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Christoph Schmid and Joel Yellin

December 5, 1968

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* These Appendices are not included in the version submitted for publication.

FINITE ENERGY SUM RULES AND THE PROCESS $0^- + 0^- \to 0^- + 0^-$*

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ABSTRACT

We discuss here pseudoscalar-pseudoscalar meson elastic scattering, using finite energy sum rules (FESR). We derive algebraic properties of the resulting equation, and give numerical values for the $1^-0^-0^-$ and $2^+0^-0^-$ couplings, computed using the physical 0^-, 1^-, and 2^+ masses.

I. INTRODUCTION

We study here the process $P + P \rightarrow P + P$ ($P = 0^-$ meson), using finite energy sum rules (FESR).[1,2] In order to avoid spin complications we have chosen to consider external pseudoscalar mesons. Internally, we take account of $1^-(V)$ and $2^+(T)$ states only. The complete self-consistent system includes processes with V and T states external (e.g., $P + V \rightarrow V + T$) and P states internal, and is not discussed here.

Provided the amplitude satisfies analyticity crossing and has Regge asymptotic behavior, its discontinuity in $\nu = \frac{1}{2}(s - u)$: $D_\nu(\nu,t)$, at fixed t satisfies

$$\frac{1}{2}\int_{-N}^{+N} D_\nu(\nu,t)\, \nu^n\, d\nu$$

$$= \int_{-N}^{+N} \left\{ \text{Background integral} + \sum_{\text{Regge cuts}} + \sum_{\text{Regge poles}} \right\} d\nu\, \nu^n \, . \tag{1.1}$$

We drastically truncate the right side of (1.1), keeping the leading Regge pole only. We then have

$$\frac{1}{2}\int_{-N}^{+N} D_\nu(\nu,t)\, \nu^n\, d\nu \cong \frac{\beta(t)\, N^{\alpha(t)+n+1}}{\alpha(t) + n + 1} \, . \tag{1.2}$$

In the following, we discuss the approximations involved in using (1.2), and we apply this relation to the process $P + P \rightarrow P + P$.

It is by no means clear, a priori, that there exists a choice of N and t such that our approximations are good. We contend that empirically such a choice exists, and we discuss how this comes about below.

On making assumptions (i)-(v), as listed in the next section, we are left with a set of linear, homogeneous equations for bilinears in the unknown couplings. These equations have certain algebraic properties which are not characteristic of the present scheme alone. $Z = 0$ field theories[3] and N/D bootstraps[4] yield equations with similar algebraic content.

In Section II, we discuss our approximations. In Section III, we show the algebraic content of (1.2). In Section IV, we give our numerical results for the VPP and TPP couplings. In Section V, we discuss exchange degeneracy. We summarize our results in Section VI.

II. DISCUSSION OF APPROXIMATIONS

In going from the exact relation (1.1) to (1.2) we have made the following approximations

(i) $\sum_{\text{Regge cuts}} \cong 0$,

(ii) Background integral $\cong 0$,

(iii) $\sum_{\text{Regge poles}} \cong$ Leading pole contribution only .

In order to evaluate (1.2) we will make two additional assumptions:

(iv) $\text{Im}\,\alpha(t) \cong 0$, for finite t. $\text{Re}\,\alpha(t) \cong \alpha(0) + \alpha'(0)t = a + bt$,

(v) $D_\nu(\nu,t)$ can be approximated by narrow resonance contributions in the s and u channels.

We now discuss (i)-(v) in detail:

(i) $\sum_{\text{Regge cuts}} \cong 0$

According to legend,[5] cuts arise in a certain class of diagrams containing multiparticle intermediate states, i.e., in the third double-spectral function region, due to the exchange, for example, of two Regge poles, or of a Regge pole plus an elementary particle. Hopefully, if we stay away from the boundary of the double spectral function, the effect of cuts will not be large enough to affect our considerations.

Without a definite detailed theory of how cuts arise, their effects are empirically inseparable from those of secondary trajectories.

(ii) Background integral $\cong 0$

In our picture, the background integral accounts for the oscillations occurring in amplitudes at low energies,[2] while the small number of Regge poles needed to fit the data at intermediate energies have only a smooth behavior.

We can roughly estimate the error we make in neglecting the background integral by computing the maximum deviation of the resonance oscillations from their average values, assuming the statement above connecting the background integral with resonance oscillations is correct. In the $\underset{\sim}{I} = 1$, $\pi\pi$, case, the error amounts to about 10%, taking N halfway between m_f^2 and m_g^2. However, more precisely this error estimate is really a lower limit, since one could always add a smooth contribution to the background integral.[6]

(iii) $\displaystyle\sum_{\text{Regge poles}}$ = leading trajectory only

Here we deal separately with: (1) neglect of secondary trajectories such as ρ'; (2) neglect of the Pomeranchon.[7]

With respect to (2), our profound ignorance of the nature of the Pomeranchon leads us to neglect it arbitrarily and completely.

With respect to (1), suppose ρ', for example, corresponds to resonances in the t channel. These resonances must occur in lower partial waves than those which would make up the ρ trajectory, which lies higher.

The ρ' contribution is suppressed for large z_t and therefore for small t and/or large N. As we shall see below, the narrow resonance approximation for the left side of (1.2) requires just the opposite: large t and/or small N. It is our contention that there exists a range of N and t for which the regions of validity of the two approximations overlap.

(iv) $\underline{\text{Im}\,\alpha(t) \cong 0. \quad \text{Re}\,\alpha(t) = \alpha(0) + \alpha'(0)t \equiv a + bt}$

If we approximate our amplitude by one Regge pole in the t channel, and stick to the right half of the J plane, unitarity implies[8]

$$\text{Im}\,\alpha(t) = K(t)\,\beta(t) \quad , \tag{2.1}$$

where $K(t)$ is the usual kinematic factor in the unitarity condition. In the narrow resonance approximation, which we use here, $\alpha(t)$ is real for real t and both sides of (2.1) are small. Equation (2.1) immediately implies $\beta(t)$ is real in this approximation. Above threshold $\alpha(t)$ and $\beta(t)/q^{2\alpha(t)}$ are real and have no right-hand cuts in t. If no trajectories cross, $\alpha(t)$ and $\beta(t)/q^{2\alpha(t)}$ also have no left-hand cuts, as pointed out by Cheng.[9]

Once we know $\alpha(t)$ and $\beta(t)/q^{2\alpha(t)}$ have no right or left-hand cuts, we conclude they are real entire functions. We assume

$$\alpha(t) = a + bt \quad , \tag{2.2}$$

and that $b \cong 1\ \text{GeV}^{-2}$ is a universal constant for all Regge trajectories. Mandelstam[10] has given arguments which make this behavior of $\alpha(t)$ plausible. The slope b then fixes the energy scale of the system.

Experimentally (2.2) seems to be fairly well satisfied.

(v) Narrow resonance approximation for $D_\nu(\nu,t)$

We note there are three sources of error here:

1. Background in low partial waves,
2. Background in high partial waves,
3. Finite resonance width effects.

Item 3. could be taken care of by, for example, using the Cheng-Sharp[8] equations as modified by Mandelstam.[10] In any case, 1. and 2. are more serious problems.

1. Since we are now in the s and u channels we must consider $(2J+1)P_J(z_s)$ or $(2J+1)P_J(z_u)$. z_s increases (decreases) with $t(N)$. In contrast to the situation considered in (iii), we want high t and/or low N to suppress the low partial waves.

2. With regard to the high partial waves, we have the following empirical information: (A) In the phase shift analysis of πN scattering, if the highest partial wave resonating has $J = J_0$, the partial wave $a(J_0+1,t)$ seems to be very small. Bareyre,[11] for example, concludes that up to the F wave (1688) resonance, all G wave phase shifts are smaller than 3 degrees. (B) The tail, for example, of the $f^0(1230)$ contributes only a few percent of the p wave amplitude at $t = s = m_\rho^2$.

A rough theoretical explanation of (A) and (B) above goes as follows: We use (1.2) at $t = m^2_{resonance} (> 0)$, so that we are outside the physical region for the s and u channels. Our partial wave expansions are therefore asymptotic and we expect some sort of divergent behavior as we hit the boundary of the double spectral function. However, Regge theory tells us that, at fixed s, the amplitude goes at most as $t^{\alpha(s)}$ as t increases. The divergence of the partial wave series for $D_\nu(\nu,t)$ at most attaches a phase to the exponential.[12]

III. FESR FOR $P + P \to P + P$ IN THE MASS DEGENERATE CASE: ALGEBRAIC PROPERTIES

We consider (1.2) in the mass degenerate case, including internal $V(1^-)$ and $T(2^+)$ states only.

As shown in Appendix A, (1.2) then takes the form,

$$\sum_s \sum_{J=1}^{2} f^{(J)} K_n^{(J)} \left[g_{abs}^{(J)} g_{cds}^{(J)} + (-1)^{n+1} g_{ads}^{(J)} g_{cbs}^{(J)} \right]$$

$$= \sum_s g_{acs}^{(n+1)} g_{bds}^{(n+1)} J_n \quad . \qquad (3.1)$$

The relation (3.1) is evaluated at fixed t, and the s channel process is $a + b \to c + d$.

In (3.1) the $g_{abc}^{(J)}$ are effective dimensionless coupling constants. The first two subscripts refer to 0^- states, the third subscript refers to a sum over intermediate spin J states. J_n and $K_n^{(J)}$ are positive kinematic factors depending on the masses (m_0, m_1, m_2) and on the limits of integration, N. The quantity $f^{(J)}$ is the fraction of the full contribution of the intermediate spin J states included on the LHS; $f^{(J)}$ is related to the choice of N. If we choose $f^{(2)} = 0$ and $f^{(1)} = 1$, N should be approximately halfway between the 1^- and 2^+ states; if $f^{(2)} = 1/2$, then $f^{(1)} = 1$, and N should be approximately at $m_2^2 \cdots$. If the term on the RHS of (3.1) is a t channel Regge pole, the terms on the left arise from resonances in s and u respectively. No crossing matrices appear in (3.1); however, the signs of the kinematic factor are shown explicitly. [These signs arise from the signs of ν^n and of $\mathrm{Disc}(x - m^2 \pm i\epsilon)^{-1}$.]

For convenience we now define

$$\lambda_0 \equiv J_0/K_0^{(1)}; \qquad \lambda_1 \equiv J_1/f^{(2)}K_1^{(2)}; \tag{3.2}$$

$$\xi_0 \equiv f^{(2)}K_0^{(2)}/K_0^{(1)}; \qquad \xi_1 \equiv K_1^{(1)}/f^{(2)}K_1^{(2)}; \tag{3.3}$$

$$f_{abc} \equiv g^{(1)}_{abc}; \qquad D_{abc} \equiv g^{(2)}_{abc}. \tag{3.4}$$

Writing out (3.1) for $n = 0$ and $n = 1$ we have, using (3.2)-(3.4), for $\underline{n = 0}$,

$$F_{abs}F_{cds} + F_{ads}F_{bcs} + \lambda_0 F_{cas}F_{bds} + \xi_0(D_{abs}D_{cds} - D_{ads}D_{cbs}) = 0; \tag{3.5}$$

for $\underline{n = 1}$

$$D_{abs}D_{cds} + D_{ads}D_{dcs} - \lambda_1 D_{cas}D_{bds} + \xi_1(F_{abs}F_{ads} - F_{ads}F_{bcs}) = 0. \tag{3.6}$$

The usual invariance arguments give $F_{abc} = -F_{bac}$, $D_{abc} = +D_{bac}$.

Summing (3.5) and (3.6) over permutations of the free indices we have

$$(F_{abs}F_{cds} + F_{ads}F_{bcs} + F_{cas}F_{bds})(\lambda_0 + 2) = 0 , \qquad (3.7)$$

$$(D_{abs}D_{cds} + D_{ads}D_{cbs} + D_{cas}D_{bds})(-\lambda_1 + 2) = 0 . \qquad (3.8)$$

Since $\lambda_0 > 0$, and assuming at least one of the D's is nonzero, (3.7) and (3.8) yield

$$F_{abs}F_{cds} + F_{ads}F_{bcs} + F_{cas}F_{bds} = 0 , \qquad (3.9)$$

$$\lambda_1 = 2 . \qquad (3.10)$$

Substituting (3.9) and (3.10) into (3.5) and (3.6) yields the two additional relations

$$D_{ads}D_{cbs} - D_{dcs}D_{abs} + \xi_1 F_{cas}F_{dbs} = 0 , \qquad (3.11)$$

$$\lambda_0 - 1 - \xi_0\xi_1 = 0 . \qquad (3.12)$$

It is immediately clear from (3.9) and (3.11) that our system allows a U(n) symmetric solution with the F's and D's identified as the Clebsch-Gordan coefficients for the antisymmetric and symmetric couplings of three representations each transforming like $\underset{\sim}{\underline{n^2 - 1}} \oplus \underline{1}$.[13] Since the antisymmetric coupling of the vector singlet vanishes, this statement has teeth with respect to the tensor singlet coupling ratio only. Computing the singlet/octet coupling ratio in the

U(3) case, one discovers that the result is equal to that conjectured by Glashow and Socolow[14] by analogy with Okubo's[15] choice for the vector nonet.

Let us explicate the remarks above.

The four conditions:

(a) Total antisymmetry of F_{abc},

(b) Equality of the 0^- and 1^- multiplicities.

(c) Equation (3.9).

(d) $F_{abs}F_{abs'} = \delta_{ss'}$.

are necessary and sufficient for the F's to be the structure constants of a compact, semisimple, Lie algebra.[16] We are unable to derive (a) through (c). We can, however, interpret (d) so that it is physically reasonable. That condition is equivalent to the statement that all V's have the same total reduced width, and that the V's couple to orthogonal combinations of PP states.[17]

If we assume (a) through (c) are satisfied the F's are then the structure constants of an arbitrary compact, semisimple, Lie algebra, and the $(0^-,1^-)$ multiplets transform as the adjoint representation.

Suppose we make the additional assumption that the 2^+, 1^-, and 0^- multiplicities are equal and that the three multiplets transform in the same way under the Lie algebra for which the F's are the structure constants. Then the D's are the Clebsch-Gordan coefficients for coupling the representation in question to itself trilinearly and symmetrically. Note that if we begin with the adjoint representation and add an arbitrary number of singlets, the F's for the singlet couplings are all zero because of the antisymmetry. The D's, however, for the singlet couplings, are nonzero. Let us set the representation $\underline{R}$ = $\underline{\text{adjoint}} \oplus$ (p $\underline{\text{singlets}}$). Since the singlets are indistinguishable we want to know if p is nonzero. Assumption (d) above must now be changed slightly to accommodate the F's which are zero, and the metric tensor $g_{ss'}$ acquires p zeroes on the diagonal. We want to see if $p \neq 0$ and if so, we want the relevant adjoint/singlet coupling ratios for the D's. Equation (3.11) contains information relative to this point. A subset of the D's must be the Clebsch-Gordan coefficients for coupling the adjoint representation to itself trilinearly and symmetrically. This fixes the Lie algebra to be of the unitary type A_n[(modulo addition of p U(1)'s]. This is because $\underline{\text{adjoint}} \otimes \underline{\text{adjoint}}$ contains adjoint only once for the orthogonal B_n, $D_{n+2}(n \geq 2)$; and symplectic $C_{n+1}(n \geq 2)$ algebras, and for G_2. We have not checked F_4, E_6, E_7, E_8.[18] We conjecture the statement is true for these also because of the completeness of the $(n + 1) \times (n + 1)$ matrix representation of A_n.

The structure constants and d's for SU(n), satisfy, in the notation of Gell-Mann,[19]

$$[\lambda_i, \lambda_j]_- = 2if'_{ijk}\lambda_k \quad , \qquad (3.13)$$

$$[\lambda_i, \lambda_j]_+ = 2d'_{ijk}\lambda_k + \frac{4}{n}\delta_{ij}\,\underset{\sim}{1} \qquad (3.14)$$

for

$$1 \leq (i, j, k) \leq n^2 - 1 \quad .$$

If we try to set $F \propto f'$, $D \propto d'$, so that $p = 0$ and there are no singlets, we immediately contradict (3.11), since the f''s and d''s satisfy

$$f'_{cas} f'_{dbs} = d'_{dcs} d'_{abs} - d'_{ads} d'_{cbs} + \frac{2}{n} (\delta_{dc} \delta_{ab} - \delta_{ad} \delta_{bc}) \;, \tag{3.15}$$

where the normalization of the λ's is chosen to be

$$\mathrm{Tr}\, (\lambda_i \lambda_j) = 2\delta_{ij} \quad . \tag{3.16}$$

Now however, if we define

$$\lambda_0 = (2/n)^{\frac{1}{2}} \underset{\sim}{1} \;, \tag{3.17}$$

we can rewrite (3.13), (3.14), and (3.15) as

$$[\lambda_i, \lambda_j]_- = 2if_{ijk} \lambda_k \;, \tag{3.18}$$

$$[\lambda_i, \lambda_j]_+ = 2d_{ijk} \lambda_k \tag{3.19}$$

for $0 \leq (i, j, k) \leq n^2 - 1$, and

$$f_{cas} f_{dbs} = d_{dcs} d_{abs} - d_{ads} d_{cbs} \;, \tag{3.20}$$

so that $p \neq 0$ explicitly gives a solution to (3.11) and

$$d_{oij} = (2/n)^{\frac{1}{2}} \delta_{ij} \quad . \tag{3.21}$$

This solution is unique. For the $n = 3$ case (3.21) corresponds to the coupling strength ratio assumed by Glashow and Socolow.[14]

IV. FESR FOR $P + P \to P + P$ IN THE NONDEGENERATE CASE: NUMERICAL RESULTS

We now apply the FESR (1.2)

$$\frac{1}{2}\int_{-N}^{+N} d\nu\, \nu^n D_\nu(\nu,t) \simeq \frac{\beta(t) N^{\alpha(t)+n+1}}{\alpha(t)+n+1}\,, \qquad (1.2)$$

to the $P + P \to P + P$ case. We consider $n = O(1)$[vector and tensor Regge terms on the right-hand side.] only. Making approximations (i)-(v), as listed in Section II, we can rewrite (1.2) as

$$\sum_{s=1} \sum_{J=1}^{2} \Big[R_{ns}^{(J)}(a,b\cdots)\, g_{abs}^{(J)}\, g_{cds}^{(J)}$$

$$+ (-1)^{n+1} R_{nu}^{(J)}(a,b\cdots)\, g_{ads}^{(J)}\, g_{cbs}^{(J)} \Big] = \sum_{s} g_{acs}^{(n+1)}\, g_{bds}^{(n+1)} J(a,b\cdots)\,, \qquad (4.1)$$

which is identical to Eq. (3.1) except that the kinematic factors are now dependent on particle labels and on the channel label, since we want to insert the known 0^-, 1^-, and 2^+ masses. We are not interested in electromagnetic splittings and therefore choose definite $\underset{\sim}{I}$, Y for the right side of (4.1), obtaining equations of the schematic form

$$\sum C_{tj}(\gamma\gamma R)_j = \gamma\gamma^{(J)}, \qquad (4.2)$$

a set of homogeneous linear equations in unknown bilinears in coupling constants which we label by e.g., $\gamma_{K\pi}^{K*}$. The isospin crossing matrix from the t to the j channel is C_{tj}, and the dependence on $(I_t, Y_t, I_s \cdots)$ etc. has been suppressed. The explicit forms of the kinematic factors (R,J) are derived in Appendix A.

In order to use (4.2) to evaluate the ratios between unknown bilinears $\gamma\gamma$, we

(a) assume there is a unique N for each process, and use the condition that the determinant of the coefficients of the unknowns must vanish to determine each N.[20]

(b) combine (ff') and ($\phi\omega$) on the right-hand side of (4.2), so that there is one $\underset{\sim}{I} = 0$, $Y = 0$ equation for each value of N. We explain this procedure in Appendix A.

In Table 1 we show the various contributions for each independent process. In Table 2, we write out the FESR in a shorthand notation, explicitly giving the crossing matrix elements and the factors of $\frac{1}{2}$ arising when a resonance contributes to one cross-channel only. In Table 3 we show the coupling constant relations arising in the degenerate case. These of course also follow from the relations of Section III. We have performed numerical computations for the equations shown in Table 2 only. We use the nonsuperconvergent equations only, since the cancellations in the superconvergent equations induce large numerical uncertainties.

As shown in Fig. 1, we test the sensitivity and self-consistency of our equations by choosing the limit of integration, N, in the following three ways: cutoff mass corresponding to

$$N = \frac{1}{2}(m_V^2 + m_T^2) \qquad \text{(Case I)} ,$$

$$N = m_T^2 \qquad \text{(Case II)} ,$$

$$N = \frac{1}{2}(m_T^2 + m_{3^-}^2) \qquad \text{(Case III)} .$$

Our specific procedure is to take for each term on the left-hand side of (4.2) a weighting factor $f^{(J)}$:

Case I: 1 x (V); 0 x (T);

Case II: 1 x (V); $\frac{1}{2}$ x (T);

Case III: 1 x (V); 1 x (T).

For each of (I, II, III) we use the determinental condition to determine N, and for each case we then find the ratios of the unknown bilinears in coupling constants.

In Table 4 we show the coupling ratios compared to experiment and to exact U(3). In Table 5 we show the calculated cutoff masses corresponding to the various self-consistent limits of integration. It will be noted that these masses are quite reasonable, and this is a strong check on the self-consistency of our system.

Though our calculated coupling ratios are reasonable we note that a good experimental test is lacking, and that we have introduced numerical ambiguities into the system by using the procedure (ii) above, as discussed in Appendix A.

V. EXCHANGE DEGENERACY

In the $PP \to PP$ system we can make some interesting statements regarding exchange degeneracy between the vector and tensor trajectories.[21] By exchange degeneracy we mean, for example,

$$\gamma_{\overline{K}K}^{\rho} : \gamma_{\overline{K}K}^{A_2} = \gamma_{\overline{K}K}^{\omega_8} : \gamma_{\overline{K}K}^{f_8 f_1} , \tag{5.1}$$

where the γ's are reduced widths. Clearly the 1^- and 2^+ octets cannot be exchange degenerate in this sense since they have F and D type coupling to the PP system, respectively. However, the addition of a 2^+ singlet of the proper strength, as in Section III, enables exchange degeneracy to be realized for $\rho - A_2$, $\omega_8 - f_1 f_8$, and $K^* - K^{**}$ in $K\pi \to K\pi$; exchange degeneracy is <u>not</u> then realized for $K^* - K^{**}$ in $K\eta \to K\eta$, and one obtains in fact

$$\gamma_{K\eta}^{K*} : \gamma_{K\eta}^{K**} = 9\gamma_{K\pi}^{K*} : \gamma_{K\pi}^{K**} . \tag{5.2}$$

Physically, we expect, and we do obtain from our bootstrap, exchange degeneracy between resonances of even and odd J, in the $\overline{K}K$ system, because there are only direct forces and no exchange forces. By contrast the K* and K** need not be exchange degenerate since the $K\pi \to K\pi (K\eta)$ channel contains not only direct forces: ρ, ff'(ff'), but also exchange forces: K*, K**(K*, K**). Unless the exchange forces cancel there is no exchange degeneracy. The K* and K** forces are always opposite in sign because of the sign of $P_\ell(z)$.

In the mass degenerate case, the superconvergent higher moment sum rule can be satisfied for a particular value of $f^{(2)}$ only. For the lower-moment case the 1^- and 2^+ contributions cancel among themselves, and if the SU(3) coupling ratios are used the equation is satisfied identically independent of $f^{(2)}$. In the higher-moment equation, however, 1^- cancels against a definite fraction, $f^{(2)}$, of 2^+. [See Table 2.] The remaining fraction of the 2^+ contribution presumably cancels against a piece of the 3^- contribution.

If the fraction $f^{(2)}$ is chosen equal to the one which produces superconvergence in the appropriate channels, the exchange forces in the higher-moment equation cancel in the $K\pi$, but not in the $K\eta$, channel.

VI. SUMMARY AND CONCLUSIONS

Our study of $P + P \rightarrow P + P$ includes an algebraic discussion of the degenerate case, and a numerical discussion of the nondegenerate case. The degenerate case yields algebraic equations. These restrict the symmetry properties of particular solutions, but do not force a unique solution. In the degenerate case, one also obtains certain kinematic relations between the input parameters, which are consistent with the actual average multiplet masses, but do not force a unique set of values for m_P^2, m_V^2, m_T^2.

In the nondegenerate case, we have in principle a bootstrap theory of SU(3) symmetry breaking.[22] However, the numerical procedure used so far induces uncertainties inasmuch we are not able to compute the splitting of the $\underset{\sim}{I} = 0$, $Y = 0$ members of the V and T nonets, and instead are forced to introduce an arbitrary assumption.

We have begun here a discussion of the self-consistency of the mesons. Our investigation is very limited however, since we make very drastic dynamical approximations, and since we consider one process only. The extension of this work to the complete system (P, V, T) is now in progress. It is our hope that somewhat more definite algebraic results will then arise in the degenerate case.

It would be of great interest if a model could be obtained, incorporating towers of meson resonances, which exactly satisfied the FESR. As yet we have not been able to exhibit such an explicit solution of the FESR, nor do we know if one exists.[23]

We would like to acknowledge discussions of the properties of Lie algebras with Professors R. Hermann and C. Fronsdal. We are indebted to Professor Stanley Mandelstam for a critical reading of the manuscript.

APPENDIX A

In this section we define our notation, give useful kinematic relations, and derive the explicit form of the kinematic factors in (4.2).

For elastic scattering of spinless particles our differential cross section is

$$\frac{d\sigma}{d\Omega} = |f|^2 \quad , \tag{A.1}$$

where f has the center-of-mass partial wave expansion

$$f = \sum_{J=0}^{\infty} (2J + 1)\, f_J(E)\, P_J(\cos \Theta) \tag{A.2}$$

and

$$f_J(E) = e^{i\delta_J} \sin \delta_J / |\underset{\sim}{q}| \quad , \tag{A.3}$$

$\underset{\sim}{q}$ being the CM spatial momentum.

In terms of f, our amplitudes $A(\nu, t)$ are given by

$$A(\nu, t) = \frac{(s)^{\frac{1}{2}}}{2} f = -\frac{F}{16\pi} \quad . \tag{A.4}$$

The ν's in each (s, t, u) channel are <u>defined</u> by:

(a) sign: $\nu_x \,\alpha (z_x = \cos \Theta_x)$.

(b) scale: $|d\nu_x| = |dt|$.

(c) origin: e.g., $\nu_u = 0$ for $s = t$.

For particles of unequal mass

$$\frac{d\sigma}{d\Omega} = \frac{|F|^2}{64\pi^2 s}\left(\frac{P_f}{P_i}\right) = |f|^2 \quad , \tag{A.5}$$

where the S matrix is

$$S = 1 - iF(2\pi)^4\delta^4(\Sigma p) \quad , \tag{A.6}$$

and the partial wave expansion is

$$f = -\frac{F}{8\pi(s)^{\frac{1}{2}}}\left(\frac{P_f}{P_i}\right)^{\frac{1}{2}} = \left(\frac{P_f}{P_i}\right)^{\frac{1}{2}} \sum_J (2J + 1)f_J P_J(\cos\theta) \ . \tag{A.7}$$

We now return to the equal mass case and discuss the narrow resonance approximation. We have

$$f_J(\text{near } E_{\text{Resonance}}) \cong -\frac{\Gamma_R M_R}{\nu - \nu_R + i\Gamma_R M_R} \cdot \frac{1}{q_R} \quad , \tag{A.8}$$

so that, near resonance, the s channel amplitude is

$$A(\nu, t) \cong (2J + 1)f_J P_J(z_s)\frac{(s)^{\frac{1}{2}}}{2}$$

$$= -\frac{(s)^{\frac{1}{2}}}{2}(2J + 1)P_J(z_s) \cdot \frac{\Gamma_R M_R}{\nu - \nu_R + i\Gamma_R M_R} \cdot \frac{1}{q_R} \ . \tag{A.9}$$

Since

$$\int_{-\infty}^{+\infty} \frac{d\nu}{\nu - \nu_R + i\Gamma_R M_R} = -i\pi \quad , \tag{A.10}$$

$$\int \mathrm{Im}\, A(\nu,\, t)\, D\nu = \frac{(s)^{\frac{1}{2}}}{2} \; \frac{2J+1}{q_R}\, P_J(z_s)\pi M_R \Gamma_R x \quad , \tag{A.11}$$

where x is the elasticity of the resonance. It is sometimes useful to have the connection between A and the coupling constants, g, defined by the two-body decays:

$$\Gamma = \frac{g^2}{4\pi}\,\frac{|\underset{\sim}{P}|}{2s}\,|\overline{M}_J|^2 \quad , \tag{A.12}$$

$$M_J \equiv (P_1 - P_2)_{\mu_1} \ldots (P_1 - P_2)_{\mu_J}\, \epsilon_{\mu_1 \cdots \mu_J} \Big/ s^{\frac{1}{2}(J-1)} \quad . \tag{A.13}$$

The bar in (A.12) indicates a spin average; $\epsilon_{\mu_1 \ldots \mu_J}$ is the polarization tensor for a spin J meson, and $s^{\frac{1}{2}(J-1)} = m_{\text{Resonance}}^{J-1}$ has been inserted for dimensional convenience. We then have

$$\Gamma = \frac{g^2}{4\pi} \cdot \frac{2^{2J}|\underset{\sim}{P}|^{2J+1}}{(2J+1)s^J} \quad . \tag{A.14}$$

The Feynman amplitude

$$F \cong - \frac{2^{2J} P_i{}^J P_f{}^J P_J(z_s) g_i g_f}{s - M^2}$$

(See Fig. A-1),

as one discovers on using the vertices defined by (A.13). This yields

$$A(\nu, t) \cong \left[\frac{g_i{}^2}{4\pi} \; \frac{g_f{}^2}{4\pi}\right]^{\frac{1}{2}} \frac{2^{2(J-1)} P_i{}^J P_f{}^J P_J(z_s)}{M^2 - s} . \tag{A.16}$$

In terms of the total width, Γ,

$$\Gamma x_i \; \alpha \; \frac{g_i}{4\pi} \; |\underset{\sim}{P}|^{2J+1} , \tag{A.17}$$

so that

$$A(\nu, t) \; \alpha \left[\frac{x_i}{P_i} \; \frac{x_f}{P_f}\right]^{\frac{1}{2}} \frac{\Gamma 2^{2J-1} \; P_J(z_s)}{M^2 - s} . \tag{A.18}$$

Our Regge amplitudes are

$$A(\nu, t) = \frac{\beta(t) \; \nu^{\alpha(t)} (\mp 1 - e^{-i\pi\alpha(t)})}{\sin \pi\alpha(t)} . \tag{A.19}$$

We take $\text{Im}\,\alpha(t) \cong 0$, $\text{Re}\,\alpha(t) \cong a + bt$, $b \cong 1\ \text{GeV}^{-2}$. For $\text{Re}\,\nu < 0$, $\text{Im}\,\nu > 0$,

$$\begin{aligned} \nu &= |\nu| \, e^{i\pi} , \\ \nu^{\alpha} &= |\nu|^{\alpha} \, e^{i\pi\alpha} . \end{aligned} \tag{A.20}$$

The discontinuity of $A(\nu, t)$ across the left-hand cut in ν is

$$D(\nu, t) = \mp \beta(t)|\nu|^{\alpha(t)} . \qquad (A.21)$$

If $A(\nu, t)$ has even (odd) signature, $D(\nu, t)$ is odd (even).

Our sum rules then read

$$\frac{1}{2} \int_{-N}^{+N} d\nu \; \nu^n \, D(\nu, t) \equiv S_n(t) = \frac{N^{\alpha(t)+n+1}}{\alpha(t)+n+1} \beta(t) , \qquad (A.22)$$

after disregarding cuts, the background integral, the Pomeranchon, and all Regge poles excepting the leading one.

In (A.22)

$$\nu = \frac{1}{2}(s - u) = s + \frac{1}{2}(t - \Sigma) , \qquad (A.23)$$

$$\Sigma \equiv \sum_{i=1}^{4} m_i^{\,2} . \qquad (A.24)$$

Our final expression for an s-channel contribution to the LHS of (A.22) is then

$$\frac{1}{2} \int_{-N}^{+N} D_s(\nu, t) \nu^n d\nu = \frac{C_{ts} \pi (2J+1)(s)^{\frac{1}{2}} P_J(z_s) \nu^n}{(4P_i P_f)^{\frac{1}{2}}} \cdot (s)^{\frac{1}{2}} (x_i x_f)^{\frac{1}{2}} , \qquad (A.25)$$

where C_{ts} is the t to s channel crossing matrix for isospin or unitary spin. To finish defining the problem we need to evaluate $\beta(t = m^2_{\text{Resonance}})$.

From (A.19), near resonance

$$A(\nu, t) \cong \frac{\beta(m^2)\nu_R{}^{\alpha(m^2)}{}_2}{-\pi\alpha'(t - m^2)}$$

$$= \lim_{|\nu| \to \infty} \cdot \frac{1}{2} \frac{(J + 1)(t)^{\frac{1}{2}} P_J(z_t) \Gamma_R M_R (x_i x_f)^{\frac{1}{2}}}{(\nu_R - \nu - i\Gamma_R M_R)(P_i P_f)^{\frac{1}{2}}} . \qquad (A.26)$$

As $|\nu| \to \infty$, $P_J(z_t) \to c_J z_t{}^J$, (A.27)

giving

$$\beta(m^2) = \frac{m\Gamma(x_i x_f)^{\frac{1}{2}} (2J + 1) \pi\alpha'(m^2) c_J (z_t)^J (t)^{\frac{1}{2}}}{2(4P_i P_f)^{\frac{1}{2}}} , \qquad (A.28)$$

where we have used, $z_t \to \nu/2P_i P_f$, (A.29)

$$\beta(m^2) = (2J + 1)\pi\alpha' (m^2) c_J \frac{m\Gamma(x_i x_f)^{\frac{1}{2}} 2^{J-1}(t)^{\frac{1}{2}}}{[(4P_i P_f)^{\frac{1}{2}}]^{2J+1}} . \qquad (A.30)$$

We are assuming $\alpha' \cong 1 \text{ GeV}^{-2}$. The cutoff, N, is given by

$$N = \frac{1}{2}(2s_N + m^2 - \sum_{i=1}^{4} m_i{}^2) . \qquad (A.31)$$

The RHS of (A.22) is then

$$\text{RHS} = \frac{(2J+1)\pi m_R^{\,2}\,\Gamma 2^{J-1} c_J (x_i x_f)^{\frac{1}{2}} N^{J+n+1}}{[(4P_i P_f)^{\frac{1}{2}}]^{2J+1}\,(J+n+1)} \quad . \tag{A.32}$$

In terms of $s = (P_1 + P_2)^2$, the CM spatial momentum is

$$\underset{\sim}{P}^2 = \frac{1}{4s}\left[s - (m_1 + m_2)^2\right]\left[s - (m_1 - m_2)^2\right] \quad . \tag{A.33}$$

The LHS of (A.22) is then given by (A.25). The CM scattering angle is

$$z = \frac{2t + s - \sum_{i=1}^{4} m_i^{\,2} + (m_1^{\,2} - m_2^{\,2})(m_3^{\,2} - m_4^{\,2})/s}{4|\underset{\sim}{P}_i||\underset{\sim}{P}_f|} \tag{A.34}$$

for the process $1 + 2 \rightarrow 3 + 4$ where $z = \cos\theta_{13}$, and the $|\underset{\sim}{P}|$'s are given by (A.33). In the mass degenerate case the two sides of (A.22) reduce to

$$\text{(RH)} \quad J_n = \frac{(2J+1)\,\pi m_J^{\,2}\Gamma^{(J)} 2^{J-1} c_J}{(m_J^{\,2} - 4m_0^{\,2})^{J+\frac{1}{2}}}\;\frac{\left[s_N - 2m_0^{\,2} + m_n^{\,2}/2\right]^{J+n+1}}{J+n+1} \;, \tag{A.35}$$

$$\text{(LH)} \quad K_n = \left[m_J^{\,2} - 2m_0^{\,2} + m_n^{\,2}/2\right]^n \frac{\pi}{2}\,\frac{(2J+1)m_J^{\,2}}{(m_J^{\,2} - 4m_0^{\,2})^{\frac{1}{2}}}\,\Gamma^{(J)}$$

$$\cdot\, P_J\left[1 + 2m_n^{\,2}/(m_J^{\,2} - 4m_0^{\,2})\right] \quad . \tag{A.36}$$

For the numerical work of Section IV, we have combined the FESR, for $I = Y = 0$, evaluated at $t = m_\omega^2$ and $t = m_\phi^2$, into one equation as follows. Let $R_H(t = m_H^2)$ be the usual Regge contribution of the particle H to the RHS of (A.22). We have assumed

$$x = R_\phi(t)/R_\omega(t) \quad , \tag{A.37}$$

is nearly constant over the range $m_\omega^2 \leq t \leq m_\phi^2$, so that the FESR at $t = m_\omega^2$ reads

$$\mathrm{LHS}(m_\omega^2) = (1 + x)\, R_\omega(m_\omega^2) \tag{A.38}$$

and at m_ϕ^2,

$$\mathrm{LHS}(m_\phi^2) = \frac{1}{x}\,(1 + x)\, R_\phi(m_\phi^2) \quad . \tag{A.39}$$

Using the physical masses and widths we have then computed $x = \mathrm{LHS}(m_\omega^2)/R_\omega(m_\omega^2) : \mathrm{LHS}(m_\phi^2)/R_\phi(m_\phi^2)$ and used (A.38) as the single $(\omega\phi)$ equation to compute broken couplings in the manner described in the test. The same procedure was carried through for (ff'). In Table A1, we give the numerical contributions corresponding to the symbolic notation of Table 2.

FOOTNOTES AND REFERENCES

* This work was supported in part by the U.S. Atomic Energy Commission.

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1. Abbreviated discussion of this work have appeared in Phys. Letters 27B, 19 (1968), and in Phys. Rev. Letters 20, 628 (1968). Discussions of related processes have been given by M. Ademollo et al., Phys. Rev. Letters 19, 1402 (1967); D. Gross, Phys. Rev. Letters 19, 1303 (1967).

2. R. Dolen, D. Horn, and C. Schmid, Phys. Rev. 166, 1768 (1968); K. Igi and S. Matsuda, Phys. Rev. Letters 18, 625 (1967); K. Igi, Phys. Rev. Letters 9, 76 (1962); A. Logunov, L. Soloviev, and A. Tavkhelidze, Phys. Letters 24B, 181 (1967).

3. See for example, P. Kaus and F. Zachariasen, preprint CALT-68-149, 1968 (unpublished).

4. Earlier work on this subject includes R. Cutkosky, Phys. Rev. 131, 1888 (1963); Chan H-M., P. DeCelles, and J. Paton, Phys. Rev. Letters 10, 312 (1963). More recently, from a somewhat different point of view, R. H. Capps (Purdue University preprint, 1968, unpublished) has extended the discussion to a wide class of meson systems. Meson bootstraps from the current algebra point of view have been studied by Chan H-M., Nuovo Cimento 43, 347 (1966).

5. S. Mandelstam, Nuovo Cimento 30, 1148 (1963).

6. The error estimate in the test depends on the assumption that oscillations about an average value actually occur. This of course need not be the case.

7. For a view of the Pomeranchon in the context of FESR, see H. Harari, Phys. Rev. Letters 20, 1395 (1968).

8. H. Cheng and D. Sharp, Ann. Phys. (N.Y.) 22, 481 (1963); Phys. Rev. 132, 1854 (1963).

9. H. Cheng, Phys. Rev. 144, 1237 (1966).

10. S. Mandelstam, Phys. Rev. 166, 1539 (1968).

11. P. Bareyre et al., Phys. Letters 18, 342 (1965).

12. We cannot prove the statement in the text. If one uses functions of the type recently proposed by G. Veneziano, Nuovo Cimento 57, 190 (1968), this behavior is exactly what one gets. For example

$$\frac{\Gamma(1 - x)\, \Gamma(1 - y)}{\Gamma(1 - x - y)} = \sum_{K=1}^{\infty} \frac{\Gamma(K + y)}{\Gamma(K)\, \Gamma(y)} \frac{1}{x - K} ,$$

and this series converges for $y < 0$. We identify $x = \alpha(s)$, $y = \alpha(t)$. The Froissart-Gribov form of the partial-wave amplitudes is then, with $\alpha(t) = a + bt$,

$$a_J(s) = \frac{1}{2q^2} \sum_{K=1}^{\infty} Q_J\left(1 + \frac{K - a}{2bq^2}\right) \frac{\Gamma[K + \alpha(s)]}{\Gamma(K)\, \Gamma[\alpha(s)]} ,$$

and this series diverges at $J = \alpha(s), \alpha(s) - 1, \cdots$.

13. For a discussion of SU(n) see R. E. Behrends et al., Rev. Mod. Phys. 34, 1 (1962).

14. S. Glashow and R. Socolow, Phys. Rev. Letters 15, 329 (1965).

15. S. Okubo, Progr. Theoret. Phys. (Kyoto) 27, 949 (1962).

16. See N. Jacobsen, Lie Algebras (Interscience, London, 1962).

17. For an alternative interpretation of the orthonormality condition (c), see R. E. Cutkosky, Phys. Rev. 131, 1888 (1963).

18. For the Cartan classification of Lie algebras, see Ref. 16. We are informed by Professor N. Burgoyne that the statement in the text is also true for E_8, the relevant product being

$$248 \oplus 248 = 1 \oplus 3875 \oplus 27000 \oplus 248 \oplus 30380 \quad ,$$

the last two terms appearing antisymmetrically. We thank Professor Burgoyne for a helpful discussion.

19. M. Gell-Mann, Phys. Rev. 125, 1067 (1962).

20. This is a strong assumption because different but well chosen N's for different channels may very well also work reasonably.

21. R. C. Arnold, Phys. Rev. Letters 14, 657 (1965).

22. R. F. Dashen et al., Phys. Rev. 143, 1185 (1966) have studied SU(3) coupling breaking extensively from the N/D point of view. The comments about this work in Ref. 1 are unfortunately unclear. We are unable to compare the possible numerical accuracy of their work with ours.

23. After the present text was written, a suggestion by Veneziano (Ref. 12) appeared, which evidently satisfies the general requirements made here.

24. A. H. Rosenfeld et al., Rev. Mod. Phys. 40, 77 (1968).

Table 1. FESR arising from Eq. (1.2). Shown are the objects which contribute to the LHS: (-) = no contribution, (X) = contribution. (y) means contributes to the y channel only. The right-hand side is always taken to be the t channel. (SC) = superconvergent.

LHS						RHS			Process
ρ	$\omega\phi$	K*	A_2	ff'	K**	I(Y)	n	Particle	
s X	-	-	-	s X	-	1	0	ρ	
s X	-	-	-	s X	-	0	1	ff'	$\pi\pi \to \pi\pi$
s X	-	-	-	s X	-	2	1	SC	
s X	-	u X	-	s X	u X	1/2	0	K*	
s X	-	u X	-	s X	u X	1/2	1	K**	
-	-	X	-	-	X	1	0	ρ	$K\pi \to K\pi$
-	-	X	-	-	X	0	1	ff'	$(\bar{K}K \to \pi\pi)$
s X	-	u X	-	s X	u X	3/2	0	SC	
s X	-	u X	-	s X	u X	3/2	1	SC	

Table 1 (Continued).

LHS						RHS			Process
ρ	ωφ	K*	A_2	ff'	K**	I(Y)	n	Particle	
t X	t X	-	t X	t X	-	1	0	ρ	
t X	t X	-	t X	t X	-	1	1	A_2	$KK \to KK$
t X	t X	-	t X	t X	-	0	1	ff'	($\bar{K}\bar{K} \to \bar{K}\bar{K}$)
t X	t X	-	t X	t X	-	0	0	φω	
X	X	-	X	X	-	0(2)	0	SC	
X	X	-	X	X	-	1(2)	1	SC	
-	-	t X	-	s X	t X	1/2	0	K*	
-	-	t X	-	s X	t X	1/2	1	K**	$K\eta \to K\eta$
-	-	X	-	-	X	0	1	ff'	($K\bar{K} \to \eta\eta$)
-	-	X	-	-	X	0	0	SC	
-	-	u X	s X	-	u X	1/2	0	K*	
-	-	u X	s X	-	u X	1/2	1	K**	$K\eta \to K\pi$
-	-	X	-	-	X	1	1	A_2	($K\bar{K} \to \pi\pi$)
-	-	X	-	-	X	1	0	SC	

Table 1 (Continued).

LHS						RHS			Process
ρ	ωØ	K*	A_2	ff'	K**	I(Y)	n	Particle	
-	-	-	u X	s X	-	1	1	A_2	
-	-	-	u X	s X	-	1	0	SC	$\pi\eta \to \pi\eta$
-	-	-	X	-	-	0	0	ff'	$(\eta\eta \to \pi\pi)$
-	-	-	X	-	-	0	0	SC	
-	-	-	-	X	-	0	1	ff'	$\eta\eta \to \eta\eta$
-	-	-	-	X	-	0	0	SC	

Table 2. Symbolic FESR for $P + P \rightarrow P + P$. Signs and magnitudes of crossing-matrix elements are shown explicitly, along with factors of 1/2 arising if a left-hand resonance contributes to one cross channel only. The definitions of the symbols are:

$$\text{Left-hand symbol:} \quad (y) = \frac{\pi(2J+1) \quad P_J(Z) \; \nu^n M_y^{\,2} (x_i x_f)^{\frac{1}{2}}}{(4P_i P_f)^{\frac{1}{2}}} \quad ;$$

$$\text{Right-hand symbol:} \quad (x) = \frac{N^{\alpha(m_x^{\,2})+n+1}}{\alpha(m_x^{\,2}) + n + 1} \beta(m_x^{\,2}) \quad .$$

(See Appendix A for details and notation.)

	Process
$1/3\ ff' + 1/2\ \rho = \rho$	$\pi\pi \rightarrow \pi\pi$
$1/3\ ff' + \rho = ff'$	
$1/3\ ff' - 1/2\ \rho = 0$	

Table 2 (Continued).

	Process
$1/2\left[\rho + \frac{1}{(6)^{\frac{1}{2}}} ff' - 1/3\ K^* + 1/3\ K^{**}\right] = K^*$	
$1/2\left[\rho + \frac{1}{(6)^{\frac{1}{2}}} ff' + 1/3\ K^* - 1/3\ K^{**}\right] = K^{**}$	
$2/3[K^* + K^{**}] = \rho$	$K\pi \to K\pi$
$(6)^{\frac{1}{2}}/3[K^* + K^{**}] = ff'$	$(\bar{K}K \to \pi\pi)$
$1/2\left[-1/2\ \rho + \frac{1}{(6)^{\frac{1}{2}}} ff' + 2/3(K^* - K^{**})\right] = 0 \quad (n = 0)$	
$1/2\left[-1/2\ \rho + \frac{1}{(6)^{\frac{1}{2}}} ff' - 2/3(K^* - K^{**})\right] = 0 \quad (n = 1)$	
$1/2[1/2\ \omega\phi + 1/2\ ff' - 1/2(\rho + A_2)] = \rho$	
Same $= A_2$	
$1/2[1/2\ \omega\phi + 1/2\ ff' + 3/2(\rho + A_2)] = ff'$	$KK \to KK$
Same $= \phi\omega$	$(K\bar{K} \to K\bar{K})$
$3/2(\rho - A_2) + 1/2\ ff' - 1/2\ \phi\omega = 0 \quad (n = 0)$	
$-1/2(\rho - A_2) + 1/2\ ff' - 1/2\ \phi\omega = 0 \quad (n = 1)$	

Table 3. Coupling-constant relations in mass degenerate limit.

[Compare Eqs. (3.9) and (3.11).]

$$3(\gamma_{\overline{K}K}{}^{\rho})^2 = (\gamma_{\overline{K}K}{}^{\phi\omega})^2$$

$$3(\gamma_{\overline{K}K}{}^{A_2})^2 = (\gamma_{\overline{K}K}{}^{ff'})^2$$

$$\xi_1\left[3(\gamma_{\overline{K}K}{}^{\rho})^2 + (\gamma_{\overline{K}K}{}^{\phi\omega})^2\right] = 3(\gamma_{\overline{K}K}{}^{A_2})^2 + (\gamma_{\overline{K}K}{}^{ff'})^2$$

$$\xi_1(\gamma_{\pi\pi}{}^{\rho})^2 = 2/3(\gamma_{\pi\pi}{}^{ff'})^2$$

$$\xi_1(\gamma_{K\eta}{}^{K^*})^2 = 9(\gamma_{K\eta}{}^{K^{**}})^2$$

$$\xi_1\,\gamma_{K\eta}{}^{K^*}\,\gamma_{K\pi}{}^{K^*} + 3\gamma_{K\eta}{}^{K^{**}}\,\gamma_{K\pi}{}^{K^{**}} - 9/4\gamma_{\eta\pi}{}^{A_2}\,\gamma_{\overline{K}K}{}^{A_2} = 0$$

$$(\gamma_{K\pi}{}^{K^{**}})^2 - \xi_1(\gamma_{K\pi}{}^{K^*})^2 = 0$$

$$(\gamma_{K\pi}{}^{K^*})^2 - 3/4\gamma_{\overline{K}K}{}^{\rho}\,\gamma_{\pi\pi}{}^{\rho} = 0$$

$$(\gamma_{K\pi}{}^{K^{**}})^2 + \xi_1(\gamma_{K\pi}{}^{K^*})^2 - 3/(6)^{\frac{1}{2}}\,\gamma_{\pi\pi}{}^{ff'}\,\gamma_{\overline{K}K}{}^{ff'} = 0$$

Table 4.

A. Independent groups of reactions corresponding to one amplitude.

B. Resonances occurring in A as intermediate states.

C. $(\text{Isoscalar factor})^2$ for SU(3), assuming magic mixing angle $\tan^2\theta = 1/2$, and the 2^+ singlet/octet coupling ratio (Ref. 14) $G^2/F^2 = 8$. (2^+ isoscalars are multiplied by 5/9).

D. Reduced experimental widths, divided by the corresponding entries in column C.

E. Our predicted theoretical reduced widths divided by the corresponding entries in column C. Shown are results for Cases I, II, and III with the corresponding self-consistent limits of integration.

F. Experimental partial widths taken from Ref. 24. In the $\pi\pi \rightarrow K\bar{K}$ case, $\Gamma_{exp}^{(i,j)} = (\Gamma_{\pi\pi} \cdot \Gamma_{K\bar{K}})^{\frac{1}{2}}$ is the effective width. Quantities used as input are underlined.

Table 4 (Continued)

A Channel	B Resonance	C Isoscalars	D $\frac{\beta(\text{exp})}{\text{Isoscalars}}$	E β theory/isoscalars I	 II	 III	F $\Gamma_{\text{exp}}^{(i,j)}$ [in GeV]
$\pi\pi \to \pi\pi$	ρ	2/3	0.73 ± 0.18	$\underline{1}$	0.91	1.04	0.124
	f } $f_1 + f_8$	1	$\underline{1}$ (±0.14)	} -	$\underline{1}$	$\underline{1}$	0.137
	f'	0	not observed				<0.010
$K\pi \to K\pi$ $\pi\pi \to K\bar{K}$	ρ	1/3	-	1.04	0.91	0.85	-
	K*	1/4	$\underline{1}$ (±0.02)	$\underline{1}$	$\underline{1}$	$\underline{1}$	0.049
	f } $f_1 + f_8$	$1/(6)^{\frac{1}{2}}$	0.83 ± 0.32	} -	1.30	1.13	0.021
	f'	0	not observed				<0.023
	K**	1/4	0.85 ± 0.05	-	0.92	0.76	0.045

Table 4 (Continued)

A Channel	B Resonance	C Isoscalars	D $\frac{\beta(\text{exp})}{\text{Isoscalars}}$	E β theory/isoscalars I	 II	 III	F $\Gamma_{\text{exp}}^{(i,j)}$ [in GeV]
$K\bar{K} \to K\bar{K}$ $KK \to KK$	ρ	1/6	-	1.06	0.94	0.92	-
	ω } ω_8	1/6	-	1	1	1	-
	ϕ } ω_8	1/3	0.85 ± 0.20				0.0029
	A_2	1/6	0.54 ± 0.14	-	1.59	1.37	0.0036
	f } $f_1 + f_8$	1/6	0.69 ± 0.24	-	1.36	1.22	0.0032
	f' } $f_1 + f_8$	1/3	1.20 ± 0.44				0.053

Table 5. Cutoff masses corresponding to self-consistent integration limits N, as in Fig. 1.

Channel	$s_n(\mathrm{GeV}^2)$		
	Case		
	I	II	III
$\pi\pi \to \pi\pi$	1.05	1.57	1.87
$K\pi \to K\pi$ $\pi\pi \to \bar{K}K$	1.17	1.75	2.08
$K\bar{K} \to K\bar{K}$ $KK \to KK$	1.27	2.00	2.40

Table A1. Numerical values of the terms shown symbolically in Table 2, for the nonsuperconvergent relations. Each equation is identified by the object appearing on the RHS. For the RHS the value of $\beta(m_x^2)$ is shown. For the LHS the entire contribution is shown. To compute the broken couplings shown in Table 4, the f, f', and ω,φ contributions have been combined as outlined in Appendix A; the determinental condition then yields the limits of integration, N, in Table 5, and the couplings of Table 4.

Equation β(RHS)	ρ	K*(890)	K**(1420)	f^0	$f^{0'}$	A_2	ω^0	ϕ^0	Process
1.93 (K*)	0.74	-0.58	1.10	1.20	0.42	-	-	-	
1.54 (K**)	2.54	2.09	-10.36	10.56	3.98	-	-	-	
1.73 (f)	-	7.38	34.15	-	-	-	-	-	$K\pi \to K\pi$
0.40 (f')	-	13.00	66.22	-	-	-	-	-	$(\bar{K}K \to \pi\pi)$
0.94 (ρ)	-	1.94	3.31	-	-	-	-	-	

Table A1 (Continued).

Equation β(RHS)	ρ	K*(890)	K**(1420)	f^0	$f^{0'}$	A_2	ω^0	ϕ^0	Process
3.60 (f)	9.34	-	-	30.35	0.76	-	-	-	
0.05 (f')	16.20	-	-	59.37	1.36	-	-	-	$\pi\pi \rightarrow \pi\pi$
1.88 (ρ)	1.58	-	-	3.90	0.09	-	-	-	
0.47 (ρ)	-0.09	-	-	0.32	2.03	-0.28	0.22	0.67	
0.66 (A_2)	-0.33	-	-	3.26	20.80	-2.86	0.81	2.64	$KK \rightarrow KK$
0.83 (f)	0.88	-	-	2.83	18.50	7.49	0.71	2.38	($K\bar{K} \rightarrow K\bar{K}$)
2.90 (f')	1.83	-	-	6.29	36.44	16.25	1.46	4.28	
2.18 (ϕ)	0.59	-	-	0.74	3.95	1.89	0.47	1.16	
1.09 (ω)	0.29	-	-	0.34	2.12	0.89	0.23	0.70	

FIGURE CAPTIONS

Fig. 1. Mandelstam diagram for $P + P \to P + P$ in degenerate case showing self-consistent limits of integration N, for Cases I, II, and III, as discussed in Section IV of the text, and also showing the location of 1^- and 2^+ poles.

Fig. A1. The process $P + P \to P + P$ with s-channel exchange of spin J object.

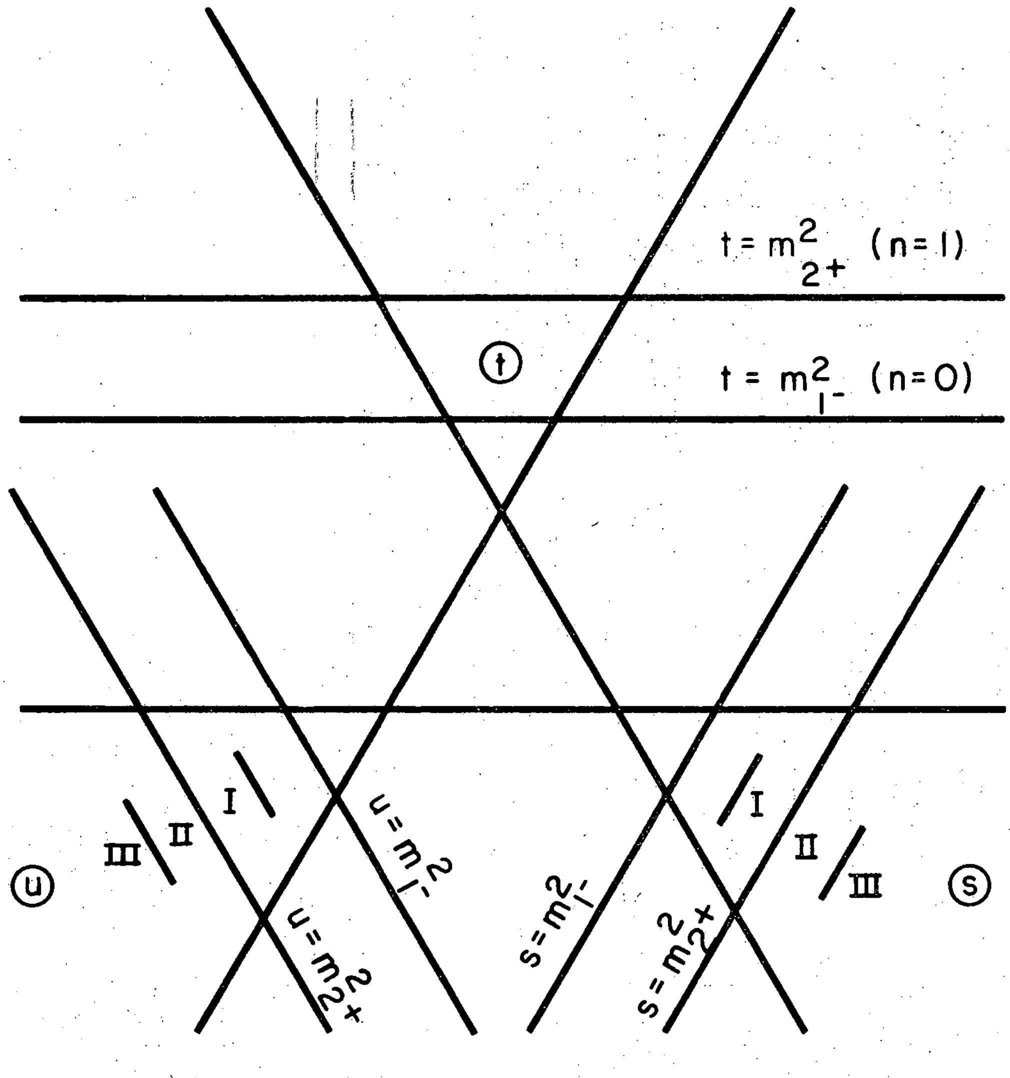
t = m$^2_{2+}$ (n=1)
t
t = m$^2_{1-}$ (n=0)
u
III
II
I
u=m$^2_{1-}$
u=m$^2_{2+}$
s=m$^2_{1-}$
s=m$^2_{2+}$
I
II
III
s
XBL6812-7296

Fig. 1.

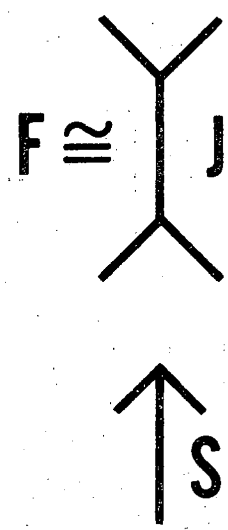

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Fig. A.1.

APPENDIX B*

Crossing Trajectories and Possible Branch Points

Following Cheng,[9] we consider the partial-wave amplitude $a(J, t)$ and the trajectory function $\alpha(t)$. We have, defining $D \equiv 1/a(J,t)$,

$$D[\alpha(t), t] = 0 \quad . \tag{B.1}$$

Suppose $D(J, t)$ is regular around (J_0, t_0) and that $D(J_0, t_0) = 0$. Then the Taylor expansion of D yields

$$\begin{aligned} D(J_0, t_0) &= 0 \\ &= \partial D/\partial J \Big|_{(J,t)=(J_0,t_0)} [\alpha(t) - J_0] + \frac{\partial D}{\partial t}\Big|_{(J,t)=(J_0,t_0)} (t - t_0) \\ &\quad + \frac{\partial^2 D}{\partial J^2}\Big|_{(J,t)=(J_0,t_0)} [\alpha(t) - J_0]^2 + \cdots \quad . \end{aligned} \tag{B.2}$$

If

$$\frac{\partial D}{\partial J}\Big|_{(J,t)=(J_0,t_0)} \neq 0 \quad ,$$

$$\alpha(t) = J_0 - R(t - t_0) + \cdots \quad , \tag{B.3}$$

---

* (Appendices B, C, and D are not included in the version of this report submitted for publication.)

where

$$R \equiv \left.\frac{\partial D/\partial t}{\partial D/\partial J}\right|_{x_0} .$$

On the other hand, if $\left.\partial D/\partial J\right|_{x_0} = 0$,

$$\alpha(t) = J_0 \pm \left(\frac{\partial D/\partial t}{\partial^2 D/\partial J^2}\right)^{\frac{1}{2}} (t - t_0)^{\frac{1}{2}} + \cdots , \qquad (B.4)$$

explicitly exhibiting the two crossing trajectories at $J = J_0$.

APPENDIX C

Some Convenient Relations for SU(n) and U(n)

The following relations are convenient for the discussion of algebraic properties as in Section III above. We define, following Gell-Mann,[19] the matrices $\{\lambda_i\}$ which generate SU(n). They satisfy

$$[\lambda_i, \lambda_j] \equiv [\lambda_i, \lambda_j]_- = 2if_{ijk}\lambda_k \quad , \tag{C.1}$$

$$\{\lambda_i, \lambda_j\} \equiv [\lambda_i, \lambda_j]_+ = 2d_{ijk}\lambda_k + (4/n)\delta_{ij} \quad , \tag{C.2}$$

for

$$1 \leqslant (i, j, k) \leqslant n^2 - 1 \quad ,$$

where the λ's are normalized so that

$$\mathrm{Tr}(\lambda_i\lambda_j) = 2\delta_{ij} \quad . \tag{C.3}$$

Under ordinary multiplication we have

$$\lambda_i\lambda_j = (2/n)\delta_{ij} + (d_{ijk} + if_{ijk})\lambda_k \tag{C.4}$$

and also

$$f_{ijk} = -\frac{i}{4}\,\mathrm{Tr}([\lambda_i, \lambda_j]\lambda_k) \quad , \tag{C.5}$$

$$d_{ijk} = +\frac{1}{4}\,\mathrm{Tr}(\{\lambda_i, \lambda_j\}\lambda_k) \quad . \tag{C.6}$$

From these relations d and f are totally antisymmetric and symmetric in their indices respectively.

The λ_i are $n \times n$ and traceless. Furthermore $\lambda_j\lambda_j$ commutes with all the λ_i and is therefore a multiple of the identity. From (C.6) we have

$$d_{iik} = \frac{1}{4}\,\mathrm{Tr}(\{\lambda_i, \lambda_i\}\lambda_k) = \frac{1}{2}\,\mathrm{Tr}(c' \underset{\sim}{1} \lambda_k) = 0 \ . \tag{C.7}$$

The Cartan condition for the f's reads

$$f_{ijk}f_{pjk} = n\delta_{ip} \ , \tag{C.8}$$

and the symmetry properties give

$$f_{ijk}d_{pjk} = 0 \ . \tag{C.9}$$

It is convenient for the derivation of further relations to employ the matrix identities

$$[[A, B], C] + [[B, C], A] + [[C, A], B] = 0, \tag{C.10}$$

$$[\{A, B\}, C] + [\{B, C\}, A] + [\{C, A\}, B] = 0 \ , \tag{C.11}$$

$$\{A, \{B, C\}\} - \{B, \{C, A\}\} + [C, [A, B]] = 0 \ . \tag{C.12}$$

Inserting the λ_i into (C.10) - (C.12), we get

$$f_{abs}f_{scd} + f_{bcs}f_{sad} + f_{cas}f_{sbd} = 0 \ , \tag{C.13}$$

$$d_{abs}f_{scd} + d_{bcs}f_{sad} + d_{cas}f_{sbd} = 0 \ , \tag{C.14}$$

$$d_{asd}d_{bcs} - d_{bsd}d_{cas} - f_{csd}f_{abs} = 0 \ . \tag{C.15}$$

From (C.8) and (C.15)

$$d_{ijl}d_{\ell jk} = \frac{n^2 - 4}{n}\,\delta_{i\ell} \quad . \tag{C.16}$$

Last, we give four relations for trilinears in f and d:

$$f_{piq}f_{qjr}f_{rkp} = -\frac{n}{2}\,f_{ijk} \quad , \tag{C.17}$$

$$d_{piq}f_{qjr}f_{rkp} = -\frac{n}{2}\,d_{ijk} \quad , \tag{C.18}$$

$$d_{fiq}d_{qjr}f_{rkp} = \frac{n^2 - 12}{2n}\,d_{ijk} \quad . \tag{C.19}$$

It is sometimes useful to define d's and f's for U(n). We set

$$\lambda_0 = (2/n)^{\frac{1}{2}}\,\underset{\sim}{1} \quad , \tag{C.20}$$

so (C.2) becomes

$$\{\lambda_i, \lambda_j\}_+ = 2d'_{ijk}\lambda_k \quad , \tag{C.21}$$

for

$$0 \leqslant (i, j, k) \leqslant n^2 - 1 \quad ;$$

then

$$d'_{oij} = (2/n)^{\frac{1}{2}}\,\delta_{ij} \quad . \tag{C.22}$$

Equation (C.4) becomes

$$\lambda_i\lambda_j = (d'_{ijk} + if'_{ijk})\lambda_k \quad , \tag{C.23}$$

(C.7) gives

$$d'_{iik} = (2/n)^{\frac{1}{2}} n^2 \quad , \tag{C.24}$$

and (C.16) becomes

$$d'_{ijk} d'_{\ell jk} = n(\delta_{i\ell} + \delta_{i0}\, \delta_{\ell 0}) \quad , \tag{C.25}$$

with the new f equation

$$f'_{ijk}\, f'_{\ell jk} = n(\delta_{i\ell} - \delta_{i0}\, \delta_{\ell 0}) \quad . \tag{C.26}$$

We also have

$$d'_{i\ell m} f'_{mjk} + d'_{j\ell m} f'_{imk} + d'_{k\ell m} f'_{ijm} \neq 0 \quad , \tag{C.27}$$

which follows immediately from (C.18).

(Most of the above relations are contained in Macfarland et al., Cambridge University preprint 1968, unpublished.)

APPENDIX D

Derivation of FESR

We derive the finite energy sum rule using the Khuri representation.

Consider the amplitude for the process $P + P \rightarrow P + P$, assuming equal masses. In the physical region for the t channel make the expansions

$$A^{\pm}(\nu, t) = \sum_{J=0}^{\infty} a^{\pm}(J, t)\, P_J(z_t)(2J + 1) \quad , \tag{D.1}$$

of the even and odd J parity amplitudes.

Noting $\nu = \frac{1}{2}(s - u)$, and defining

$$a_{(1)}(J, t) = \frac{1}{\pi} \int_{4m^2}^{\infty} \frac{ds}{2q^2}\, Q_J\left(1 + \frac{s}{2q^2}\right) D_s(t, s) \quad , \tag{D.2}$$

$$a_{(2)}(J, t) = \frac{1}{\pi} \int_{4m^2}^{\infty} \frac{du}{2q^2}\, Q_J\left(1 + \frac{u}{2q^2}\right) D_u(t, u) \tag{D.3}$$

$(4q^2 = t - 4m^2)$

$$a^{\pm}(J, t) = a_{(1)}(J, t) \pm a_{(2)}(J, t) \quad , \tag{D.4}$$

where $D_{s,u}$ are the s,u discontinuities in the dispersion relation

$$A(\nu, t) = \frac{1}{\pi}\int_{4m^2}^{\infty} ds' \frac{D_s(t, s')}{s' - s} + \frac{1}{\pi}\int_{4m^2}^{\infty} du' \frac{D_u(t, u')}{u' - u} . \tag{D.5}$$

Note

$$z_t = \frac{s - u}{t - 4m^2} = \frac{\nu}{2q^2} = -1 - u/2q^2 = 1 + s/2q^2 .$$

If there are no exchange forces $D_u = a_{(2)} = 0$, $a^+ = a^-$, and we have what is usually called exchange degeneracy.

We prefer to expand

$$A(\nu, t) = \sum_{k=0}^{\infty} C_1(k, t)s^k + C_2(k, t)_u{}^k = \sum_{k=0}^{\infty} C(k, t)\nu^k , \tag{D.6}$$

in a power series in ν (i.e., in s and u). Khuri, [N. N. Khuri, Phys. Rev. 132, 914 (1963); 130, 429 (1963)] has proved theorems relating the analytic behavior of $a_{(i)}(J, t)$ in the complex J plane to that of $C_{(1,2)}(J, t)$, under the assumptions (i) $a(J, t)$ has a unique, meromorphic interpolation for $\text{Re } J > -\frac{1}{2}$ with a finite number of poles in $\text{Re } J > -\frac{1}{2}$; (ii) $\lim_{J\to\infty} a(J, t) \to c(t)\sqrt{J}e^{-|J|\,\xi(t)}$,

$$\text{Re } \xi(t) > 0, \quad \text{Im } \xi(t) = 0 .$$

Under these conditions $C(K,t)$ is such that we can make a Watson-Sommerfeld transformation on (D.6),

$$A^{\pm}(\nu,t) = \int_{J_0-i\infty}^{J_{0+i\infty}} dJ\; C_{\pm}(J, t)(-\nu)^J \; \frac{\pm 1 - e^{-i\pi J}}{\sin(\pi J)} \;, \tag{D.7}$$

where we have separated $A(\nu,t)$ into even and odd parts in ν. We have generalized Khuri's result slightly here, assuming $A(\nu, t)$ has no singularities for $\operatorname{Re} J > J_0$. Note that (D.7) is true provided analyticity, crossing, and unitarity hold. When t is chosen such that the positions of the poles in the J plane, $\alpha(t)$, satisfy $\operatorname{Im} \alpha \equiv 0$, $\operatorname{Re} \alpha \approx n$ (n any integer), the pole contributions which appear on moving the contour in (D.7) have the usual Regge form

$$\sim \beta\left(\frac{\nu}{\nu_0}\right)^{\alpha} \; \frac{\pm 1 - e^{-i\pi\alpha}}{\sin \pi\alpha} \quad . \tag{D.8}$$

As we move the contour to the left, uncovering the right half of the J plane, we will get, in general, a sum over pole contributions, a sum over cut contributions, and a remainder integral, usually referred to in the literature as the background integral. Our approximation will be to neglect all these terms, except the leading Regge pole, so that (D.7) becomes

$$A^{\pm}(\nu, t) = (\text{Background Integral}) + \sum_{\text{cuts}} + \sum_{\text{poles}}$$

$$\approx \beta(t)\left(\frac{\nu}{\nu_0}\right)^{\alpha(t)} \; \frac{\pm 1 - e^{-i\pi\alpha(t)}}{\sin \pi\alpha(t)} \quad . \tag{D.9}$$

We have here two kinds of amplitudes with $J^P = (1^-, 3^-, \cdots)$ or $(0^+, 2^+, \cdots)$, corresponding to odd or even $A(\nu, t)$. We assume here also $\text{Im}\,\alpha \approx 0$, $\text{Re}\,\alpha(t) \approx at + b$, with $b \approx 1\ \text{GeV}^{-2}$ for all trajectories.

Taking the imaginary part of (B.9) for $t = (m_{\text{resonance}})^2$, multiplying by ν^n, and integrating from $-N$ to $+N$, we get

$$\frac{1}{2}\int_{-N}^{+N} \text{Im}\,A(\nu, t)\, d\nu\, \nu^n \approx \frac{\beta(t)\, N^{\alpha(t)+n+1}}{\alpha(t) + n + 1} . \tag{D.10}$$

Since taking the imaginary part changes the symmetry in ν, n = even in (D.10) corresponds to the trajectory with J^P = odd, and n = odd to $J^P(\text{even}^+)$ [V(T) trajectories].

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