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FINITE FORMULATION OF THE ELECTROMAGNETIC FIELD

E. Tonti

Università di Trieste, 34127 Trieste, Italy

Abstract—The objective of this paper is to present an approach to electromagnetic field simulation based on the systematic use of the global (i.e. integral) quantities. In this approach, the equations of electromagnetism are obtained directly in a finite form starting from experimental laws without resorting to the differential formulation. This finite formulation is the natural extension of the network theory to electromagnetic field and it is suitable for computational electromagnetics.

1. INTRODUCTION

The laws of electromagnetic phenomena were first formulated by their discoverers using global quantities, such as charge, current, electric and magnetic flux, electromotive and magnetomotive force. The Kirchhoff's network equations were also expressed using current and voltage.

After the publication of Maxwell's treatise, electromagnetic laws were commonly written using differential formulation. From that moment, electromagnetic field equations were identified with the "Maxwell equations", i.e. with partial differential equations.

When applied to field theories, numerical methods require the solution of a system of algebraic equations. It is standard practice to derive these equations starting from the differential equations resorting one of many discretization methods. This is the case, for instance, of finite difference methods, finite element methods, edge element methods, etc. This is summarized in the upper part of Fig. 1.

Even when an integral formulation is used, as in the finite volume method or in the finite integration theory (an extension of the finite-difference time-domain method), standard practice is to use integrals of field functions. Field functions are an indispensable ingredient of differential formulation. At this point, one can pose the following question: *is it possible to express the laws of electromagnetism directly by a set of algebraic equations, instead of obtaining them from a discretization process applied to differential equations?*

In this paper, we show that such a finite formulation is possible, it is simple, and that it is useful for numerical computation.

In such formulation, the classical procedure of writing the laws of physics in differential form is inverted. Instead, we start from finite formulation and deduce differential formulation whenever it is required. In traditional methods, one is forced to select one of many discretization procedures. This is not the case of the finite formulation as illustrated in the lower part of Fig. 1.

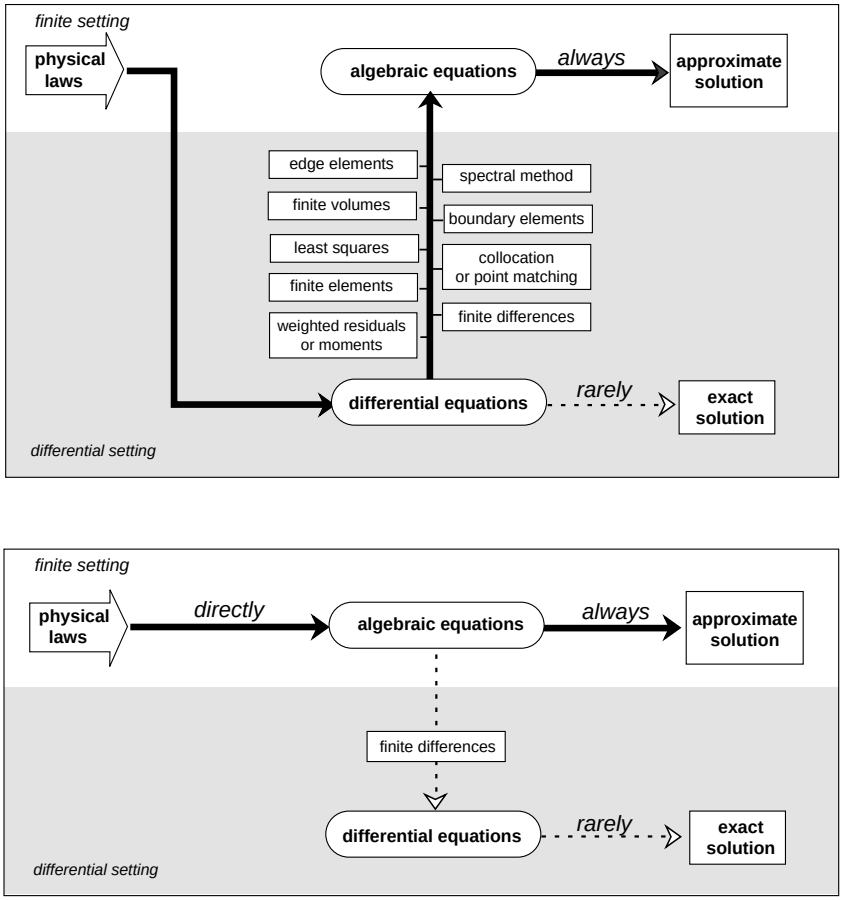


Figure 1. *(above)* In traditional differential formulation to obtain an approximate solution one is forced to pass through one of many methods of discretization. *(below)* On the contrary, using global variables and complexes, one obtain a finite formulation directly.

What we propose in this paper *is not* a refusal of the differential formulation of electromagnetic laws but *an alternative* to it. Our aim is to show that, for numerical purposes, it is more convenient to describe electromagnetism in a finite form from the beginning and later to obtain differential formulation as a consequence.

Exact and approximate solutions. To avoid differential formulation as starting point we need to completely revise our attitude.

In the paradigm formed by three centuries of differential formulation of physical laws, we find the differential formulation so prevalent that we are led to think that it is *the natural* formulation for physics. Moreover, we are convinced that differential formulation leads to an *exact* solution to physical problems.

However, we know full well that only in a few elementary cases we can obtain a solution in *closed* form: hence the "exact solution" promised by differential formulation, *is almost never attained in practice.* Moreover the great scientific and technological advancement obtained in our days by numerical solution of physical problems that do not admit a solution in closed form, suggests that this progress arises mainly because we have found the way to obtain *approximate* solutions to our problems. In our culture, modelled on mathematical analysis, the term "approximate" sounds flawed. Nevertheless the goal of a numerical simulation is agreement with *experimental measurements*.

To reduce error of an approximate solution does not mean to make the error *as small as we like*, as a limit process requires, but to make error *smaller than a preassigned tolerance*.

We are well aware that all measurements are affected by a tolerance: every measuring instrument belongs to a given class of precision. In measurements an "infinite" precision, in the sense of a limit process of mathematics, is not attainable. The same positioning of the measuring probe in a field implies a tolerance.

The notion of *precision* in measuring apparatus plays the same role of the notion of *tolerance* in manufacturing and of the notion of *error* in numerical analysis.

In conclusion one cannot deny the satisfaction of knowing the exact solution of a physical problem *when the latter is available*. What we deny is *the need* to refer to an idealized exact solution when this is not available in order to compare a numerical result with experience.

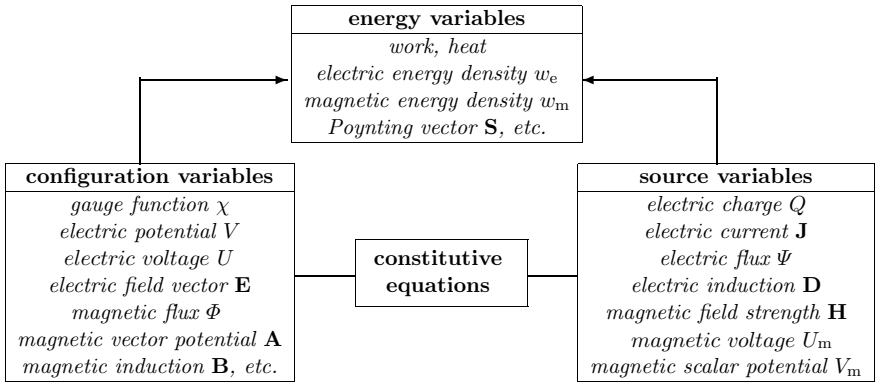
2. FINITE FORMULATION: THE PREMISES

A reformulation of field laws in a direct finite formulation must start with an analysis of physical quantities in order to make explicit the maximum of information content that is implicit in definition and in measurement of physical quantities. To this end it is convenient to introduce two classifications of physical quantities.

2.1. Configuration, Source and Energy Variables

A first classification criterion of great usefulness in teaching and in research is that based on the *role* that every physical variable plays in

Table 1. A classification of physical variables of electromagnetism.



a theory. Analysis of the role of physical variables in a theory leads to three classes of variables: *configuration*, *source* and *energy* variables. These three classes for electromagnetism are shown in Table 1. In every field of physics one can find:

- *Configuration variables* that describe the configuration of the field or of the system. These variables are linked one to another by operations of sum, of difference, of limit, of derivative and integral.
- *Source variables* that describe the sources of the field or the forces acting on the system. These variables are linked one to another by operations of sum, of difference, of limit, of derivative and integral.
- *Energy variables* that are obtained as the product of a configuration for a source variable. These variables are linked one to another by operations of sum and difference, of limit, of derivative and integration.

This classification has a pivotal role in physical theories. One consequence is the fact that it permits *constitutive* equations to be defined: *they are equations that link configuration with source variables of a physical field and contain material and system parameters*. This classification has been given by Hallen in 1947 [9, p.1]; by Penfield and Haus in 1967 [21, p.155] and in 1972 by the present author [29, p.49].

2.2. Global Variables and Field Variables

To introduce a finite formulation for electromagnetics we take a radical viewpoint: *we search for a formulation completely independent from the differential one.* To this end we avoid introducing field functions, and, as a consequence, we avoid the integration process. For this reason

instead of the term "integral" quantity we shall use the equivalent term *global* quantity.

We must emphasize that physical measurements deal mainly with *global variables*, not with field variables. Field variables are needed in a differential formulation because the very notion of derivative refers to a point function. On the contrary a global quantity refers to a system, to a space or time element like a line, a surface, a volume, an interval, i.e. is a *domain function.* Thus a flow meter measures the electric charge that crosses a given surface in a given time interval. A flux meter measures the flux (=flow rate) associated with a surface at given time instant. The corresponding physical quantities are associated with space and time elements, not only with points and instants.

One fundamental advantage of global variables is that they are continuous through the separation surface of two materials while the field variables suffer discontinuity. This implies that the differential formulation is restricted to regions of material homogeneity: one must break the domain in subdomains, one for every material and introduce jump conditions. If one reflects on the great number of different materials present in a real device, one can see that the idealization required by differential formulation is too restrictive.

This shows that *differential formulation imposes differentiability conditions on field functions that are restrictive from the physical point of view*.

Contrary to this, a direct finite formulation based on global variables accepts material discontinuities, i.e. does not add regularity conditions to those requested by the physical nature of the variable.

To help the reader, accustomed to thinking in terms of traditional field variables $\rho, \mathbf{J}, \mathbf{B}, \mathbf{D}, \mathbf{E}, \mathbf{H}$, we first examine corresponding integral variables $Q^{\mathrm{c}}, Q^{\mathrm{f}}$, Φ, $\Psi, \mathcal{U}, \mathcal{U}_{\mathrm{m}}$: these are collected in Table 2. This table shows that integral variables arise by integration of field functions on space domains i.e. lines, surfaces, volumes and on time intervals. The time integral of a physical variable, say F, will be called its *impulse* and will be denoted by the corresponding calligraphic letter, say $\mathcal{F}$. The last three variables of the left side, $\mathcal{K}, G, \Lambda$ deals with the hypothetical magnetic monopole charge, monopole flow, monopole production. The role of these variables and of the corresponding ones $\tau, \mathcal{V}_{\mathrm{m}}, \eta$ of the right side is clarified in Table 2.

It is remarkable that the *integral* configuration variables all have the dimension of a magnetic flux and that *integral* source variables all have the dimension of a charge. The product of a global configuration variable and a global source variable has the dimension of an action (energy $\times$ time).

Table 3 shows the six integral variables that are measurable and

Table 2. Integral physical variables of electromagnetism (global variables) and corresponding field functions. Underlined variables are the measurable ones.

configuration variables (*SI units: weber=volt × second*)		*source variables* (*SI units: coulomb=ampere × second*)	
gauge function	χ	elec. charge prod.	$Q^p = \int_{\tilde{\mathbf{T}}} \int_{\tilde{\mathbf{V}}} \sigma \, \mathrm{d}V \, \mathrm{d}t$
elec. potential impulse	$\mathcal{V} = \int_{\mathbf{T}} V \, \mathrm{d}t$	<u>elec. charge content</u>	$Q^{\mathrm{c}} = \int_{\tilde{\mathbf{V}}} \rho \, \mathrm{d}V$
electrokinetic momentum	$p = \int_{\mathbf{L}} \mathbf{A} \cdot \mathrm{d}\mathbf{L}$	<u>elec. charge flow</u>	$Q^{\mathrm{f}} = \int_{\tilde{\mathbf{T}}} \int_{\tilde{\mathbf{S}}} \mathbf{J} \cdot \mathrm{d}\mathbf{S} \, \mathrm{d}t$
<u>electric voltage impulse</u>	$\mathcal{U} = \int_{\mathbf{T}} \int_{\mathbf{L}} \mathbf{E} \cdot \mathrm{d}\mathbf{L} \, \mathrm{d}t$	<u>electric flux</u>	$\Psi = \int_{\tilde{\mathbf{S}}} \mathbf{D} \cdot \mathrm{d}\mathbf{S}$
<u>magnetic flux</u>	$\Phi = \int_{\mathbf{S}} \mathbf{B} \cdot \mathrm{d}\mathbf{S}$	<u>magnetic voltage impulse</u>	$\mathcal{U}_{\mathrm{m}} = \int_{\tilde{\mathbf{T}}} \int_{\tilde{\mathbf{L}}} \mathbf{H} \cdot \mathrm{d}\mathbf{L} \, \mathrm{d}t$
(magn. charge flow)	$\mathcal{K} = \int_{\mathbf{T}} \int_{\mathbf{S}} \mathbf{k} \cdot \mathrm{d}\mathbf{S} \, \mathrm{d}t$	(nameless)	$\tau = \int_{\tilde{\mathbf{L}}} \mathbf{T} \cdot \mathrm{d}\mathbf{L}$
(magn. charge content)	$G = \int_{\mathbf{V}} g \, \mathrm{d}V$	magn. pot. imp.	$\mathcal{V}_{\mathrm{m}} = \int_{\tilde{\mathbf{T}}} V_{\mathrm{m}} \, \mathrm{d}t$
(magn. charge prod.)	$\Lambda = \int_{\mathbf{T}} \int_{\mathbf{V}} \lambda \, \mathrm{d}V \, \mathrm{d}t$	(nameless)	η

Table 3. The global variables of electromagnetism to be used in finite formulation and corresponding field functions of differential formulation.

finite formulation global variables			*differential formulation field functions*	
electric charge content	Q^{c}	$\rightarrow$	ρ	electric charge density
electric charge flow	Q^{f}	$\rightarrow$	$\mathbf{J}$	electric current density
magnetic flux	Φ	$\rightarrow$	$\mathbf{B}$	magnetic induction
electric flux	Ψ	$\rightarrow$	$\mathbf{D}$	electric induction
electric voltage impulse	$\mathcal{U}$	$\rightarrow$	$\mathbf{E}$	electric field strength
magnetic voltage impulse	$\mathcal{U}_{\mathrm{m}}$	$\rightarrow$	$\mathbf{H}$	magnetic field strength

the corresponding field functions.

3. PHYSICAL VARIABLES AND GEOMETRY

There is a strict link between physics and geometry. This is well known. What does not seem to be well known is that global physical variables are naturally associated with space and time elements, i.e. points, lines, surfaces, volumes, instants and intervals. In order to examine such association we need the notion of orientation of a space element.

In differential formulation a fundamental role is played by points: field functions are point functions. In order to associate points with numbers we introduce *coordinate systems*.

In finite formulation we need to consider not only points ($\mathbf{P}$) but also lines ($\mathbf{L}$), surfaces ($\mathbf{S}$) and volumes ($\mathbf{V}$). We shall call these *space elements*. We use a boldface characters for reasons that will be explained later. The natural substitute of coordinate systems are *cell complexes*. They exhibit vertices, edges, faces and cells. The latter are representative of the four spatial elements $\mathbf{P}, \mathbf{L}, \mathbf{S}, \mathbf{V}$.

3.1. Inner and Outer Orientation

The notions of inner and outer orientation of a space element play a pivotal role in electromagnetism as well as in all physical theories. We shall refer to the left side of Fig. 2.

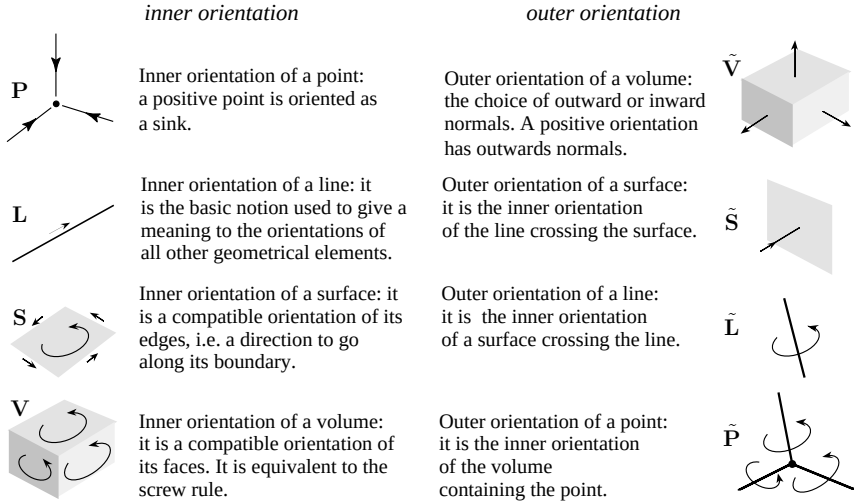


Figure 2. The two notions of inner and outer orientations in three-dimensional space.

Inner orientation. We shall refer to Fig. 2. Points can be oriented as "sources" or "sinks". The notion of source and sink, borrowed from fluid dynamics, can be used to define an inner orientation of points because it permits us to maintain the notion of incidence number from lines and points. In particular we note that points are usually oriented as *sinks*. This is never explicitly stated but it can be inferred from the fact that space differences of a point function between two points

$\mathbf{P}$ and $\mathbf{Q}$ are given by $(+1)f(\mathbf{Q}) + (-1)f(\mathbf{P})$. This means that the line segment $\mathbf{PQ}$, oriented from $\mathbf{P}$ to $\mathbf{Q}$, is positively incident in $\mathbf{Q}$ (incidence number $+1$) and negatively incident in $\mathbf{P}$ (incidence number -1). In other words: in the expression $(\mathbf{Q}-\mathbf{P})$ signs can be interpreted as incidence numbers between the orientation of the line segment and those of its terminal points.

A line is endowed of inner orientation when a direction has been chosen on the line. A surface is endowed with inner orientation when its boundary has an inner orientation. A volume is endowed with inner orientation when its boundary is so.

Outer orientation. To write a balance we need a notion of exterior of a volume, because we speak of charge *contained* in the volume. This is usually done by fixing outwards or inwards normals to its boundary, as shown in Fig. 2 (*right*). A surface is equipped with outer orientation when one of its faces has been chosen as positive and the other negative: this is equivalent to fixing the direction of an arrow crossing the surface from the negative to the positive face, as shown in Fig. 2 (*right*). We need the outer orientation of a surface when we consider a flow crossing the surface. A line is endowed with outer orientation when a direction of rotation *around* the line has been defined: think to the rotation of the plane of polarization of a light beam. A point is endowed with outer orientation when all line segment with origin in the point have an outer orientation. Think, for example, to the sign of the scalar magnetic potential of a coil at a point: its sign depends on the direction of the current in the coil.

The four space elements endowed with outer orientation will be denoted $\tilde{\mathbf{P}}, \tilde{\mathbf{L}}, \tilde{\mathbf{S}}, \tilde{\mathbf{V}}$.

Contrary to inner orientation, outer orientation depends on the dimension of the space in which the element is embedded, as shown in Fig. 3. Hence exterior orientation of a line segment embedded in a three-dimensional space is a direction of rotation *around* the segment; in a two-dimensional space it is an arrow that *crosses* the line and when the segment is embedded in a one-dimensional space, it is represented by two arrows as if the segment were compressed or extended. This is typical orientation used in mechanics to denote compression or traction of a bar.

3.2. Time Elements

Let us consider a given interval of the time axis and divide it into small intervals, as shown in Table 4. The primal instants, we shall denote $\mathbf{t}_0, \mathbf{t}_1, ..., \mathbf{t}_{n-1}, \mathbf{t}_n, \mathbf{t}_{n+1}, ...$ are oriented as sinks, such as space points.

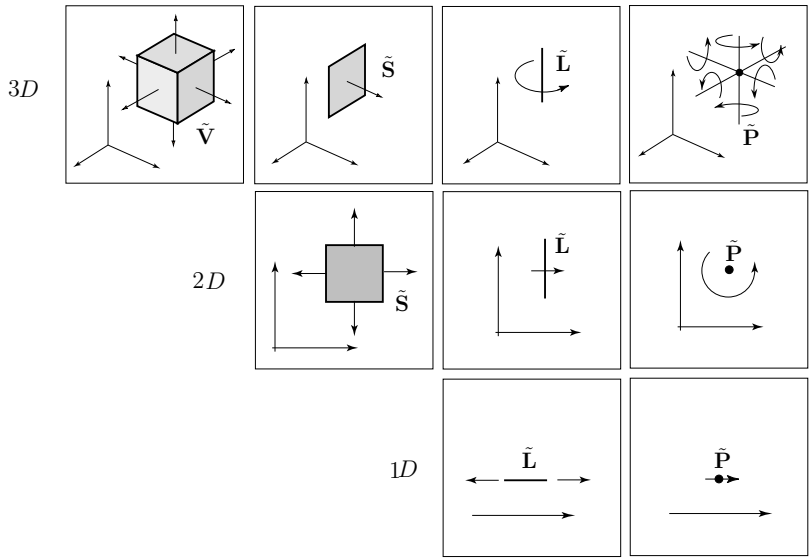


Figure 3. The outer orientation of a space element depends on the dimensions of the embedding space.

The primal intervals, we shall denote by $\boldsymbol{\tau}_1, ..., \boldsymbol{\tau}_n, \boldsymbol{\tau}_{n+1}, ...$ will be endowed with *inner* orientation, i.e. they are oriented towards increasing time. The dual instants $\tilde{\mathbf{t}}_1, ..., \tilde{\mathbf{t}}_n, \tilde{\mathbf{t}}_{n+1}, ...$ are endowed with *outer* orientation, i.e. they have the same orientation as primal intervals. The dual intervals $\tilde{\boldsymbol{\tau}}_1, ..., \tilde{\boldsymbol{\tau}}_n, \tilde{\boldsymbol{\tau}}_{n+1}, ...$ are endowed with *outer* orientation that is, by definition, the inner orientation of the primal instants.

3.3. Global Variables and Space-time Elements

From the analysis of a great number of physical variables of classical fields one can infer the

First Principle. *In a spatial description, global configuration variables are associated with space and time elements endowed with inner orientation. On the contrary, global source variables and global energy variables are associated with space and time elements endowed with outer orientation.*

The reason for associating source and energy variables with outer orientation is that they are used in balance equations and a balance require a volume with outer orientation (outwards or inwards normals).

Table 4. A time cell complex and its dual.

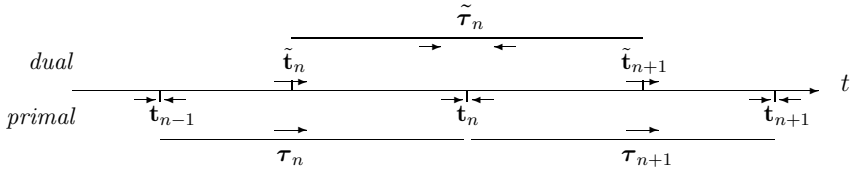


Table 5. The global variables of electromagnetism and the associated space and time elements.

global physical variable	*symbol*	*time element (orientation)*	*space element (orientation)*	*symbol*
electric charge content	Q^{c}	instant (outer)	volume(outer)	$\tilde{\mathbf{I}}\tilde{\mathbf{V}}$
electric charge flow	Q^{f}	interval (outer)	surface (outer)	$\tilde{\mathbf{T}}\tilde{\mathbf{S}}$
electric voltage impulse	$\mathcal{U}$	interval(inner)	line (inner)	$\mathbf{TL}$
magnetic voltage impulse	$\mathcal{U}_{\mathrm{m}}$	interval (outer)	line(outer)	$\tilde{\mathbf{T}}\tilde{\mathbf{L}}$
magnetic flux	Φ	instant (inner)	surface(inner)	$\mathbf{IS}$
electric flux	Ψ	instant (outer)	surface(outer)	$\tilde{\mathbf{I}}\tilde{\mathbf{S}}$
electric potential impulse	$\mathcal{V}$	interval (inner)	point(inner)	$\mathbf{TP}$
magnetic potential impulse	$\mathcal{V}_{\mathrm{m}}$	interval (outer)	point(outer)	$\tilde{\mathbf{T}}\tilde{\mathbf{P}}$

Table 6. The "descriptive" and the "formal" notations we use for space and time elements.

descriptive inner orientation		*formal primal complex*	
point	$\mathbf{P}$	$\mathbf{p}_h$	vertex
line	$\mathbf{L}$	$\mathbf{l}_\alpha$	edge
surface	$\mathbf{S}$	$\mathbf{s}_\beta$	face
volume	$\mathbf{V}$	$\mathbf{v}_k$	cell
instant	$\mathbf{I}$	$\mathbf{t}_n$	instant
interval	$\mathbf{T}$	$\boldsymbol{\tau}_m$	interval

descriptive outer orientation		*formal dual complex*	
volume	$\tilde{\mathbf{V}}$	$\tilde{\mathbf{v}}_h$	cell
surface	$\tilde{\mathbf{S}}$	$\tilde{\mathbf{s}}_\alpha$	face
line	$\tilde{\mathbf{L}}$	$\tilde{\mathbf{l}}_\beta$	edge
point	$\tilde{\mathbf{P}}$	$\tilde{\mathbf{p}}_k$	vertex
interval	$\tilde{\mathbf{T}}$	$\tilde{\boldsymbol{\tau}}_n$	interval
instant	$\tilde{\mathbf{I}}$	$\tilde{\mathbf{t}}_m$	instant

In short:

configuration variables → inner orientation
source and energy variables → outer orientation.

This principle offers a rational criterion to associate global variables of every physical theory to space and time elements and, as such, it is useful in computational electromagnetism. Figure 4

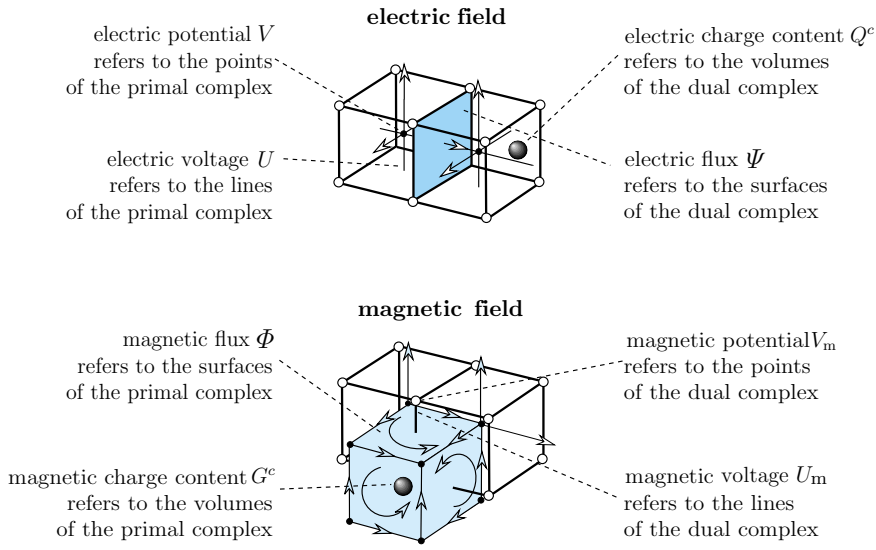


Figure 4. Global physical variables of electromagnetism and space elements of primal and dual cell complex with which they are associated.

shows this association for physical variables of electromagnetism. It shows that a single cell complex is not sufficient but it is necessary to introduce a *dual complex*.

To analyze this association we consider, first of all, the six measurable global variables of electromagnetism. It is important to note that each one of these six variables admits an operational definition.

3.4. Operational Definition of Six Global Variables

Since we take a new approach to electromagnetism starting from global variables rather than field functions, we are obliged to give an operational definition of global variables as we do for field functions in differential formulation. Fig. 5 shows the operational definitions of the six global quantities.

Doing this we stress the fact that a finite formulation of the electromagnetic field uses those global variables that can be measured. In this way there is a direct link between measurements and computational electromagnetism without the intermediation of field functions and of differential equations.

Electric charge content Q^{c}. Electric charge is an extensive quantity: from the material viewpoint it is associated with a system, a body, a particle. From the spatial viewpoint we must distinguish three aspects of charge: *content* Q^{c}, *outflow* Q^{f} and *production* Q^{p}. It is a basic physical law that electric charge cannot be produced, i.e. $Q^{\mathrm{p}} = 0$. Charge content Q^{c} is the amount of charge contained inside a volume at a given instant. The notion of "inside" and "outside" presupposes an *outer* orientation of volumes: for this reason we write $Q^{\mathrm{c}}[\tilde{\mathbf{V}}]$: see Fig. 5a. We put into square brackets the space and time element to which global variables are referred because global variables are *domain functions* not point functions.

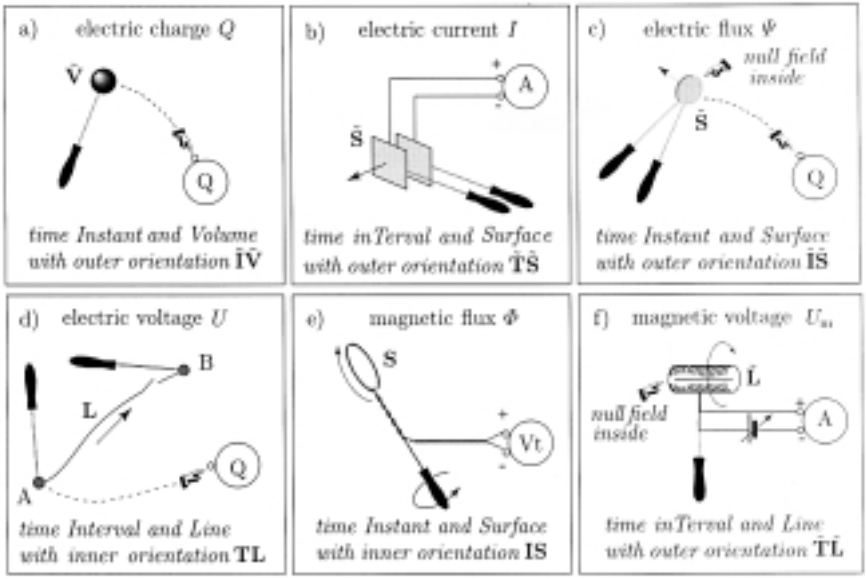


Figure 5. The operational definition of the six measurable variables of electromagnetism.

Electric charge flow Q^{f}. Let us consider electrical conduction in a medium. If we put in the medium two flat metal surfaces separated by a dielectric and connected to an amperometer, as shown in Fig. 5b, we obtain a device called a *rheometer*. In this way we can measure the electric charge flow that enters one disk and leaves the other in a given time interval. Since the notion of "entering" or "leaving" a surface presupposes its outer orientation, we shall denote the surface of the disk endowed with outer orientation by $\tilde{\mathbf{S}}$ and we shall write

$Q^{\mathrm{f}}[\tilde{\mathbf{S}}]$. The rate of this quantity is the electric current I.

Electric flux Ψ. Let us consider an electrostatic field. If we put a small metal disk somewhere in the field then charges of opposite sign will be collected on the two faces as a consequence of electrical induction. After selection of one face as positive we call *electric flux* Ψ the charge collected on this positive face of the disk. The electric flux is then related to an outer oriented surface. If we change the outer orientation of the surface, the sign of the flux changes. As we see from this definition, electric flux requires the notion of the outer orientation of a surface and hence we shall write $\Psi[\tilde{\mathbf{S}}]$.

To measure electric flux, instead of one metal disk, it is better to use two small metal disks. The disks will be held by an insulated handle and brought into contact, as shown in Fig. 5c. If we separate the two disks also the electric charges will be separated and each one can be measured with an electrometer. The charge collected on a prefixed disk is, by definition, electric flux (this direct measurement of electric flux is often ignored in books of electromagnetism. It can be found in Maxwell [18, p.47] and in [8, p.71]; [7, p.61]; [25, p.230]; [26, p.25]; [12, p.80; p.225]).

Electromotive force E, voltage U. In an electrostatic field we can measure the voltage along a line from point A to point B with a method devised by Faraday. This runs as follows: let us put at A and B two small metal spheres, as shown in Fig. 5d, say of radii r_A and r_B. If we connect them by a wire of very small section, the charges move from one sphere to another to maintain the whole set, spheres and wire, at the same potential.

If the capacity of the wire can be neglected in comparison with the capacities of the spheres we can neglect the charge on the wire. In turn the spheres are small enough to make negligible the influence of charges collected on the spheres on the sources of the surrounding electric field. In these hypotheses let us denote q_A the charge collected on the sphere in A and q_B the one collected on the sphere in B: it will be $q_A = -q_B$.

If we break the connection between the two spheres the charges remain trapped. In the center of a sphere the potential of the charges q collected on its surfaces is $q/(4\pi\epsilon r)$. The fact that the potential of the two spheres connected by the wire are equal implies that

$$V_A + \frac{q_A}{4\pi\epsilon r_A} = V_B + \frac{q_B}{4\pi\epsilon r_B} \tag{1}$$

from which we obtain

$$V_{AB} \equiv V_B - V_A = \frac{-q_A}{4\pi\epsilon}\left(\frac{1}{r_A} + \frac{1}{r_B}\right). \tag{2}$$

Hence we can measure voltage from measuring the charge collected on one sphere.

In particular if we choose B on the grounds the "sphere" B becomes the Earth and then $V_B = 0$ and $1/r_B = 0$: it follows [22, p.519]

$$V_A = \frac{-q_A}{4\pi\epsilon r_A}. \tag{3}$$

The voltage refers to a line endowed with inner orientation: $V[\mathbf{L}]$ as shown in Fig. 5d.

Magnetic flux Φ. A magnetic field is completely described by two global variables: *magnetic flux* and *magnetic voltage*. Magnetic flux refers to surfaces while magnetic voltage refers to lines.

Magnetic flux is linked to a surface endowed with an inner orientation and is defined as the magnetic voltage impulse induced in a coil that binds the surface [23, p.67] when the magnetic field is switched off. If the coil is connected with a ballistic voltmeter we can measure the magnetic voltage impulse produced. The sign of the magnetic flux depends on the direction chosen for the boundary of the surface, as shown in Fig. 5e. Then $\Phi[\mathbf{S}]$.

Magnetomotive force F_{m}, magnetic voltage U_{m}. We want to introduce a global physical variable that gives a measure of the magnetic field along a line. To this end we consider a long solenoid with a small cross section that has the line as its axis. Let N be the number of turns and i the current. The magnetic field inside such a solenoid is almost uniform and almost null outside it. The *magnetic voltage* U_{m} along the axis of the solenoid can be defined as $N\,i$: this is a global variable in space.

The sign of this variable depends on the direction of the current in the solenoid, i.e. it requires an outer orientation of the line. Accordingly, *magnetic tension is associated with lines endowed with outer orientation.*

To measure the magnetic tension along a line segment in a static magnetic field we introduce a small solenoid with N loops with a section much smaller than its length, as shown in Fig. 5f.

We can adjust the direction and the intensity of the current i' in the solenoid in such a way that the component of the magnetic field along the line vanishes. In such a way we have compensated the field

in the interior region. Let us put $I' = N\,i'$: the magnetomotive force along the line is then $F_{\rm m} = -I'$. This procedure is known as the method of *compensating coil* [8, p.224]; [23, p.66]; [26, p.41].

This shows that magnetic tension is associated with a line with the direction of rotation around it: the direction is *opposite* to the one of the compensating current. Denoting by $\tilde{\mathbf{L}}$ a line segment endowed with an outer orientation we can write

$$U_{\rm m}[\tilde{\mathbf{L}}] \stackrel{\rm def}{=} -I'. \tag{4}$$

An equivalent way to do the test is to consider a small tube of superconducting material: the tube will be crossed by a uniform current I' that automatically makes the interior field vanish [16, p.494].

It is obvious that physical variables that are global in space and time are also associated with time elements such as instants and intervals. Thus electric charge content Q, electric flux Ψ and magnetic flux Φ refer to instants.

On the contrary the electric charge flow $Q^{\rm f}$ refers to time intervals. Electric voltage U can be integrated in time by giving the electric voltage *impulse* $\mathcal{E}$ and for this reason it is associated with time intervals. One argument for the introduction of electric voltage impulse is that this quantity is used to measure magnetic flux via Faraday's law. Another argument is that Ohm law $U = RI$ can be written in an integrated form as $\mathcal{U} = R\,Q^{\rm f}$.

Since magnetic voltage $U_{\rm m} = Ni$ can be integrated in time the corresponding global time variable $\mathcal{U}_{\rm m} = N\,Q^{\rm f}$, the *magnetic voltage impulse*, will refer to time intervals.

These associations do not specify, up to now, the kind of orientation, inner or outer, of the time elements. This association becomes clear if we consider a space-time complex and its dual. It is obvious that if a physical variable refers to spatial elements of a space-time cell complex it must also refer to time elements *of the same* cell complex as shown in Table 5.

Classical "time reversal", i.e. the operation of reversing the order of events in time, corresponds to inversion of the orientation of the primal time intervals and it coincides, by definition, with inversion of the orientation of dual instants. It follows that *if a physical variable refers to primal time intervals or to dual time instants it changes sign under time reversal*. Inversely, if a physical variable refers to dual intervals or to primal instants it does not change sign under time reversal. An example is the impulse of a force: if a body A impacts a body B the impulse that A gives to B is directed from A to B. When we see the backward motion, as a movie running backward, we see

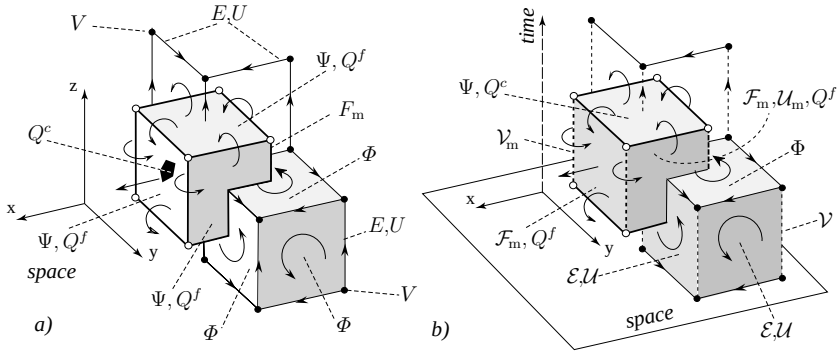


Figure 6. *a)* A space complex and associated variables; *b)* a three-dimensional space-time and associated variables.

that velocities are inverted but the impulse that A gives to B is always directed from A to B.

The space and time association of global electromagnetic variables is summarized in Table 5.

The space and time association is made clearer from a geometrical viewpoint, if we use a three-dimensional projection of four-dimensional cube, as shown in Fig. 7. The two draws of the central level show that the four variables $\Phi, \Psi, \mathcal{U}, \mathcal{U}_\mathrm{m}$ are referred to surfaces: the first two to space-like surfaces, the last two to space-time surfaces. The two draws on the lower level shows that the eight Maxwell equations express a balance on a volume: two of them (Gauss' laws) express a balance on a space volume, the other six express a balance on a space-time volume.

3.5. Physical Laws and Space-time Elements

The first Principle states that global physical variables refer to the oriented space and time elements. From the analysis of a great number of physical variables of classical fields one can infer [31]:

Second Principle: *In every physical theory there are physical laws that link global variables referred to an oriented space-time element with others referred to its oriented boundary.*

We shall show later that the fundamental laws of electromagnetism satisfy this principle. To give an example from outside electromagnetism, we mention the equilibrium of a body that links the volume forces acting on a region of the body with the surface forces acting on the boundary of the region. This principle gives the reason of the

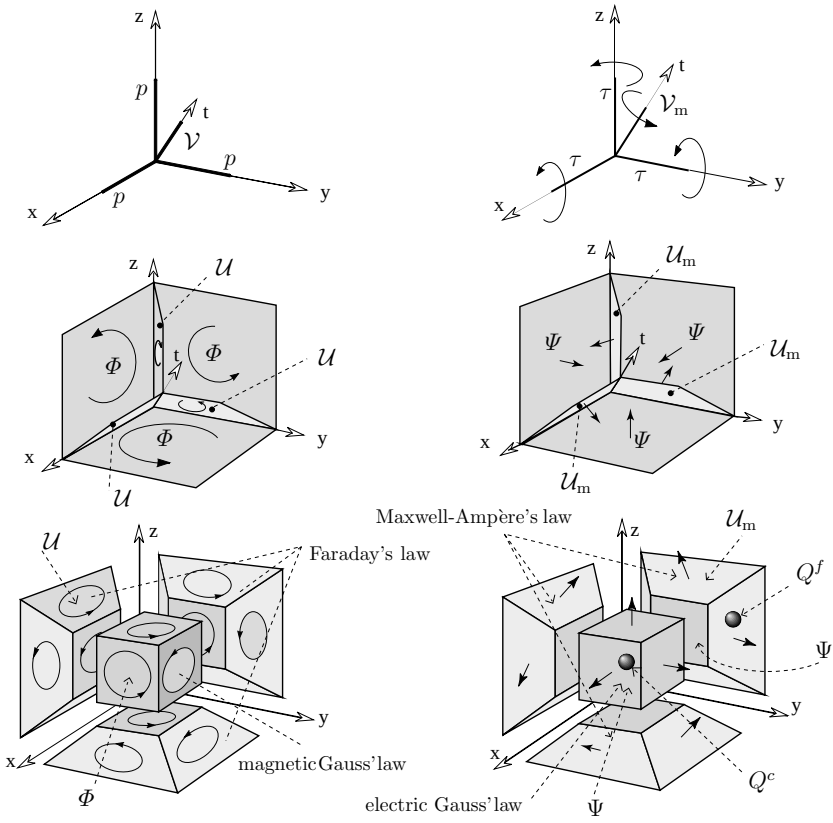


Figure 7. Space-time elements and global variables associated with them. The picture in the last row is a four-dimensional cube exploded.

ubiquitous appearance of the exterior differential on differential forms.

3.6. The Field Laws in Finite Form

Experiments lead us to infer the following laws of electromagnetism:

- The *magnetic flux* referred to the *boundary* of a volume endowed with inner orientation at any instant vanishes (magnetic Gauss' law).
- The *electromotive force impulse* referred to the *boundary* of a surface endowed with inner orientation during a time interval is opposite to the *magnetic flux* variation across the surface in the same interval (Faraday's electromagnetic induction law).

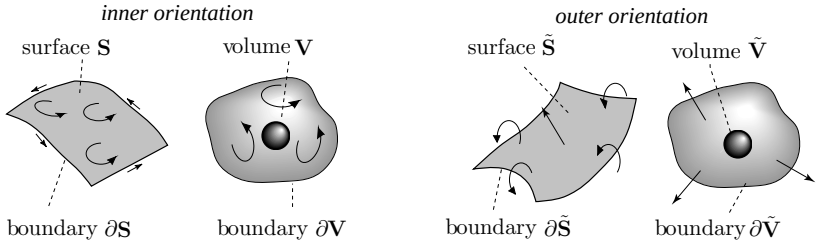


Figure 8. The four manifolds to which the four Maxwell equations make reference.

- The *electric flux* across the *boundary* of a volume endowed with outer orientation at any instant is equal to the *electric charge* contained inside the volume at that instant (Faraday's electrostatic induction law = electric Gauss' law).
- The *magnetomotive force impulse* referred to the *boundary* of a surface endowed with outer orientation in a time interval is equal to the sum of the *electric charge flow* across the surface in that time interval and the *electric flux* variation across the surface in that interval (Maxwell-Ampère's law).
- The *electric charge flow* across the *boundary* of a volume endowed with outer orientation in an interval is opposite to the variation of the *electric charge content* inside the volume in the same interval (conservation of charge).

These 4+1 laws can be written

$$\left\{\begin{array}{rcl} \Phi[\partial\mathbf{V},\mathbf{I}] &=& 0 \\ \mathcal{E}[\partial\mathbf{S},\mathbf{T}] &=& \Phi[\mathbf{S},\mathbf{I}^-] - \Phi[\mathbf{S},\mathbf{I}^+] \\ \Psi[\partial\tilde{\mathbf{V}},\tilde{\mathbf{I}}] &=& Q^{\mathrm{c}}[\tilde{\mathbf{V}},\tilde{\mathbf{I}}] \\ \mathcal{F}_{\mathrm{m}}[\partial\tilde{\mathbf{S}},\tilde{\mathbf{T}}] &=& \Psi[\tilde{\mathbf{S}},\tilde{\mathbf{I}}^+] - \Psi[\tilde{\mathbf{S}},\tilde{\mathbf{I}}^-] + Q^{\mathrm{f}}[\tilde{\mathbf{S}},\tilde{\mathbf{T}}] \\ Q^{\mathrm{f}}[\partial\tilde{\mathbf{V}},\tilde{\mathbf{I}}] &=& Q^{\mathrm{c}}[\tilde{\mathbf{V}},\tilde{\mathbf{I}}^-] - Q^{\mathrm{c}}[\tilde{\mathbf{V}},\tilde{\mathbf{I}}^+]. \end{array}\right. \tag{5}$$

Equations (5) are the 4+1 laws of electromagnetism in a finite formulation we are searching for. These are algebraic equations that enjoy the following properties:

- they link physical variables of the same kind, i.e. configuration variables with configuration variables and source variables with source variables;

- they are valid *in whatever medium* and then are free from any material parameter;
- they *do not involve metrical notions*, i.e. lengths, areas, measures of volumes and durations are not required [37].

These five equations, that are equivalent to the integral formulation describe the "structure" of the field and we shall call them *equation of structure*. Since they are valid for whatever volume and whatever surface respectively they are of topological nature and we can name them also *topological equations* [20, p.20] [36].

4. CELL COMPLEXES IN SPACE AND TIME

The equations (5) are the finite formulation of the electromagnetic laws. How to apply them to solve field problems? The principle is a very simple one: we build up a cell complex in the region in which the field is considered and then apply the equations in finite form to all cells of the complex. Some equations must be applied to the cells others to their faces; some equations must be applied to the cells and faces of the primal, some other to those of the dual complex. Doing so we obtain a system of algebraic equations whose solution gives the space and time distribution of the global variables of the field. In this way we solve the fundamental problem of electromagnetism: *given the space and time distribution of charges and currents to find the resulting field*.

To pursue this goal we must introduce the notion of cell complex and of its dual. Let us consider, first of all, a cell complex formed by cubic cells, as shown in Fig. $9c$.

The elements of the same dimension can be numbered according to any criterion. The number is a label that permits us to specify the space element and play the same role of coordinates of a point in a coordinate system. We shall consider cell complexes with a finite number N_0 of vertices. Since vertices are points we shall denote the typical vertex by $\mathbf{p}_h$. At first it seems convenient to assign to every edge a pair of numbers, the labels of its bounding points. Thus the edge that connects the vertex $\mathbf{p}_h$ with the vertex $\mathbf{p}_k$ can be denoted $\mathbf{l}_{hk}$. But this notation becomes cumbersome so that we have chosen to denote the edge with a single Greek index, e.g. $\mathbf{l}_\alpha$. If N_1 is the number of edges the Greek index takes the values $1, 2, ...N_1$. We shall denote with a Greek index also the face, e.g. $\mathbf{s}_\beta$ while the typical cell will be denoted with a Latin index, e.g. $\mathbf{v}_k$.

As in space we have four elements, so in time we have two elements: instants $\mathbf{I}$ and intervals $\mathbf{T}$. When we consider a cell complex on the time axis we shall denote by $\mathbf{t}_n$ the time instants and $\boldsymbol{\tau}_m$ the intervals.

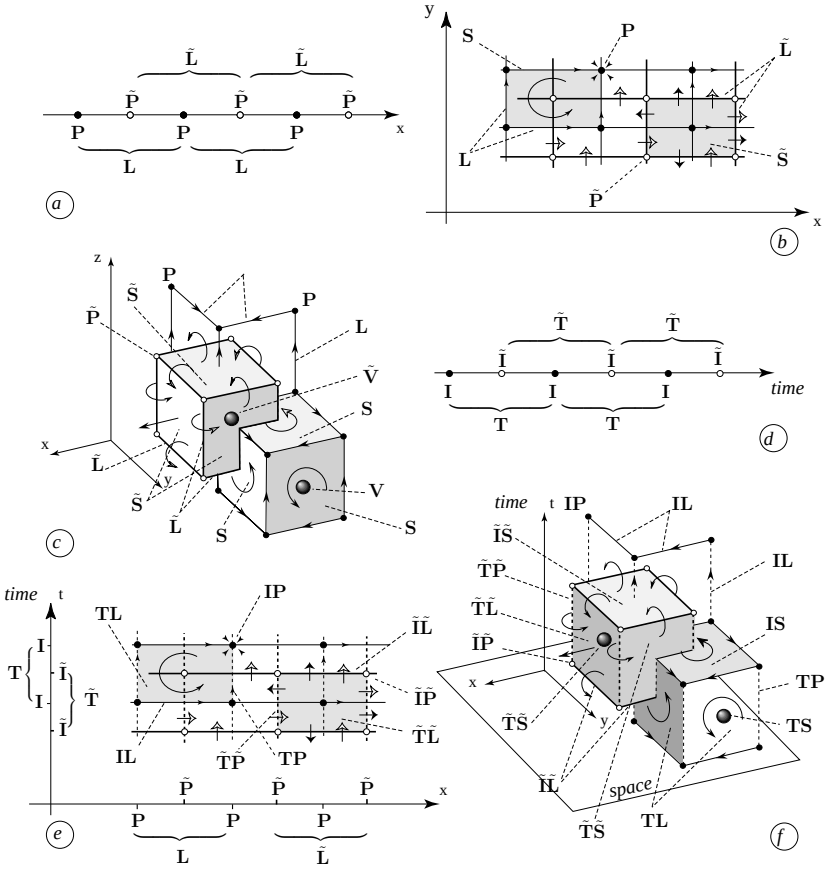


Figure 9. *(a)* A one-dimensional cell complex; *(b)* a two-dimensional cell complex; *(c)* a three-dimensional cell complex; *(d)* a one-dimensional cell complex on a time axis; *(e)* a cell complex in two-dimensional space-time; *(f)* a cell complex in three-dimensional space-time.

We shall use boldface letters to denote the elements of a cell complex for two reasons: the first is to distinguish between the element and its measure. Thus $\mathbf{l}_\alpha$ denotes an edge while l_α denotes its length; $\mathbf{s}_\beta$ denotes a face while s_β denotes its area; $\mathbf{v}_k$ denotes a cell while v_k denotes its volume. On the time axis $\boldsymbol{\tau}_n$ denotes a time interval while τ_n denotes its extension (duration).

The second reason will be explained in connection with orientation.

Cell complexes are basic tools of *algebraic topology*. In this

branch of topology many notions were developed around cell complexes including the notions of orientation, duality and incidence numbers. In algebraic topology vertices, edges and faces of cells are considered as cells of a lower dimension. The vertices are called 0-dimensional cells or briefly *0-cells*, edges *1-cells*, faces *2-cells* and original cells *3-cells*. It follows that a cell complex in space is not only a set of 3-cells but a set of p-cells with $p = 0, 1, 2, 3$. In four-dimensional space-time a cell complex is formed by cells of dimension $p = 0, 1, 2, 3, 4$.

Table 6 (left) collects the notations we use for space elements: when we must mention points, lines, surfaces and volumes without reference to a cell complex we shall use a "descriptive" notation with boldface, uppercase letters. On the contrary, when we refer to a cell complex we must specify the labels of the elements involved and accordingly we shall use a "formal" notation with boldface, lowercase letters with indices.

A cell complex can be based on a coordinate system: in such a case the edges of the cells lie on the coordinate lines and the faces on the coordinate surfaces. An example is shown in Fig. 12 (left). A coordinate-based cell complex is useful when one aims to deduce the differential formulation from a finite one.

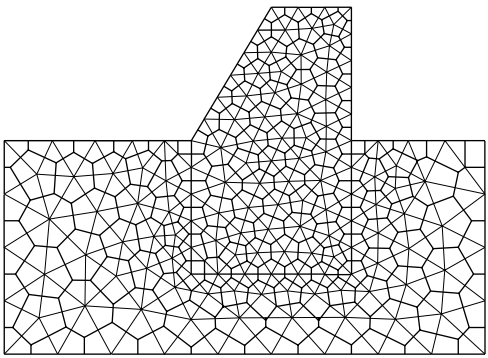


Figure 10. Finite formulation permits different materials to be treated assuring continuity at the separation surface automatically.

Conversely, for numerical applications it is convenient to give up the coordinate based cell complex and to use simplicial complexes, i.e. the ones formed by triangles in 2D and tetrahedra in 3D. Simplicial complex have many advantages over the coordinate-based complexes. A first advantage is that simplexes can be adapted to the boundary of the domain, as shown in Fig. 10. A second advantage is that, when

we have two or more subregions that contain different materials, the vertices of the simplexes can be put on the separation surface, as shown in Fig. 10. A third reason is that simplexes can change in size from one region to another. This allows to adopt smaller simplexes in the regions of large variations of the field.

Once we have introduced a cell complex we can consider the *dual* complex. In a coordinated-based complex one can consider the barycenter of every coordinate-cell as shown in Fig. 9. Connecting the barycentres of the adjacent cells one obtains a dual complex. The term "dual" refers to the fact that not only every barycenter (dual vertex) corresponds to a cell (primal volume) but also every edge of the dual complex (dual edge) intersects a face of the primal one (primal face). Conversely, every primal edge intersects a dual face. Lastly, every vertex of the primal lies inside a cell of the dual. In a simplicial

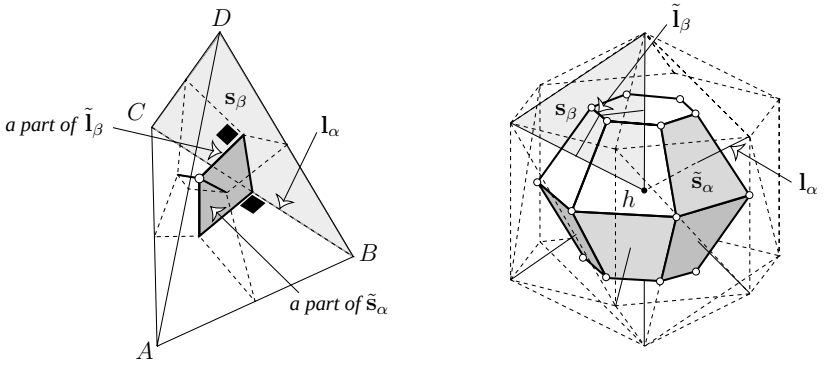


Figure 11. *a)* The six faces of a Voronoi cell contained in a tetraedron; *b)* the dual Voronoi cell $\tilde{\mathbf{v}}_h$ of a cluster of tetrahedra with a common vertex.

complex the commonest choice are either the barycentres of every simplex or the circumcentres (in 2D) and the circumspheres (in 3D): *in this paper we consider only circumcentres and circumspheres.* Since the straight line connecting the circumcentres of two adjacent simplexes in 2D is orthogonal to the common edge, the dual polygon thus obtained has its sides orthogonal to the common edge. This is called *Voronoi polygon* in 2D and *Voronoi polyhedron* in 3D. The circumcentres have the disadvantage that in triangles with obtuse angles they lie outside the triangle. This is inconvenient when the circumcentre of one obtuse triangle goes beyond the one of the adjacent triangle with the common sides. This is avoided when the triangulation satisfies the *Delaunay condition*. This leads us to consider only Delaunay-Voronoi complexes,

as we shall do in this paper. As in coordinate systems it is preferable to deal with orthogonal coordinate systems, so in a simplicial complex it is preferable to deal with a Delaunay complex and its associated Voronoi complex as dual, as shown in Figure 12 (right).

The same can be done when we introduce a cell complex on a time axis, as shown in Fig. $9d$: the elements of time are instants ($\mathbf{I}$) and intervals ($\mathbf{T}$). If we take the middle instants of intervals we can call these *dual* instants ($\tilde{\mathbf{I}}$) and the corresponding intervals as *dual* intervals ($\tilde{\mathbf{T}}$). It is evident that to every instant of the primal complex there corresponds an interval of the dual and to every interval of the primal there corresponds an instant of the dual. Thus we have the correspondence $\mathbf{I} \leftrightarrow \tilde{\mathbf{T}}$ and $\tilde{\mathbf{I}} \leftrightarrow \mathbf{T}$ and this is a duality map.

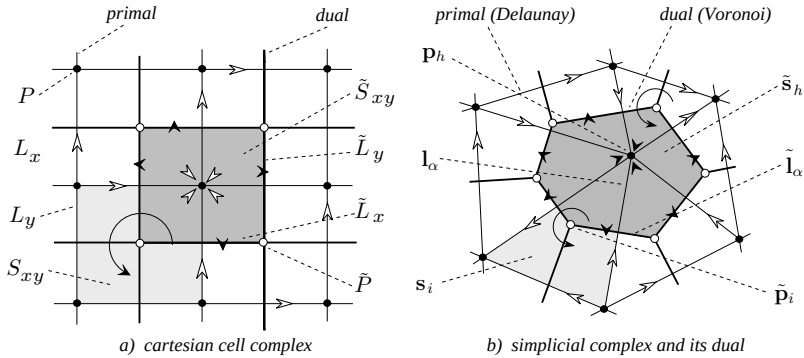


Figure 12. A two-dimensional cell complex (thin lines) and its dual (thick lines). In the simplicial complex the vertices of dual complex are the intersections of three axes of primal 1-cells. This gives the advantage that 1-cells of dual are orthogonal to primal 1-cells.

A cell complex and its dual enjoy a peculiar property: once the vertices, edges, faces and cells of the primal complex has been endowed with inner orientation, this inner orientation induces an outer orientation on the cells, faces, edges and vertices of its dual. It follows that a pair formed by a cell complex and its dual are the natural frames to exhibit all space elements with the two kind of orientations.

Since we have stated that the configuration variables are associated with the space elements endowed with inner orientation, it follows that the configuration variables can be associated with the vertices, edges, faces and cells of the primal complex. Moreover since the source and energy variables are associated with space elements endowed with outer orientation, it follows that these variables can be

associated with cells, faces, edges and vertices of the dual complex. One can say that the role of the dual complex is to form a background structure to which source and energy variables can be referred.

4.1. Classification Diagram of Space-time Elements

A cell complex and its dual in a space of dimension n permits a classification of space elements of $\mathbb{R}^n$, as shown in Fig. 9. Let us start with Fig. 9*a* that shows a cell complex in $\mathbb{R}^1$. The primal complex is formed by points **P** and lines **L**; the dual one is formed by dual points $\tilde{\mathbf{P}}$ and dual lines $\tilde{\mathbf{L}}$. The two complexes are shifted and to a dual point there corresponds a primal line: $\tilde{\mathbf{P}} \leftrightarrow \mathbf{L}$. Moreover $\tilde{\mathbf{L}} \leftrightarrow \mathbf{P}$. These 2×2 elements can be collected in a diagram shown in Table 7*a*.

Fig. 9*b* shows a cell complex in $\mathbb{R}^2$ and its dual. The primal complex exhibits points, lines and surfaces and its dual exhibits the same elements in reverse order. These 3×2 elements can be collected in a diagram shown in Table 7*b*. From Fig. 9*c* we can infer the corresponding diagram for $\mathbb{R}^3$ that is shown in Table 7*c*.

Fig. 9*d* shows a cell complex on a time axis: the corresponding diagram is shown in Table 7*d*.

Fig. 9*e* shows a two-dimensional space-time complex whose corresponding classification diagram is reported in Table 7*e*. In the case of space-time complexes we shift the columns so as to obtain a kind of assonometric view that will make the diagrams we shall present later more readable. The points of these space-time diagrams, which in relativity are called *events*, will be denoted **IP** to mean that they combine an instant **I** with a space point **P**.

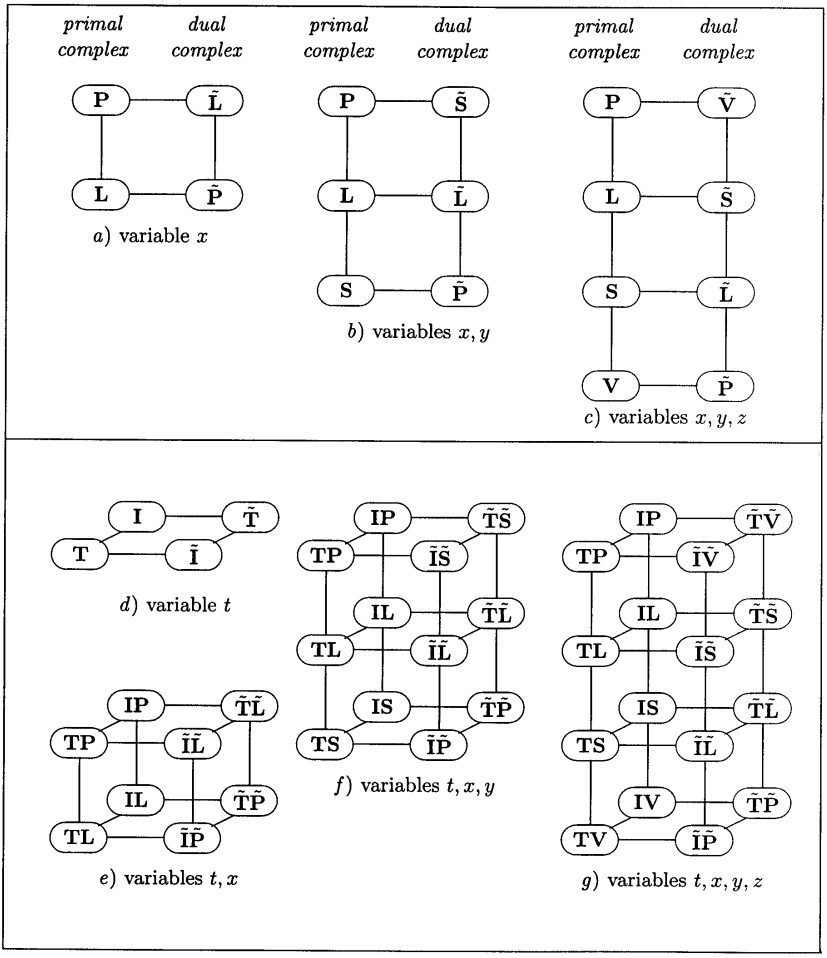
A three-dimensional space-time is shown in Fig. 9*f*: the corresponding diagram is shown in Table 7*f*. A complete four-dimensional space-time diagram is shown in Table 7*g*.

These classification diagrams play a remarkable role in the description of physical properties. In fact *the natural association of configuration variables to elements of a complex and of source and energy variables to its dual respectively, lead to an analogous classification diagram for physical variables*, as we shall show later.

4.2. Incidence Numbers

In network theory one introduces the node-edge and edge-loop incidence matrices with their dual. Following the notations of Fig. 13 we are now in a position to define the *incidence number* of a p-cell $\mathbf{c}_h$ with a (p-1)-cell $\mathbf{b}_k$. This is a relative integer $i_{hk}=[\mathbf{c}_h : \mathbf{b}_k]$ whose values are:

Table 7. Classification of geometrical elements of spaces and space-time of various dimensions.



- +1 if $\mathbf{b}_k$ is a face of $\mathbf{c}_h$ and the orientations of $\mathbf{b}_k$ and $\mathbf{c}_h$ are compatible;
- -1 if $\mathbf{b}_k$ is a face of $\mathbf{c}_h$ and the orientation of $\mathbf{b}_k$ and $\mathbf{c}_h$ are not compatible;
- 0 if $\mathbf{b}_k$ is not a face of $\mathbf{c}_h$.

We point out that in the notation i_{kh} the first index k refers to the cell of *greatest* dimension.

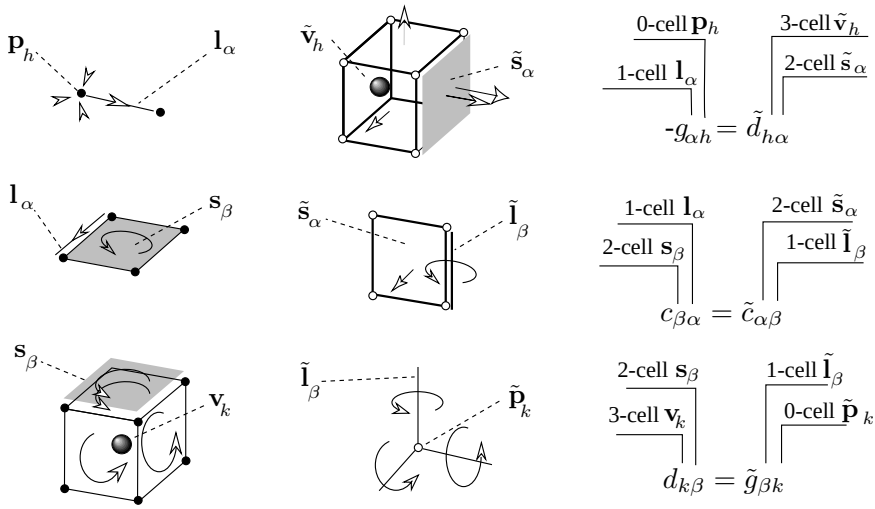


Figure 13. The incidence numbers of a pair of cells are equal to those of the dual pair.

In three-dimensional space three matrices $\boldsymbol{G}, \boldsymbol{C}, \boldsymbol{D}$ can be introduced for the primal complex K and three matrices $\tilde{\boldsymbol{G}}, \tilde{\boldsymbol{C}}, \tilde{\boldsymbol{D}}$ for the dual complex $\tilde{K}$. We choose these three letters because they are the initial of the names of the three formal differential operators *gradient*, *curl* and *divergence* to which they reduce in the differential. In summary:

$$\begin{cases} \boldsymbol{G} \stackrel{\text{def}}{=} ||g_{\alpha h}|| & \boldsymbol{C} \stackrel{\text{def}}{=} ||c_{\beta\alpha}|| & \boldsymbol{D} \stackrel{\text{def}}{=} ||d_{k\beta}|| \\ \tilde{\boldsymbol{D}} \stackrel{\text{def}}{=} ||\tilde{d}_{h\alpha}|| & \tilde{\boldsymbol{C}} \stackrel{\text{def}}{=} ||\tilde{c}_{\alpha\beta}|| & \tilde{\boldsymbol{G}} \stackrel{\text{def}}{=} ||\tilde{g}_{\beta k}||. \end{cases} \tag{6}$$

From Fig. 13 we can see the important fact that, apart from the case point-line, *the incidence number between a p-cell and a (p-1)-cell of the primal cell complex is equal to the incidence number between the corresponding dual cells*. The exception of the incidence point-line is due to historical reasons: points are implicitly considered as sinks (inwards normals) while volumes have outwards normals.

Note that the indices of the matrix elements $\tilde{d}_{h\alpha}$ and $g_{\alpha h}$ are reversed and then the corresponding matrices are transpose to one

another. We have

$$\left\{\begin{array}{rcccccc} -g_{\alpha h} & \stackrel{\text{def}}{=} & -[\mathbf{l}_\alpha : \mathbf{p}_h] & = & [\tilde{\mathbf{v}}_h : \tilde{\mathbf{s}}_\alpha] & = & \tilde{d}_{h\alpha} \quad \rightarrow \quad -\boldsymbol{G} = \tilde{\boldsymbol{D}}^{\mathsf{T}} \\ c_{\beta\alpha} & \stackrel{\text{def}}{=} & [\mathbf{s}_\beta : \mathbf{l}_\alpha] & = & [\tilde{\mathbf{s}}_\alpha : \tilde{\mathbf{l}}_\beta] & = & \tilde{c}_{\alpha\beta} \quad \rightarrow \quad \boldsymbol{C} = \tilde{\boldsymbol{C}}^{\mathsf{T}} \\ d_{k\beta} & \stackrel{\text{def}}{=} & [\mathbf{v}_k : \mathbf{s}_\beta] & = & [\tilde{\mathbf{l}}_\beta : \tilde{\mathbf{p}}_k] & = & \tilde{g}_{\beta k} \quad \rightarrow \quad \boldsymbol{D} = \tilde{\boldsymbol{G}}^{\mathsf{T}}. \end{array}\right. \tag{7}$$

When the equations (5) are applied to the corresponding cells of the two complexes, we obtain a *local* form of the field equations of the electromagnetic field in a finite setting, i.e.

$$\left\{\begin{array}{rcl} \displaystyle\sum_\alpha c_{\beta\alpha}\, \mathcal{U}[\boldsymbol{\tau}_{n+1}, \mathbf{l}_\alpha] + \Big\{ \Phi[\mathbf{t}_{n+1}, \mathbf{s}_\beta] - \Phi[\mathbf{t}_n, \mathbf{s}_\beta] \Big\} & = & 0 \\ \displaystyle\sum_\beta d_{k\beta}\ \Phi[\mathbf{t}_n, \mathbf{s}_\beta] & = & 0 \\ \displaystyle\sum_\beta \tilde{c}_{\alpha\beta}\, \mathcal{U}_{\mathrm{m}}[\tilde{\boldsymbol{\tau}}_n, \tilde{\mathbf{l}}_\beta] - \Big\{ \Psi[\tilde{\mathbf{t}}_{n+1}, \tilde{\mathbf{s}}_\alpha] - \Psi[\tilde{\mathbf{t}}_n, \tilde{\mathbf{s}}_\alpha] \Big\} & = & Q^{\mathrm{f}}[\tilde{\boldsymbol{\tau}}_n, \tilde{\mathbf{s}}_\alpha] \\ \displaystyle\sum_\alpha \tilde{d}_{h\alpha}\ \Psi[\tilde{\mathbf{t}}_n, \tilde{\mathbf{s}}_\alpha] & = & Q^{\mathrm{c}}[\tilde{\mathbf{t}}_n, \tilde{\mathbf{v}}_h] \\ \displaystyle\sum_\alpha \tilde{d}_{h\alpha}\, Q^{\mathrm{f}}[\tilde{\boldsymbol{\tau}}_n, \tilde{\mathbf{s}}_\alpha] + \Big\{ Q^{\mathrm{c}}[\tilde{\mathbf{t}}_{n+1}, \tilde{\mathbf{v}}_h] - Q^{\mathrm{c}}[\tilde{\mathbf{t}}_n, \tilde{\mathbf{v}}_h] \Big\} & = & 0. \end{array}\right. \tag{8}$$

For computational purposes it is useful to make the following changes of symbols: $t_n \rightarrow n$; $\tilde{t}_n \rightarrow n + 1/2$; $\Phi[\mathbf{t}_n, \mathbf{s}_\beta] \rightarrow \Phi^n_\beta$; *etc.* In particular the two evolution equations can be written as (remember that $\tilde{c}_{\alpha\beta} = c_{\beta\alpha}$)

$$\left\{\begin{array}{rcl} \Phi^{n+1}_\beta & = & \Phi^n_\beta - \displaystyle\sum_\alpha c_{\beta\alpha}\, \mathcal{U}^{n+1/2}_\alpha \\ \Psi^{n+1/2}_\alpha & = & \Psi^{n-1/2}_\alpha + \displaystyle\sum_\beta c_{\beta\alpha}\, (\mathcal{U}_{\mathrm{m}})^n_\beta - (Q^{\mathrm{f}})^n_\alpha. \end{array}\right. \tag{9}$$

It is convenient to introduce the rates of the five global variables $\mathcal{U}, \mathcal{U}_{\mathrm{m}}, Q^{\mathrm{f}}, \mathcal{V}, \mathcal{V}_{\mathrm{m}}$ that are associated with time intervals.

The ratio of a global variable, associated with a time interval, with the duration of the interval gives a mean rate. If the interval is small the global variable can be considered to depend linearly on the duration and then the mean rate approximate the value of the instantaneous rate at the middle instant of the interval. Since the middle instant of

an interval is the instant of the dual time complex one can write

$$\frac{\mathcal{U}[\boldsymbol{\tau}_n, \mathbf{l}_\alpha]}{\tau_n} \approx U_\alpha(\tilde{t}_n) \quad \frac{\mathcal{U}_{\mathrm{m}}[\tilde{\boldsymbol{\tau}}_n, \tilde{\mathbf{l}}_\beta]}{\tilde{\tau}_n} \approx U_{\mathrm{m}\beta}(t_n) \quad \frac{Q^{\mathrm{f}}[\tilde{\boldsymbol{\tau}}_n, \tilde{\mathbf{s}}_\alpha]}{\tilde{\tau}_n} \approx I_\alpha(t_n) \quad etc. \tag{10}$$

The round brackets denote that the rates are functions of the time instants.

Voltages and fluxes are the most natural variables to be used in computational electromagnetism. In particular the equations (9) are simple to use in numerical solutions. They can be written in a simpler

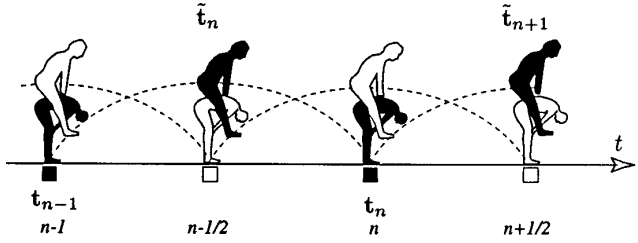


Figure 14. The *leapfrog* algorithm is a general algorithm to be used in finite formulation for every field of physics, not only in electromagnetism.

form as

$$\begin{cases} \displaystyle\sum_\alpha c_{\beta\alpha}\, U_\alpha(\tilde{t}_n) + \frac{\Phi_\beta(t_n) - \Phi_\beta(t_{n-1})}{\tau_n} = 0 \\ \displaystyle\sum_\beta d_{k\beta}\, \Phi_\beta(t_n) = 0 \\ \displaystyle\sum_\beta \tilde{c}_{\alpha\beta}\, U_{\mathrm{m}\beta}(t_n) - \frac{\Psi_\alpha(\tilde{t}_{n+1}) - \Psi_\alpha(\tilde{t}_n)}{\tilde{\tau}_n} = I_\alpha(t_n) \\ \displaystyle\sum_\alpha \tilde{d}_{h\alpha}\, \Psi_\alpha(\tilde{t}_n) = Q^{\mathrm{c}}_h(\tilde{t}_n) \\ \displaystyle\sum_\alpha \tilde{d}_{h\alpha}\, I_\alpha(t_n) + \frac{Q^{\mathrm{c}}_h(\tilde{t}_{n+1}) - Q^{\mathrm{c}}_h(\tilde{t}_n)}{\tilde{\tau}_n} = 0. \end{cases} \tag{11}$$

This shows that while magnetic voltage, magnetic flux and electric current must be evaluated on the instants of the primal time cell complex, electric voltage and electric flux must be evaluated in the

intermediate instants i.e. the dual instants. This is the "leapfrog" algorithm shown in Fig. 14.

4.3. Constitutive Laws in Finite Form

The equations that link the source variables with the configuration ones are the *constitutive* or *material* equations. In a region of uniform field the three material equations of electromagnetism in finite form are

$$\begin{cases} \dfrac{\Psi[\tilde{\mathbf{t}}_n, \tilde{\mathbf{s}}_\alpha]}{\tilde{s}_\alpha} \approx \epsilon \dfrac{\mathcal{U}[\boldsymbol{\tau}_n, \mathbf{l}_\alpha]}{\tau_n\, l_\alpha} & \textit{when} \quad \tilde{\mathbf{s}}_\alpha \perp \mathbf{l}_\alpha \\ \dfrac{\Phi[\mathbf{t}_n, \mathbf{s}_\beta]}{s_\beta} \approx \mu \dfrac{\mathcal{U}_{\mathrm{m}}[\tilde{\boldsymbol{\tau}}_n, \tilde{\mathbf{l}}_\beta]}{\tilde{\tau}_n\, \tilde{l}_\beta} & \textit{when} \quad \tilde{\mathbf{l}}_\beta \perp \mathbf{s}_\beta \\ \dfrac{Q^{\mathrm{f}}[\tilde{\boldsymbol{\tau}}_n, \tilde{\mathbf{s}}_\alpha]}{\tilde{\tau}_n\, \tilde{s}_\alpha} \approx \sigma \dfrac{1}{2}\left(\dfrac{\mathcal{U}[\boldsymbol{\tau}_n, \mathbf{l}_\alpha]}{\tau_n\, l_\alpha} + \dfrac{\mathcal{U}[\boldsymbol{\tau}_{n+1}, \mathbf{l}_\alpha]}{\tau_{n+1}\, l_\alpha}\right) & \textit{when} \quad \tilde{\mathbf{s}}_\alpha \perp \mathbf{l}_\alpha \end{cases} \tag{12}$$

in which $\tau_n, \tilde{\tau}_n, l_\alpha, \tilde{l}_\beta, s_\beta, \tilde{s}_\alpha$ are the extensions of the corresponding cells. We note that the notion of uniformity of a field does not imply the introduction of vectors: a field is uniform when the global variables associated with space elements are invariant under translation of the element.

To explain the particular form of Ohm's law let us remark that while the electric current $I_\alpha(t_n)$ is function of the primal instant t_n the voltage is function of the dual instant $\tilde{t}_n$, i.e. $U_\alpha(\tilde{t}_n)$, as shown in Eq. (10). Since the constitutive equations link variables referred to the same instant we need to evaluate the voltage at the primal instant t_n. Then we write

$$U_\alpha(t_n) \approx \frac{U_\alpha(\tilde{t}_n) + U_\alpha(\tilde{t}_{n+1})}{2}. \tag{13}$$

These equations are valid if cells are cubes or if the simplicial complex is a *Delaunay* complex and its dual a *Voronoi* complex, as is shown in Fig. 11. In these cases 1-cells of the dual are orthogonal to the primal 2-cells and vice versa. It is possible to avoid the orthogonality condition and then to avoid the Voronoi complex using the barycenter [17].

With reference to Fig. 15 the main properties are:

- They are valid in regions in which the field is *uniform* because these are the experimental conditions under which they can be tested;

- They link a variable referred to a p-cell of a complex with the dual $(n-p)$-cell of the dual complex. This geometrical property is not clear in differential formulation.
- They contain *material parameters*.
- They require *metrical notions* such as length, areas, volumes and orthogonality.

We emphasize that Ohm's law, written in terms of global variables, links two variables that refer to the primal and dual time intervals respectively. This implies that under time reversal ($\boldsymbol{\tau}_n \to -\boldsymbol{\tau}_n$) e.m.f. impulses change sign while electric charge flow does not. It follows that Ohm's law is not invariant under time reversal and this corresponds to the fact that electric conduction is an irreversible phenomenon.

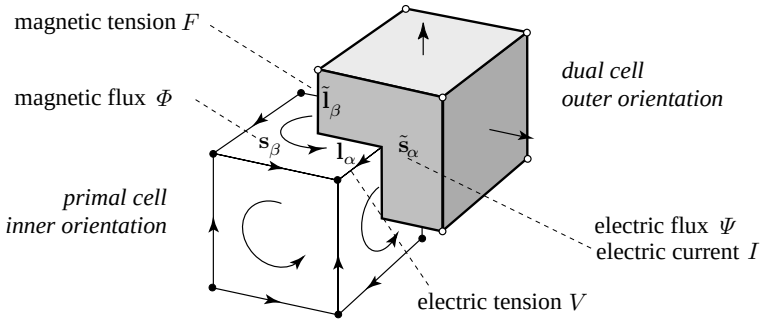


Figure 15. Constitutive equations link a variable associated with a cell of primal complex with a variable associated with the dual cell.

While the field equations in finite form describe the corresponding physical laws *exactly*, the constitutive ones in finite form describe the corresponding physical laws *approximately* because they are experienced only in regions of uniform field.

4.4. Computational Procedure

When one combines the equations of structure (9) with the constitutive equations (12), one obtains the fundamental system, i.e. the system whose solution is solution of the fundamental problem of the electromagnetic field (to find the field given its sources).

The computational procedure is collected in Table 8. The notation "II ord" means that with these approximations the convergence in time is of second order.

Table 8. Computational sequence using a Delaunay-Voronoi complex.

notations: $U_{\mathrm{m}} \to F \;\; \mathcal{U}_{\mathrm{m}} \to \mathcal{F} \;\; Q^{\mathrm{f}} \to \mathcal{J} \;\; A(t_n) \to A^n; \; A(\tilde{t}_n) \to A^{n-1/2}$

$\Psi_\alpha^{-1/2} = 0; \; U_\alpha^{-1/2} = 0; \;\; \Phi_\beta^0 = 0; \; F_\beta^0 = 0; \;\; (I_{\mathrm{c}})_\alpha^0 = 0; \; (I_{\mathrm{i}})_\alpha^0 \neq 0; \;\; \mathcal{J}_\alpha^0 = \tilde{\tau}_0 \, (I_{\mathrm{i}})_\alpha^0$

1 $\Psi_\alpha^{n+1/2} = \Psi_\alpha^{n-1/2} + \sum_\beta \tilde{c}_{\alpha\beta} \, \mathcal{F}_\beta^n - \mathcal{J}_\alpha^n$

2 $U_\alpha^{n+1/2} = \dfrac{1}{\epsilon} \dfrac{l_\alpha}{\tilde{s}_\alpha} \Psi_\alpha^{n+1/2} \quad (\mathbf{l}_\alpha \perp \tilde{\mathbf{s}}_\alpha)$

3 $\mathcal{U}_\alpha^{n+1/2} = \tau_{n+1} \, U_\alpha^{n+1/2}$

4 $\Phi_\beta^{n+1} = \Phi_\beta^n - \sum_\beta c_{\beta\alpha} \, \mathcal{U}_\alpha^{n+1/2}$

5 $F_\beta^{n+1} = \dfrac{1}{\mu} \dfrac{\tilde{l}_\beta}{s_\beta} \Phi_\beta^{n+1} \quad (\tilde{\mathbf{l}}_\beta \perp \mathbf{s}_\beta)$

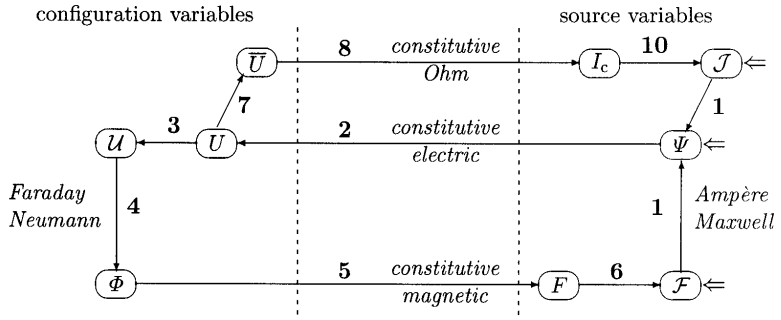
6 $\mathcal{F}_\beta^{n+1} = \tilde{\tau}_{n+1} \, F_\beta^{n+1}$

7 $\overline{U}_\alpha^n = \dfrac{1}{2} \left[U_\alpha^{n+1/2} + U_\alpha^{n-1/2} \right]$

8 $(I_{\mathrm{c}})_\alpha^n = \sigma \dfrac{\tilde{s}_\alpha}{l_\alpha} \overline{U}_\alpha^n \quad (\mathbf{l}_\alpha \perp \tilde{\mathbf{s}}_\alpha)$

9 $I_\alpha^n = (I_{\mathrm{i}})_\alpha^n + (I_{\mathrm{c}})_\alpha^n$ { i = *impressed*; c = *conduction* }

10 $\mathcal{J}_\alpha^n = \tilde{\tau}_n \, I_\alpha^n$



4.5. Classification Diagrams of Physical Variables

As we have seen by using a cell complex and its dual we can classify space elements, time elements and space-time elements, as shown in the diagrams of Tables 9 and 10. Since configuration and source variables of a physical theory naturally refer to space and time elements, it follows that *we may use the same classification diagram for physical variables*.

The diagram is valid for finite and differential formulation. It clearly separates the *field* equations that link the variables of the same

vertical column, from the *material* or constitutive equations that link the two columns. The horizontal links describe reversible phenomena while the oblique ones describe irreversible phenomena.

The space-time diagrams can be conceived as an assonometric view of a building whose "pillars" are the vertical columns and whose "beams" are the material equations. In the space-time diagrams we can see a front and a back. The links from back to front, which are horizontal in the assonometric view, contain time variations. In the diagrams we can see that boxes at the front describe electrostatics while those at the back describe magnetostatics.

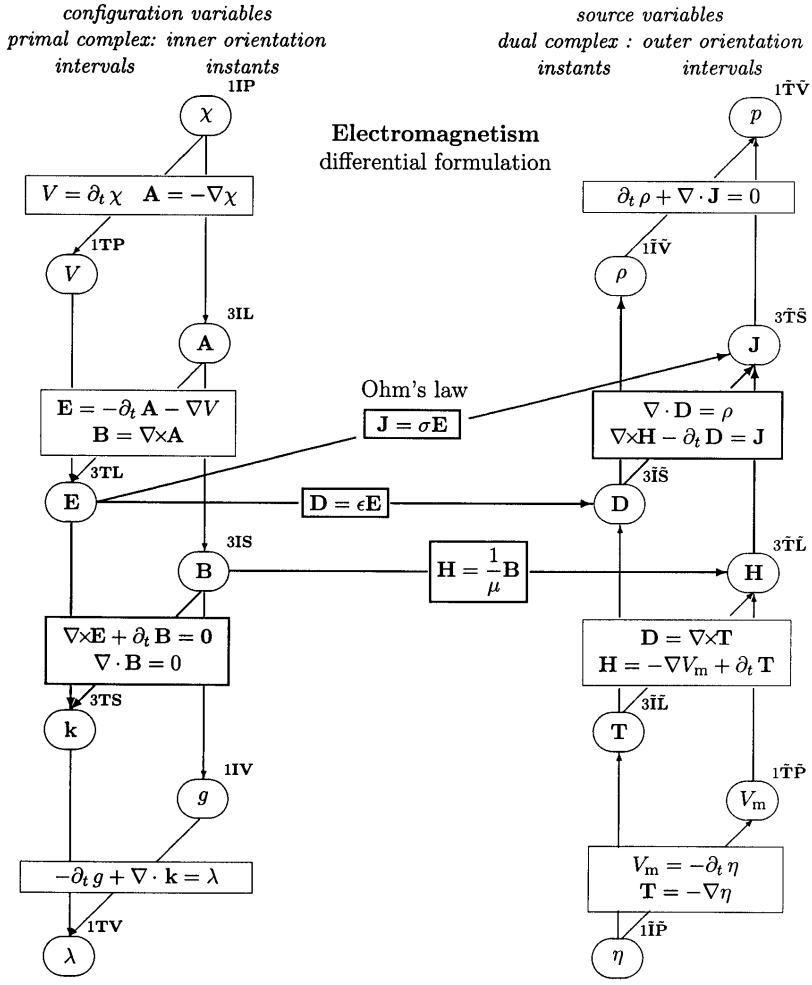
The variables on the same horizontal link are conjugated with respect to energy. This classification diagram valid for many physical theories has been presented in [29, 30, 32, 31, 33]. A similar diagram for electromagnetism, without a topological basis, appears in the papers by Deschamps [5, 6].

5. THE RELATION WITH DIFFERENTIAL FORMULATION

The differential formulation of Maxwell equations does not require two cell complexes in space or in time. Balance equations are applied to an infinitesimal cell bounded by coordinate surfaces and circuital equations are written on an infinitesimal circuit formed by coordinate lines. This is easily forgotten because infinitesimal dimensions permit the use of partial derivatives. The notion of derivative of a function of one variable at a point presupposes the existence of the limit of the incremental ratio on the *left* and on the *right* and their *equalities.* This property is violated in the space region in the direction normal to the surface of separation of two different media. It is for this reason that *Maxwell equations are valid only in regions in which the properties of material media are differential functions of the position.* It follows that, in the differential approach, the study of electromagnetic fields in regions that contain different materials requires the separation into subregions and the use of *jump conditions*.

Finite formulation requires the introduction of a primal cell-complex in such a way that on the separation surface between two media the 2-cells (faces) lie on the surface, as shown in Fig. 10. Doing so, the very fact that we consider the e.m.f. on the edges that bound the faces and that lie also on the separation surface, assures continuity of the e.m.f.: this corresponds to the continuity of the tangential component E_t of differential formulation. At the same time considering the magnetic flux referred to the faces we assure continuity of the magnetic flux that amounts to continuity of the normal component B_n

Table 9. The differential structure of electromagnetism.

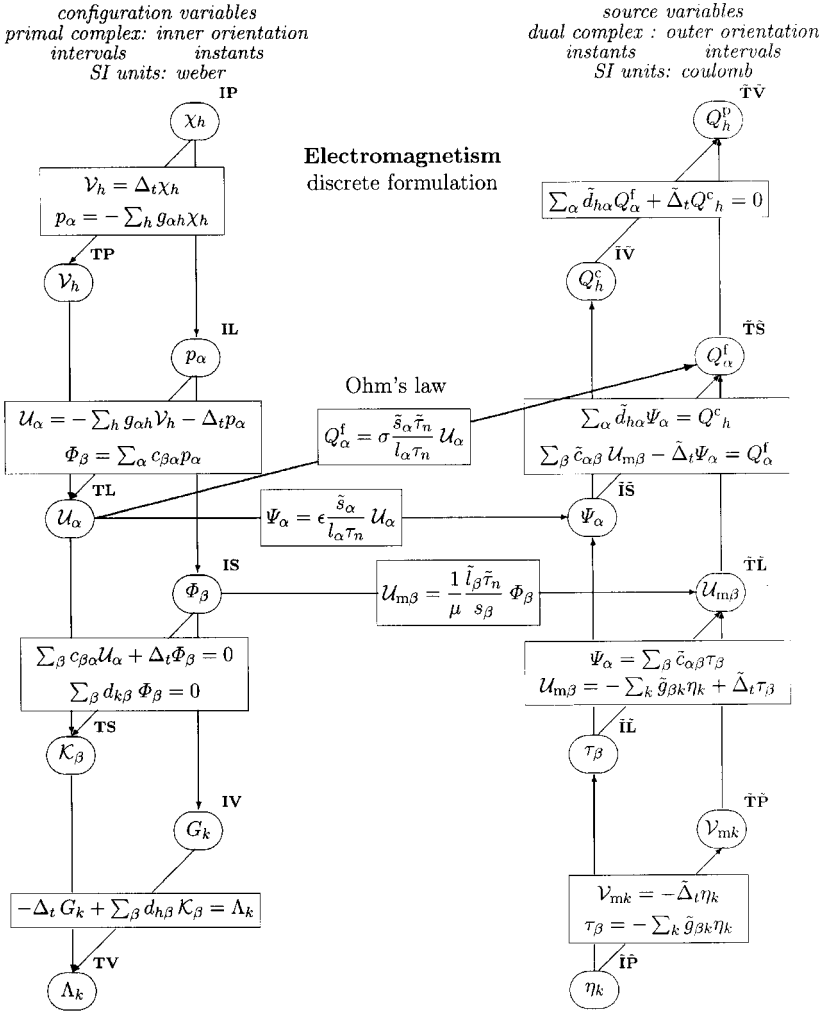


of differential formulation.

Thus finite formulation avoids jump conditions and hence permits a unified treatment of field equations and of material discontinuities. This is a significant advantage over differential formulation.

We may note that the description of physical laws in finite form contains information which is normally ignored in differential form. Differential formulation, by ignoring the association of physical variables with space elements, consequently ignores the distinction between two orientations and accordingly does not need a pair of cell complexes.

Table 10. The discrete structure of electromagnetism.



The laws of electromagnetic field can thus be expressed in finite form without losing any physical content and without adding any differentiability condition to the physical phenomenon described [34].

5.1. Relation with Other Numerical Methods

Finite element method (FEM) was first developed in the sixties in the field of solid mechanics: the unknown were the nodal displacements. FEM was introduced in electromagnetism around 1969 by Silvester

Table 11. Correspondence between finite and differential formulation of the electromagnetic equations.

	finite formulation *domain functions*	*differential formulation* *field functions*
	field laws	
1	Faraday's law	
	$\sum_\alpha c_{\beta\alpha}\, \mathcal{E}[\boldsymbol{\tau}_n, \mathbf{l}_\alpha] + \left\{\Phi[\mathbf{t}_n, \mathbf{s}_\beta] - \Phi[\mathbf{t}_{n-1}, \mathbf{s}_\beta]\right\} = 0$	$\begin{cases} \operatorname{curl} \mathbf{E} + \partial_t \mathbf{B} = 0 \\ \mathbf{n}\times(\mathbf{E}^+ - \mathbf{E}^-) = 0 \end{cases}$
2	magnetic Gauss' law	
	$\sum_\beta d_{k\beta}\, \Phi[\mathbf{t}_n, \mathbf{s}_\beta] = 0$	$\begin{cases} \operatorname{div} \mathbf{B} = 0 \\ \mathbf{n}\cdot(\mathbf{B}^+ - \mathbf{B}^-) = 0 \end{cases}$
3	Maxwell-Ampère's law	
	$\sum_\beta \tilde{c}_{\alpha\beta}\, \mathcal{F}_{\mathrm{m}}[\tilde{\boldsymbol{\tau}}_n, \tilde{\mathbf{l}}_\beta] - \left\{\Psi[\tilde{\mathbf{t}}_{n+1}, \tilde{\mathbf{s}}_\alpha] - \Psi[\tilde{\mathbf{t}}_n, \tilde{\mathbf{s}}_\alpha]\right\} = Q^{\mathrm{f}}[\tilde{\boldsymbol{\tau}}_n, \tilde{\mathbf{s}}_\alpha]$	$\begin{cases} \operatorname{curl} \mathbf{H} - \partial_t \mathbf{D} = \mathbf{J} \\ \mathbf{n}\times(\mathbf{H}^+ - \mathbf{H}^-) = \mathbf{K} \end{cases}$
4	electric Gauss' law (electrostatic induction)	
	$\sum_\alpha \tilde{d}_{h\alpha}\, \Psi[\tilde{\mathbf{t}}_n, \tilde{\mathbf{s}}_\alpha] = Q^{\mathrm{c}}[\tilde{\mathbf{t}}_n, \tilde{\mathbf{v}}_h]$	$\begin{cases} \operatorname{div} \mathbf{D} = \rho \\ \mathbf{n}\cdot(\mathbf{D}^+ - \mathbf{D}^-) = \sigma \end{cases}$
5	charge conservation law	
	$\sum_\alpha \tilde{d}_{h\alpha}\, Q^{\mathrm{f}}[\tilde{\boldsymbol{\tau}}_n, \tilde{\mathbf{s}}_\alpha] + \left\{Q^{\mathrm{c}}[\tilde{\mathbf{t}}_{n+1}, \tilde{\mathbf{v}}_h] - Q^{\mathrm{c}}[\tilde{\mathbf{t}}_n, \tilde{\mathbf{v}}_h]\right\} = 0$	$\begin{cases} \operatorname{div} \mathbf{J} + \partial_t \rho = 0 \\ \mathbf{n}\cdot(\mathbf{J}^+ - \mathbf{J}^-) = 0 \end{cases}$
6	general solution of magnetic Gauss' law	
	$\Phi[\mathbf{t}_n, \mathbf{s}_\beta] = \sum_\alpha c_{\beta\alpha}\, p[\mathbf{t}_n, \mathbf{l}_\alpha]$	$\begin{cases} \mathbf{B} = \operatorname{curl} \mathbf{A} \\ \mathbf{n}\times(\mathbf{A}^+ - \mathbf{A}^-) = 0 \end{cases}$
7	general solution of Faraday's law	
	$\mathcal{E}[\boldsymbol{\tau}_n, \mathbf{l}_\alpha] = -\sum_k g_{\alpha k}\, \mathcal{V}[\boldsymbol{\tau}_n, \mathbf{p}_k] - \left\{p[\mathbf{t}_n, \mathbf{l}_\alpha] - p[\mathbf{t}_{n-1}, \mathbf{l}_\alpha]\right\}$	$\begin{cases} \mathbf{E} = -\operatorname{grad} V - \partial_t \mathbf{A} \\ V^+ - V^- = 0 \end{cases}$
	material laws	
8	$\Psi[\tilde{\mathbf{t}}_n, \tilde{\mathbf{s}}_\alpha] = \epsilon \frac{\tilde{s}_\alpha}{\tau_n l_\alpha}\, \mathcal{E}[\boldsymbol{\tau}_n, \mathbf{l}_\alpha]$	$\mathbf{D} = \epsilon \mathbf{E}$
9	$\mathcal{F}_{\mathrm{m}}[\tilde{\boldsymbol{\tau}}_n, \tilde{\mathbf{l}}_\beta] = \frac{1}{\mu} \frac{\tilde{\tau}_n \tilde{L}_\beta}{s_\beta}\, \Phi[\mathbf{t}_n, \mathbf{s}_\beta]$	$\mathbf{H} = \frac{1}{\mu} \mathbf{B}$
10	Ohm's law	
	$\frac{Q^{\mathrm{f}}[\tilde{\boldsymbol{\tau}}_n, \tilde{\mathbf{s}}_\alpha]}{\tilde{\tau}_n\, \tilde{s}_\alpha} = \sigma \left(\frac{\mathcal{E}[\boldsymbol{\tau}_n, \mathbf{l}_\alpha]}{2\tau_n\, l_\alpha} + \frac{\mathcal{E}[\boldsymbol{\tau}_{n+1}, \mathbf{l}_\alpha]}{2\tau_{n+1}\, l_\alpha}\right)$	$\mathbf{J} = \sigma \mathbf{E}$

[27]. The application of FEM to electromagnetism followed this line of thought: since in continuum mechanics the displacements, i.e. vectors, refer to nodes it appeared natural to consider the vectors **E** and **H** as homologous and hence considered as nodal unknowns.

This identification can be criticized for the following reasons. Since the sources of the electromagnetic field are charges and these are scalar quantities, it follows that all the integral quantities of electromagnetism are scalars. These are charge, current, electric and magnetic fluxes, electric and magnetic voltages. The laws of electromagnetism, when one uses integral quantities, are all relation between scalar variables

and then they are expressed by scalar equations. If this is so, why do we commonly use vector quantities? The reason can be found in the fact that there are physical variables that refer to lines and surfaces: this is the case of electric and magnetic voltages, electric and magnetic fluxes alongside with currents. Since for every space point there is an infinity of directions one is led to introduce at every space point a vector to evaluate the integral variable referred to a line and to a surface by the scalar product of the field vector at the point and the vectors $\mathrm{d}\mathbf{L}$ and $\mathrm{d}\mathbf{S}$ that describe the geometrical elements, i.e,

$$\begin{cases} U[L] = \displaystyle\int_L \mathbf{E} \cdot \mathrm{d}\mathbf{L} & \Phi[S] = \displaystyle\int_S \mathbf{B} \cdot \mathrm{d}\mathbf{S} \\ U_{\mathrm{m}}[\tilde{L}] = \displaystyle\int_{\tilde{L}} \mathbf{H} \cdot \mathrm{d}\mathbf{L} & \Psi[\tilde{S}] = \displaystyle\int_{\tilde{S}} \mathbf{D} \cdot \mathrm{d}\mathbf{S} \qquad I[\tilde{S}] = \displaystyle\int_{\tilde{S}} \mathbf{J} \cdot \mathrm{d}\mathbf{S}. \end{cases} \tag{14}$$

On the contrary in continuum mechanics the sources of the field are forces, i.e. vectors. This implies that all global variables of continuum mechanics are vectors. Such are displacements, velocities, relative displacements, relative velocities, surface and volume forces, momenta, etc. The relative displacements of two points depend on the vector connecting the points. The force across a surface depends on the space orientation of the surface. This fact leads us to introduce second rank tensors to express the dependence of such vector quantities on the vectors that describe the lines and the surfaces: such are the strain tensor, the strain-rate tensor and the stress tensor. In continuum mechanics, where finite elements were born, displacements $\mathbf{u}$ and forces $\mathbf{f}$ are vectors associated with points (mesh nodes). In electromagnetism the vectors $\mathbf{E}$ and $\mathbf{H}$, which are associated with lines are not homologous to the vectors of continuum mechanics. *The vectors in electromagnetism play the same role as tensors in continuum mechanics.* Stated in other words: the vectors of electromagnetism *are not* the homologous of the vectors of continuous mechanics.

A numerical treatment of physical phenomena that does not take into account these differences is an "act of violence" on the physics of the problem and gives rise to inconvenients. This was the case of numerical treatment of electromagnetism by the finite element method: the vectors $\mathbf{E}$ and $\mathbf{H}$ have been applied to nodes. This gives rise to spurious solutions in electromagnetic guide waves with two dielectrics as well as in three-dimensional electromagnetic problems [28, 11]. Much time and ingenuity has been spent on finding the reason for such spurious solutions.

A further negative feature connected with field vectors is encountered in the solution of diffraction problems, where it is found that the electromagnetic field vectors may become infinite at sharp

edges of a diffracting obstacle while electromagnetic energy in any finite domain must be finite: this is the so called *edge condition* [19].

Faced with the appearance of spurious solutions, in 1982 Bossavit and Vérité [1] suggested abandoning the nodal values of field vectors $\mathbf{E}, \mathbf{H}$. Using tetrahedral meshes, they introduced, in electromagnetic computations, electromotive and m.m.f. along the edges of the tetrahedra [4, p.XV]. This was the birth of the **edge element** method (EEM) introduced by Bossavit in 1988 [2]. This method is an extension of FEM and it uses a *single mesh*. The same authors realized that the use of such global variables is in harmony with the use of the *exterior differential forms*. Taking the latter as a starting point, the finite form was achieved via special differential forms, the so called *Whitney forms* following a suggestion of Kotiuga [13].

A completely different method to give a finite formulation to Maxwell equations was introduced in 1966 by Yee [43]. He started considering a Cartesian mesh and associating the three components E_x, E_y, E_z located in the middle of the edges and the three components H_x, H_y, H_z located in the center of the faces. Doing so he introduced a pair of dual grids, G and $\tilde{G}$, later called the *electric* and *magnetic* grids. He considered the two differential equations containing $\operatorname{curl}\mathbf{E}$ and $\operatorname{curl}\mathbf{H}$ and discretized them using finite differences. Yee's method was called Finite Differences in Time Domain (FDTD). It is a refinement of FDM based on a pair of dual Cartesian meshes and on an *ad hoc* association of physical variables to the two meshes.

The FDTD method of Yee has been developed by Weiland since 1977 [39]. The author, in a paper of 1984 [40], [41] initiated the use of Maxwell equations in *integral* form. The method has been named Finite Integration Theory (FIT) and is the method used in the program MAxwell Finite Integration Algoritm (MAFIA). The integral form was *approximated* considering the tangential component E_t in the middle of the edges of G and the normal component B_n in the middle of the faces of G. The dual mesh $\tilde{G}$, called the *magnetic grid*, appeared as a natural completion of the mesh G, called the *electric grid* to make the normal component B_n, which is normal to the faces of G, tangential to the edges of $\tilde{G}$ [40, p.250]. In the FIT method spurious solutions do not appear [41, p.229].

In 1996 Weiland [42] takes an important step forward when he introduces integrated field as state variables rather than directly field components. In other words the unknowns are now vectors like $(e_1, e_2, ...e_n)$ where e_k are *voltages* along the edges. Doing so he obtained an *exact* implementation of Maxwell's equations. In the words of Weiland "*The outstanding features of Maxwell's Grid Equations (MGE) when compared with other numerical methods for*

solving field problems is that this set of matrix equations is a consistent finite representation of the original field equations in that sense that basic properties of analytical fields are maintained when moving from $\mathbb{R}^3$ *to* $\{G, \tilde{G}\}$." Weiland went on in 1996 to consider electric and magnetic voltages using a pair of dual meshes while Bossavit [1] did the same in 1982 but on a single mesh. We can see that, starting from different points of departure, computational electromagnetism evolved towards the use of a *pair of dual meshes* and the use of *global variables*.

Outside electromagnetism, in fluid dynamics, the Finite Volume Method (FVM) evolved in the same direction. Here one use a pair of *dual meshes* and make use of *global variables*. We remark that FDM, FDTD, FIT and FVM use mainly cartesian meshes while EEM uses simplicial complexes.

The finite formulation of electromagnetism we have presented in this paper, starting from an analysis of global physical variables, joins the methods inaugurated by Yee and Bossavit.

5.2. The Cell Method

The main feature of finite formulation we have presented *is the use of global variables* that are domain functions instead of local variables, i.e. field functions. This implies that we do not use differential equations or differential forms. The solutions to field problems can be achieved considering a simplicial complex K of Delaunay type and its orthogonal dual $\tilde{K}$ of Voronoi. This presupposes a Delaunay-Voronoi mesh generator in two or in three dimensions. This choice permits a simple implementation of material equations, at least for isotropic media, because they link a physical variable associated with a p-cell with another variable associated with the dual $(n-p)$-cell when the cells are orthogonal.

Variables. The variables used are: magnetic flux Φ; electric flux Ψ; electric voltage impulse $\mathcal{U}$ and magnetic voltage impulse $\mathcal{U}_{\mathrm{m}}$.

Field equations. Field equations are implemented as follows:

- the Maxwell-Ampère's and Gauss' magnetic laws on dual cells;
- the Faraday's law and Gauss' electric laws on primal cells.

Field equations have an *exact* implementation because we use global variables in space and time.

Constitutive equations. The field inside every simplex is supposed to be *uniform* and the material *homogeneous* so that the

material equations are exact in every simplex: *the only approximation we carry out is to assume uniformity of the field inside simplexes.*

Material discontinuities. The working region can be filled with different materials. It is always possible to construct a simplicial complex whose simplexes do not cross the separation surfaces but have a face that laying on them, as shown in Fig. 10. This automatically assures that jump conditions will be satisfied because voltages and fluxes on the edges and faces respectively are the same for two adjacent cells.

A numerical method based on these rules will be called the **cell method**.

6. CONCLUSION

We have shown that it is possible to build a finite formulation to the electromagnetic equations directly based on experimental laws, i.e. without passing from the differential formulation. This put into evidence some features that are not commonly considered in the differential formulation. First of all we use *global variables* instead of (local) field variables. The global variables are associated with space and time elements endowed with inner and outer orientation. These two kind of orientations are not explicit in the differential formulation. As a consequence of this association, global variables are *domain functions* while field functions are point functions. Coordinate systems, that are a natural framework for field functions, are substituted by cell-complexes. The role of orthogonal coordinate systems is played by a pair of Delaunay-Voronoi complexes for which the p-cells of one complex are orthogonal to the $(n-p)$-cells of the other.

The fact that some global variables are referred to space and time elements endowed with inner orientation and other to outer orientation implies that some global variables are naturally referred to the elements of the primal complex and other to the elements of the dual one. This gives a fundamental role to a pair of cell complexes, a notion that is lacking in the differential formulation.

The analysis we have done permits to clearly separate *topological* equations from *phenomenological* equations.

The topological equations link physical variables referred to the same cell complex, primal or dual, and are expressed by the operators gradient, curl and divergence in the differential setting. Topological equations are independent of metrical notions: a fact that is not usually stressed in the differential formulation. In fact the notion of gradient, curl and divergence are usually presented in a metrical context. This metric-independence, on the contrary, it stressed using

exterior differential forms.

The phenomenological equations, i.e. the so called material or constitutive equations, link a variable associated with a p-dimensional cell with another variable associated with the dual $(n-p)$-dimensional cell. Contrary to the topological equations they depend on metrical notions. This is another feature hidden by differential formulation.

In the theory of differential forms the topological equations on the primal complex (inner orientation) are expressed by exterior differential forms while those on the dual complex (outer orientation) are expressed by *twisted* or *odd* or *impair* exterior differential forms.

The natural role of two cell complexes is of primary importance in computational electromagnetism and gives a justification to the use of staggered meshes as done in FDTD.

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