Finite Energy Solutions of Nonlinear Schrödinger Equations of Derivative Type

N. Hayashi and T. Ozawa

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Finite Energy Solutions of Nonlinear Schrödinger Equations of Derivative Type

NAKAO HAYASHI[1] AND TOHRU OZAWA

Laboratoire de Physique Théorique et Hautes Energies
Bâtiment 211, Université de
Paris Sud, 91405 Orsay, France
and
Department of Mathematics, Hokkaido University
Sapporo 060, Japan

Abstract. This paper is concerned with the initial value problem for nonlinear Schrödinger equations of the form

$$(*)\qquad \begin{cases} i\partial_t\psi + \partial^2\psi = i\lambda\partial(|\psi|^2\psi) + \lambda_1|\psi|^{p_1-1}\psi + \lambda_2|\psi|^{p_2-1}\psi, & (t,x)\in\mathbb{R}\times\mathbb{R}, \\ \psi(0,x) = \phi(x), & x\in\mathbb{R}, \end{cases}$$

where $\partial = \partial_x = \partial/\partial x$, λ, λ_1, $\lambda_2 \in \mathbb{R}$, and $2 \leqq p_1 < p_2 < 5$. It is shown that if $\phi \in H^1(\mathbb{R})$ and

$$||\phi||_2^2 < \frac{2\pi}{|\lambda|},$$

then there exists a unique global solution ψ of $(*)$ such that

$$\psi \in C(\mathbb{R}; H^1(\mathbb{R})).$$

In this paper we introduce a new method to obtain the result.

Key words. Derivative Nonlinear Schrödinger Equations, Gauge Transformations.
AMS(MOS) subject classifications. 35Q55, 35Q60.

[1]On leave of absence from Department of Mathematics, Faculty of Engineering, Gunma University, Kiryu 376, Japan

§1. Introduction. In this paper we study the Cauchy problem for nonlinear Schrödinger equations of the form

$$
(1.1)\qquad \begin{cases} i\partial_t\psi + \partial^2\psi = i\lambda\partial(|\psi|^2\psi) + F(\psi), & (t,x)\in\mathbb{R}\times\mathbb{R},\\ \psi(0,x) = \phi(x), & x\in\mathbb{R}, \end{cases}
$$

where ψ is a complex valued function of $(t,x)\in\mathbb{R}\times\mathbb{R}$, $\partial = \partial/\partial x$, $\lambda\in\mathbb{R}$ and $F\in C(\mathbb{R}^2;\mathbb{R}^2)$ satisfies the gauge condition $F(e^{i\theta}\zeta) = e^{i\theta}F(\zeta)$, $\theta\in\mathbb{R}$.

We assume that F can be written as $F = F_1 + F_2$ with $F_j\in C^2(\mathbb{R}^2\setminus\{0\};\mathbb{R}^2)$, $j = 1,2$, satisfying $F_j''\in L^\infty_{\mathrm{loc}}(\mathbb{R}^2)$, $F_j(0) = F_j'(0) = 0$ and there exist positive constants C_j, D_j such that

$$
(1.2)\qquad \begin{cases} |F_j(\zeta) - F_j(\zeta')| \leqq C_j(|\zeta|^{p_j-1} + |\zeta'|^{p_j-1})|\zeta-\zeta'|,\\ |F_j'(\zeta) - F_j'(\zeta')| \leqq D_j(|\zeta|^{p_j-2} + |\zeta'|^{p_j-2})|\zeta-\zeta'|, \end{cases}
$$

where $2\leqq p_1 < p_2 < 5$. Furthermore we assume that there exists a function $H\in C^1(\mathbb{R}^2;\mathbb{R})$ satsfying $H(0) = 0$, $F = \partial H/\partial\bar{\zeta}$ and

$$
(1.3)\qquad \int_{\mathbb{R}} H(f)dx \geqq -M(\|f\|_2) - \mu\|f\|_6^6 \quad \text{for all} \quad f\in H^1(\mathbb{R})
$$

where $\mu\in\mathbb{R}^+$, $M\in C(\mathbb{R}^+;\mathbb{R}^+)$. Under the above conditions we prove

Theorem 1. *We assume that (1.2), (1.3) are satisfied, $\phi\in H^1(\mathbb{R})$ and $\|\phi\|_2^2 < \frac{2\pi}{\sqrt{\lambda^2+16\mu^2}}$. Then there exists a unique global solution ψ of (1.1) such that*

$$
\psi\in C(\mathbb{R};H^1(\mathbb{R}))\cap L^4_{\mathrm{loc}}(\mathbb{R};W^{1,4}(\mathbb{R})).
$$

Moreover, the map $\phi\mapsto\psi$ is continuous from $\{\phi\in H^1(\mathbb{R}); \|\phi\|_2^2 < 2\pi/\sqrt{\lambda^2+16\mu^2}\}$ with topology indued from $H^1(\mathbb{R})$ to $C(\mathbb{R};H^1(\mathbb{R}))\cap L^4_{\mathrm{loc}}(\mathbb{R};W^{1,4}(\mathbb{R}))$.

We prove that our result applies to the example

$$
(1.4)\qquad F(\psi) = \lambda_1|\psi|^{p_1-1}\psi + \lambda_2|\psi|^{p_2-1}\psi,
$$

where $2 \leqq p_1 < p_2 < 5$, $\lambda_1, \lambda_2 \in \mathbb{R}$. First it is clear that (1.4) satisfies (1.2) (see, e.g., [6]). Secondly we have

$$\int_{\mathbb{R}} H(\psi)dx = \frac{2\lambda_1}{p_1+1}||\psi||_{p_1+1}^{p_1+1} + \frac{2\lambda_2}{p_2+1}||\psi||_{p_2+1}^{p_2+1}. \tag{1.5}$$

We deduce (1.3) from (1.5). By Hölder's inequality we have

$$||f||_{p+1}^{p+1} \leqq ||f||_6^{\frac{3(p-1)}{2}} ||f||_2^{\frac{5-p}{2}},$$

from which we see that for any $\varepsilon > 0$, there exists a positive constant C_ε such that

$$||f||_{p+1}^{p+1} \leqq \varepsilon ||f||_6^6 + C_\varepsilon ||f||_2^2.$$

We obtain by this and (1.5)

$$\int_{\mathbb{R}} H(\psi)dx \geqq -C_{\varepsilon'}||f||_2^2 - \varepsilon'||f||_6^6,$$

where $\varepsilon' = \sum_{j=1}^{2} \frac{2|\lambda_j|}{p_j+1}\varepsilon$, $C_{\varepsilon'} = \sum_{j=1}^{2} \frac{2|\lambda_j|}{p_j+1} C_\varepsilon$, which implies (1.3) with $\mu = \varepsilon'$. Hence we have by Theorem 1

Corollary 1. *We let $F(\psi) = \sum_{j=1}^{2} \lambda_j |\psi|^{p_j - 1}\psi$, where $2 \leqq p_1 < p_2 < 5$. We assume that $\phi \in H^1(\mathbb{R})$ and $||\phi||_2^2 < \frac{2\pi}{|\lambda|}$. Then the same result of Theorem 1 holds valid.*

In the case of $F(\psi) \equiv 0$, Theorem 1 was proved in [4] (see also [3]) by using the result of [9, Theorem 3] which ensures the existence of global smooth solutions of (1.1) with $F(\psi) \equiv 0$. When $F(\psi) \not\equiv 0$, the result [9, Theorem 3] is not applicable, and therefore we need a different method from the previous ones (see [3], [4]). We state our strategy of the proof of Theorem 1 for the convenience of the reader. To obtain the result we first consider the system of nonlinear Schrödinger equations

$$\begin{cases} Lu = -i\lambda u^2 \bar{v} + F(u), \\ Lv = i\lambda v^2 \bar{u} + \partial_u F(u) \cdot v + \partial_{\bar{u}} F(u) \cdot \bar{v} \\ u(0) = u_0, \ v(0) = v_0. \end{cases} \tag{1.6}$$

with the constraint

$$v_0 = \partial u_0 + i\frac{\lambda}{2}|u_0|^2 u_0. \tag{1.7}$$

The initial value problem of (1.6) without constraint (1.7) fits in the framework of the previous methods as in [7], [8], [10], [11] by making use of the space-time estimates of solutions to the linear Schrödinger equation. We first give the basic results about (1.6) in section 2 (see Propositions 2.1 - 2.3). Secondly we prove the existence of solutions to (1.6) with constraint (1.7) in Propositions 2.4 - 2.5. In Proposition 2.4, we prove the unique existence of $H^2 \times H^2$ solutions to (1.6) and the invariance of the constraint $v = \partial u + i(\lambda/2)|u|^2 u$. The proof of the invariance requires the gauge condition of F. In Proposition 2.5, we prove that the unique existence of solutions to (1.6) still holds for the data with minimal regularity assumption $(u_0, v_0) \in H^1 \times L^2$ under constraint (1.7) which is to be invariant under the time evolution. The proof of Proposition 2.5 requires the smallness condition of u_0 in the L^2-norm.

In Theorem 2 we show that Propositon 2.5 implies the existence of a unique local solution to (1.1) in H^1 through the relation

$$u_0 = \exp(-i\lambda \int_{-\infty}^{x} |\phi| dy)\phi.$$

Finally we prove Theorem 1 by a priori estimates of local solutions to (1.1).

In [5], we studied the nonlinear Schrödinger equations of the form

$$\begin{cases} i\partial_t u + \partial^2 u = F(u, \partial u, \bar{u}, \partial \bar{u}), \\ u(0) = u_0, \end{cases} \tag{1.8}$$

where $F : \mathbb{C}^4 \to \mathbb{C}$ is a polynomial having neither constant nor linear terms. We showed the existence of a unique local solution of (1.8) when the data satisfy conditions such as $u_0 \in H^3$ and $xu_0 \in H^2$. The mixed nonlinear Schrödinger equations

$$i\partial_t u + \partial^2 u = i\beta u^2 \partial \bar{u} + i\gamma |u|^2 \partial u + g(|u|^2)u$$

was studied by [1] by energy methods, where β, $\gamma \in \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ is a function satisfying some regularity conditions. They established existence and uniqueness of global smooth solutions in H^s for $s \geqq 3$ under some smallness condition on the data.

In the case where the underlying space is a bounded domain $\Omega = (0, \ell)$ with $\ell > 0$, the global existence of $H_0^1(\Omega)$ solutions of the equation

$$i\partial_t u + \partial^2 u = i\lambda \partial(|u|^2 u) + \alpha |u|^\rho u, \quad t > 0, \ x \in \Omega,$$

with Dirichlet zero condition was shown in [2] under the smallness condition of the $H_0^1(\Omega)$ norm of the data, where $\lambda, \alpha \in \mathbb{R}$, and $\rho \geqq 2$.

We conclude this section by giving notations. We abbreviate $\partial/\partial u$ to ∂_u. By $\bar{u}$ we denote the complex conjugate of u. We let $L^p = \{f; f$ is measurable on $\mathbb{R}, ||f||_p < \infty\}$, where $||f||_p^p = \int_{\mathbb{R}} |f(x)|^p dx$ if $1 \leqq p < \infty$ and $||f||_\infty = \text{ess.sup}\{|f(x)|; x \in \mathbb{R}\}$ if $p = \infty$, and we let $W^{m,p} = \{f \in L^p; ||f||_{W^{m,p}} = \sum_{j=0}^{m} ||\partial^j f||_p\}$. For simplicity we put $W^{m,2} = H^m$. We denote by $(\cdot,\cdot)$ the inner product in L^2. For any interval I of $\mathbb{R}$ and a Banach space B with norm $||\cdot||_B$, we let $C(I; B)$ be the space of continuous functions from I to B and $L^p(I; B)$ the space consisting of strongly measurable B valued functions $u(\cdot)$ defined on I such that $\int_I ||u(t)||_B^p dt < \infty$. Different positive constants will be denoted by the same letter C. If necessary, by $C(*, \cdots, *)$ we denote constants depending on the quantities appearing in parentheses.

§2. Proof of Theorem 1.

We consider the system of nonlinear Schrödinger equations of the form:

$$
\begin{cases} Lu = -i\lambda u^2 \bar{v} + F(u), \\ Lv = i\lambda v^2 \bar{u} + \partial_u F(u) \cdot v + \partial_{\bar{u}} F(u) \bar{v}, \\ u(0) = u_0, \ v(0) = v_0, \end{cases} \tag{2.1}
$$

where $L = i\partial_t + \partial_x^2$. In what follows we assume that (1.2) and (1.3) are satisfied. For simplicity we restrict our attention to positive times since the problem is treated analogously for nagative times.

To obtain Theorem 2 stated below we need the following Propositions.

Proposition 2.1. *We assume that $u_0 \in L^2$ and $v_0 \in L^2$. Then there exist unique solutions u, v of (2.1) and a positive constant T such that*

$$u, v \in C([0, T]; L^2) \cap L^4(0, T; L^\infty).$$

For the proof, see [8, Theorem IV].

Proposition 2.2. *We assume that $u_0 \in H^2$ and $v_0 \in H^2$. Then there exist unique solutions u, v of (2.1) such that*

$$u, v \in C(0, T]; H^2) \cap L^4(0, T; W^{2,\infty})$$

for the same T as that given in Proposition 2.1.

For the proof, see [8, Theorem V]

Propositon 2.3. *We assume that $u_0^{(n)}, v_0^{(n)} \in H^2$, $u_0, v_0 \in L^2$ and $||u_0^{(n)} - u_0||_2 + ||v_0^{(n)} - v_0||_2 \to 0$ as $n \to \infty$. We let $u^{(n)}$ and $v^{(n)}$ be the solutions constructed in Proposition 2.2 with data $u_0^{(n)}$ and $v_0^{(n)}$, respectively, and we let u and v be the solutions constructed in Proposition 2.1 with data u_0 and v_0, respectively, where T is the same one as that given in Proposition 2.1. Then we have*

$$\begin{aligned}&\sup_{0\leqq t\leqq T} ||u^{(n)}(t) - u(t)||_2 + \sup_{0\leqq t\leqq T} ||v^{(n)}(t) - v(t)||_2 \\ &+ (\int_0^T ||u^{(n)} - u(t)||_\infty^4 dt)^{1/4} + (\int_0^T ||v^{(n)}(t) - v(t)||_\infty^4 dt)^{1/4} \\ &\to 0 \quad \text{as} \quad n \to \infty.\end{aligned}$$

For the proof, see [8, Theorem I′]

By using Proposition 2.2 we now prove

Proposition 2.4. *We assume that $u_0 \in H^2$, $v_0 \in H^2$ satisfty the condition*

$$v_0 = \partial u_0 + i\frac{\lambda}{2}|u_0|^2 u_0.$$

Then there exist unique solutions u, v of (2.1) such that

$$u, v \in C([0,T]; H^2) \cap L^4(0,T; W^{2,\infty})$$

and

$$v = \partial u + i\frac{\lambda}{2}|u|^2 u \quad \text{in} \quad C([0,T]; L^2)$$

for the same time T as that given in Propositon 2.1.

Proof. By Proposition 2.2 it is sufficient to prove that

$$\sup_{0\leqq t\leqq T} ||v(t) - (\partial u(t) + i\frac{\lambda}{2}|u|^2 u(t))||_2 = 0. \tag{2.3}$$

A direct calculation gives

$$L(\partial u) = \partial L u = -2i\lambda u\bar{v}\partial u - i\lambda u^2 \partial\bar{v} + \partial_u F(u)\cdot\partial u + \partial_{\bar{u}} F(u)\cdot\partial\bar{u} \tag{2.4}$$

$$\begin{aligned} L(|u|^2 u) &= 2|u|^2 Lu + 2(\partial u)^2\bar{u} + 4u|\partial u|^2 + u^2(-\overline{Lu} + 2\partial^2\bar{u}) \\ &= 2|u|^2(-i\lambda u^2\bar{v} + F(u)) + 2(\partial u)^2\bar{u} + 4u|\partial u|^2 \\ &\quad + 2u^2\partial^2\bar{u} - u^2(i\lambda\bar{u}^2 v + \overline{F(u)}) \qquad (\text{by } (2.1)) \\ &= -2i\lambda|u|^2u^2\bar{v} + 2(\partial u)^2\bar{u} + 4u|\partial u|^2 + 2u^2\partial^2\bar{u} \\ &\quad - i\lambda|u|^4 v + (2|u|^2F(u) - u^2\overline{F(u)}). \end{aligned} \tag{2.5}$$

We put $w = \partial u + \frac{i\lambda}{2}|u|^2 u$, then we have by (2.4) and (2.5)

$$\begin{aligned} Lw &= -2i\lambda u\bar{v}\partial u - i\lambda u^2\partial\bar{v} + \lambda^2|u|^2u^2\bar{v} + i\lambda(\partial u)^2\bar{u} \\ &\quad + 2i\lambda u|\partial u|^2 + i\lambda u^2\partial^2\bar{u} + \frac{\lambda^2}{2}|u|^4 v \\ &\quad + [\frac{i}{2}\lambda(2|u|^2F - u^2\bar{F}) + \partial_u F\cdot\partial u + \partial_{\bar{u}}F\cdot\partial\bar{u}]. \end{aligned}$$

Hence by the second identity of (2.1) we get

$$\begin{aligned} L(w - v) &= -2i\lambda u\bar{v}\partial u - i\lambda u^2\partial\bar{v} + \lambda^2|u|^2u^2\bar{v} + i\lambda(\partial u)^2\bar{u} \\ &\quad + 2i\lambda u|\partial u|^2 + i\lambda u^2\partial^2\bar{u} + \frac{\lambda^2}{2}|u|^4 v - i\lambda v^2\bar{u} \\ &\quad + [\frac{i}{2}\lambda(2|u|^2F - u^2\bar{F}) + \partial_u F(\partial u - v) + \partial_{\bar{u}}F(\partial\bar{u} - \bar{v})]. \end{aligned} \tag{2.6}$$

We denote the j-th term of the right hand side of (2.6) by I_j. By using the definition $w = \partial u + \frac{i\lambda}{2}|u|^2 u$ we obtain

$$\begin{aligned} I_1 &= -2i\lambda u\bar{v}\partial u \\ &= 2i\lambda u(\bar{w} - \bar{v})\partial u - 2i\lambda u\overline{(\partial u + \frac{i\lambda}{2}|u|^2 u)}\partial u \\ &= -2i\lambda u(\bar{w} - \bar{v})\partial u - I_5 - \lambda^2|u|^4\partial u. \end{aligned}$$

This implies

$$I_1 + I_5 = 2i\lambda u(\bar{w} - \bar{v})\partial u - \lambda^2|u|^4\partial u. \tag{2.7}$$

We have

$$
\begin{aligned}
(2.8)\qquad I_2 &= -i\lambda u^2\partial\bar v\\
&= i\lambda u^2(\partial\bar w - \partial\bar v) - i\lambda u^2\partial\overline{(\partial u + \frac{i\lambda}{2}|u|^2u)}\\
&= i\lambda u^2\partial(\bar w - \bar v) - I_6 - \frac{\lambda^2}{2}u^2\partial(|u|^2\bar u).
\end{aligned}
$$

Since

$$
\begin{aligned}
-\frac{\lambda^2}{2}u^2\partial(|u|^2\bar u) &= -\frac{\lambda^2}{2}|u|^4\partial u - \lambda^2|u|^2u^2\partial\bar u\\
&= -\frac{\lambda^2}{2}|u|^4(w-v) + \frac{\lambda^2}{2}|u|^4(\frac{i\lambda}{2}|u|^2u - v) - \lambda^2|u|^2u^2\partial\bar u\\
&= -\frac{\lambda^2}{2}|u|^2(w-v) + \frac{i\lambda^3}{4}|u|^6u - \lambda^2|u|^2u^2\partial\bar u - I_7,
\end{aligned}
$$

we obtain from (2.8)

$$
(2.9)\qquad I_2 + I_6 + I_7 = i\lambda u^2\partial(\bar w - \bar u) - \frac{\lambda^2}{2}|u|^2(w-v) + \frac{i\lambda^3}{4}|u|^6u - \lambda^2|u|^2u^2\partial\bar u.
$$

We next consider the contribution of the terms I_3, I_4, I_8, of the last term of the right hand side of (2.7), and of the last two terms of the right hand side of (2.9). We have

(2.10)

$$
\begin{aligned}
&\lambda^2|u|^2u^2\bar v + i\lambda(\partial u)^2\bar u - i\lambda v^2\bar u - \lambda^2|u|^4\partial u + \frac{i\lambda^3}{4}|u|^6u - \lambda^2|u|^2u^2\partial\bar u\\
&= i\lambda((\partial u)^2 + i\lambda|u|^2u\partial u - v^2)\bar u - \lambda^2|u|^2u^2(\partial\bar u - \bar v) + \frac{i\lambda^3}{4}|u|^6u\\
&= i\lambda((\partial u + \frac{i\lambda}{2}|u|^2u)^2 - v^2)\bar u - \lambda^2|u|^2u^2(\overline{\partial u + \frac{i\lambda}{2}|u|^2u} - \bar v)\\
&= i\lambda(w^2 - v^2)\bar u - \lambda^2|u|^2u^2(\bar w - \bar v).
\end{aligned}
$$

By the gauge condition $F(e^{i\theta}\psi) = e^{i\theta}F(\psi)$, $\theta\in\mathbb{R}$, we see that $F(|\psi|) = F(\frac{|\psi|}{\psi}\psi) = \frac{|\psi|}{\psi}F(\psi)$. Hence $F(\psi)$ is written as

$$
(2.11)\qquad F(\psi) = G(|\psi|^2)\psi
$$

if we put $G(s^2) = \begin{cases} F(s)/s & \text{for}\quad s>0,\\ 0 & \text{for}\quad s=0.\end{cases}$

By using (2.11) the last term of the right hand side of (2.6) I_9 is rewritten as

$$
\begin{aligned}
I_9 = \frac{i}{2}\lambda(&2G(|u|^2)|u|^2u - \bar G(|u|^2)|u|^2u)\\
&+ (G'(|u|^2)u\cdot\bar u + G(|u|^2))(\partial u - v) + G'(|u|^2)u\cdot u(\partial\bar u - \bar v),
\end{aligned}
$$

where $G'(|u|^2) = \partial_{|u|^2} G(|u|^2)$.

From the condition that there exists a function $H \in C^1(\mathbb{R}^2;\mathbb{R})$ satisfying $F(\zeta) = \partial H/\partial\bar{\zeta}$, it follows that $G = \bar{G}$. Hence

$$
\begin{aligned}
(2.12)\qquad I_9 &= G'(|u|^2)|u|^2(\partial u + \frac{i}{2}\lambda|u|^2u - v) \\
&\quad + G(|u|^2)(\partial u + \frac{i}{2}\lambda|u|^2u - v) + G'(|u|^2)u^2\overline{(\partial u + \frac{i}{2}\lambda|u|^2u} - \bar{v}) \\
&= G'(|u|^2)|u|^2(w - v) + G(|u|^2)(w - v) + G'(|u|^2)u^2(\bar{w} - \bar{v}) \\
&= \partial_u F(w - v) + \partial_{\bar{u}} F(\bar{w} - \bar{v}).
\end{aligned}
$$

By (2.6), (2.7), (2.9), (2.10) and (2.12)

$$
\begin{aligned}
(2.13)\quad L(w - v) &= \sum_{j=1}^{9} I_j \\
&= 2i\lambda u\partial u(\bar{w} - \bar{v}) + i\lambda u^2\partial(\bar{w} - \bar{v}) - \frac{\lambda^2}{2}|u|^2(w - v) + i\lambda(w^2 - v^2)\bar{u} \\
&\quad - \lambda^2|u|^2u^2(\bar{w} - \bar{v}) + \partial_u F(w - v) + \partial_{\bar{u}} F(\bar{w} - \bar{v}).
\end{aligned}
$$

Multiplying both sides of (2.13) by $\bar{w} - \bar{v}$, taking the imaginary part, and integrating over space with integration by parts for the second term of the right hand side of (2.13), we obtain

$$
\begin{aligned}
\frac{d}{dt}||w - v||_2^2 \leqq{}& (2|\lambda|\ ||u||_\infty||\partial u||_\infty + 2|\lambda|\ ||w + v||_\infty||u||_\infty \\
&+ 2\lambda^2||u||_\infty^4 + ||\partial_u F||_\infty + ||\partial_{\bar{u}} F||_\infty)||w - v||_\infty
\end{aligned}
$$

from which we have

$$
\begin{aligned}
||w - v||_2^2 \leqq{}& ||w(0) - v(0)||_2^2 \\
&\times \exp(C\int_0^t (||u||_\infty||\partial u||_\infty + ||u||_\infty||v||_\infty + ||u||_\infty^4 + ||u||_\infty^{p_1-1} + ||u||_\infty^{p_2-1})ds)
\end{aligned}
$$

This implies the desired identity (2.3). Therefore we have the proposition.

We next prove

Proposition 2.5. *We assume that $u_0 \in H^1$, $v_0 \in L^2$ satisfy the conditions*

$$||u_0||_2^2 < 2\pi/|\lambda|$$

and

$$v_0 = \partial u_0 + i\frac{\lambda}{2}|u_0|^2 u_0.$$

Then there exist unique solutions u, v of (2.1) such that

$$\begin{cases} u \in C([0,T]; H^1) \cap L^4(0,T; W^{1,\infty}) \\ v \in C([0,T]; L^2) \cap L^4(0,T; L^\infty) \\ v = \partial u + i\frac{\lambda}{2}|u|^2 u \quad in \quad C([0,T]; L^2) \end{cases} \tag{2.14}$$

for the same time T as that given in Proposition 2.1. Moreover, the map $u_0 \mapsto u$ is continuous from $\{u_0 \in H^1; ||u_0||_2^2 < 2\pi/|\lambda|\}$ with topology induced from H^1 to $C([0,T]; H^1) \cap L^4(0,T; W^{1,\infty})$.

Proof. We let $u_0^{(n)} \in H^2$, $v_0^{(n)} \in H^2$ satisfy $||u_0^{(n)}||_2 = ||u_0||_2$ for any $n \in \mathbb{N}$ and $||u_0^{(n)} - u_0||_2 + ||v_0^{(n)} - v_0||_2 \to 0$ as $n \to \infty$. Then by Proposition 2.4 we see that there exist unique solutions $u^{(n)}$, $v^{(n)}$ of (2.1) with the data $u^{(n)}(0) = u_0^{(n)}$, $v^{(n)}(0) = v_0^{(n)}$ satisfying

$$\begin{cases} u^{(n)}, v^{(n)} \in C([0,T]; H^2) \cap L^4(0,T; W^{2,\infty}) \\ v^{(n)} = \partial u^{(n)} + i\frac{\lambda}{2}|u^{(n)}|^2 u^{(n)} \quad \text{in} \quad C([0,T]; L^2) \end{cases} \tag{2.15}$$

From Proposition 2.3 it follows that

$$\lim_{n\to\infty} u^{(n)} = u \text{ and } \lim_{n\to\infty} v^{(n)} = v \text{ strongly in } C([0,T]; L^2) \cap L^4(0,T; L^\infty) \tag{2.16}$$

where u, v are the unique solutions to (2.1) with the data $u(0) = u_0$, $v(0) = v_0$. Hence Proposition 2.5 is obtained by showing

$$\begin{cases} u \in C([0,T]; H^1) \cap L^4(0,T; W^{1,\infty}) \\ v = \partial u + i\frac{\lambda}{2}|u|^2 u \quad \text{in} \quad C([0,T]; L^2). \end{cases} \tag{2.17}$$

We now prove (2.17).

By the second line of (2.15) and the first identity of (2.1) we have

$$Lu^{(n)} = -i\lambda(u^{(n)})^2 \partial \bar{u}^{(n)} - \frac{\lambda^2}{2}|u^{(n)}|^4 u^{(n)} + F(u^{(n)}). \tag{2.18}$$

Multiplying both sides of (2.18) by $\bar{u}^{(n)}$, taking the imaginary part, and integrating in x, we obtain

$$||u^{(n)}(t)||_2 = ||u_0^{(n)}||_2 = ||u_0||_2, \tag{2.19}$$

where we have used the condition (1.3).

We put

$$\psi_j^{(n)} = G_n^{-(6-j)/4} u^{(n)}, \quad j = 2, 3, \cdots, 6$$

with

$$G_n = \exp(-i\frac{\lambda}{2}\int_{-\infty}^{x} |u^{(n)}|^2 dy),$$

then it is clear that

$$\psi_j^{(n)} = G_n^{-1/4}\psi_{j+1}^{(n)}.$$

Hence

$$\partial\psi_j^{(n)} = G_n^{-1/4}(i\frac{\lambda}{8}|u^{(n)}|^2\psi_{j+1}^{(n)} + \partial\psi_{j+1}^{(n)}), \tag{2.20}$$

from which it follows that

$$\begin{aligned} ||\partial\psi_j^{(n)}||_2^2 &= ||\partial\psi_{j+1}^{(n)}||_2^2 + \frac{\lambda}{4}\operatorname{Im}(|u^{(n)}|^2\psi_{j+1}^{(n)}, \partial\psi_{j+1}^{(n)}) \\ &\quad + \frac{\lambda^2}{4}||\,|u^{(n)}|^2\psi_{j+1}^{(n)}||_2^2 \\ &\geqq ||\partial\psi_{j+1}^{(n)}||_2^2 - \frac{|\lambda|}{4}||\psi_{j+1}^{(n)}||_6^3||\partial\psi_{j+1}^{(n)}||_2, \end{aligned} \tag{2.21}$$

since

$$|u^{(n)}| = |\psi_j^{(n)}| = |\psi_{j+1}^{(n)}|. \tag{2.22}$$

We apply the Gagliardo-Nirenberg inequality of the form

$$||f||_6^6 \leqq \frac{4}{\pi^2}||\partial f||_2^2||f||_2^4 \tag{2.23}$$

to (2.21) to obtain

$$||\partial\psi_j^{(n)}||_2^2 \geqq (1 - \frac{|\lambda|}{2\pi}||\psi_{j+1}^{(n)}||_2^2)||\partial\psi_{j+1}^{(n)}||_2^2.$$

We use (2.19) and (2.21) to see that

$$||\partial\psi_j^{(n)}||_2^2 \geqq (1 - \frac{|\lambda|}{2\pi}||u_0||_2^2)||\partial\psi_{j+1}^{(n)}||_2^2$$

from which and the condition $||u_0||_2^2 < 2\pi/|\lambda|$ it follows that

$$||\partial\psi_{j+1}^{(n)}||_2^2 \leqq (1 - \frac{|\lambda|}{2\pi}||u_0||_2^2)^{-1}||\partial\psi_j^{(n)}||_2^2. \tag{2.24}$$

The definition of $\psi_j^{(n)}$ implies

$$\partial\psi_2^{(n)} = G_n^{-1}v^{(n)} \quad \text{and} \quad \psi_6^{(n)} = u^{(n)}. \tag{2.25}$$

By (2.24) and (2.25)

$$||\partial\psi_{j+1}^{(n)}||_2 \leqq (1 - \frac{|\lambda|}{2\pi}||u_0||_2^2)^{-(j-1)/2}||v^{(n)}||_2 \tag{2.26}$$

for $j = 1, 2, \cdots, 5$.

We now prove $\{u^{(n)}\}$ is a Cauchy sequence in $C([0,T]; H^1)$. By using (2.20) and (2.22) we write

$$\begin{aligned}\partial\psi_j^{(n)} - \partial\psi_j^{(m)} = G_n^{-1/4}(i\frac{\lambda}{8}(|\psi_{j+1}^{(n)}|^2\psi_{j+1}^{(n)} - |\psi_{j+1}^{(m)}|^2\psi_{j+1}^{(m)}) + \partial\psi_{j+1}^{(n)} - \partial\psi_{j+1}^{(m)})\\ + (G_n^{-1/4} - G_m^{-1/4})(i\frac{\lambda}{8}|\psi_{j+1}^{(m)}|^2\psi_{j+1}^{(m)} + \partial\psi_{j+1}^{(m)}).\end{aligned}$$

From the mean value theorem and the Schwarz inequality we have

$$\begin{aligned}&||\partial\psi_{j+1}^{(n)} - \partial\psi_{j+1}^{(m)}||_2\\ &\leqq ||\partial\psi_j^{(n)} - \partial\psi_j^{(m)}||_2\\ &\quad + C(||\psi_{j+1}^{(n)}||_\infty^2 + ||\psi_{j+1}^{(m)}||_\infty^2)||\psi_{j+1}^{(n)} - \psi_{j+1}^{(m)}||_2\\ &\quad + C(||\psi_{j+1}^{(m)}||_6^3 + ||\partial\psi_{j+1}^{(m)}||_2)(||u^{(m)}||_2 + ||u^{(m)}||_2)||u^{(n)} - u^{(m)}||_2.\end{aligned} \tag{2.27}$$

We apply the Gagliardo-Nirenberg inequality, (2.26) and (2.22) to (2.27) to obtain

$$\begin{aligned}&||\partial\psi_{j+1}^{(n)} - \partial\psi_{j+1}^{(m)}||_2\\ &\leqq ||\partial\psi_j^{(n)} - \partial\psi_j^{(m)}||_2\\ &\quad + C(||v^{(n)}||_2||u^{(n)}||_2 + ||v^{(m)}||_2||u^{(m)}||_2)||\psi_{j+1}^{(n)} - \psi_{j+1}^{(m)}||_2\\ &\quad + C(||v^{(m)}||_2||u^{(m)}||_2^2 + ||v^{(m)}||_2)(||u^{(n)}||_2 + ||u^{(m)}||_2)||u^{(n)} - u^{(m)}||_2.\end{aligned} \tag{2.28}$$

On the other hand we have from

$$\psi_j^{(n)} - \psi_j^{(m)} = G_n^{-(6-j)/4}(u^{(n)} - u^{(m)}) + (G_n^{-(6-j)/4} - G_m^{-(6-j)/4})u^{(m)}$$

$$(2.29)\quad ||\psi_j^{(n)} - \psi_j^{(m)}||_2 \leqq ||u^{(n)} - u^{(m)}||_2 + C||u^{(m)}||_2(||u^{(n)}||_2 + ||u^{(m)}||_2)||u^{(n)} - u^{(m)}||_2$$

for $j = 2, 3, \cdots, 6$.

Hence (2.26), (2.28) and (2.29) imply that there exists a positive constant C which does not depend on n, m, j and satisfies

$$||\partial\psi_{j+1}^{(n)} - \partial\psi_{j+1}^{(m)}||_2 \leqq ||\partial\psi_j^{(n)} - \partial\psi_j^{(m)}||_2 + C||u^{(n)} - u^{(m)}||_2$$

for $j = 2, 3, \cdots, 5$,

from which and (2.25) it follows that

$$(2.30)\qquad ||\partial u^{(n)} - \partial u^{(m)}||_2 \leqq ||\partial\psi_2^{(n)} - \partial\psi_2^{(m)}||_2 + C||u^{(n)} - u^{(m)}||_2.$$

In the same way as in the proof of (2.29) we have

$$(2.31)\quad ||\partial\psi_2^{(n)} - \partial\psi_2^{(m)}||_2 \leqq ||v^{(n)} - v^{(m)}||_2 + C||v^{(n)}||_2(||u^{(n)}||_2 + ||u^{(m)}||_2)||u^{(n)} - u^{(m)}||_2.$$

Therefore we obtain by (2.30) and (2.31)

$$(2.32)\qquad ||\partial u^{(n)} - \partial u^{(m)}||_2 \leqq ||v^{(n)} - v^{(m)}||_2 + C||u^{(n)} - u^{(m)}||_2.$$

This and (2.16) imply $\{u^{(n)}\}$ is a Cauchy sequence in $C([0,T]; H^1)$. Hence,

$$(2.33)\qquad \lim_{n\to\infty} u^{(n)} = u \quad \text{strongly in} \quad C([0,T]; H^1).$$

From (2,33) and the Gagliardo-Nirenberg inequality it is clear that

$$(2.34)\qquad \lim_{n\to\infty} |u^{(n)}|^2u^{(n)} = |u|^2u \quad \text{strongly in} \quad C([0,T]; L^2).$$

By (2.16), (2.33) and (2.34) we have the second line of (2.17). The proof of (2.17) is completed by showing

$$(2.35)\qquad \partial u \in L^4(0,T; L^\infty).$$

Since $u \in C([0,T]; H^1) \cap L^4(0,T; L^\infty)$, the Gagliardo-Nirenberg inequality gives $|u|^2u \in L^4(0,T; L^\infty)$. Hence the second line of (2.17) and (2.16) yield (2.35). This completes the proof of Proposition 2.5.

We prove local existence of solutions to the original equation (1.1).

Theorem 2. *We assume that $\phi \in H^1(\mathbb{R})$ and $||\phi||_2^2 < 2\pi/|\lambda|$. Then there exist a unique solution ψ of (1.1) and a positive constant T such that*

$$\psi \in C([0,T]; H^1) \cap L^4(0,T; W^{1,\infty}).$$

Moreover, the map $\phi \mapsto \psi$ is continuous from $\{\phi \in H^1; ||\phi||_2^2 < 2\pi/|\lambda|\}$ with topology induced from H^1 to $C([0,T]; H^1) \cap L^4(0,T; W^{1,\infty})$.

Proof. For any $\phi \in H^1(\mathbb{R})$ we define u_0 and v_0 as follows:

$$u_0 = \exp(-i\lambda \int_{-\infty}^{x} |\phi|^2 dy)\phi,$$
$$v_0 = \partial u_0 + \frac{i}{2}\lambda |u_0|^2 u_0.$$

Then it is clear that $u_0 \in H^1$, $v_0 \in L^2$ and $||u_0||_2^2 = ||\phi||_2^2 < 2\pi/|\lambda|$. Hence by Proposition 2.5 there exist unique solutions u, v of (2.1) satisfying (2.14). We define ψ by

$$\psi(t,x) = \exp(i\lambda \int_{-\infty}^{x} |u(t,y)|^2 dy) u(t,x). \tag{2.36}$$

Theorem 2 is established if we prove that ψ satisfies (1.1). By (2.14) and the first equation of (2.1) for u we obtain

$$Lu = -i\lambda u^2 \partial \bar{u} - \frac{\lambda^2}{2}|u|^4 u + F(u). \tag{2.37}$$

A direct calculation and (2.36) give

$$L\psi = \exp(i\lambda \int_{-\infty}^{x} |u|^2 dy)(Lu + i\lambda(L\int_{-\infty}^{x} |u|^2 dy)u + 2i\lambda |u|^2 \partial u - \lambda^2 |u|^4 u). \tag{2.38}$$

By (2.37)

$$\begin{aligned} \partial_t |u|^2 &= 2\,\mathrm{Im}(\bar{u}(-\partial^2 u - i\lambda u^2 \partial \bar{u})) \\ &= -\partial(2\,\mathrm{Im}(\bar{u}\partial u)) - 2\lambda |u|^2 \,\mathrm{Re}(u \partial \bar{u}) \\ &= -\partial(2\,\mathrm{Im}(\bar{u}\partial u) + \frac{\lambda}{2}|u|^4), \end{aligned} \tag{2.39}$$

$$\begin{aligned} i\lambda L \int_{-\infty}^{x} |u|^2 dy &= -\lambda \int_{-\infty}^{x} \partial_t |u|^2 dy + i\lambda \partial |u|^2 \\ &= 2\lambda\,\mathrm{Im}(\bar{u}\partial u) + \frac{\lambda^2}{2}|u|^4 + i\lambda\partial|u|^2 \quad \text{(by (2.37))} \\ &= -i\lambda(\bar{u}\partial u - u\partial\bar{u}) + \frac{\lambda^2}{2}|u|^4 + i\lambda(\bar{u}\partial u + u\partial\bar{u}) \\ &= 2i\lambda u \partial \bar{u} + \frac{\lambda^2}{2}|u|^4. \end{aligned} \tag{2.40}$$

From (2.37)-(2.40) we have

$$\begin{aligned}(2.41)\ L\psi &= \exp(i\lambda\int_{-\infty}^{x}|u|^2dy)(-i\lambda u^2\partial\bar{u} - \frac{\lambda^2}{2}|u|^4u + F(u)\\ &\qquad + 2i\lambda u^2\partial\bar{u} + \frac{\lambda^2}{2}|u|^4u + 2i\lambda|u|^2\partial u - \lambda^2|u|^4u)\\ &= \exp(i\lambda\int_{-\infty}^{x}|u|^2dy)(i\lambda u^2\partial\bar{u} + 2i\lambda|u|^2\partial u - \lambda^2|u|^4u + F(u))\\ &= \exp(i\lambda\int_{-\infty}^{x}|u|^2dy)(2i\lambda|u|^2(\partial u + i\lambda|u|^2) + i\lambda u^2(\partial\bar{u} - i\lambda|u|^2\bar{u}) + F(u)).\end{aligned}$$

Since

$$\partial\psi = \exp(i\lambda\int_{-\infty}^{x}|u|^2dy)(\partial u + i\lambda|u|^2u),$$

$$|u|^2 = |\psi|^2 \quad \text{and} \quad u^2 = \psi^2\exp(-2i\lambda\int_{-\infty}^{x}|u|^2dy),$$

we have by (2.41)

$$L\psi = 2i\lambda|\psi|^2\partial\psi + i\lambda\psi^2\partial\bar{\psi} + \exp(i\lambda\int_{-\infty}^{x}|u|^2dy)F(u)$$

Gauge condition of F implies the desired identity

$$L\psi = i\lambda\partial(|\psi|^2\psi) + F(\psi).$$

The last equation makes sense in $C([0,T];H^{-1})$ and all the computations above are justified in $C([0,T];H^{-2})$, or we could go though with the computations by using approximate solutions $u^{(n)}$ as in the proof of Proposition 2.5 before limiting procedure at the final stage. This completes the proof of Theorem 2.

We are now in a position to prove Theorem 1.

Proof of Theorem 1. The iterative use of the same argument as that in the proof of Theorem 2 and a priori estimates of solution ψ of (1.1) in H^1 yield Theorem 1. Then it is sufficient to prove

$$\sup_{t\in[0,T]}||\psi(t)||_{H^1} \leqq C(||\phi||_{H^1}). \tag{2.42}$$

We prove (2.42). After a long tedious calculation (see Appendix) we arrive at

$$\frac{d}{dt}(||\partial\psi||_2^2 + \frac{3}{2}\lambda\,\mathrm{Im}(|\psi|^2\psi, \partial\psi) + \frac{1}{2}\lambda^2||\psi||_6^6 + \int H(\psi)dx) = 0$$

from which and (1.3) we have

$$(2.43)\qquad \begin{aligned} ||\psi(t)||_2 &= ||\phi||_2 \\ E \equiv ||\partial\psi||_2^2 &+ \frac{3}{2}\lambda\,\mathrm{Im}(|\psi|^2\psi, \partial\psi) + \frac{1}{2}\lambda^2||\psi||_6^6 + \int H(\psi)dx \\ &= ||\partial\phi||_2^2 + \frac{3}{2}\lambda\,\mathrm{Im}(|\phi|^2\phi, \partial\phi) + \frac{1}{2}\lambda^2||\phi||_6^6 + \int H(\phi)dx. \end{aligned}$$

By the Gagliardo-Nirenberg inequality and (1.3) we get

$$(2.44)\qquad E \leqq C(||\phi||_{H^1}).$$

We put

$$G = \exp(-i\frac{\lambda}{2}\int_{-\infty}^{x} |\psi(t,y)|^2 dy).$$

Then

$$\begin{aligned} ||\partial\psi||_2^2 + \frac{3}{2}\lambda\,\mathrm{Im}(|\psi|^2\psi, \partial\psi) &= ||\partial\psi||_2^2 - \frac{3}{2}\lambda\,\mathrm{Re}(i|\psi|^2\psi, \partial\psi) \\ &= ||\partial\psi - \frac{3}{4}i\lambda|\psi|^2\psi||_2^2 - \frac{9}{16}\lambda^2||\psi||_6^6 \\ &= ||\partial(G^{3/2}\psi)||_2^2 - \frac{9}{16}\lambda^2||\psi||_6^6. \end{aligned}$$

Hence

$$E = ||\partial(G^{3/2}\psi)||_2^2 - \frac{1}{16}\lambda^2||\psi||_6^6 + \int H(\psi)dx.$$

By (1.3) we see that

$$E \geqq ||\partial(G^{3/2}\psi)||_2^2 - M(||\psi||_2) - (\frac{\lambda^2}{16} + \mu)||G^{3/2}\psi||_6^6.$$

We apply (2.23) and (2.43) to the above to obtain

$$E + M(||\phi||_2) \geqq (1 - \frac{4}{\pi^2}(\frac{\lambda^2}{16} + \mu)||\phi||_2^4)||\partial(G^{3/2}\psi)||_2^2.$$

From this and (2.44) it follows that

$$||\partial(G^{3/2}\psi)||_2^2 \leqq C(||\phi||_{H^1}). \tag{2.45}$$

We let $\psi_j = G^{(6-j)/4}\psi$ for $0 \leq j \leq 6$. We have by (2.23) and (2.43)

$$\begin{aligned} ||\partial\psi_j||_2^2 &= ||\partial(G^{1/4}\psi_{j+1})||_2^2 \\ &= ||G^{1/4}(\partial\psi_{j+1} - \frac{1}{8}i\lambda|\psi|^2\psi_{j+1})||_2^2 \\ &= ||\partial\psi_{j+1}||_2^2 - \frac{1}{4}\lambda\,\mathrm{Im}(\partial\psi_{j+1}, |\psi|^2\psi_{j+1}) + \frac{\lambda^2}{64}||\psi||_6^6 \\ &\geqq (1 - \frac{|\lambda|}{2\pi}||\phi||_2^2)||\partial\psi_{j+1}||_2^2. \end{aligned}$$

Therefore

$$||\partial\psi_0||_2^2 \geqq (1 - \frac{|\lambda|}{2\pi}||\phi||_2^2)^6||\partial\psi_6||_2^2. \tag{2.46}$$

Since $\psi = \psi_6$, $G^{3/2}\psi = \psi_0$, by (2.45) and (2.46) we have the desired estimate (2.42). This completes the proof of Theorem 1.

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Appendix.

Lemma A. *Let ψ be the solution of (1.1) constructed in Theorem 2. Then we have*

(a) $$\frac{d}{dt}||\psi||_2^2 = 0,$$

(b) $$\frac{d}{dt}(||\partial\psi||_2^2 + \frac{3}{2}\lambda\,\mathrm{Im}(|\psi|^2\psi, \partial\psi) + \frac{1}{2}\lambda^2||\psi||_6^6 + \int H(\psi)dx) = 0.$$

Proof. We first note that computation stated below is rather formal, but it can be justified by using H^2-solutions ψ_k with the data $\phi_k \in H^2$ which satisfy the following continuous dependence such that

$$\lim_{k\to\infty} ||\phi_k - \phi||_{H^1} = 0 \quad \text{implies} \quad \lim_{k\to\infty} ||\psi_k(t) - \psi(t)||_{H^1} = 0,$$

(see Proposition 2.5). In what follows we use the integration by parts without particular comments. We have by (1.1)

$$\text{(A.1)} \qquad \begin{aligned} \partial_t|\psi|^2 &= 2\,\mathrm{Im}(\bar{\psi}i\partial_t\psi) \\ &= 2\,\mathrm{Im}(\bar{\psi}(-\partial^2\psi + i\lambda\partial(|\psi|^2\psi) + F(\psi))) \\ &= -\partial(2\,\mathrm{Im}(\bar{\psi}\partial\psi)) + 2\lambda\,\mathrm{Re}(\bar{\psi}\partial(|\psi|^2\psi)). \end{aligned}$$

Since $\mathrm{Re}(\bar{\psi}\partial(|\psi|^2\psi)) = \frac{3}{4}\partial|\psi|^4$ we have by (A.1)

$$\text{(A.2)} \qquad \partial_t|\psi|^2 = \partial(-\partial\,\mathrm{Im}(\bar{\psi}\partial\psi) + \frac{3}{2}\lambda|\psi|^4),$$

from which (a) follows.

We have

$$\text{(A.3)} \qquad \begin{aligned} \frac{d}{dt}\|\partial\psi\|_2^2 &= 2\,\mathrm{Re}(\partial\psi, \partial_t\partial\psi) \\ &= -2\,\mathrm{Re}(\partial^2\psi, \partial_t\psi) \\ &= -2\,\mathrm{Re}(-i\partial_t\psi + i\lambda\partial(|\psi|^2\psi) + F(\psi), \partial_t\psi) \qquad \text{(by (1.1))} \\ &= 2\lambda\,\mathrm{Im}(\partial(|\psi|^2\psi), \partial_t\psi) - 2\,\mathrm{Re}(F(\psi), \partial_t\psi) \\ &= -2\lambda\,\mathrm{Im}(|\psi|^2\psi, \partial_t\partial_\psi) - \frac{d}{dt}\int H(\psi)dx \\ &\qquad \text{(by the condition of } F). \end{aligned}$$

We next consider the first term of the right hand side of (A.3). We have

$$\text{(A.4)} \qquad \begin{aligned} -\mathrm{Im}(|\psi|^2\psi, \partial_t\partial\psi) &= -\frac{d}{dt}(\mathrm{Im}(|\psi|^2\psi, \partial\psi)) \\ &\quad + \mathrm{Im}((\partial_t|\psi|^2)\psi, \partial\psi) + \mathrm{Im}(|\psi|^2\partial_t\psi, \partial\psi). \end{aligned}$$

We apply (A.2) to the second term of the right hand side of (A.4) to obtain

$$\text{(A.5)} \qquad \begin{aligned} \mathrm{Im}((\partial_t|\psi|^2)\psi, \partial\psi) &= \mathrm{Im}((\partial(-2\,\mathrm{Im}(\bar{\psi}\partial\psi) + \frac{3}{2}\lambda|\psi|^4))\psi, \partial\psi)) \\ &= 2(\partial(\mathrm{Im}\,\bar{\psi}\partial\psi), \mathrm{Im}\,\bar{\psi}\partial\psi) + \frac{3}{2}\lambda\,\mathrm{Im}((\partial|\psi|^4)\psi, \partial\psi) \\ &= -\frac{3}{2}\lambda\,\mathrm{Im}(|\psi|^4\psi, \partial^2\psi) \end{aligned}$$

By using (1.1) in the third term of the right hand side of (A.4), we have

$$\text{(A.6)}\quad \begin{aligned}&\operatorname{Im}(|\psi|^2\partial_t\psi,\partial\psi)\\ &= -\operatorname{Im}((\partial|\psi|^2)\partial_t\psi,\psi) - \operatorname{Im}(|\psi|^2\partial_t\partial\psi,\partial\psi)\\ &= \operatorname{Re}(\partial|\psi|^2\cdot(-\partial^2\psi + i\lambda\partial(|\psi|^2\psi) + F(\psi)),\psi) + \operatorname{Im}(|\psi|^2\psi,\partial_t\partial\psi)\\ &= -\operatorname{Re}(\partial|\psi|^2\cdot\partial^2\psi,\psi) - \lambda\operatorname{Im}(\partial|\psi|^2\cdot\partial(|\psi|^2\psi),\psi) + \operatorname{Im}(|\psi|^2\psi,\partial_t\partial\psi)\end{aligned}$$

Since

$$\begin{aligned}&\operatorname{Re}(\partial(|\psi|^2\psi),\partial^2\psi)\\ &= \operatorname{Re}(\partial|\psi|^2\cdot\psi,\partial^2\psi) + \operatorname{Re}(|\psi|^2\partial\psi,\partial^2\psi)\\ &= \operatorname{Re}(\partial|\psi|^2\cdot\partial^2\psi,\psi) - \frac{1}{2}(\partial|\psi|^2,|\partial\psi|^2)\\ &= \operatorname{Re}(\partial|\psi|^2\cdot\partial^2\psi,\psi) - \frac{1}{2}(\partial|\psi|^2,\frac{1}{2}\partial^2|\psi|^2 - \operatorname{Re}(\bar\psi\partial^2\psi))\\ &= \frac{3}{2}\operatorname{Re}(\partial|\psi|^2\cdot\partial^2\psi,\psi),\end{aligned}$$

the first term of the right hand side of (A.6) is written as

$$\text{(A.7)}\quad \begin{aligned}&-\operatorname{Re}(\partial|\psi|^2\cdot\partial_2\psi,\psi)\\ &= -\frac{2}{3}\operatorname{Re}(\partial(|\psi|^2\psi),\partial^2\psi)\\ &= -\frac{2}{3}\operatorname{Re}(\partial(|\psi|^2\psi),-i\partial_t\psi + i\lambda(|\psi|^2\psi) + F(\psi)) \qquad \text{(by (1.1))}\\ &= \frac{2}{3}\operatorname{Im}(\partial(|\psi|^2\psi),\partial_t\psi) \qquad \text{(by the condition of } F)\\ &= -\frac{2}{3}\operatorname{Im}(|\psi|^2\psi,\partial_t\partial\psi)\end{aligned}$$

By (A.4), (A.6) and (A.7)

$$\begin{aligned}-\operatorname{Im}(|\psi|^2\psi,\partial_t\partial\psi) &= -\frac{d}{dt}(\operatorname{Im}(|\psi|^2\psi,\partial\psi)) - \frac{3}{2}\lambda\operatorname{Im}(|\psi|^4\psi,\partial^2\psi)\\ &\quad - \lambda\operatorname{Im}(\partial|\psi|^2\cdot\partial(|\psi|^2\psi),\psi) + \frac{1}{3}\operatorname{Im}(|\psi|^2\psi,\partial_t\partial\psi)\\ &= -\frac{d}{dt}(\operatorname{Im}(|\psi|^2\psi,\partial\psi)) + 2\lambda\operatorname{Im}(|\psi|^4\partial^2\psi,\psi) + \frac{1}{3}\operatorname{Im}(|\psi|^2\psi,\partial_t\partial\psi).\end{aligned}$$

From above it follows that

$$\text{(A.8)}\quad -\operatorname{Im}(|\psi|^2\psi,\partial_t\partial\psi) = -\frac{3}{4}\frac{d}{dt}(\operatorname{Im}(|\psi|^2\psi,\partial\psi)) + \frac{3}{2}\lambda\operatorname{Im}(|\psi|^4\partial^2\psi,\psi).$$

From (A.3) and (A.8) it follows that

$$\frac{d}{dt}\|\partial\psi\|_2^2 = -\frac{3}{2}\lambda\frac{d}{dt}(\mathrm{Im}(|\psi|^2\psi, \partial\psi)) + 3\lambda^2\,\mathrm{Im}(|\psi|^4\partial^2\psi, \psi) + \frac{d}{dt}\int H(\psi)dx. \tag{A.9}$$

On the other hand we have by (1.1)

$$\begin{aligned} \mathrm{Im}(|\psi|^4\partial^2\psi, \psi) &= \mathrm{Im}(|\psi|^4(-i\partial_t\psi + i\lambda\partial(|\psi|^2\psi) + F(\psi)), \psi) \\ &= -\,\mathrm{Re}(|\psi|^4\partial_t\psi, \psi) + \lambda\,\mathrm{Re}(|\psi|^4(\partial|\psi|^2\cdot\psi + |\psi|^2\partial\psi), \psi) \\ &= -\frac{1}{6}\frac{d}{dt}\|\psi\|_6^6. \end{aligned} \tag{A.10}$$

Hence (A.9) and (A.10) yield the desired identity (b).

References

1 G. Boling and T. Shaobin, *On smooth solutions to the initial value problem for the mixed nonlinear Schrödinger equations*, Proc. Royal Soc. Edinburgh **119** A (1991), 31-45.

2 Y.M. Chen, *The initial-boundary value problem for a class of nonlinear Schrödinger equations*, Acta Math. Sci. **6** (1986), 405-418.

3 N. Hayashi, *The initial value problem for the derivative nonlinear Schrödinger equation*, Nonlinear Anal. TMA, in press.

4 N. Hayashi and T. Ozawa, *On the derivative nonlinear Schrödinger equation*, Physica D **55** (1992), 14-36.

5 N. Hayashi and T. Ozawa, *Remarks on nonlinear Schrödinger equations in one space dimension*, preprint, Hokkaido (1992).

6 N. Hayashi and Y. Tsutsumi, *Remarks on the scattering problem for nonlinear Schrödinger equations*, Lecture Notes in Math. Vol. **1285**, 162-168, Springer (1987).

7 T. Kato, *On nonlinear Schrödinger equations*, Ann. Inst. Henri Poincaré, Phys. Théor., **46** (1987), 113-129.

8 T. Kato, *Nonlinear Schrödinger equations*, in "Schrödinger Operators," Lecture Notes in Phys., Vol. **345**, Springer (1989).

9 M. Tsutsumi and I. Fukuda, *On solutions of the derivative nonlinear Schrödinger equation II*, Funkcialaj Ekvacioj **24** (1981), 85-94.

10 Y. Tsutsumi, *L^2-solutions for nonlinear Schrödinger equations and nonlinear groups*, Funkcialaj Ekvacioj **30** (1987), 115-125.

11 Y. Tsutsumi, *Global strong soluitons for nonlinear Schrödinger equations*, Nonlinear Anal. TMA **11** (1987), 1143-1154.