorphic polar coordinates around r, compatible with the asymptotic marker and defined as follows. We take a holomorphic coordinate system $\sigma : D \to \mathcal{D}$ around r which is compatible with the asymptotic marker as described above and define the holomorphic map $h : \mathbb{R}^+ \times S^1 \to \mathcal{D} \setminus \{r\}$ by

$$h(s,t) = \sigma\big(e^{-2\pi(s+it)}\big).$$

If one considers the composition $v \circ h$, then $v \circ h(s,t)$ will converge in $C^\infty(\mathbb{R})$ as $s \to \infty$ to a parametrization of the asymptotic limit. The limiting loop $[t \mapsto y(t)]$ is independent of the choice of σ as long as σ is compatible with the asymptotic marker.

If the puncture r is negative, we take the holomorphic polar coordinates $h : \mathbb{R}^- \times S^1 \to \mathcal{D} \setminus \{r\}$ defined by

$$h(s,t) = \sigma\big(e^{2\pi(s+it)}\big).$$

We consider now a special finite energy map $\widetilde{u} = (a, u) : S \setminus \{o\} \to \mathbb{R} \times M$, where S is the noded surface introduced above with the node $o = o_- = o_+$, where $o_+ \in S_+$ and $o_- \in S_-$. We assume that o_+ is a negative and o_- positive puncture. Moreover, the negative asymptotic limit of $u|_{S_+ \setminus \{o_+\}}$ coincides with the positive asymptotic limit of $u|_{S_- \setminus \{o_-\}}$. We assume, in addition, that $\widehat{o}_\pm$ are asymptotic marked points. If h_+ are negative holomorphic polar coordinates at o_+ compatible with $\widehat{o}_+$, and if h_- are positive holomorphic polar coordinates at o_- compatible with $\widehat{o}_-$, we finally require that

$$\lim_{s \to \infty} u \circ h_-(s,t) = \lim_{s \to -\infty} u \circ h_+(s,t)$$

for all $t \in S^1$. A finite energy surface $\widetilde{u} : S \setminus \{o\} \to \mathbb{R} \times M$ having all these properties is called compatible with the asymptotic markers.

Definition 4.3 (Directional convergence modulo $\mathbb{R}$).
Assume that the finite energy surface $\widetilde{u}=(a,u):S\setminus\{o\}\to\mathbb{R}\times M$ is compatible with the asymptotic markers as described above. The sequence $\widetilde{u}_n:(S_n,j_n)\to\mathbb{R}\times M$ is called directionally convergent to $\widetilde{u}:S\setminus\{o\}\to\mathbb{R}\times M$ for the given asymptotic marked points $\widehat{o}^{\pm}$ if there exists a sequence $f_n:S_n\to S$ of deformations onto the nodal surface S and a sequence $\Psi_n:[-R_n,R_n]\times S^1\to S_n$ of biholomorphic maps, where $2R_n=mod(S_n,j_n)\to\infty$, with the following properties,

(i) $u_n\circ f_n^{-1}\to u$

(ii) $(f_n)_*j_n\to j_\pm$

in $C^\infty_{loc}\big(S_\pm\setminus\{o_\pm\}\big)$. Moreover, we require that the sequences of mappings e_n^+, e_n^-, defined by

$$
\begin{aligned}
e_n^+(z)&=\Psi_n^{-1}\circ f_n^{-1}(z)-(R_n,0)\quad \textit{for } z\in S_+\setminus\{o_+\}\\
e_n^-(z)&=\Psi_n^{-1}\circ f_n^{-1}(z)+(R_n,0)\quad \textit{for } z\in S_-\setminus\{o_-\}
\end{aligned}
$$

do converge as $n\to\infty$, in the sense that

$$
\begin{aligned}
e_n^+&\to e^+\quad \textit{in } C^\infty_{loc}\big(S_+\setminus\{o_+\},\mathbb{R}^-\times S^1\big).\\
e_n^-&\to e^-\quad \textit{in } C^\infty_{loc}\big(S_-\setminus\{o_-\},\mathbb{R}^+\times S^1\big).
\end{aligned}
$$

Then the limit maps $e^+:S_+\setminus\{o_+\}\to\mathbb{R}^-\times S^1$ and $e^-:S_-\setminus\{o_-\}\to\mathbb{R}^+\times S^1$ are necessarily biholomorphic, and in particular, inverse maps of holomorphic polar coordinates. Defining the associated holomorphic coordinate systems $v_\pm:D\to S_\pm$ by setting $v_\pm(0)=o_\pm$ and $\big(e^\pm\big)^{-1}(s,t)=v_\pm\big(e^{\pm2\pi(s+it)}\big)$ on $\mathbb{R}^\mp\times S^1$, we require that $v_\pm$ are compatible with the asymptotic markers.

Since $e_n^\pm$ are convergent, it follows from the properties of holomorphic mappings that the limiting maps $e^\pm$ are biholomorphic. This implies, in particular, that given any $h>0$, the preimage $\Psi_n^{-1}\circ f_n^{-1}(o)$ of the node, which a priori is a circle in $[-R_n,R_n]\times S^1$, is actually contained in $[-R_n+h,R_n-h]\times S^1$ if n is sufficiently large. As a consequence, the composition $f_n\circ\Psi_n$ is defined on the complement of $[-R_n+h,R_n-h]\times S^1$ if n is sufficiently large. Consequently,

$$
u_n\circ\Psi_n(-R_n+s,t)=u_n\circ f_n^{-1}\circ\big(f_n\circ\Psi_n(-R_n+s,t)\big)
$$

is well defined for every $s\geq 0$ and sufficiently large n. Moreover, the right hand side converges to the map $u\circ(e^-)^{-1}:\mathbb{R}^+\times S^1\to M$. Hence, introducing the translations $\tau_a(s,t)=(s+a,t)$, we have

$$
u_n\circ\Psi_n\circ\tau_{-R_n}\to u\circ(e^-)^{-1}\quad\text{in } C^\infty_{\text{loc}}(\mathbb{R}^+\times S^1,M).
$$

Similarly,

$$
u_n\circ\Psi_n\circ\tau_{R_n}\to u\circ(e^+)^{-1}\quad\text{in } C^\infty_{\text{loc}}(\mathbb{R}^-\times S^1,M).
$$

We will make use of all these considerations in order to prove the following uniqueness statement.

Proposition 4.4. *If $\widetilde{u}_n$ converges directionally to $\widetilde{u}$ and to $\widetilde{v}$, then*

$$
u=v\circ\varphi_\pm\quad \textit{on } S^\pm\setminus\{o_\pm\}
$$

for two biholomorphic mappings $\varphi_\pm : S_\pm \to S_\pm$ *satisfying* $\varphi_\pm(o_\pm) = o_\pm$ *and*

$$T\varphi_\pm(o_\pm)\widehat{o}_\pm = e^{\pm i\theta} \cdot \widehat{o}_\pm$$

for some $\theta \in \mathbb{R}$.

Proof. Denote the data for the convergence to $\widetilde{u}$ by $(f_n, \Psi_n, e^\pm)$ and the corresponding data for the convergence to $\widetilde{v}$ by $(g_n, \Phi_n, d^\pm)$. We note that $\Phi_n^{-1} \circ \Psi_n$ is a biholomorphic map of the cylinder $[-R_n, R_n] \times S^1$. As such it has necessarily the form $(s,t) \mapsto (s, t + c_n)$, considered mod 1 in the second argument. Hence it is a rotation σ_n of the cylinder. After taking a subsequence we may assume that $c_n \to c$. Recall that

$$u_n \circ \Psi_n \circ \tau_{-R_n} \to u \circ (e^-)^{-1}$$
$$u_n \circ \Phi_n \circ \tau_{-R_n} \to v \circ (d^-)^{-1}$$

in $C^\infty_{\mathrm{loc}}(\mathbb{R} \times S^1, M)$. On the other hand, since $\Phi_n^{-1} \circ \Psi_n$ converges to a limiting rotation σ of the cylinder, we deduce from $u_n \circ \Psi_n \circ \tau_{-R_n} = (u_n \circ \Phi_n \circ \tau_{-R_n}) \circ (\tau_{R_n} \circ \Phi_n^{-1} \circ \Psi_n \circ \tau_{-R_n})$ in the limit as $n \to \infty$,

$$u \circ (e^-)^{-1} = v \circ (d^-)^{-1} \circ \sigma.$$

Similarly,

$$u \circ (e^+)^{-1} = v \circ (d^+)^{-1} \circ \sigma.$$

The desired result now follows from the compatibility of the holomorphic coordinate systems defined by $e^\pm$ and $d^\pm$ with the asymptotic markers. □

The main result of this section is the following compactness result.

Theorem 4.5. *A convergent sequence* $\widetilde{u}_n$ *has a directionally convergent subsequence.*

Proof. By hypothesis, there is a sequence of deformations $f_n : S_n \to S$ so that $u_n \circ f_n^{-1} \to u$ in $C^\infty_{\mathrm{loc}}(S \setminus \{o\})$. Fix a point $q \in \partial S_+$ and choose biholomorphic maps $\Psi_n : [-R_n, R_n] \times S^1 \to S_n$ satisfying $q = f_n \circ \Psi_n(R_n, 0)$. The maps $\Psi_n^{-1} \circ f_n^{-1} : S_+ \setminus \{o_+\} \to [-R_n, R_n] \times S^1$ are injective and holomorphic if the target is equipped with the standard complex structure and the domain with the structure induced from j_n on S_n via the push forward by f_n. The boundary ∂S_+ is mapped to the upper boundary of the cylinder. A standard bubbling-off analysis will show that the gradients of $\Psi_n^{-1} \circ f_n^{-1}$ are uniformly bounded and hence the higher derivatives as well.

Lemma 4.6. *The gradients of* $\Psi_n^{-1} \circ f_n^{-1}$ *are uniformly bounded on compact subsets of* $S_+ \setminus \{o_+\}$

Proof of Lemma 4.6. In order to simplify the notation we may assume that S_+ is the closed unit disk $D \subset \mathbb{C}$ equipped with the standard complex structure, that $o_+ = 0$, and $\Psi_n^{-1} \circ f_n^{-1} : D \setminus \{0\} \to [-R_n, R_n] \times S^1$. Indeed, take a biholomorphic map $\varphi : S^+ \to D$ satisfying $\varphi(o_+) = 0$. If D is equipped with the standard complex structure i and S_+ with j, we consider the composition $\Psi_n^{-1} \circ f_n^{-1} \circ \varphi$ defined on $D \setminus \{0\}$. Abbreviating by $\widetilde{j}_n$ the complex structure induced from $(f_n)_* j_n$ via the push forward by φ, we conclude from $(f_n)_* j_n \to j$ in $C^\infty_{\rm loc}(S_+ \setminus \{o_+\})$ that $\widetilde{j}_n \to i$ in $C^\infty_{\rm loc}(D \setminus \{0\})$.

Hence we may assume that the maps $\Psi_n^{-1} \circ f_n^{-1} : D \setminus \{0\} \to [-R_n, R_n] \times S^1$ are embeddings and holomorphic if the target is equipped with the standard complex structure and the domain with $\widetilde{j}_n$ satisfying $\widetilde{j}_n \to i$ in $C^\infty_{\rm loc}(D \setminus \{0\})$. Define $g_n : D \setminus \{0\} \to [-2R_n, 0] \times S^1$ by

$$g_n(z) = \Psi_n^{-1} \circ f_n^{-1}(z) - (R_n, 0).$$

The boundary ∂D is mapped by g_n onto $\{0\} \times S^1$. To prove the lemma we argue by contradiction and assume the existence of a sequence $z_n \in D \setminus \{0\}$ satisfying $|\nabla g_n(z_n)| \to \infty$. We first assume $z_n \to z_0$ for an interior point of $z_0 \in D \setminus \{0\}$. Proceeding as in [5] we find a sequence of positive numbers $\varepsilon_n \to 0$ satisfying $\varepsilon_n |\nabla g_n(z_n)| \to \infty$ and

$$|\nabla g_n(z)| \leqslant 2 |\nabla g_n(z_n)| \quad \text{for } |z_n - z| < \varepsilon_n.$$

If the $\mathbb{R}$-components of $g_n(z_n)$ are bounded, we introduce the rescaled maps

$$\widetilde{g}_n(z) = g_n(z_n + z/r_n) \quad \text{for } |z| < \varepsilon_n r_n,$$

where $r_n = |\nabla g_n(z_n)|$. The maps $\widetilde{g}_n$ satisfy

$$\begin{aligned} |\nabla \widetilde{g}_n(0)| &= 1 \\ |\nabla \widetilde{g}_n(z)| &\leqslant 2 \quad \text{for } |z| < \varepsilon_n r_n \\ T\widetilde{g}_n \circ \widetilde{j}_n &= i \circ T\widetilde{g}_n. \end{aligned} \tag{81}$$

In view of the gradient bounds for $\widetilde{g}_n$ and in view of the convergence $\widetilde{j}_n \to i$ in $C^\infty_{\rm loc}(D \setminus \{0\})$ it follows from the last equation, that we have uniform bounds also for the higher derivatives of $\widetilde{g}_n$. Hence, using Ascoli-Arzela's theorem, we may assume after taking a subsequence that the sequence $\widetilde{g}_n$ converges to a non constant holomorphic map $v : \mathbb{C} \to \mathbb{R}^- \times S^1$ in $C^\infty_{\rm loc}(\mathbb{C}, \mathbb{R}^- \times S^1)$. The map v is holomorphic with respect to the standard complex structure on the domain and the target. Since the gradient of v is bounded, it has a removable singularity at ∞. Identifying the half-cylinder $\mathbb{R}^- \times S^1$ biholomorphically with $D \setminus \{0\}$ and denoting the extension again by v we have a non constant holomorphic map $v : S^2 \to D \subset \mathbb{C}$. However, in view of maximum principle for holomorphic maps, the map v has to be constant.

If the $\mathbb{R}$-components $g_{n,1}(z_n)$ of $g_n(z_n)$ are unbounded, we consider the following rescaled maps $\widetilde{g}_n$,

$$\widetilde{g}_n(z) = g_n\big(z_n + z/r_n\big) - \big(g_{n,1}(z_n), 0\big) \quad \text{for } |z| < \varepsilon_n r_n.$$

Again the maps $\widetilde{g}_n$ satisfy the properties listed in (81). Application of Ascoli-Arzela's theorem produces this time a subsequence $\widetilde{g}_n$ which converges to a non constant holomorphic map $v : \mathbb{C} \to \mathbb{R} \times S^1$. Identifying the cylinder $\mathbb{R} \times S^1$ biholomorphically with the 2-punctured Riemann sphere and denoting by v also the extension over the removable singularity at ∞, we get a non constant holomorphic map $v : S^2 \to S^2 \setminus \{\text{point}\} \equiv \mathbb{C}$, which is not possible.

Assume next that the bubbling off sequence z_n converges to a boundary point so that $z_n \to z_0 \in \partial D$. We have to distinguish two cases depending on the behavior of the sequence $r_n \cdot \operatorname{dist}(z_n, \partial D)$. If the sequence $r_n \cdot \operatorname{dist}(z_n, \partial D)$ is not bounded, then the same bubbling off analysis as above leads to a contradiction. To treat the second case, we assume that $r_n \cdot \operatorname{dist}(z_n, \partial D) \to d \in [0, \infty)$ and make a biholomorphic change of coordinates. Let $\varphi : H_+ \to D \setminus \{-z_0\}$ be the biholomorphic map defined on the upper half-plane by

$$\varphi(z) = z_0 \frac{i-z}{i+z}, \quad z \in H_+.$$

Then $\varphi(0) = z_0$ and $\varphi(\mathbb{R}) = \partial D \setminus \{-z_0\}$. Using the old notation g_n for the composition $g_n \circ \varphi$ we consider a sequence of embeddings $g_n : H_+ \setminus \{i\} \to [-2R_n, 0] \times S^1$ having the property that $g_n(\mathbb{R}) \subset \{0\} \times S^1$. The bubbling off sequence z_n now satisfies $z_n \to 0$ and $r_n \cdot \mathrm{dist}(z_n, \partial H_+) \to d \in [0, \infty)$. Abbreviating $d_n = r_n \mathrm{Im} z_n$, the rescaled maps $\widetilde{g}_n(z)$ defined by

$$\widetilde{g}_n(z) = g_n\big(z_n + (z - i \cdot d_n)/r_n\big) \quad \text{ for } z \in B_{\varepsilon_n r_n}(i \cdot d_n) \cap H_+,$$

have bounded gradients and so, using Ascoli-Arzela's theorem as above we obtain a non constant holomorphic map $v : H_+ \to \mathbb{R}^- \times S^1$ satisfying $v(\mathbb{R}) \subset \{0\} \times S^1$. With the biholomorphic map $\psi : \mathbb{R}^- \times S^1 \to D \setminus \{0\}$ we find the holomorphic embedding $w = \psi \circ v \circ \varphi^{-1} : D \setminus \{-z_0\} \to D \setminus \{0\}$ satisfying $|w(z)| = 1$ for $|z| = 1$, $z \neq -z_0$. Since the gradient of w is bounded, the singularity at $-z_0$ is removable and the extension w satisfies $|w(-z_0)| = 1$. Therefore, we arrive at a non constant holomorphic map $w : D \to D \setminus \{0\}$ satisfying $w(\partial D) \subset \partial D$. Applying the maximum principle to the holomorphic map $z \mapsto 1/w(z)$ on D we conclude $|w(z)| = 1$ on D so that w must be constant. With this contradiction, the proof of Lemma 4.6 is complete.

□

In view of Lemma 4.6, perhaps going over to a subsequence, the maps $z \mapsto \Psi_n^{-1} \circ f_n^{-1}(z) - (R_n, 0)$ converge in $C^\infty_{\mathrm{loc}}(S_+ \setminus \{o_+\})$. The limit mapping $e^+ : S_+ \setminus \{o_+\} \to \mathbb{R}^- \times S^1$ is biholomorphic and maps q to the point $(0,0)$. Similarly, we find a subsequence of the sequence of maps $z \mapsto \Psi_n^{-1} \circ f_n^{-1}(z) + (R_n, 0)$ which converges in $C^\infty_{\mathrm{loc}}(S_- \setminus \{o_-\})$ to a biholomorphic map $e^- : S_- \setminus \{o_-\} \to \mathbb{R}^+ \times S^1$. These maps determine the asymptotic markers as follows. We define the associated holomorphic coordinates $v_\pm : D \to S_\pm$ by $v_\pm(0) = o_\pm$ and $(e^\pm)^{-1}(s,t) = v_\pm\big(e^{\pm 2\pi(s+it)}\big)$ on $\mathbb{R}^\mp \times S^1$ and set $\widehat{o}_\pm = T v_\pm(0)\mathbb{R}$. It remains to show that the asymptotic limits coincide in the sense that

$$\lim_{s\to\infty} u \circ (e^-)^{-1}(s,t) = \lim_{s\to-\infty} u \circ (e^+)^{-1}(s,t). \tag{82}$$

For this we observe that the right hand side of

$$u_n \circ \Psi_n = u_n \circ f_n^{-1} \circ \big(f_n \circ \Psi_n\big)$$

is well defined for points in a given distance from the boundary of the cylinder $[-R_n, R_n] \times S^1$ if n is sufficiently large. By the above discussion we conclude the convergence

$$u_n \circ \Psi_n \circ \tau_{R_n} = \big(u_n \circ f_n^{-1}\big) \circ \big(f_n \circ \Psi_n \circ \tau_{R_n}\big) \to u \circ (e^+)^{-1}$$

in $C^\infty_{\mathrm{loc}}(\mathbb{R}^- \times S^1, M)$ and

$$u_n \circ \Psi_n \circ \tau_{-R_n} = \big(u_n \circ f_n^{-1}\big) \circ \big(f_n \circ \Psi_n \circ \tau_{-R_n}\big) \to u \circ (e^-)^{-1}$$

in $C^\infty_{\mathrm{loc}}(\mathbb{R}^+ \times S^1, M)$. From this it follows that the energies of the cylinders $\widetilde{u}_n \circ \Psi_n : [-R_n, R_n] \times S^1 \to \mathbb{R} \times M$ are bounded and, moreover, that for given h_1 sufficiently large and $0 < \gamma < \gamma_0$,

$$\int_{[-R_n+h_1,R_n-h_1]\times S^1} \big(u_n \circ \Psi_n\big)^* d\lambda \leqslant \gamma$$

for all n. In addition, $A(\widetilde{u}_n \circ \Psi_n) > 0$. Having verified all the hypotheses of Theorem 1.4 we conclude from the estimates in Theorem 1.4 that for given $\varepsilon > 0$ there exists a constant $h > 0$, so that for all n, $d\big(u_n \circ \Psi_n(s,t), u_n \circ \Psi_n(0,t)\big) \leqslant \varepsilon$ for all $s \in [-R_n + h, R_n - h]$. This implies after passing to the limit, that for large R,

$$d\big(u \circ (e^-)^{-1}(R,t), u \circ (e^+)^{-1}(-R,t)\big) \leqslant 2\varepsilon.$$

Since ε was arbitrary we have proved the required identity (82) and the proof of Theorem 4.5 is complete. $\square$

Remark 4.7. *We should finally point out, that the notion of directional convergence does not depend on the choice of the disks near the node. Indeed, consider a sequence $\widetilde{u}_n : (S_n, j_n) \to \mathbb{R} \times M$ converging directionally to $\widetilde{u} : S \setminus \{o\} \to \mathbb{R} \times M$ for the given asymptotic marked points $\widehat{o}^{\pm}$. Let S' be obtained by taking in $S^{\pm}$ disk-like neighborhoods $\mathcal{D}^{\pm}$ of $o^{\pm}$ having smooth boundaries and gluing them at the node $o = o^+ = o^-$, so that $S' = \mathcal{D}^+ \cup_{o^+=o^-} \mathcal{D}^-$. In addition, assume that S and S' have the same asymptotic marked points $\widehat{o}^{\pm}$. Denoting by $S'_n \subset S_n$ the annulus type region having the boundaries $f_n^{-1}(\partial\mathcal{D}^+)$ and $f_n^{-1}(\partial\mathcal{D}^-)$ we define $\widetilde{v}_n = \widetilde{u}_n|_{S'_n}$ and $\widetilde{v} = \widetilde{u}|_{S'}$. Using the methods developed in this section one can verify that the directional convergence of $\widetilde{u}_n$ to $(\widetilde{u}, \widehat{o}^{\pm})$ implies also the directional convergence of $\widetilde{v}_n$ to $(\widetilde{v}, \widehat{o}^{\pm})$.*

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