FINITE-FUEL SINGULAR CONTROL WITH DISCRETIONARY STOPPING

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September 30, 2005

Dedicated to Dr. Václav E. Beneš on the occasion of his 70th birthday

AMS Subject Classifications: Primary 93E20, 60G40; Secondary 60J65.
Key Words: Singular stochastic control, optimal stopping, finite-fuel constraint, variational inequality, free-boundary problem, generalized Itô formula, principle of smooth-fit.
*Research supported in part by the National Science Foundation under Grant NSF-DMS-97-32810.

Abstract

We discuss the finite-fuel, singular stochastic control problem of optimally tracking the standard Brownian motion $x+W(\cdot)$ started at $x \in \mathbb{R}$, by an adapted process $\xi(\cdot) = \xi^+(\cdot) - \xi^-(\cdot)$ of bounded total variation $\check{\xi}(t) \triangleq \xi^+(t) + \xi^-(t) \leq y, \ \forall\, 0 \leq t < \infty$, so as to minimize the total expected discounted cost

$$\mathbb{E}\left[\int_0^\tau e^{-\alpha t}\lambda X^2(t)\,dt + \int_{[0,\tau]} e^{-\alpha t}\,d\check{\xi}(t) + e^{-\alpha\tau}\delta X^2(\tau)\cdot 1_{\{\tau<\infty\}}\right]$$

over such processes $\xi(\cdot)$ and stopping times τ. Here $X(\cdot) = x+W(\cdot)+\xi(\cdot)$, and $\alpha > 0$, $\delta \geq 0$, $\lambda > 0$ are given real numbers. In its form $\delta = 0$, $\tau \equiv \infty$, this problem goes back to the seminal paper of Beneš, Shepp & Witsenhausen (1980). For fixed $\alpha > 0$ and $\delta > 0$ we characterize explicitly the optimal policy in the case $\lambda \geq \alpha\delta$ (of the "act–or–stop" type, since the continuation cost is relatively large), and in the case $0 < \lambda \leq \lambda^*$ with

$$\lambda^* \triangleq \frac{\alpha\delta}{1+\frac{\delta/\alpha}{\frac{1}{4\delta}+\frac{1}{\sqrt{2\alpha}}}}$$

(of the "act, stop, or wait" type, since the relative continuation cost is relatively small). In the latter case, an associated free-boundary problem is solved exactly. The case $\lambda^* < \lambda < \alpha\delta$, of "moderate" relative continuation cost, is suggested as an open question.

1 INTRODUCTION

Stochastic optimization problems that combine features of both *continuous control* and *stopping* are relatively new in the applied probability literature. Krylov (1980, section 6.4) establishes some general conditions for optimality, Beneš (1992) provides explicit solutions to LQG-type problems with control of the drift and with discretionary stopping, Karatzas & Sudderth (1999) solve the optimal stopping problem for a diffusion on an interval with absorption at the endpoints and control of both drift and diffusion, while Karatzas & Wang (1999) treat the classical consumption/investment problem of financial economics for an investor who can decide when to "exit" from the market.

In an important relatively recent paper, Davis & Zervos (1994) analyze such problems when the controlling effort (or "fuel") can take the form of a *bounded variation* process, as opposed to the absolutely continuous control of the drift; one incurs costs proportional to the amount of fuel being used, and to the quadratic deviation from the origin (both up to, and at, termination). The question then, is to find an optimal stopping time, and a control strategy leading up to it, so as to minimize expected discounted total cost over an infinite time-horizon.

Discretionary stopping in stochastic control arises naturally in *target tracking* problems, where one has to stay close to a target by spending fuel, declare when one has arrived "sufficiently close" to the target, and then reach a decision about whether to engage the target or not. Combined stochastic control / optimal stopping problems also arise in mathematical finance, namely, in the context of computing the upper- and lower-hedging prices of *American contingent claims under constraints*; these computations lead to stochastic control of the absolutely continuous or the singular type (as in Karatzas & Kou (1998) or Karatzas & Wang (1998), respectively.)

Our aim in this paper is to treat the Davis & Zervos (1994) target-tracking problem, when the available supply of fuel is limited. The problem is set up in Section 2 and is linked to the existing literature of singular stochastic control, in particular to the work of Beneš, Shepp & Witsenhausen (1980). This seminal paper is largely responsible for the rapid growth of interest in so-called *singular stochastic control* problems during the last twenty years, which has encompassed applications as diverse as queueing networks (e.g. Harrison (1985)) and portfolio optimization under transaction costs (e.g. Davis & Norman (1990), Shreve & Soner (1994)).

The two extreme cases, of "no fuel at all" and of infinite fuel, are reviewed in Sections 6 and 7, respectively. Section 3 provides a heuristic discussion of the finite-fuel problem, leading to a suitable Variational Inequality that its value function has to satisfy. It is then shown, in Section 4, under what conditions a solution of this Variational Inequality will coincide with the value function of the stochastic control problem. The Variational Inequality is then solved exactly, when the relative "continuation cost" is either relatively large (Section 5) or relatively small (Sections 8-9, as well as Section 10).

The optimal strategy has qualitatively different behavior in each of these two cases. In the first case it is of the *"act-or-stop"* type. In the second case an intermediate region appears, which becomes narrower as the supply of available fuel diminishes. In the interior of this region, one simply does not exert any control; and when the "inner boundary" of the region is reached, it becomes optimal to stop. On the "outer boundary" of this region, and as long as the amount of available fuel exceeds a certain critical level, one exerts control in a "singular" manner, by spending just as much fuel as is necessary in order to keep the controlled process within the region (reflecting boundary); but as soon as the amount of available fuel falls at or below the critical level, one spends it all at once ("exit", or repelling, boundary). Thus, the optimal policy is of the *"act, continue, or stop"* variety in this second case. Finally, we suggest as an interesting open problem the computation of the value function and of the optimal policy in the third case, of "moderate continuation cost".

2 THE PROBLEM

Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $\mathbb{F} = \{\mathcal{F}_t, \quad 0 \le t < \infty\}$ which satisfies the usual condition of right-continuity and augmentation by null sets. We denote by $\mathcal{S}$ the class of all $\mathbb{F}$-stopping times. We also denote by $\mathcal{A}$ the class of $\mathbb{F}$-adapted processes $\xi = \{\xi_t, \quad 0 \le t < \infty\}$ with paths that are right-continuous and have finite total variation on any compact interval, as well as $\xi_{0-} = 0$. A process $\xi \in \mathcal{A}$ will be considered in its minimal decomposition

$$\xi_t = \xi_t^+ - \xi_t^-, \qquad 0 \le t < \infty \tag{2.1}$$

as the difference of two *non-decreasing* process $\xi^\pm \in \mathcal{A}$, so that the total variation of the function $s \mapsto \xi_s$ on the interval $[0, t]$ is given by

$$\check{\xi}_t = \xi_t^+ + \xi_t^-, \qquad t \in [0, \infty]. \tag{2.2}$$

In addition to the class $\mathcal{A}$, we shall consider its nested subclasses

$$\mathcal{A}(y) = \big\{\xi \in \mathcal{A} \;/\; \check{\xi}_\infty \le y, \text{ a.s.}\big\} \tag{2.3}$$

for $0 \le y \le \infty$, with $\mathcal{A}(0) = \{\mathbf{0}\}$ and $\mathcal{A}(\infty) = \mathcal{A}$.

Let us also assume that the probability space carries a standard, one-dimensional $\mathbb{F}$-Brownian motion $W = \{W_t, \quad 0 \le t < \infty\}$. Corresponding to any given *initial position* $x \in \mathbb{R}$ and *control process* $\xi \in \mathcal{A}$, we consider the two-dimensional state process

$$(X_t, Y_t) = (x + W_t + \xi_t,\ y - \check{\xi}_t), \qquad 0 \le t < \infty. \tag{2.4}$$

The first component X_t of this random vector can be interpreted as the position at time t of a "particle" started at x, subjected to the random motion W, and controlled by the process ξ through the cumulative effect of a "push" ξ_t^+ (respectively, ξ_t^-) to the right (respectively, to the left). Then the quantity $\check{\xi}_t$ of (2.2) represents the total amount of fuel expended by time t, so that Y_t, the second component in (2.4), can be interpreted as the "remaining fuel at time t". The objective of the control is to "keep the particle as close to the origin as possible, for as long as the controlling action lasts" (that is, up to a stopping time τ), and its success is measured by the functional

$$J(x, y; \xi, \tau) = \mathbb{E}\left[\int_0^\tau e^{-\alpha t}\lambda X_t^2\, dt + \int_{[0,\tau]} e^{-\alpha t}\, d\check{\xi}_t + e^{-\alpha\tau}\delta X_\tau^2 \cdot 1_{\{\tau<\infty\}}\right] \tag{2.5}$$

for any given $x \in \mathbb{R},\ y \in [0, \infty]$ and $\tau \in \mathcal{S},\ \xi \in \mathcal{A}(y)$.

In other words, there is a quadratic "running cost" for missing the "target state", the origin; this cost can be reduced by "exerting fuel", in the form of an increase in ξ^+ or ξ^- (i.e., of a rightward or a leftward push) in the right direction. The total supply of fuel, however, is limited, namely $\check{\xi}_\infty \le y$, a.s.; and spending an amount of fuel $\check{\xi}_{t+h} - \check{\xi}_t$ over a small interval $(t, t + h]$ costs proportionally to the amount spent. On the other hand, the controller has the option to disengage completely, at a stopping time τ of his choice, by incurring a quadratic "terminal cost". What should be the optimal disengagement or *stopping rule* $\tau^* \in \mathcal{S}$, and what should be the optimal *control strategy* $\xi^* \in \mathcal{A}(y)$ up to disengagement, if one is trying to minimize the expected discounted total cost of (2.5)? And how can we compute the minimal such expected cost

$$V(x,y) \triangleq \inf_{\xi\in\mathcal{A}(y),\ \tau\in\mathcal{S}} \mathbb{E}\left[\int_0^\tau e^{-\alpha t}\lambda X_t^2\,dt + \int_{[0,\tau]} e^{-\alpha t}\,d\check{\xi}_t + e^{-\alpha\tau}\delta X_\tau^2\cdot 1_{\{\tau<\infty\}}\right] \tag{2.6}$$

when starting in position $x \in \mathbb{R}$ and with a given amount $y \in [0,\infty]$ of available fuel?

This is a stochastic optimization problem that incorporates features of *bounded-variation control*, of *optimal stopping*, and of *finite-fuel constraints*. In its version $\tau \equiv \infty,\ \delta = 0$ (i.e., without discretionary stopping) it goes back to the seminal work of Beneš, Shepp & Witsenhausen (1980); see also Karatzas & Shreve (1986). The case $\tau \equiv \infty,\ \delta = 0,\ y = \infty$ (infinite fuel, no stopping) was treated in Karatzas (1983), whereas the problem with discretionary stopping and infinite fuel ($\delta > 0,\ y = \infty$) was solved by Davis & Zervos (1994).

With fixed values for the parameters $\alpha > 0$ and $\delta > 0$, we shall solve the problem (2.6) explicitly when $\lambda > 0$ is *sufficiently large*, namely $\lambda \geq \alpha\delta$ (cf. Section 5), as well as when $\lambda > 0$ is *sufficiently small*, first with $0 < \lambda \leq \lambda_*$ for some

$$\lambda_* \in (0,\lambda^*], \qquad \lambda^* \triangleq \frac{\alpha\delta}{1+\frac{\delta/\alpha}{\frac{1}{4\delta}+\frac{1}{\sqrt{2\alpha}}}} < \alpha\delta \tag{2.7}$$

(cf. Sections 8, 9) and then with $\lambda_* < \lambda \leq \lambda^*$ (Section 10). We shall leave the case $\lambda^* < \lambda < \alpha\delta$ as an open problem.

The methodology of Sections 8-10 relies heavily on the familiar "principle of smooth-fit", which postulates sufficient smoothness on the part of the value-function $V(\cdot,\cdot)$ of (2.6) across suitable *absorbing*, *reflecting* (for $y > \bar{y}$) and *repelling* (for $0 < y \leq \bar{y}$) free-boundaries; here $\bar{y} \geq 0$ is a critical fuel-level. Broadly speaking, it turns out that we have $\mathcal{C}^1-$smooth-fit across absorbing or repelling boundaries (cf. (8.11), (8.12), (10.4)(a), or (10.4)(c)), and $\mathcal{C}^2-$smooth-fit across reflecting boundaries (cf. (8.14), (8.15) and (10.4)(b)). With the help of the Verification Theorem of Section 4, this principle then leads to the computation of the value function in the form (9.1) for the case $0 < \lambda \leq \lambda_*$, and in the form (10.26) for the case $\lambda_* < \lambda \leq \lambda^*$. We do not have yet a good guess for the optimal policy in the case $\lambda \in (\lambda^*, \alpha\delta)$, or for the smoothness of the value-function in that case, so a new methodology may have to be developed for treating it.

Along with those of Beneš et al. (1980), the computations presented in Sections 8-10 provide rare examples of free-boundary problems arising in sequential analysis or stochastic control that can be solved exactly, and for which the resulting free-boundaries are neither parabolas nor straight lines.

3 HEURISTIC DISCUSSION

It is clear from (2.6), (2.4) that, if the state-process X ever finds itself at the origin, then we should stop the first time this happens. It is also clear that we may consider control processes $\xi \in \mathcal{A}(y)$, such that

$$X_t \cdot X_{t-} \geq 0, \qquad 0 \leq t < \infty \tag{3.1}$$

$$\int_0^\infty 1_{\{X_t > 0\}}\, d\xi_t^+ = \int_0^\infty 1_{\{X_t < 0\}}\, d\xi_t^- = 0 \tag{3.2}$$

hold almost surely. In other words, it is never optimal to jump across the origin, or to push to the right (resp., to the left) when X_t is positive (resp., negative). All these claims can be verified easily, after the fact. Thus we may compute the function $V(\cdot, y)$ only on $[0,\infty)$, and then extend it by *even symmetry*

$$V(-x,y) = V(x,y); \qquad x \geq 0, \quad y \geq 0. \tag{3.3}$$

We may also take $x \geq 0$ and monotone control $\xi = -\xi^-$, with

$$X_t = x + W_t - \xi_t^- \; \in [0,\infty). \tag{3.4}$$

Let us consider the value function $V(x,y)$ of (2.6) in the two extreme cases of "no fuel at all" ($y=0$) and of "infinite available fuel" ($y=\infty$), namely,

$$V_0(x) \;\triangleq\; \inf_{\tau \in \mathcal{S}} \;\mathbb{E}\left[\int_0^\tau e^{-\alpha t}\lambda(x+W_t)^2\, dt + e^{-\alpha\tau}\delta(x+W_\tau)^2 \cdot 1_{\{\tau<\infty\}}\right] \qquad \text{and} \tag{3.5}$$

$$V_\infty(x) \;\triangleq\; \inf_{\xi\in\mathcal{A},\; \tau\in\mathcal{S}} \;\mathbb{E}\left[\int_0^\tau e^{-\alpha t}\,\lambda X_t^2\, dt + \int_{[0,\tau]} e^{-\alpha t}\, d\check{\xi}_t + e^{-\alpha\tau}\delta X_\tau^2 \cdot 1_{\{\tau<\infty\}}\right], \tag{3.6}$$

respectively. The function of (3.5) is the optimal risk in a problem of pure *optimal stopping* which can be solved explicitly (cf. Section 6). The function of (3.6) is the value of the problem considered, and solved explicitly, by Davis & Zervos (1994); see Section 7. It is also intuitively clear that

$$V(x,\cdot) \;\text{ should be decreasing, with }\; \lim_{y\downarrow 0} V(x,y) = V_0(x) \;\text{ and }\; \lim_{y\to\infty} V(x,y) = V_\infty(x). \tag{3.7}$$

In order to proceed further, we shall make an additional assumption about the value function $V(x,y)$ of (2.6). This property will also be verified after the fact.

$$\left\{\begin{array}{c}\text{The function } V:\ \mathbb{R}\times[0,\infty)\to[0,\infty) \text{ is continuous and continuously}\\ \text{differentiable, as well as twice continuously differentiable, with locally}\\ \text{bounded second derivatives away from } \{(x,y)/x=\pm f(y) \text{ or } x=\pm h(y)\},\\ \text{for some continuous functions}\\[1ex] f:[0,\infty)\to(0,\infty),\; h:[0,\infty)\to(0,\infty) \quad \text{with} \quad f\in\mathcal{C}^1((0,\infty)),\\[1ex] h\in\mathcal{C}^1((0,\infty)/\{\hat{y}\}) \quad \text{for some } \hat{y}>0, \quad f'(\cdot)\leq 0, \quad h'(\cdot)>0,\\ f(0)\leq h(0), \qquad \text{and} \qquad f(y)<g(y),\ \forall\, y>0.\end{array}\right\} \tag{3.8}$$

We claim then that $V(x,y)$ should satisfy the following conditions

$$V(x,y) \leq \delta x^2 \quad ; \qquad x\in\mathbb{R},\; y\geq 0 \tag{3.9}$$

$$|V_x(x,y)| + V_y(x,y) \leq 1 \quad ; \qquad x\in\mathbb{R},\; y>0 \tag{3.10}$$

$$\alpha V(x,y) \leq \frac{1}{2}V_{xx}(x,y) + \lambda x^2 \quad ; \qquad x\neq\pm f(y),\; x\neq\pm h(y),\; y\geq 0, \quad \text{and} \tag{3.11}$$

(3.12) $$[\delta x^2 - V(x,y)] \cdot [1 - |V_x(x,y)| - V_y(x,y)] \cdot \left[\frac{1}{2}V_{xx}(x,y) + \lambda x^2 - \alpha V(x,y)\right] = 0$$

for $x \neq \pm f(y),\ x \neq \pm h(y),\ y > 0$.

The property (3.9) is obvious since we can always *stop immediately*, i.e. take $\tau = 0$, in (2.6). For (3.10) in the case $x > 0$, observe that if we *act* by spending immediately a small amount $\vartheta \in (0, y)$ of fuel and move to the new position $(x - \vartheta, y - \vartheta)$, we obtain

$$V(x,y) \leq \vartheta + V(x - \vartheta, y - \vartheta).$$

Subtracting $V(x, y - \vartheta)$ from both sides of this inequality, dividing by ϑ, and then letting $\vartheta \downarrow 0$, we obtain (3.10) in the form $V_x(x,y) + V_y(x,y) \leq 1$. In a similar manner we obtain the inequality (3.10) for $x < 0$, namely $V_y(x,y) - V_x(x,y) \leq 1$.

Finally, suppose that we start at $x \notin \{\pm f(y),\ \pm h(y)\}$, that we *continue* for a short while, and that we then behave optimally; such a policy gives

$$V(x,y) \leq \mathbb{E}\left[\int_0^{\epsilon\wedge\sigma} e^{-\alpha t}\lambda(x + W_t)^2\, dt + e^{-\alpha(\epsilon\wedge\sigma)}V(x + W_{\epsilon\wedge\sigma}, y)\right]$$

for every $\epsilon > 0$ and $\sigma \triangleq \inf\{t \geq 0 \ /\ x + W_t = \pm f(y) \ \text{ or } \ x + W_t = \pm h(y)\}$. Under the assumption (3.8) we may apply Itô's rule to the last term, and then divide by $\epsilon > 0$ before letting $\epsilon \downarrow 0$, to obtain (3.11).

Now we have *equality* in (3.9), (3.10) or (3.11) if, at position x and with available fuel y, it is optimal to stop, to act, or to wait, respectively. Because at any given pair (x, y) we *must* do one of these things, at least one of (3.9)–(3.11) should hold as equality, whence (3.12). The conditions of (3.9)–(3.12) constitute a *Variational Inequality.* For $x \neq \pm f(y),\ x \neq \pm h(y),\ y > 0$, this inequality may be written in the more compact from

(3.13) $$\min\left[\delta x^2 - V(x,y),\ 1 - |V_x(x,y)| - V_y(x,y),\ \frac{1}{2}V_{xx}(x,y) + \lambda x^2 - \alpha V(x,y)\right] = 0.$$

4 A VERIFICATION THEOREM

Motivated by the heuristic discussion of the previous section, let us state now conditions that are sufficient for optimality in our problem (2.6).

4.1 Theorem: *Consider a function $Q : \mathbb{R} \times [0,\infty) \to [0,\infty)$ which satisfies the properties (3.8), namely is continuous and continuously differentiable, as well as twice continuously differentiable, with locally bounded second derivatives away from $\{(x,y)/x = \pm f(y)$ or $x = \pm h(y)\}$. Here $f : [0,\infty) \to (0,\infty)$ and $h : [0,\infty) \to (0,\infty)$ are suitable functions with the properties of (3.8). Suppose that $Q(\cdot, y)$ is* evenly symmetric

(4.1) $$Q(-x,y) = Q(x,y), \qquad \forall\ x \geq 0,\ y \geq 0$$

and satisfies the growth condition

(4.2) $$|Q_x(x,y)| \leq K(y)(1 + |x|), \qquad \forall\ x \geq 0,\ y \geq 0$$

for some continuous and increasing function $K : [0,\infty) \to (0,\infty)$. Suppose also that Q satisfies the conditions of the Variational Inequality *(3.9)–(3.12), namely:*

$$Q(x,y) \le \delta x^2 \quad , \qquad x \in \mathbb{R},\ y \ge 0\ ; \tag{4.3}$$

$$|Q_x(x,y)| + Q_y(x,y) \le 1 \quad , \qquad x \in \mathbb{R},\ y > 0\ ; \tag{4.4}$$

$$\alpha Q(x,y) \le \frac{1}{2} Q_{xx}(x,y) + \lambda x^2 \quad , \qquad x \ne \pm f(y),\ x \ne \pm h(y),\ y \ge 0\ ; \quad \textit{and} \tag{4.5}$$

$$[\delta x^2 - Q(x,y)] \cdot [1 - |Q_x(x,y)| - Q_y(x,y)] \cdot \left[\frac{1}{2} Q_{xx}(x,y) + \lambda x^2 - \alpha Q(x,y) \right] = 0 \tag{4.6}$$

for $x \ne \pm f(y),\ x \ne \pm h(y),\ y > 0$. Then the function Q is a lower bound on the attainable expected cost (2.5), namely

$$V(x,y) \ge Q(x,y), \qquad \forall\ x \in \mathbb{R},\ y \ge 0 \tag{4.7}$$

in the notation of (2.6).

Proof: For fixed $x \in \mathbb{R},\ y \ge 0$ and arbitrary process $\xi \in \mathcal{A}(y)$ and stopping time $\tau \in \mathcal{S}$, we have to show

$$J(x,y;\xi,\tau) \ge Q(x,y). \tag{4.8}$$

To do this, let us consider the two-dimensional process $\{(X_t,Y_t),\ 0 \le t < \infty\}$ of (2.4). An application of the generalized Itô rule to the process $\{e^{-\alpha t} Q(X_t,Y_t),\ 0 \le t < \infty\}$ yields

$$\begin{aligned} e^{-\alpha T} Q(X_T,Y_T) \;&=\; Q(x,y) + \int_0^T e^{-\alpha t} Q_x(X_t,Y_t)\, dW_t + \int_{[0,T]} e^{-\alpha t} Q_x(X_{t-},Y_{t-})\, d\xi_t \\ &- \int_{[0,T]} e^{-\alpha t} Q_y(X_{t-},Y_{t-})\, d\check{\xi}_t + \int_0^T e^{-\alpha t} \left\{ \frac{1}{2} Q_{xx}(X_t,Y_t) - \alpha Q(X_t,Y_t) \right\} dt \\ &+ \sum_{0 \le t \le T} e^{-\alpha t} \left[Q(X_t,Y_t) - Q(X_{t-},Y_{t-}) - \triangle X_t \cdot Q_x(X_{t-},Y_{t-}) - \triangle Y_t \cdot Q_y(X_{t-},Y_{t-}) \right]; \end{aligned} \tag{4.9}$$

see, for instance , Protter (1990), p. 74. Here $\triangle \xi_t \stackrel{\triangle}{=} \xi_t - \xi_{t-}$ and the series $\sum_{0 \le t \le T} e^{-\alpha t} Q_x(X_{t-},Y_{t-}) \cdot \triangle X_t$, $\sum_{0 \le t \le T} e^{-\alpha t} Q_y(X_{t-},Y_{t-}) \cdot \triangle Y_t$ are both absolutely convergent and bounded from above in each finite interval $[0,T]$, almost surely. (The possible lack of $\mathcal{C}^2-$smoothness of Q across the boundaries $\{(x,y)/x = \pm f(y) \text{ or } x = \pm h(y)\}$ can be dealt with by mollification, as in the proof of Theorem 2.7.9, pp. 74–76 in Karatzas & Shreve (1998), leading to the formula (4.9).) Let us denote by $\eta \stackrel{\triangle}{=} \xi^c$ the *continuous part* of the process ξ, so that

$$\xi_t = \eta_t^+ - \eta_t^- + \sum_{0 \le s \le t} (\triangle X_s), \quad 0 \le t < \infty, \tag{4.10}$$

$$\check{\xi}_t = \eta_t^+ + \eta_t^- + \sum_{0 \le s \le t} (-\triangle Y_s), \quad 0 \le t < \infty. \tag{4.11}$$

Substituting these expressions in it, we may re-write (4.9) in the equivalent form

$$\text{(4.12)}\quad \int_0^T e^{-\alpha t}\lambda X_t^2\,dt+\int_{[0,T]} e^{-\alpha t}\,d\check{\xi}_t+\delta e^{-\alpha T}X_T^2 = Q(x,y)+\int_0^T e^{-\alpha t}Q_x(X_t,Y_t)\,dW_t+\sum_{i=1}^5 I_j(T),$$

where we have set

$$\text{(4.13)}\quad I_1(T) \ \stackrel{\triangle}{=}\ e^{-\alpha T}\Big(\delta X_T^2 - Q(X_T,Y_T)\Big)$$

$$\text{(4.14)}\quad I_2(T) \ \stackrel{\triangle}{=}\ \int_0^T e^{-\alpha t}\left(\frac{1}{2}Q_{xx}(X_t,Y_t)+\lambda X_t^2-\alpha Q(X_t,Y_t)\right)dt$$

$$\text{(4.15)}\quad I_3(T) \ \stackrel{\triangle}{=}\ \int_{[0,T]} e^{-\alpha t}\Big(1+Q_x(X_{t-},Y_{t-})-Q_y(X_{t-},Y_{t-})\Big)\,d\eta_t^+$$

$$\text{(4.16)}\quad I_4(T) \ \stackrel{\triangle}{=}\ \int_{[0,T]} e^{-\alpha t}\Big(1-Q_x(X_{t-},Y_{t-})-Q_y(X_{t-},Y_{t-})\Big)\,d\eta_t^-$$

$$\text{(4.17)}\quad I_5(T) \ \stackrel{\triangle}{=}\ \sum_{0\le t\le T} e^{-\alpha t}\Big(Q(x\pm\vartheta,y-\vartheta)-Q(x,y)+\vartheta\Big)\Big|_{(x,y)=(X_{t-},Y_{t-}),\ \vartheta=-\triangle Y_t=|\triangle X_t|>0}$$

(as the referee points out, Shreve & Soner (1994) carry out an argument similar to our reduction of the equation (4.9) by use of (4.10), (4.11)). The conditions (4.3)-(4.5) guarantee that each of the terms $I_j(T),\ j=1,\ldots,5$ is *non-negative* (for $I_5(T)$, use the mean-value theorem, along with the condition (4.4)). On the other hand, thanks to the growth condition (4.2), we have

$$\mathbb{E}\int_0^\infty e^{-2\alpha t}\big(Q_x(X_t,Y_t)\big)^2\,dt \le 2K^2(y)\int_0^\infty e^{-2\alpha t}\big(1+x^2+\mathbb{E}W_t^2+y^2\big)\,dt<\infty,$$

so that the stochastic integral on the right-hand side of (4.12) has expectation equal to zero.

Now we can take expectations in (4.12), and conclude that

(4.18)

$$Q(x,y)\le \mathbb{E}\left[\int_0^{\tau\wedge n} e^{-\alpha t}\lambda X_t^2\,dt+\int_{[0,\tau\wedge n]} e^{-\alpha t}\,d\check{\xi}_t+\delta e^{-\alpha\tau}X_\tau^2\cdot 1_{\{\tau<n\}}\right]+\delta e^{-\alpha n}\,\mathbb{E}\left[X_n^2\cdot 1_{\{\tau\ge n\}}\right]$$

holds for every $\tau\in\mathcal{S}$, $n\in\mathbb{N}$. Clearly

$$0\le e^{-\alpha n}\,\mathbb{E}\left[X_n^2\cdot 1_{\{\tau\ge n\}}\right]\le 4e^{-\alpha n}\,\mathbb{E}\left(x^2+W_n^2+\check{\xi}_n^2\right)\le 4e^{-\alpha n}(x^2+n+y^2)\longrightarrow 0$$

and we obtain

$$\text{(4.19)}\qquad Q(x,y)\le \mathbb{E}\left[\int_0^\tau e^{-\alpha t}\lambda X_t^2\,dt+\int_{[0,\tau]} e^{-\alpha t}\,d\check{\xi}_t+\delta e^{-\alpha\tau}X_\tau^2\cdot 1_{\{\tau<\infty\}}\right],$$

in the limit as $n\longrightarrow\infty$, thanks to the Monotone Convergence Theorem. □

4.2 Corollary: *Suppose the pair $(\xi,\tau)\in\mathcal{A}(y)\times\mathcal{S}$ is such that*

$$\text{(4.20)}\qquad I_j(\tau\wedge n)\equiv 0,\qquad \forall\ j=1,\ldots,5,\quad \forall n\in\mathbb{N}\qquad a.s.,$$

in the notation of (4.13)–(4.17). Then (4.8) and (4.7) hold as equalities, namely

$$\text{(4.21)}\qquad J(x,y;\xi,\tau)=Q(x,y)=V(x,y),$$

and the pair (ξ,τ) is optimal for the problem of (2.6).

5 THE CASE $\lambda \geq \alpha\delta$

In this case it is "too expensive to wait and do nothing". The optimal policy consists of a combination of "act-and-stop" moves, depicted graphically in Figure 1 for $x \geq 0$, whereby one

$$(5.1) \qquad \left\{ \begin{array}{l} \textit{(i) stops immediately, if} \quad 0 \leq x \leq \frac{1}{2\delta}: \quad \xi^* \equiv 0,\ \tau^* = 0, \\ \\ \textit{(ii) spends all fuel at once, and then stops immediately, if} \quad x \geq y + \frac{1}{2\delta}: \\ \qquad \xi_0^* = -y,\ \tau^* = 0,\ \textit{or} \\ \\ \textit{(iii) spends the amount } x - \frac{1}{2\delta} \textit{ of fuel at } t = 0 \textit{ to move to the position } \frac{1}{2\delta}, \\ \qquad \textit{and then stops, if } \frac{1}{2\delta} < x < y + \frac{1}{2\delta}: \quad \xi_0^* = -(x - \frac{1}{2\delta}),\ \tau^* = 0. \end{array} \right\}$$

All these moves assume an initial position $x \geq 0$; analogous (symmetric) moves take place for $x < 0$. The function

$$(5.2) \qquad Q(x,y) \stackrel{\triangle}{=} \min_{0 \leq a \leq y} \left[\delta(|x| - a)^2 + a\right] = \left\{ \begin{array}{rcl} \delta x^2 & ; & 0 \leq x \leq \frac{1}{2\delta},\ y \geq 0 \\ x - \frac{1}{4\delta} & ; & \frac{1}{2\delta} < x < y + \frac{1}{2\delta},\ y > 0 \\ \delta(x-y)^2 + y & ; & x \geq y + \frac{1}{2\delta},\ y \geq 0 \\ Q(-x,y) & ; & x < 0,\ y \geq 0 \end{array} \right\}$$

is easily seen to satisfy the conditions (4.1)–(4.6), in particular with $f(y) \equiv \frac{1}{2\delta}$ and $h(y) = \frac{1}{2\delta} + y$ for $0 \leq y < \infty$. We leave the details of this verification to the reader. It is also fairly clear that the strategy (ξ^*, τ^*), described in (5.1) above, satisfies the conditions of Corollary 4.2 and is thus optimal for the problem of (2.6), namely

$$(5.3) \qquad J(x,y;\xi^*,\tau^*) = V(x,y) = Q(x,y).$$

5.1 Remark: For $\lambda \geq \alpha\delta$, it can be checked easily that the process

$$\int_0^t e^{-\alpha s} \lambda (x + W_s)^2 \, ds + e^{-\alpha t} \delta (x + W_t)^2, \quad 0 \leq t < \infty$$

is a submartingale; this shows that $V_0(x) = \delta x^2$, and that the rule $\tau^* = 0$ ("stop at once") is optimal for the problem of (3.5).

6 THE CASE $\lambda < \alpha\delta$, $y = 0$: NO FUEL

In this case we face the pure *optimal stopping problem* of (3.5). A reasonable guess for this problem is that its optimal stopping region should be a closed interval around the origin of the form $\mathbf{\Sigma} = [-f_0,\ f_0]$ ("stop, when you have come sufficiently close to the target"), and that the open set $\mathbb{R} \setminus \mathbf{\Sigma} = (-\infty, -f_0) \bigcup (f_0, \infty)$ should be the optimal continuation region. Then the value $V_0(x)$ of this problem can be sought as the solution of the following *free–boundary problem*

$$\frac{1}{2}Q_0''(x) + \lambda x^2 - \alpha Q_0(x) > 0 \quad ; \qquad 0 < x < f_0 \tag{6.1}$$

$$\frac{1}{2}Q_0''(x) + \lambda x^2 - \alpha Q_0(x) = 0 \quad ; \qquad x > f_0 \tag{6.2}$$

$$Q_0(x) < \delta x^2 \quad ; \qquad x > f_0 \tag{6.3}$$

$$Q_0(x) = \delta x^2 \quad ; \qquad 0 \le x \le f_0 \tag{6.4}$$

$$Q_0(-x) = Q_0(x) \quad ; \qquad x < 0 \tag{6.5}$$

in the space of functions $\mathcal{C}^1(\mathbb{R}) \cap \mathcal{C}^2(\mathbb{R} \setminus \{\pm f_0\})$, for a suitable $f_0 > 0$ that has to be determined along with the function $Q_0(\cdot)$. The postulated continuity of $Q_0(\cdot)$ and $Q_0'(\cdot)$ across the point f_0 leads to the *smooth-fit conditions*

$$Q_0(f_0+) = \delta f_0^2, \qquad Q_0'(f_0+) = 2\delta f_0. \tag{6.6}$$

(Salminen (1985), p.93 provides necessary and sufficient conditions for the C^1-smoothness of the optimal expected reward from stopping, for one-dimensional diffusions; see also Shiryaev (1978). Friedman (1976), p. 447, Theorem 4.2 has similar sufficient conditions in several dimensions.)

It is possible to solve the system of (6.1)-(6.5) for a constant $f_0 > 0$ and for a function $Q_0(\cdot)$ in $\mathcal{C}^1(\mathbb{R}) \cap \mathcal{C}^2(\mathbb{R} \setminus \{\pm f_0\})$, as follows:

$$Q_0(x) = \left\{ \begin{array}{ccl} \frac{\lambda}{\alpha}x^2 + \frac{\lambda}{\alpha^2} + B_0 e^{-x\sqrt{2\alpha}} & ; & x > f_0 \\ \delta x^2 & ; & 0 \le x \le f_0 \\ Q_0(-x) & ; & x < 0 \end{array} \right\}. \tag{6.7}$$

Here the two free constants $B_0,\ f_0$ are determined by the two conditions of (6.6), as

$$B_0 = -\frac{2f_0}{\alpha\sqrt{2\alpha}}(\alpha\delta - \lambda)e^{f_0\sqrt{2\alpha}} < 0, \tag{6.8}$$

and

$$f_0 \equiv f_0(\lambda) \stackrel{\triangle}{=} \frac{1}{\sqrt{2\alpha}}\left(\sqrt{\frac{\alpha\delta + \lambda}{\alpha\delta - \lambda}} - 1\right) > 0, \tag{6.9}$$

respectively. Note that f_0 is the unique positive solution of the equation

$$\rho(f_0) = 0, \quad \text{ with } \quad \rho(x) \stackrel{\triangle}{=} x^2 + \frac{2x}{\sqrt{2\alpha}} - \frac{\lambda/\alpha}{\alpha\delta - \lambda}, \tag{6.10}$$

and that the function $\lambda \mapsto f_0(\lambda)$ of (6.9) is strictly increasing on $(0, \alpha\delta)$. Having thus determined the solution of the system (6.1)–(6.6), one can then check that the process

$$Y_t = e^{-\alpha t}Q_0(x + W_t) + \int_0^t e^{-\alpha s}\lambda(x + W_s)^2\, ds, \qquad 0 \le t < \infty$$

is a submartingale, whereas the stopped process $\{Y_{t\wedge\tau_0^*},\ 0 \le t < \infty\}$, with

$$\tau_0^* \stackrel{\triangle}{=} \inf\left\{t \ge 0\ /\ |x + W_t| \le f_0\right\} < \infty \qquad \text{a.s.}, \tag{6.11}$$

is a martingale. These two properties lead to the computation of the optimal risk, to the optimality of the stopping time τ_0^* for the problem of (3.5), namely

$$V_0(x) = Q_0(x) = \mathbb{E}\left[\int_0^{\tau_0^*} e^{-\alpha t}\lambda(x+W_t)^2\,dt + \delta e^{-\alpha\tau_0^*}(x+W_{\tau_0^*})^2\right], \tag{6.12}$$

and thus also to the justification of the guess that $[-f_0,\ f_0]$ should be the optimal stopping region. We leave the details of these derivations to the care of the reader.

7 THE CASE $\lambda < \alpha\delta$, $y = \infty$: INFINITE FUEL

This is the problem of (3.6), which was solved by Davis & Zervos (1994). These authors showed that the value-function $V_\infty(\cdot)$ can be sought as the solution of the *free-boundary problem*

$$\frac{1}{2}Q''_\infty(x) + \lambda x^2 - \alpha Q_\infty(x) > 0 \quad ; \qquad x \in [0, f_\infty) \cup (g_\infty, \infty) \tag{7.1}$$

$$\frac{1}{2}Q''_\infty(x) + \lambda x^2 - \alpha Q_\infty(x) = 0 \quad ; \qquad x \in (f_\infty, g_\infty) \tag{7.2}$$

$$Q_\infty(x) < \delta x^2 \quad ; \qquad x \in (f_\infty, \infty) \tag{7.3}$$

$$Q_\infty(x) = \delta x^2 \quad ; \qquad x \in [0, f_\infty] \tag{7.4}$$

$$Q'_\infty(x) < 1 \quad ; \qquad x \in [0, g_\infty) \tag{7.5}$$

$$Q'_\infty(x) = 1 \quad ; \qquad x \in [g_\infty, \infty) \tag{7.6}$$

$$Q_\infty(-x) = Q_\infty(x) \quad ; \qquad x < 0 \tag{7.7}$$

in the space of functions $\mathcal{C}^1(\mathbb{R}) \cap \mathcal{C}^2(\mathbb{R} \setminus \{\pm f_\infty\})$, for suitable constants $0 < f_\infty < g_\infty < \infty$ that have to be determined along with the function $Q_\infty(\cdot)$. As will be explained in more detail in (7.17)–(7.19) below, the constant f_∞ (respectively, g_∞) plays the rôle of an absorbing (respectively, reflecting) barrier. Together, the two constants f_∞, g_∞ determine the nature of the optimal policy for the problem (3.6), in the form of an optimal stopping region $\mathbf{\Sigma} = [-f_\infty, f_\infty]$, an optimal continuation region $\mathbf{C} = (-g_\infty, -f_\infty) \bigcup (f_\infty, g_\infty)$, and a region $\mathbf{A} = (-\infty, -g_\infty] \bigcup [g_\infty, \infty)$ where immediate action is optimal. The postulated continuity of $Q_\infty(\cdot)$, $Q'_\infty(\cdot)$ across the points f_∞ and g_∞, and of $Q''_\infty(\cdot)$ across the point g_∞, leads to the *smooth-fit conditions*

$$Q_\infty(f_\infty+) = \delta f_\infty^2, \quad Q'_\infty(f_\infty+) = 2\delta f_\infty \tag{7.8}$$

$$Q'_\infty(g_\infty-) = 1, \quad Q''_\infty(g_\infty-) = 0. \tag{7.9}$$

As Davis & Zervos (1994) demonstrate, the system (7.1)–(7.9) can be solved for two constants f_∞, g_∞ and for a function $Q_\infty(\cdot)$ in $\mathcal{C}^1(\mathbb{R}) \cap \mathcal{C}^2(\mathbb{R} \setminus \{\pm f_\infty\})$, in the form

$$Q_\infty(x) = \left\{ \begin{array}{ccl} \frac{\lambda}{\alpha}x^2 + \frac{\lambda}{\alpha^2} + A_\infty e^{x\sqrt{2\alpha}} + B_\infty e^{-x\sqrt{2\alpha}} & ; & f_\infty \le x \le g_\infty \\ \delta x^2 & ; & 0 \le x \le f_\infty \\ Q_\infty(g_\infty) + x - g_\infty & ; & g_\infty \le x < \infty \\ Q_\infty(-x) & ; & x < 0 \end{array} \right\}. \tag{7.10}$$

The four free constants $A_\infty, B_\infty, f_\infty, g_\infty$ of this expression, can be determined by the four conditions of (7.8), (7.9); in particular,

$$A_\infty = h_1(f_\infty), \qquad B_\infty = h_2(f_\infty), \tag{7.11}$$

and the constants $f_\infty,\ g_\infty$ are uniquely characterized by the equations

$$h_3(g_\infty) = h_1(f_\infty)\sqrt{2\alpha}, \qquad h_4(g_\infty) = h_2(f_\infty)\sqrt{2\alpha}. \tag{7.12}$$

We have set

$$h_1(x) \triangleq \frac{\alpha\delta - \lambda}{2\alpha}\, e^{-x\sqrt{2\alpha}}\, \rho(x) \tag{7.13}$$

$$h_2(x) \triangleq \frac{\alpha\delta - \lambda}{2\alpha}\, e^{x\sqrt{2\alpha}} \left(\rho(x) - \frac{4x}{\sqrt{2\alpha}}\right) \tag{7.14}$$

in the notation of (6.10), and

$$h_3(x) \triangleq \frac{\lambda}{\alpha}\left[\frac{\alpha}{2\lambda} - \left(x + \frac{1}{\sqrt{2\alpha}}\right)\right] e^{-x\sqrt{2\alpha}} \tag{7.15}$$

$$h_4(x) \triangleq \frac{\lambda}{\alpha}\left[x - \left(\frac{\alpha}{2\lambda} + \frac{1}{\sqrt{2\alpha}}\right)\right] e^{x\sqrt{2\alpha}}. \tag{7.16}$$

With the solution of the system (7.1)–(7.9) thus determined, the optimal policy takes the form of a combination of *"act, continue, or stop"* moves, whereby one:

(7.17)

- *(i) Stops immediately, if* $|x| \le f_\infty$: $\quad \tau^* = 0, \quad \xi^* \equiv 0.$
- *(ii) Erects "reflecting barriers" at the endpoints of the interval* $(-g_\infty, g_\infty)$*, spends no fuel as long as the state process* X *is in* $(-g_\infty, -f_\infty) \cup (f_\infty, g_\infty)$*, and stops the first time* $|X|$ *hits* f_∞*, if* $f_\infty < |x| < g_\infty$.
- *(iii) Spends the amount* $|x| - g_\infty$ *of fuel at* $t = 0$*, in order to move to the position* $g_\infty \cdot \mathrm{sgn}(x)$ *at once, and then continues as in (ii) above, if* $|x| \ge g_\infty$.

In other words, the optimal policy is of the form $\xi^* = (\xi^*)^+ - (\xi^*)^-$, with

$$\begin{aligned} &(\xi^*)^+_t \triangleq \max\left[0, \max\nolimits_{0\le s\le t}(-x - W_s - g_\infty)\right], \\ &(\xi^*)^-_t \triangleq \max\left[0, \max\nolimits_{0\le s\le t}(x + W_s - g_\infty)\right], \qquad X^*_t = x + W_t - \xi^*_t \end{aligned} \tag{7.18}$$

and

$$\tau^* \triangleq \inf\left\{t \ge 0 \ / \ |X^*_t| \le f_\infty\right\} < \infty, \qquad \text{a.s.} \tag{7.19}$$

The processes $(\xi^*)^\pm$ of (7.18) act only as much as is necessary to bring (at time $t = 0$) or keep (at all times $t > 0$) the state-process X^* inside the interval $[-g_\infty, g_\infty]$, up until X^* enters $[-f_\infty, f_\infty]$, at which time it is optimal to stop. Davis & Zervos (1994) also show that $f_\infty < f_0$, where f_0 is the optimal stopping boundary of (6.9) in the case $y = 0$.

8 THE CASE $\lambda < \alpha\delta,\ 0 < y < \infty$: ANALYSIS

Motivated by the solutions to the problems with no fuel at all (Section 6) and with infinite fuel (Section 7), let us *postulate* now a solution of the *finite-fuel problem (2.6)*, in the form suggested by Figure 2.

In particular, we guess that for every finite fuel-level $y > 0$, there exist an *absorbing barrier* $f(y)$ and a *reflecting barrier* $g(y)$ with $0 < f(y) < g(y) < \infty$; these function much like f_∞ and g_∞ of Section 7. We also guess that the functions $y \mapsto f(y)$ and $y \mapsto g(y)$ are continuous and monotone (decreasing and increasing, respectively), with $f(\infty) \equiv f_\infty$, $g(\infty) \equiv g_\infty$ as in Section 7 and $f(0+) = f_0$ as in Section 6. (These guesses will be vindicated in Section 9, but only for values of the parameter λ in the smaller range $(0, \lambda_*]$ as in (2.7); a more elaborate guess is made, and then justified, in Section 10 for the case $\lambda \in (\lambda_*, \lambda^*]$, whereas the case $\lambda \in (\lambda^*, \alpha\delta)$ is still open.)

The graphs of the functions $f(\cdot)$, $g(\cdot)$ should then describe the optimal policy for the problem of (2.6), by determining an optimal stopping-region $\mathbf{\Sigma} = \{(x,y)\,/\ 0 \le y < \infty,\ \ |x| \le f(y)\}$, an optimal continuation-region $\mathbf{C} = \{(x,y)\,/\,0 < y < \infty,\ \ f(y) < |x| < g(y)\} \cup \{(x,0)\,/\,|x| > f(0)\}$, and an optimal action-region $\mathbf{A} = \{(x,y)\,/\,0 < y < \infty,\ \ |x| \ge g(y)\}$; see Figure 2, as well as (9.6)–(9.13) for a detailed description of the optimal policy (ξ^*, τ^*).

We shall assume that the value $V(x,y)$ of (2.6) can be characterized in terms of the following *moving–boundary problem*

$$\frac{1}{2}Q_{xx}(x,y) + \lambda x^2 - \alpha Q(x,y) > 0 \quad ; \qquad 0 \le x < f(y) \ \text{ or } \ x > g(y),\ y \ge 0 \tag{8.1}$$

$$\frac{1}{2}Q_{xx}(x,y) + \lambda x^2 - \alpha Q(x,y) = 0 \quad ; \qquad f(y) < x < g(y),\ y \ge 0 \tag{8.2}$$

$$Q(x,y) < \delta x^2 \quad ; \qquad x > f(y),\ y \ge 0 \tag{8.3}$$

$$Q(x,y) = \delta x^2 \quad ; \qquad 0 \le x \le f(y),\ y \ge 0 \tag{8.4}$$

$$Q_x(x,y) + Q_y(x,y) < 1 \quad ; \qquad 0 \le x < g(y),\ y \ge 0 \tag{8.5}$$

$$Q_x(x,y) + Q_y(x,y) = 1 \quad ; \qquad x \ge g(y),\ y \ge 0 \tag{8.6}$$

$$Q(-x,y) = Q(x,y) \quad ; \qquad x < 0,\ y \ge 0 \tag{8.7}$$

$$Q(x,0) = Q_0(x) \quad ; \qquad x \in \mathbb{R},\ y = 0 \tag{8.8}$$

$$|Q_x(x,y)| \le K(y)(1+|x|) \quad ; \qquad x \in \mathbb{R},\ y \ge 0 \tag{8.9}$$

for two suitable "moving-boundary" functions $f:\ [0,\infty) \to (0,\infty)$, $\ g:\ [0,\infty) \to (0,\infty)$ of class $\mathcal{C}^1$ with $f(\cdot) < g(\cdot)$, and for some function $Q:\ \mathbb{R} \times [0,\infty) \to [0,\infty)$ of class $\mathcal{C}^1\big(\mathbb{R} \times [0,\infty)\big) \cap \mathcal{C}^2\left((\mathbb{R} \times [0,\infty)) \setminus \{(x,y)/x = \pm f(y)\}\right)$. In the last two equations, we have used the notation of (4.2) and (6.7). Notice that if the triple (f, g, Q) solves the moving-boundary problem of (8.1)–(8.9), then Q solves the variational inequality (4.1)–(4.6).

To make headway with solving the moving-boundary problem of (8.1)–(8.9), let us fix a number $y \in [0,\infty)$ and write the solution $Q(\cdot, y)$ of the equation $\frac{1}{2}Q_{xx} + \lambda x^2 - \alpha Q = 0$ as

$$Q(x,y) = \frac{\lambda}{\alpha}x^2 + \frac{\lambda}{\alpha^2} + A(y)e^{x\sqrt{2\alpha}} + B(y)e^{-x\sqrt{2\alpha}}, \quad \text{for } \ f(y) < x < g(y),\ \ y \ge 0. \tag{8.10}$$

Here $A:\ [0,\infty) \to \mathbb{R}$, $B:\ [0,\infty) \to \mathbb{R}$ are suitable functions of class $\mathcal{C}^1$, to be determined below. The postulated continuity of $Q(\cdot, y)$, $Q_x(\cdot, y)$ across the absorbing-barrier $x = f(y)$, gives the *smooth-fit conditions*

$$\frac{\lambda}{\alpha}x^2 + \frac{\lambda}{\alpha^2} + A(y)e^{x\sqrt{2\alpha}} + B(y)e^{-x\sqrt{2\alpha}} = \delta x^2 \quad ; \qquad \text{at } x = f(y) \tag{8.11}$$

$$\frac{2\lambda}{\alpha}x + \sqrt{2\alpha}\left[A(y)e^{x\sqrt{2\alpha}} - B(y)e^{-x\sqrt{2\alpha}}\right] = 2\delta x \quad ; \qquad \text{at } x = f(y) \tag{8.12}$$

or equivalently

$$A(y) = h_1\big(f(y)\big), \qquad B(y) = h_2\big(f(y)\big); \qquad y \geq 0 \tag{8.13}$$

in the notation of (7.13), (7.14). From these and (6.7), we expect to have $f(0) = f_0$, as well as $A(0) = 0,\ B(0) = B_0$.

On the other hand, the postulated continuity of the directional derivative $U \triangleq Q_x + Q_y$ and of $U_x = Q_{xx} + Q_{xy}$ across the reflecting-barrier $x = g(y)$, leads to the additional *smooth-fit conditions*

$$\frac{2\lambda}{\alpha}x + \sqrt{2\alpha}\left[A(y)e^{x\sqrt{2\alpha}} - B(y)e^{-x\sqrt{2\alpha}}\right] + A'(y)e^{x\sqrt{2\alpha}} + B'(y)e^{-x\sqrt{2\alpha}} \ = \ 1 \tag{8.14}$$

$$\frac{2\lambda}{\alpha} + 2\alpha\left[A(y)e^{x\sqrt{2\alpha}} + B(y)e^{-x\sqrt{2\alpha}}\right] + \sqrt{2\alpha}\left[A'(y)e^{x\sqrt{2\alpha}} - B'(y)e^{-x\sqrt{2\alpha}}\right] \ = \ 0 \tag{8.15}$$

at $x = g(y)$, or equivalently

$$A'(y) + \sqrt{2\alpha}A(y) = h_3\big(g(y)\big) \quad ; \qquad y \geq 0 \tag{8.16}$$

$$\sqrt{2\alpha}B(y) - B'(y) = h_4\big(g(y)\big) \quad ; \qquad y \geq 0 \tag{8.17}$$

in the notation of (7.15), (7.16). (The continuity of the directional derivative DQ, and the continuity of its first spatial derivative U_x, were used as smooth-fit conditions to similar effect in the paper of Beneš, Shepp & Witsenhausen (1980).)

8.1 Lemma: *For every $\lambda \in (0, \alpha\delta)$, the function $h_1(\cdot)$ of (7.13) (respectively, $h_2(\cdot)$ of (7.14)) is strictly increasing (respectively, decreasing) on $\left[0, \sqrt{\delta/(\alpha\delta - \lambda)}\right]$, with $0 < f_0 < \sqrt{\delta/(\alpha\delta - \lambda)}$ and*

$$h_1(0) = h_2(0) = \frac{-\lambda}{2\alpha^2}, \qquad h_1(f_0) = 0, \qquad h_2(f_0) = -\frac{2(\alpha\delta - \lambda)f_0}{\alpha\sqrt{2\alpha}} \cdot e^{f_0\sqrt{2\alpha}} \equiv B_0, \tag{8.18}$$

$$h_2'(x) = -h_1'(x) \cdot e^{2x\sqrt{2\alpha}}, \qquad x \in \mathbb{R}. \tag{8.19}$$

We collect in Appendix A the proofs of those results in Sections 8,9 that are not developed fully in the text.

Let us assume, for a moment, that the absorbing moving-boundary $f(\cdot)$ *is strictly decreasing, with $f(0) = f_0$ as in (6.9) and* $\lim_{y\to\infty} f(y) = f_\infty$ *as in (7.12)*; see Proposition 8.5 below for justification. Under this assumption, and in conjunction with the properties of the function $h_1(\cdot)$ from Lemma 8.1, we can re-write the equations (8.13) in the form

$$f(y) = h_1^{-1}\big(A(y)\big), \qquad B(y) = H\big(A(y)\big) \tag{8.20}$$

with $H \triangleq h_2 \circ h_1^{-1}$, so that (8.19) implies $H'\big(h_1(z)\big) = h_2'(z)/h_1'(z) = -e^{2z\sqrt{2\alpha}}$. Substituting the expression for $B(\cdot)$ from (8.20) into (8.17), and using (8.16), (8.13), we obtain

$$\begin{aligned} h_4\big(g(y)\big) &= \sqrt{2\alpha}\cdot H\big(A(y)\big) - H'\big(A(y)\big)\cdot A'(y) \\ &= \sqrt{2\alpha}\cdot h_2\big(f(y)\big) - H'\big(h_1\left(f(y)\right)\big)\cdot \Big[h_3\big(g(y)\big) - \sqrt{2\alpha}\cdot h_1\big(f(y)\big)\Big] \end{aligned}$$

or equivalently

$$q(g(y); f(y)) = 0, \tag{8.21}$$

where we have set

$$\begin{aligned} q(x;z) &\triangleq \sqrt{2\alpha}\left[h_2(z) - h_1(z)\cdot e^{2z\sqrt{2\alpha}}\right] + h_3(x)e^{2z\sqrt{2\alpha}} - h_4(x) \\ &= -2z\frac{\alpha\delta - \lambda}{\alpha}e^{z\sqrt{2\alpha}} + \frac{\lambda}{\alpha}e^{x\sqrt{2\alpha}}\left[\left(\frac{\alpha}{2\lambda} - x - \frac{1}{\sqrt{2\alpha}}\right)e^{2(z-x)\sqrt{2\alpha}} + \left(\frac{\alpha}{2\lambda} - x + \frac{1}{\sqrt{2\alpha}}\right)\right]. \end{aligned} \tag{8.22}$$

8.2 Lemma: *Assume that $0 < \lambda \le \lambda^*$ in the notation of (2.7). Then for every $z \in [0, f_0]$ there exists a unique number $x = \mathcal{X}(z) \in \big(\frac{\alpha}{2\lambda}, \infty\big)$ such that $q(\mathcal{X}(z); z) = 0$ holds, and we have*

$$z \le f_0 \le \frac{1}{2\delta} < \frac{\alpha}{2\lambda} < \mathcal{X}(z) < \frac{\alpha}{2\lambda} + \frac{1}{\sqrt{2\alpha}} \;. \tag{8.23}$$

Proof: For fixed $z \in [0, f_0]$, consider the function $q(\cdot\,; z)$ of (8.22) on $[z, \infty)$. We have $q(z; z) = e^{z\sqrt{2\alpha}}(1 - 2\delta z) \ge 0$, since $z \le f_0 \le \frac{1}{2\delta}$; this is because the function $\lambda \mapsto f_0(\lambda)$ of (6.9) is strictly increasing, and $f_0 \le \frac{1}{2\delta}$ amounts to the condition $\rho\big(\frac{1}{2\delta}\big) \ge 0$ in the notation of (6.10). This, in turn is equivalent to the condition

$$0 < \lambda \le \lambda^* \triangleq \frac{\alpha\delta}{1 + \frac{\delta/\alpha}{\frac{1}{4\delta} + \frac{1}{\sqrt{2\alpha}}}} \;, \tag{8.24}$$

in the notation of (2.7). It is easy to verify that $q(\infty; z) = -\infty$; furthermore, the quantity

$$\frac{\partial}{\partial x}q(x;z) = \lambda\sqrt{\frac{2}{\alpha}}\left(\frac{\alpha}{2\lambda} - x\right)e^{x\sqrt{2\alpha}}\left(1 - e^{2(z-x)\sqrt{2\alpha}}\right), \quad z \le x < \infty \tag{8.25}$$

has the sign of $\frac{\alpha}{2\lambda} - x$. Now $f_0 < f_0(\lambda^*) = \frac{1}{2\delta} < \frac{\alpha}{2\lambda}$, so the function $q(\cdot\,; z)$ is strictly increasing on $\big(z, \frac{\alpha}{2\lambda}\big)$ and strictly decreasing on $\big(\frac{\alpha}{2\lambda}, \infty\big)$. Thus, there exists a *unique* $\mathcal{X}(z)$, as in (8.23), that satisfies $q(\mathcal{X}(z); z) = 0$. The last inequality of (8.23) follows from the simple observation $q(\frac{\alpha}{2\lambda} + \frac{1}{\sqrt{2\alpha}}; z) < 0$. $\square$

8.3 Lemma: *Under the assumption $0 < \lambda \le \lambda^*$, the function $z \mapsto \mathcal{X}(z)$ of Lemma 8.2 is of class $\mathcal{C}^2$, strictly decreasing and strictly concave.*

We shall impose for the remainder of this Section the assumption (8.24), as it seems to be critical for the validity of Lemmata 8.2 and 8.3. Thanks to these two results, we deduce that the two moving-boundary functions $f(\cdot)$ and $g(\cdot)$ are related by

$$g(y) = \mathcal{X}\big(f(y)\big); \qquad 0 \le y < \infty. \tag{8.26}$$

On the other hand, substituting $A(y) = h_1\big(f(y)\big)$ and its consequence $A'(y) = f'(y) \cdot h_1'\big(f(y)\big)$ from (8.13) into (8.16), we obtain for the moving-boundary function $f(\cdot)$ the first-order *differential equation*

$$f'(y) = \frac{m(f(y))}{h_1'(f(y))}, \quad 0 \le y < \infty \qquad \text{where} \quad m(z) \stackrel{\triangle}{=} h_3(\mathcal{X}(z)) - \sqrt{2\alpha}\, h_1(z). \tag{8.27}$$

8.4 Remark: The relation (8.26) is valid, at least formally, also for $y = \infty$. Indeed, $g(\infty) \equiv g_\infty$ and $f(\infty) \equiv f_\infty$ satisfy the equation $g_\infty = \mathcal{X}(f_\infty)$ trivially, thanks to (7.12) and (8.21), (8.22).

8.5 Proposition: *The differential equation of (8.27) has a unique solution* $f : [0, \infty) \to (0, \infty)$ *with* $f(0) = f_0$*, in* $\mathcal{C}^1([0, \infty))$. *This function is strictly decreasing, strictly convex, of class* $\mathcal{C}^2$*, and satisfies* $\lim_{y\to\infty} f(y) = f_\infty$*; whereas the function* $g : [0, \infty) \to (0, \infty)$ *given by (8.26) is of class* $\mathcal{C}^2$*, strictly increasing, strictly concave, and satisfies* $\lim_{y\to\infty} g(y) = g_\infty$.

8.6 Lemma: *The function* Q *of (8.10) satisfies* $Q(x, 0) = Q_0(x)$ *for all* $x \in \mathbb{R}$*, and for any* $y > 0$

$$Q_y(x, y) = f'(y) h_1'\big(f(y)\big) e^{-x\sqrt{2\alpha}} \left(e^{2x\sqrt{2\alpha}} - e^{2f(y)\sqrt{2\alpha}}\right) < 0 \tag{8.28}$$

$$Q_x(x, y) = \frac{2\lambda}{\alpha} x + \sqrt{2\alpha}\left[h_1\big(f(y)\big) e^{x\sqrt{2\alpha}} - h_2\big(f(y)\big) e^{-x\sqrt{2\alpha}}\right] > 0 \tag{8.29}$$

$$\frac{1}{2} Q_{xx}(x, y) + \lambda x^2 = \alpha Q(x, y), \qquad Q(x, y) < \delta x^2 \tag{8.30}$$

$$\frac{1}{2} Q_{xx}(x, y) = \frac{\lambda}{\alpha} + \alpha\left[h_1\big(f(y)\big) e^{x\sqrt{2\alpha}} + h_2\big(f(y)\big) e^{-x\sqrt{2\alpha}}\right] > 0 \tag{8.31}$$

$$Q_{xy}(x, y) = m\big(f(y)\big)\sqrt{2\alpha}\, e^{-x\sqrt{2\alpha}}\left[e^{2x\sqrt{2\alpha}} + e^{2f(y)\sqrt{2\alpha}}\right] < 0 \tag{8.32}$$

$$Q_{yy}(x, y) = f'(y) e^{-x\sqrt{2\alpha}}\left[m'\big(f(y)\big)\left(e^{2x\sqrt{2\alpha}} - e^{2f(y)\sqrt{2\alpha}}\right) - 2\sqrt{2\alpha} \cdot m\big(f(y)\big)\right] \tag{8.33}$$

for $f(y) < x \le g(y)$*, and*

$$U(x, y) \stackrel{\triangle}{=} Q_x(x, y) + Q_y(x, y) < 1, \quad \textit{for} \quad f(y) \le x < g(y). \tag{8.34}$$

In particular, $Q(x, \cdot)$ *is* not *convex, since* $Q_{yy}(x, y) < 0$ *for* $x - f(y) > 0$ *sufficiently small.*

The strict concavity of the function $g(\cdot)$, established in Proposition 8.5, implies

$$g'(0) \le 1 \quad \Longleftrightarrow \quad g'(y) < 1, \quad \forall\, 0 < y < \infty. \tag{8.35}$$

This property will be crucial in the next section, when we construct the optimal policy for the problem of (2.6), so we shall need conditions to ensure it.

8.7 Proposition: *For given* $\alpha > 0$, $\delta > 0$*, there exists a constant* $\lambda_* = \lambda_*(\alpha, \delta)$ *with*

$$0 < \lambda_* \le \lambda^* = \frac{\alpha\delta}{1 + \frac{\delta/\alpha}{\frac{1}{4\delta} + \frac{1}{\sqrt{2\alpha}}}} \tag{8.36}$$

as in (8.24), such that

$$g'(0) \le 1 \quad \Longleftrightarrow \quad 0 < \lambda \le \lambda_*. \tag{8.37}$$

This proposition will be proved in Appendix B. It should be noted here that *the second inequality in (8.36) can be strict*. For instance, with $\alpha = 1/\delta = 2$, we have $\lambda^* = 0.8$ from (8.24), while the methodology of Appendix B computes the constant λ_* of (8.36) as $\lambda_* \cong 0.7885$.

9 THE CASE $0 < \lambda \le \lambda_*,\ 0 < y < \infty$: SYNTHESIS

We are now in a position to reverse the steps of the analysis carried out in Section 8, and to verify the guesses made in the beginning of that Section, at least for $0 < \lambda \le \lambda_*$. Let us fix the parameters $\alpha > 0,\ \delta > 0$ and $\lambda \in (0, \lambda_*]$ as in Proposition 8.7, construct the solution $f(\cdot)$ of the differential equation (8.25) and from it the function $g(\cdot)$ of (8.26), as in Proposition 8.5, and *define* the function

$$
(9.1) \quad Q(x,y) \triangleq \left\{ \begin{array}{rcl} \delta x^2 & ; & 0 \le x \le f(y) \\ \frac{\lambda}{\alpha}x^2 + \frac{\lambda}{\alpha^2} + h_1\big(f(y)\big)e^{x\sqrt{2\alpha}} + h_2\big(f(y)\big)e^{-x\sqrt{2\alpha}} & ; & f(y) < x \le g(y) \\ \zeta + Q(x-\zeta, y-\zeta) & ; & g(y) < x < y + g(0) \\ y + Q_0(x-y) & ; & y + g(0) \le x < \infty \\ Q_0(x) & ; & x \ge 0,\ y = 0 \\ Q(-x,y) & ; & -\infty < x < 0 \end{array} \right\}
$$

for $0 \le y < \infty$. We have used here the notation of (7.13), (7.14), (6.7) and have defined $\zeta \equiv \zeta(x,y) \in (0,y)$ *uniquely*, via

$$
(9.2) \qquad x - \zeta = g(y - \zeta), \qquad \text{for} \quad g(y) < x < y + g(0),
$$

thanks to (8.37) and (8.35). These properties ensure that the 45°-line emanating from the point $(g_0, 0)$ touches the graph of $g(\cdot)$ at that point only; see Figure 2.

9.1 Theorem: *The function* $Q : \mathbb{R} \times [0,\infty) \to [0,\infty)$*, defined in (9.1), is of class* $\mathcal{C}^1(\mathbb{R} \times [0,\infty)) \cap \mathcal{C}^2(\mathbb{R} \times [0,\infty) \setminus \{(x,y)/x = \pm f(y)\})$*, and solves the moving-boundary problem (8.1)–(8.9) as well as the variational inequality (4.3)–(4.6). Furthermore, this function has the properties*

$$
(9.3) \qquad Q_y(x,y) \le 0, \quad \textit{for} \quad 0 < y < \infty; \qquad \lim_{y \to \infty} Q(x,y) = Q_\infty(x)
$$

for any $x \in \mathbb{R}$*, in the notation of (7.10).*

¿From Theorem 4.1, we know then that $Q(x,y)$ is a lower-bound on the attainable expected cost $V(x,y)$ of (2.5); namely, that (4.7) holds. We shall show that, in fact, we have equality in (4.7), namely

$$
(9.4) \quad Q(x,y) = \mathbb{E}\left[\int_0^{\tau_*} e^{-\alpha t} \lambda (X_t^*)^2\, dt + \int_{[0,\tau_*]} e^{-\alpha t} \left(d\xi_*^+(t) + d\xi_-^*(t) \right) + \delta e^{-\alpha \tau_*} \left(X_{\tau_*}^* \right)^2 \right] = V(x,y)
$$

for a policy $\xi_* = \xi_*^+ - \xi_*^- \in \mathcal{A}(y),\ \tau_* \in \mathcal{S}$ with $\tau_* < \infty$, a.s.. Here

$$
(9.5) \qquad X_t^* = x + W_t + \xi_*(t), \quad Y_t^* = y - \left(\xi_*^+(t) + \xi_*^-(t) \right); \quad 0 \le t < \infty
$$

are the optimal "position" and "remaining-fuel" processes. By analogy with (7.17), this policy takes again the form of a combination of *"act, continue, or stop"* moves. In terms of the regions

$$
(9.6) \qquad \mathbf{\Sigma} \triangleq \left\{ (x,y) \ / \ 0 \le y < \infty, \quad |x| \le f(y) \right\}
$$

$$\mathbf{C} \triangleq \{(x,y) \ / \ 0 < y < \infty, \ f(y) < |x| < g(y)\} \cup \{(x,0) \ / \ |x| > f(0)\} \tag{9.7}$$

$$\mathbf{A} \triangleq \mathbf{A}_1 \cup \mathbf{A}_2 \tag{9.8}$$

depicted in Figure 2, where

$$\mathbf{A}_1 \triangleq \{(x,y) \ / \ 0 < y < \infty, \ g(y) < |x| < g(0) + y\}, \quad \mathbf{A}_2 \triangleq \{(x,y) \ / \ 0 < y < \infty, \ |x| \geq g(0) + y\},$$

the moves of the optimal (ξ_*, τ_*) can be described as follows:

(9.9)

(i) Stops immediately; if $(x,y) \in \mathbf{\Sigma}$.

(ii) Erects "reflecting barriers" along the moving boundaries $\pm g(\cdot)$, spends no fuel as long as $(X_t^*, Y_t^*) \in \mathbf{C}$, spends just enough fuel along the "reflecting barriers" to ensure $|X_\cdot^*| \leq g\big(y - \xi_*^+(\cdot) - \xi_*^-(\cdot)\big)$ at all times, and stops the first time $(X_t^*, Y_t^*) \in \mathbf{\Sigma}$; if $(x,y) \in \mathbf{C}$.

(iii) Spends at $t = 0$ as much fuel as necessary to move to the closest "reflecting barrier", and then continues as in (ii) above; if $(x,y) \in \mathbf{A}_1$.

(iv) Spends all available fuel at once, moves to the position $x' = x - y \cdot sgn(x)$, and stops the first time t such that $|x' + W_t| = f_0$; if $(x,y) \in \mathbf{A}_2$.

More precisely, we can construct a pair of non-decreasing processes $\xi_*^{\pm}(\cdot) \in \mathcal{A}$, such that

$$\xi_*^-(t) = \max\left[0, \max_{0 \leq u \leq t} \big(x + W_u - g\big(y - \xi_*^-(u)\big)\big)\right] \wedge y, \quad 0 \leq t \leq \tau_* \tag{9.10}$$

$$\xi_*^+(t) = \max\left[0, \max_{0 \leq u \leq t} \big(-x - W_u - g\big(y - \xi_*^+(u)\big)\big)\right] \wedge y, \quad 0 \leq t \leq \tau_* \tag{9.11}$$

and $\xi_*^{\pm}(t) \triangleq \xi_*^{\pm}(\tau_*)$ for $t \geq \tau_*$, where

$$\tau_* \triangleq \inf\{t \geq 0 \ / \ |X_t^*| \leq f(Y_t^*)\} < \infty, \qquad \text{a.s.} \tag{9.12}$$

$$\xi_* \triangleq \xi_*^+ - \xi_*^- \in \mathcal{A}(y), \quad (X^*, Y^*) \text{ as in (9.5).} \tag{9.13}$$

The reader should consult Beneš, Shepp & Witsenhausen (1980) for the actual construction of the processes $\xi_*^{\pm}(\cdot)$ in (9.10) and (9.11), and should argue the a.s. finiteness of the stopping time τ_*, as indicated in (9.12).

9.2 Theorem: *The policy $(\xi_*, \tau_*) \in (\mathcal{A}(y) \times \mathcal{S})$ of (9.10)-(9.13) is optimal for the problem of (2.6), whose value is then given by the expression of (9.1); in other words, (9.4) holds.*

Proof: Let us consider the policy (ξ_*, τ_*) of (9.10)-(9.13), recall (9.5), set $\eta \equiv \xi_*^c$, and observe that

$$(X_{\tau_*}^*, Y_{\tau_*}^*) \in \mathbf{\Sigma} \quad \Leftrightarrow \quad Q(X_{\tau_*}^*, Y_{\tau_*}^*) = \delta(X_{\tau_*}^*)^2, \text{ a.s.} \tag{9.14}$$

$$(X_t^*, Y_t^*) \in \mathbf{C} \quad \Leftrightarrow \quad \frac{1}{2}Q_{xx}(x,y) + \lambda x^2 - \alpha Q(x,y)\Big|_{(x,y)=(X_t^*,Y_t^*)} = 0 \tag{9.15}$$

for every $0 < t < \tau_*$, a.s. on $\{\tau_* > 0\}$,

$$
\text{(9.16)} \qquad \left\{ \begin{array}{l} \eta^{+} \text{ is flat away from } \{t \geq 0 \ / \ Y^*_{t-} > 0, \ X^*_{t-} \leq -g(Y^*_{t-})\} = \\ \qquad\qquad\qquad \{t \geq 0 \ / \ Y^*_{t-} > 0, \ (-Q_x + Q_y)(X^*_{t-}, Y^*_{t-}) = 1\} \end{array} \right\},
$$

$$
\text{(9.17)} \qquad \left\{ \begin{array}{l} \eta^{-} \text{ is flat away from the set } \{t \geq 0 \ / \ Y^*_{t-} > 0, \ X^*_{t-} \geq g(Y^*_{t-})\} = \\ \qquad\qquad\qquad \{t \geq 0 \ / \ Y^*_{t-} > 0, \ (Q_x + Q_y)(X^*_{t-}, Y^*_{t-}) = 1\} \end{array} \right\},
$$

$$
\text{(9.18)} \qquad \left\{ \begin{array}{l} \text{at any time } t \in [0, \tau_*] \text{ with } \triangle \eta^{\pm}_t > 0, \text{ we have} \\ \\ \qquad Q(x \pm \vartheta, y - \vartheta) - Q(x, y) + \vartheta \Big|_{(x,y) = (X^*_{t-}, Y^*_{t-}), \ \vartheta = \triangle \eta^{\pm}_t} = 0, \quad \text{a.s.} \end{array} \right\}.
$$

It follows from (9.14)–(9.18) that we have $I_j(\tau_*) = 0$ a.s., for $j = 1, \cdots, 5$ in the proof of Theorem 4.1, and (9.4) is then a consequence of Corollary 4.2. $\square$

We discuss the situation $\lambda_* < \lambda \leq \lambda^*$ in the next Section 10. By contrast, the case $\lambda^* < \lambda < \alpha\delta$ has so far eluded our efforts. It may require the development of a new methodology; we suggest its analysis as an interesting open problem.

10 THE CASE $\lambda_* < \lambda \leq \lambda_*, \ 0 < y < \infty$.

When $\lambda_* < \lambda \leq \lambda^*$, one expects that the optimal policy will be characterized by two curves $\{(x, y) \ / \ x = F(y), \ \ 0 \leq y < \infty\}$ and $\{(x, y) \ / \ x = G(y), \ \ 0 \leq y < \infty\}$ with $F(\cdot) < G(\cdot)$, and will obey the rules:

(10.1)

- (a) In region $\mathrm{I} \triangleq \{(x, y) \ / \ 0 \leq x \leq F(y), \ \ 0 \leq y < \infty\}$, *stop* immediately.
- (b) In region $\mathrm{II} \triangleq \{(x, y) \ / \ F(y) < x < G(y), \ \ 0 \leq y < \infty\}$, *continue* without exercise of control (expenditure of fuel).
- (c) If the starting point (x_0, y_0) is in region $\mathrm{III} \triangleq \{(x, y) \ / \ x > G(y), \ \ 0 \leq y < \infty\}$, expend fuel immediately and bring the state to the boundary $\{(x, y) \ / \ x = G(y), \ \ y > 0\} \bigcup \{(x, 0) \ / \ x \geq G(0)\}$ of the region.
- (d) Use the optimal stopping rule of Section 6, if $y = 0$.

Because of the definition of λ_* we guess that, if the optimal strategy has the suggested form, then there should exist a *critical level of fuel* $\bar{y} \in (0, \infty)$, such that

$$
\text{(10.2)} \qquad G'(y) > 1 \ \text{ for } \ 0 \leq y < \bar{y}, \qquad \text{and} \qquad 0 < G'(y) \leq 1 \ \text{ for } \ y \geq \bar{y}.
$$

Along the part of the curve $\{(x, y) \ / \ x = G(y), \ \ y > \bar{y}\}$ where $G'(y) \leq 1$, the strategy is to use fuel to reflect the vector-process (X_t, Y_t) so as to keep it in the region $\{(x, y) \ / \ F(y) \leq x \leq G(y), \ \ 0 \leq y < \infty\}$, just as was done in the previous case of $0 < \lambda \leq \lambda_*$ (Section 9). However, along the part of the curve $\{(x, y) \ / \ x = G(y), \ \ 0 < y \leq \bar{y}\}$ where $G'(y) > 1$, this is no longer possible. *We propose that the policy of (10.1)(c) extend to this portion of the boundary:*

$$
\text{(10.1)(e)} \qquad \left\{ \begin{array}{l} \text{From any initial condition } (x_0, y_0) \in \big\{(x,y) \ / \ x = G(y), \ \ 0 < y \le \bar{y}\big\}, \\ \text{one expends all available fuel to bring the state to the } x\text{-axis.} \end{array} \right\}
$$

This guess is natural, because the 45°-line segment connecting any such (x_0, y_0) to the axis, remains in region $\mathbb{III}$; cf. Figure 3.

Let $\pi(F;G)$ denote the control/stopping policy outlined in the above discussion of (10.1)(a-e). This strategy treats always $F(\cdot)$ as an *absorbing boundary* (stops, as soon as $|X_t| \le F(Y_t)$). As long as $Y_t > \bar{y}$, this policy treats $G(\cdot)$ as a *reflecting boundary*, in the manner of (9.9)(iii); but as soon as $Y_t \in (0, \bar{y}]$, it treats $G(\cdot)$ as an *exit* ("repelling") *boundary*, i.e. spends all available fuel at once if $|X_t| \ge G(Y_t)$. In particular, with initial fuel $Y_0 = y_0$, the state-process $\{(X_t, Y_t); \ 0 \le t < \infty\}$ never visits the region $\big\{(x,y) \ / \ |x| \le G(y), \ \ 0 < y < \bar{y} \wedge y_0\big\}$ if the policy $\pi(F;G)$ is used.

We shall show that, for $\lambda_* < \lambda \le \lambda^*$, there are curves $F(\cdot)$ and $G(\cdot)$ as in Figure 3 such that $\pi(F;G)$ is optimal, and $F(\cdot)$, $G(\cdot)$ are uniquely determined by the *smooth-fit conditions* of (10.4) below. Let $Q(x,y)$ denote the expected cost associated with $\pi(F;G)$ and initial condition $(X_0, Y_0) = (x,y)$, and assume as an Ansatz:

$$
\text{(10.3)} \qquad G(y) > y + \frac{\alpha}{2\lambda}, \qquad \text{for } \ 0 \le y \le \bar{y} \ ;
$$

$$
\text{(10.4)} \qquad \left\{ \begin{array}{l} \text{(a) } \ Q(\cdot, y),\, Q_x(\cdot, y) \text{ are continuous across } x = F(y), \ \ \forall \ y \in (0,\infty); \\[6pt] \text{(b) } \ Q(\cdot, y),\, (Q_x + Q_y)(\cdot, y) \text{ and } (Q_{xx} + Q_{xy})(\cdot, y) \text{ are continuous across } x = G(y), \\ \qquad \forall \ y \in (\bar{y}, \infty); \\[6pt] \text{(c) } \ Q(\cdot, y) \text{ and } Q_x(\cdot, y) \text{ are continuous across } x = G(y), \ \ \forall \ y \in (0, \bar{y}); \\[6pt] \text{(d) } \ Q \in \mathcal{C}\,(\mathbb{R} \times [0, \infty)). \end{array} \right\}
$$

10.1 Remark: The requirement (10.4)(b) is the same smooth-fit condition used in the case $0 < \lambda \le \lambda_*$ of Sections 8 and 9 (for which, from the perspective of our more general framework, $\bar{y} = 0$). Condition (10.4)(c) is very natural, since the policy $\pi(F;G)$ determines the value of $Q(G(y), y)$ for $0 \le y \le \bar{y}$; thus, only one more smooth-fit condition is needed to determine $F(\cdot)$ and $G(\cdot)$, namely, the continuity of $Q_x(\cdot, y)$ across $x = G(y)$.

The proof will be developed in two steps. First, we shall identify the critical fuel-level $\bar{y}$ and the portions $\big\{(F(y), G(y)) \ / \ 0 \le y \le \bar{y}\big\}$ of the moving-boundaries below it, by imposing (10.4)(c) under the Ansatz (10.3). We shall construct $Q(\cdot,\cdot)$ on $\mathbb{R} \times [0, \bar{y}]$ as the expected-cost-function under this policy, and show by the Verification Theorem 4.1 that it consitutes the value function for our problem (2.6).

Next, we shall extend $F(\cdot)$, $G(\cdot)$ to the region $y > \bar{y}$ above the critical level, using (10.4)(b). This can be done quite easily, using the results of Sections 8 and 9, once certain facts about $F(\cdot)$ and $G(\cdot)$ at $y = \bar{y}$ have been established.

Before presenting the results, we state formulae for the expected-cost-function $\{Q(x,y) \ / \ x \in \mathbb{R}, \ y \in [0, \bar{y}]\}$ associated with $\pi(F;G)$ and for the smooth-fit conditions (10.4)(a,c). With the notation of (6.7)–(6.9) for $Q_0(\cdot) \equiv Q(\cdot, 0)$ and $f_0 \equiv f_0(\lambda)$, with given $F(\cdot)$ and $G(\cdot)$ as above, and with suitable function $\mathcal{A}(\cdot)$, $\mathcal{B}(\cdot)$ (to be determined), we have

$$Q(x,y) = \begin{cases} \delta x^2 & ; \quad 0 \le x \le F(y) \\ \frac{\lambda}{\alpha}x^2 + \frac{\lambda}{\alpha^2} + \mathcal{A}(y)e^{x\sqrt{2\alpha}} + \mathcal{B}(y)e^{-x\sqrt{2\alpha}} & ; \quad F(y) < x < G(y) \\ Q(x-y, 0) + y & ; \quad x \ge G(y) \\ Q(-x, y) & ; \quad x < 0 \end{cases} \tag{10.5}$$

for $0 \le y \le \bar{y}$. Because we are assuming (10.3), we have $G(y) > y + \frac{\alpha}{2\lambda} > y + f_0$, and thus

$$Q(x,y) = y + \frac{\lambda}{\alpha}(x-y)^2 + \frac{\lambda}{\alpha^2} + h_2(f_0)e^{-(x-y)\sqrt{2\alpha}}; \qquad x \ge G(y),\ 0 \le y \le \bar{y}. \tag{10.6}$$

As shown in (8.13), the smooth-fit condition (10.4)(a) implies

$$\mathcal{A}(y) = h_1\big(F(y)\big), \quad \mathcal{B}(y) = h_2\big(G(y)\big); \qquad 0 \le y \le \bar{y}. \tag{10.7}$$

On the other hand, the smooth-fit condition (10.4)(c) requires

$$\begin{aligned} \frac{\lambda}{\alpha}x^2 + \frac{\lambda}{\alpha^2} + h_1\big(F(y)\big)e^{x\sqrt{2\alpha}} + h_2\big(F(y)\big)e^{-x\sqrt{2\alpha}} &= Q(x-y,0) + y \\ \frac{2\lambda}{\alpha}x + \sqrt{2\alpha}\left[h_1\big(F(y)\big)e^{x\sqrt{2\alpha}} - h_2\big(F(y)\big)e^{-x\sqrt{2\alpha}}\right] &= Q_x(x-y,0) \end{aligned}$$

at $x = G(y)$. Algebraic manipulation reduces these equations to

$$h_1\big(F(y)\big) = \mathcal{H}_3\big(G(y), y\big), \qquad h_2\big(F(y)\big) = \mathcal{H}_4\big(G(y), y\big) \tag{10.8}$$

where

$$\begin{aligned} \mathcal{H}_3(x,y) &\triangleq yh_3(x) + \frac{\lambda}{2\alpha}y^2 e^{-x\sqrt{2\alpha}} \\ \mathcal{H}_4(x,y) &\triangleq -yh_4(x) + \frac{\lambda}{2\alpha}y^2 e^{x\sqrt{2\alpha}} + h_2(f_0)e^{y\sqrt{2\alpha}}. \end{aligned} \tag{10.9}$$

Let us denote again by $f(\cdot)$, $g(\cdot)$ the solutions provided by Proposition 8.5 to the system of (8.26), (8.27) with $f(0) = f_0$, $g(0) = g_0$. Both these functions depend on the value of the parameter λ (though we shall not indicate this dependence explicitly), and satisfy the relation $q\big(g(y); f(y)\big) = 0$ of (8.21). Here $q(\cdot;\cdot)$ is the function of (8.23), and will play a central role in the analysis that follows.

10.2 Proposition: *For $\lambda_* < \lambda \le \lambda^*$, there exists a real number $Y > 0$ and a unique solution $\big\{\big(F_\lambda(y), G_\lambda(y)\big);\ \ 0 \le y \le Y\big\}$ to the equations of (10.8), for which $G_\lambda(\cdot)$ sastifies (10.3) and*

$$F_\lambda(\cdot),\ \ G_\lambda(\cdot) \quad \textit{are in} \quad \mathcal{C}^1\left([0,Y]\right)\ ; \tag{10.10}$$

$$F_\lambda(0) = f_0,\ \ F'_\lambda(0) = f'(0) \ \ \textit{and}\ \ F_\lambda(\cdot)\ \ \textit{is decreasing}\,; \tag{10.11}$$

$$G_\lambda(0) = g_0 \ \ \textit{and}\ \ G'_\lambda(0) = \tfrac{1}{2}\left[1 + g'(0)\right] \in \big(1, g'(0)\big)\ ; \tag{10.12}$$

$$G'_\lambda(y) = 1 + \frac{q\big(G_\lambda(y); F_\lambda(y)\big)}{\frac{2\lambda y}{\sqrt{2\alpha}}\left[G_\lambda(y) - \left(\frac{y}{2} + \frac{\alpha}{2\lambda}\right)\right]e^{G_\lambda(y)\sqrt{2\alpha}}\left(1 - e^{2\left(F_\lambda(y) - G_\lambda(y)\right)\sqrt{2\alpha}}\right)}, \quad 0 < y \le Y. \tag{10.13}$$

The set $\{y \in [0,Y] \;/\; G'_\lambda(y) \le 1\}$ *is not empty; and with* $\bar{y} \triangleq \sup\{y > 0 \;/\; G'_\lambda(y) > 1\}$, *we have*

$$0 < \bar{y} \le Y,\; G'_\lambda(\bar{y}) = 1 < G'_\lambda(y) \text{ for } 0 < y < \bar{y},\; \bar{y} = \sup\{y > 0 \;/\; q\big(G_\lambda(y); F_\lambda(y)\big) > 0\}, \tag{10.14}$$

$$q\big(G_\lambda(\bar{y}); F_\lambda(\bar{y})\big) = 0, \quad \text{or equivalently} \quad G_\lambda(\bar{y}) = \mathcal{X}\big(F_\lambda(\bar{y})\big). \tag{10.15}$$

10.3 Theorem: *Let* $F_\lambda(\cdot)$, $G_\lambda(\cdot)$, $\bar{y}$ *be given as in Proposition 10.2, and* Q *be defined on* $\mathbb{R}\times[0,\bar{y}]$ *as in (10.5), (10.7). Then* Q *is of class* $\mathcal{C}\big(\mathbb{R}\times[0,\bar{y}]\big)\cap\mathcal{C}^1\big(\mathbb{R}\times(0,\bar{y})\big)$, *is twice continuously differentiable with locally bounded second derivatives away from* $\{(x,y) \in \mathbb{R}\times(0,\bar{y}) \;/\; x = \pm F_\lambda(y) \text{ or } x = \pm G_\lambda(y)\}$, *and satisfies (4.2) and the Variational Inequality (4.3)-(4.6) on* $\mathbb{R}\times(0,\bar{y})$.

10.4 Corollary: *For* $\lambda_* < \lambda \le \lambda^*$, *the function* Q *of Theorem 10.2 coincides with the value function* V *on* $\mathbb{R}\times[0,\bar{y}]$, *and the strategy* $\pi(F_\lambda; G_\lambda)$ *(of (10.1) and subsequent discussion) is optimal for the problem (2.6) for* $x \in \mathbb{R}$, $y \in [0,\bar{y}]$.

Proof: By Theorems 10.2 and 4.1, we have $Q \le V$ on $\mathbb{R}\times[0,\bar{y}]$. Since Q is the expected cost associated with the policy $\pi(F_\lambda; G_\lambda)$ we also have $Q \ge V$ on $\mathbb{R}\times[0,\bar{y}]$, and the claims follow. □

To complete the solution, it remains to determine the functions $F(\cdot)$, $G(\cdot)$ on $[\bar{y},\infty)$ and show that the expected cost Q associated with the policy $\pi(F;G)$ satisfies the Variational Inequality of (4.3)–(4.6). In accordance with (10.2) we want $G'(\cdot) \le 1$ on $[\bar{y},\infty)$. If this is the case, then we should first apply (10.4)(a,b) as a smooth-fit principle to determine $F(\cdot)$ and $G(\cdot)$ on $[\bar{y},\infty)$, and then check the validity of $G'(\cdot) \le 1$ on this interval.

The first of these tasks has already been carried out in Section 8 (Lemmata 8.2, 8.3 and Proposition 8.5). Let $\tilde{f}(\cdot)$ be defined on $[\bar{y},\infty)$ as the solution to the equation

$$\tilde{f}'(y) = \left.\frac{h_3\big(\mathcal{X}(z)\big) - \sqrt{2\alpha}\; h_1(z)}{h'_1(z)}\right|_{z=\tilde{f}(y)} \quad \text{for} \;\; y > \bar{y}, \;\; \text{and} \;\; \tilde{f}(\bar{y}) = F_\lambda(\bar{y}). \tag{10.16}$$

To proceed with the second task, it is important to note the following.

10.5 Lemma: *Let* $f_*(\cdot) \equiv f_{\lambda_*}(\cdot)$ *be the function of (8.27) defined on* $[0,\infty)$ *corresponding to the parameter value* λ_* *of (8.36). Then with* $\lambda_* < \lambda \le \lambda^*$ *we have* $F_\lambda(\cdot) \ge f_*(\cdot)$ *on* $[0,\bar{y}]$*; in particular,* $F_\lambda(\bar{y}) > f_\infty$.

Proof: The optimal stopping region for the problem of (2.6) with value $V(\cdot,\bar{y})$ corresponding to λ_*, is given by $\{(x,y) \;/\; |x| \le f_*(y),\;\; 0 \le y \le \bar{y}\}$. On the other hand, the set $\{(x,y) \;/\; |x| \le F_\lambda(y),\;\; 0 \le y \le \bar{y}\}$ is the optimal stopping region for the problem of (2.6) with value function $V(\cdot,\bar{y})$ corresponding to $\lambda \in (\lambda_*, \lambda^*]$. The optimal stopping region increases with λ, and the claims follow. □

¿From Proposition 8.5, the equation (10.16) admits a unique solution; this is a strictly decreasing, strictly convex function, of class $\mathcal{C}^2\big((\bar{y},\infty)\big)$, and satisfies $\lim_{y\to\infty} \tilde{f}(y) = f_\infty$. Now define

$$F(y) \triangleq \left\{ \begin{array}{ccc} F_\lambda(y) & ; & 0 \le y \le \bar{y} \\ \tilde{f}(y) & ; & \bar{y} < y < \infty \end{array} \right\} \tag{10.17}$$

as well as

$$G(y) \triangleq \left\{ \begin{array}{ccc} G_\lambda(y) & ; & 0 \le y \le \bar{y} \\ \mathcal{X}\big(F(y)\big) & ; & \bar{y} < y < \infty \end{array} \right\}. \tag{10.18}$$

In particular,

$$q\big(G(y);\ F(y)\big) = 0, \qquad \bar{y} < y < \infty. \tag{10.19}$$

Since $F(\cdot)$ is continuous at $y = \bar{y}$ by (10.16), it follows from (10.15), (10.19) that $G(\cdot)$ is also continuous at $y = \bar{y}$. Furthermore, we have $\lim_{y\to\infty} G(y) = g_\infty$ from Proposition 8.5.

10.6 Theorem: *The function $F(\cdot)$ of (10.17) is continuously differentiable at $y = \bar{y}$:*

$$F'(y+) \triangleq \lim_{y\downarrow\bar{y}} \frac{F(y) - F(\bar{y})}{y - \bar{y}} = F'(y-) \triangleq \lim_{y\uparrow\bar{y}} \frac{F(y) - F(\bar{y})}{y - \bar{y}}. \tag{10.20}$$

On the other hand, the function $G(\cdot)$ of (10.18) satisfies

$$G'(\bar{y}+) \le 1 \qquad \textit{and} \tag{10.21}$$

$$G'(y) < 1, \qquad \textit{for all} \;\; y > \bar{y}. \tag{10.22}$$

Proof: Let us recall $G'(\bar{y}) = 1$ from (10.14); differentiation in (10.8) with respect to y, and then calculation at $y = \bar{y}$, gives

$$h_1'\big(F(\bar{y})\big) \cdot F'(\bar{y}-) = \left(\frac{\partial \mathcal{H}_3}{\partial x} + \frac{\partial \mathcal{H}_3}{\partial y} \right) \big(G(\bar{y}), \bar{y}\big). \tag{10.23}$$

Now an easy calculation in (10.9) shows

(10.24)

$$\left(\frac{\partial \mathcal{H}_3}{\partial x} + \frac{\partial \mathcal{H}_3}{\partial y} \right)(x, y) = h_3(x) - \sqrt{2\alpha}\mathcal{H}_3(x, y), \qquad \left(\frac{\partial \mathcal{H}_4}{\partial x} + \frac{\partial \mathcal{H}_4}{\partial y} \right)(x, y) = \sqrt{2\alpha}\mathcal{H}_4(x, y) - h_4(x).$$

¿From (10.18) and (10.8) we obtain $G(\bar{y}) = \mathcal{X}(F(\bar{y}))$, $h_1(F(\bar{y})) = \mathcal{H}_3\big(G(\bar{y}), \bar{y}\big)$ and thus (10.23) reads

$$F'(\bar{y}-) = \left. \frac{h_3\big(\mathcal{X}(z)\big) - \sqrt{2\alpha}h_1(z)}{h_1'(z)} \right|_{z = F(\bar{y})}.$$

But the right-hand side of this expression coincides with $F'(y+)$ thanks to the definitions (10.17) and (10.16), so (10.20) follows. Now use (10.14) to obtain

$$0 \ge \lim_{y\uparrow\bar{y}} \frac{q\big(G(y); F(y)\big) - q\big(G(\bar{y}); F(\bar{y})\big)}{y - \bar{y}} = \frac{\partial q}{\partial x}\big(G(\bar{y}); F(\bar{y})\big) + F'(\bar{y}) \cdot \frac{\partial q}{\partial z}\big(G(\bar{y}); F(\bar{y})\big) \tag{10.25}$$

since $G'(\bar{y}-) = 1$, and recall (8.25) to observe

$$\frac{\partial q}{\partial x}\big(G(\bar{y}); F(\bar{y})\big) = \lambda \sqrt{\frac{2}{\alpha}} \left(\frac{\alpha}{2\lambda} - G(\bar{y}) \right) e^{G(\bar{y})\sqrt{2\alpha}} \left[1 - e^{2\left(F(\bar{y}) - G(\bar{y})\right)\sqrt{2\alpha}} \right] < 0$$

thanks to (10.3). Therefore, (10.25) yields

$$1 \geq -\frac{\partial q/\partial z}{\partial q/\partial x}\big(G(\bar{y}); F(\bar{y})\big) \cdot F'(\bar{y}+) = G'(\bar{y}+)$$

in conjunction with (10.19) and (10.20). This proves (10.21). Proposition 8.5 implies that $G(\cdot)$ is strictly concave on $(\bar{y}, \infty)$ and so (10.22) follows immediately. □

In terms of the functions $F(\cdot)$ and $G(\cdot)$ of (10.17), (10.18), let us define now for $y > \bar{y}$ the function

(10.26)

$$Q(x,y) \triangleq \left\{ \begin{array}{cll} \delta x^2 & ; & 0 \leq x \leq F(y) \\ \frac{\lambda}{\alpha}x^2 + \frac{\lambda}{\alpha} + h_1\big(F(y)\big)e^{x\sqrt{2\alpha}} + h_2\big(F(y)\big)e^{-x\sqrt{2\alpha}} & ; & F(y) < x \leq G(y) \\ \zeta + Q(x-\zeta, y-\zeta) & ; & G(y) < x < G(\bar{y}) + (y-\bar{y}) \\ y + Q(x-y, 0) & ; & x \geq G(\bar{y}) + (y-\bar{y}) \\ Q(-x,y) & ; & x < 0 \end{array} \right\}$$

where $\zeta = \zeta(x,y)$ is defined uniquely via $x - \zeta = G(y-\zeta)$, and observe $\lim_{x\to x_0,\ y\downarrow\bar{y}} Q(x,y) = Q(x_0,\bar{y})$. Hence Q is continuous at all points of $\big\{(x,y)\ /\ y = \bar{y}\big\}$. It is also clear that Q is $\mathcal{C}^2$ except possibly on $\big\{(x,y)\ /\ x = \pm F(y) \text{ or } x = \pm G(y)\big\}$ (by the analysis of the proof of Theorem 9.1, Q is $\mathcal{C}^2$ across the boundary $\big\{(x,y)\ /\ x = \pm G(y),\ y > \bar{y}\big\}$).

It can be shown that Q is of class $\mathcal{C}^1$ on $\mathbb{R} \times [0,\infty)$. Indeed, we know from Theorem 10.2 that $Q \in \mathcal{C}^1(\mathbb{R} \times [0,\bar{y}])$, and from the analysis of Theorem 9.1 that $Q \in \mathcal{C}^1(\mathbb{R} \times (\bar{y},\infty))$. Hence it suffices to show

$$\lim_{x\to x_0,\ y\downarrow y_0} Q_x(x,y) = Q_x(x_0,\bar{y})$$

and similarly for Q_y. This is easily checked using straightforward differentiation and part (iii) of the proof of Theorem 9.1.

The proof of Theorem 9.1 shows that Q satisfies the Variational Inequalities (4.3)–(4.6) for $y > \bar{y}$, and Q satisfies (9.3). Since we have verified that Q satisfies the conditions of (4.3)–(4.6) for $y \leq \bar{y}$, it follows that Q solves them everywhere. Thus we have proved

10.7 Theorem: *Let $\lambda_* < \lambda \leq \lambda^*$. The function Q defined in (10.26) is the value function for the problem of (2.6), and $\pi(F;G)$ is the optimal strategy.*

Proof of Proposition 10.2: Abusing notation slightly, we shall denote $F_\lambda(\cdot)$ by $F(\cdot)$ and $G_\lambda(\cdot)$ by $G(\cdot)$, throughout this proof.

Step 1: *Reduction of the equations (10.8) to an implicit equation for $G(\cdot)$.* From Lemma 8.1, the function $h_1(\cdot)$ restricted to $[0,f_0]$ admits an inverse $h_1^{-1} : \big[-\frac{\lambda}{2\alpha^2}, 0\big] \to [0,f_0]$. It is shown in Appendix C that $\mathcal{H}_3(x,y) \in \big(-\frac{\lambda}{2\alpha^2}, 0\big)$, for $y > 0$ and $x > \Big(\frac{\alpha}{2\lambda} + \frac{y}{2} - \frac{1}{\sqrt{2\alpha}}\Big)^+$. Therefore, $H(\mathcal{H}_3(x,y)) = (h_2 \circ h_1^{-1})(\mathcal{H}_3(x,y))$ is defined on this domain, with $H = h_2 \circ h_1^{-1}$ as in Section 8. Set

$$L(x,y) \triangleq \mathcal{H}_4(x,y) - H\big(\mathcal{H}_3(x,y)\big); \qquad y \geq 0,\ \ x > \left(\frac{\alpha}{2\lambda} + \frac{y}{2} - \frac{1}{\sqrt{2\alpha}}\right)^+ \tag{10.27}$$

and note that, for $y = 0$, we have $\mathcal{H}_3(x,0) \equiv 0$ and $H\big(\mathcal{H}_3(x,0)\big) = h_2(f_0)$, thus $L(x,0) \equiv 0$.

We claim that if $G(\cdot)$ solves the equation

$$L\big(G(y), y\big) = 0 \ , \tag{10.28}$$

then $F(\cdot) \triangleq h_1^{-1}\left(\mathcal{H}_3\left(G(\cdot), \cdot\right)\right)$ and $G(\cdot)$ solve the equations of (10.8). Indeed, $F(\cdot), G(\cdot)$ satisfy the first equation in (10.8) by definition, and from (10.28): $\mathcal{H}_4(G(y), y) = h_2\left(h_1^{-1}(G(y), y)\right) = h_2(F(y))$.

Step 2: *Solution of (10.28).* In Appendix C it is shown that

$$\left\{\begin{array}{l} \text{there is a } Y > 0 \text{ such that, for every } 0 < y \leq Y, \text{ the equation (10.28)} \\ \text{has solution } G(y) \geq y + \frac{\alpha}{2\lambda}. \text{ Moreover, } G(y) > y + \frac{\alpha}{2\lambda} \text{ for } 0 < y < Y, \\ \text{and } \ G(Y) = Y + \frac{\alpha}{2\lambda}. \text{ Also, } \ G(0) \triangleq \lim_{y\downarrow 0} G(y) = g_0. \end{array}\right\} \tag{10.29}$$

Two immediate consequences of (10.29) are $F(Y) = h_1^{-1}\left(\mathcal{H}_3(G(y), y)\right) < f_0$ if $y > 0$, and $F(0) = \lim_{y\downarrow 0} F(y) = h_1^{-1}\left(\mathcal{H}_3(g_0, 0)\right) = h_1^{-1}(0) = f_0$.

Step 3: *Proof of (10.10)–(10.13).* From (C.8), (C.4) in Appendix C, we obtain

$$\left(\frac{\partial L}{\partial x} + \frac{\partial L}{\partial y}\right)(G(y), y) = q(G(y); F(y)), \tag{10.30}$$

(10.31)
$$\frac{\partial L}{\partial x}(G(y), y) = -\frac{2\lambda y}{\sqrt{2\alpha}}\left(G(y) - \left(\frac{y}{2} + \frac{\alpha}{2\lambda}\right)\right) e^{G(y)\sqrt{2\alpha}}\left[1 - e^{2\sqrt{2\alpha}(F(y)-G(y))}\right] < 0, \quad \text{for } 0 < y \leq Y.$$

Therefore, $G(\cdot) \in \mathcal{C}^1\left((0, y]\right)$ by the implicit function theorem, and

$$G'(y) = -\left(\frac{L_y}{L_x}\right)(G(y), y) = 1 - \frac{q(G(y); F(y))}{L_x(G(y), y)}, \qquad 0 < y \leq Y, \tag{10.32}$$

which proves (10.13).

By differentiation of the first equation in (10.8), and the fact that $h_1'(\cdot) > 0$ on $[0, f_0]$, we deduce that $F(\cdot) \in \mathcal{C}^1\left((0, y]\right)$ also. An explicit formula for $F'(\cdot)$ is given in (C.9); using that formula, it is shown that $\lim_{y\downarrow 0} F'(y) = f'(0)$ and that $F(\cdot)$ is decreasing. Thus $F(\cdot) \in \mathcal{C}^1\left([0, y]\right)$ with $F'(0) = f'(0)$, as claimed in (10.10) and (10.11).

The proof that $G(\cdot) \in \mathcal{C}^1\left([0, y]\right)$ and $G'(0) = \frac{1}{2}(1 + g'(0))$ is carried out in Appendix C.

Step 4: *Proof of (10.14) and (10.15).* Since $G(Y) = Y + \frac{\alpha}{2\lambda}$ and $G(0) = g_0 > \frac{\alpha}{2\lambda}$, we have $\frac{G(Y)-G(0)}{Y} < 1$. Therefore, by the Mean-Value Theorem, there is a $\tilde{y} \in (0, Y)$ with $G'(\tilde{y}) < 1$. Since $G'(0) > 1$, there must exist a $\bar{y} \in (0, \tilde{y})$ with $G'(\bar{y}) = 1$. The alternative characterization of $\bar{y}$ and (10.15) follow immediately from equation (10.13) for $G'(\cdot)$. □

The major remaining technical work is summarized in the next lemma, proved in Appendix C.

10.8 Lemma: *The function $Q(\cdot,\cdot)$ of (10.5), (10.7) is of class $\mathcal{C}^1\left(\mathbb{R} \times [0, \bar{y}]\right)$ and satisfies*

$$(Q_x + Q_y)(x, y) < 1; \qquad \textit{for } \ 0 \leq x < G(y), \quad 0 < y \leq \bar{y}, \tag{10.33}$$

$$(Q_x + Q_y)(x, y) = 1; \qquad \textit{for } \ x \geq G(y), \quad 0 < y \leq \bar{y}. \tag{10.34}$$

Proof of Theorem 10.3: Lemma 10.8 establishes the fact that Q is $\mathcal{C}^1$. It is easy to see that Q is $\mathcal{C}^2$ except at the curves $\{(x,y)\,/\,x=F(y)\}$ and $\{(x,y)\,/\,x=G(y)\}$, and that its second derivative are locally bounded. It remains to check the Variational Inequality (4.3)–(4.6) and (4.2). The condition (4.2) is immediate from the construction of Q. Lemma 10.8 takes care of that part of the Variational Inequality having to do with the directional derivative Q_x+Q_y. To conclude, it remains to verify

$$\begin{array}{ll} (i) & \frac{1}{2}Q_{xx}+\lambda x^2-\alpha Q>0; \qquad \text{in regions I, III}, \\ (ii) & Q<\delta x^2; \qquad \text{in regions II, III}. \end{array}$$

Now $\frac{1}{2}Q_{xx}+\lambda x^2-\alpha Q=\delta-(\alpha\delta-\lambda)x^2>\delta-(\alpha\delta-\lambda)f_0^2>0$ is checked, for $|x|\le F(y)$, exactly as in the proof of Theorem 9.1. On the other hand, we have in region III:

$$\frac{1}{2}Q_{xx}+\lambda x^2-\alpha Q=\lambda\left[x^2-(x-y)^2-\alpha y\right]=2\lambda y\left[x-\left(\frac{y}{2}+\frac{\alpha}{2\lambda}\right)\right]>0.$$

To check (ii), observe $Q_y(x,y)=h_1'(F(y))F'(y)e^{x\sqrt{2\alpha}}\cdot\left[1-e^{2(F(y)-x)\sqrt{2\alpha}}\right]<0$ in region II, because $F'(y)<0$, $h_1'(F(y))>0$ and $x>F(y)$ there; likewise,

$$Q_y(x,y)=1-\frac{2\lambda}{\alpha}(x-y)+\sqrt{2\alpha}\cdot h_2(f_0)\,e^{(y-x)\sqrt{2\alpha}}<0$$

in region III, because $h_2(f_0)<0$ and $x>G(y)>y+\frac{\alpha}{2\lambda}$ there. $\square$

A. APPENDIX: PROOFS OF RESULTS IN SECTIONS 8, 9

Proof of Lemma 8.1: It is straightforward to verify (8.18). It is checked from (6.10) and (7.13), (7.14) that

$$\text{(A.1)}\quad h_1'(x) = \frac{\alpha\delta-\lambda}{\sqrt{2\alpha}}\left(\frac{\delta}{\alpha\delta-\lambda} - x^2\right)e^{-x\sqrt{2\alpha}} > 0, \qquad h_2'(x) = \frac{\alpha\delta-\lambda}{\sqrt{2\alpha}}\left(x^2 - \frac{\delta}{\alpha\delta-\lambda}\right)e^{x\sqrt{2\alpha}} < 0$$

for $0 \le x < \sqrt{\frac{\delta}{\alpha\delta-\lambda}}$, and (8.19) follows. On the other hand we have $\rho\left(\sqrt{\frac{\delta}{\alpha\delta-\lambda}}\right) > 0$, which implies $f_0 < \sqrt{\frac{\delta}{\alpha\delta-\lambda}}$, so $h_1'(\cdot) > 0$ and $h_2'(\cdot) < 0$ on $(0, f_0)$. □

Proof of Lemma 8.3: The first claim follows from the implicit function theorem, since

$$\text{(A.2)}\qquad \mathcal{X}'(z) = -\left(\frac{\partial}{\partial z}q(x;z)\Big/\frac{\partial}{\partial x}q(x;z)\right)\Bigg|_{x=\mathcal{X}(z)} < 0.$$

Indeed, both $\frac{\partial}{\partial x}q(x;z)$ from (8.25), and

$$\text{(A.3)}\qquad \frac{\partial}{\partial z}q(x;z) = -2\frac{\alpha\delta-\lambda}{\alpha}\cdot\frac{d}{dz}\left(ze^{z\sqrt{2\alpha}}\right) + \frac{2\lambda}{\alpha}\sqrt{2\alpha}\left(\frac{\alpha}{2\lambda} - x - \frac{1}{\sqrt{2\alpha}}\right)e^{(2z-x)\sqrt{2\alpha}},$$

are negative at $x = \mathcal{X}(z) > \frac{\alpha}{2\lambda}$. On the other hand, differentiating further in (A.2), we obtain

$$\text{(A.4)}\qquad \frac{\partial^2 q}{\partial z^2}(x;z) + 2\frac{\partial^2 q}{\partial x\partial z}(x;z)\cdot\mathcal{X}'(z) + \frac{\partial^2 q}{\partial x^2}(x;z)\cdot\mathcal{X}''(z) + \frac{\partial^2 q}{\partial x^2}(x;z)\cdot\left(\mathcal{X}'(z)\right)^2 = 0$$

with $x = \mathcal{X}(z)$. Now from (8.25), (A.3) we have the computations

$$\begin{aligned}
&\frac{\sqrt{\alpha/2}}{\lambda e^{x\sqrt{2\alpha}}}\cdot\frac{\partial^2 q}{\partial x^2}(x;z) = -\left(x - \frac{\alpha}{2\lambda}\right)\sqrt{2\alpha}\left[1 + e^{2(z-x)\sqrt{2\alpha}}\right] - \left(1 - e^{2(z-x)\sqrt{2\alpha}}\right) < 0\\
&\frac{\partial^2 q}{\partial x\partial z}(x;z) = 4\lambda\left(x - \frac{\alpha}{2\lambda}\right)e^{(2z-x)\sqrt{2\alpha}} > 0\\
&\frac{\partial^2 q}{\partial z^2}(x;z) = -2\frac{\alpha\delta-\lambda}{\alpha}\cdot\frac{d^2}{dz^2}\left(ze^{z\sqrt{2\alpha}}\right) + 8\lambda\left(\frac{\alpha}{2\lambda} - x - \frac{1}{\sqrt{2\alpha}}\right)e^{(2z-x)\sqrt{2\alpha}} < 0
\end{aligned}$$

for $x = \mathcal{X}(z) > \frac{\alpha}{2\lambda}$, which lead in (A.4) to $\mathcal{X}''(z) < 0$. □

Proof of Proposition 8.5: Let us start by observing that the function $m(\cdot)$ of (8.27) is strictly decreasing. This is because we have $m'(z) = h_3'(\mathcal{X}(z))\cdot\mathcal{X}'(z) - \sqrt{2\alpha}\cdot h_1'(z) < 0$, thanks to Lemmata 8.1-8.3 and $h_3'(x) = \sqrt{\frac{\alpha}{2}}\left(\frac{2\lambda}{\alpha}x - 1\right)e^{-x\sqrt{2\alpha}} > 0$ for $x > \frac{\alpha}{2\lambda}$. In particular, we obtain then

$$\text{(A.5)}\qquad m(z) < 0 \quad \text{for} \quad f_\infty < z \le f_0$$

from the first equality in (7.12), which amounts to $m(f_\infty) = 0$. Let us consider also the strictly decreasing function

$$\text{(A.6)}\qquad F(z) \triangleq -\int_z^{f_0}\frac{h_1'(u)}{m(u)}du, \quad f_\infty < z \le f_0 \quad \text{with} \quad F(f_\infty+) = \infty,\ F(f_0) = 0$$

and $F'(z) = h_1'(z)/m(z)$. In view of (8.27) and (A.5), this gives $F'\big(f(y)\big) = 1/f'(y)$, which, combined with the fact that $F(f_0) = 0$, implies $F\big(f(y)\big) = y,\ 0 \le y < \infty$. This way, we may construct the solution of the differential equation (8.27) with the initial condition $f(0) = f_0$ by taking the inverse $f(\cdot) \triangleq F^{-1}(\cdot)$ of the function $F(\cdot)$ in (A.6); this solution also satisfies $f(\infty) = f_\infty$. For this function, (8.27) gives

$$f'(y) = \frac{m(z)}{h_1'(z)}\bigg|_{z=f(y)} < 0, \qquad \frac{f''(y)}{f'(y)} = \frac{m'(z)h_1'(z) - m(z)h_1''(z)}{\big(h_1'(z)\big)^2}\Bigg|_{z=f(y)}.$$

For $f_\infty < z \le f_0$, we have $m(z) < 0,\ h_1'(z) > 0,\ m'(z) < 0$ and

$$h_1''(z) = \frac{\alpha\delta - \lambda}{\sqrt{2\alpha}} e^{-z\sqrt{2\alpha}} \left[-\sqrt{2\alpha}\left(\eta + \frac{1}{\alpha} - z^2\right) - 2z \right] < 0$$

from the proof of Lemma 8.1, so $f''(y) > 0$. In other words, the function $f(\cdot)$ is strictly convex. ¿From these facts, the relation $g(y) = \mathcal{X}\big(f(y)\big)$ of (8.26) and Lemma 8.3 imply

$$g'(y) = f'(y)\mathcal{X}'\big(f(y)\big) > 0, \quad g''(y) = f''(y)\mathcal{X}'\big(f(y)\big) + \big(f'(y)\big)^2 \mathcal{X}''\big(f(y)\big) < 0, \tag{A.7}$$

and the strict increase and strict concavity of $g(\cdot)$ follow. □

Proof of (8.28), (8.29): The first follows from Lemma 8.1 and $f'(y) < 0$. For the second, fix $y > 0$, and observe that we have $Q_x\big(f(y), y)\big) = 2\delta f(y) > 2\delta f_\infty > 0$ and $Q_x\big(g(y), y)\big) = 1 - Q_y\big(g(y), y)\big) \ge 1$, so the maximum principle implies $Q_x(x, y) \ge 1 \wedge (2\delta f_\infty)$ for $f(y) \le x \le g(y)$.

Proof of (8.34): For fixed $y \in (0, \infty)$, the function $d(x) \triangleq U(x, y) = \frac{2\lambda}{\alpha}x + h_3\big(g(y)\big)e^{x\sqrt{2\alpha}} - h_4\big(g(y)\big)e^{-x\sqrt{2\alpha}}$, $f(y) \le x \le g(y)$ of (8.34) satisfies

$$d\big(f(y)\big) = 2\delta f(y) < 1 = d\big(g(y)\big), \tag{A.8}$$

as well as

$$d'(x) = (DQ_x)(x, y) = (Q_{xx} + Q_{xy})(x, y) = \frac{2\lambda}{\alpha} + \sqrt{2\alpha}\left[h_3\big(g(y)\big)e^{x\sqrt{2\alpha}} + h_4\big(g(y)\big)e^{-x\sqrt{2\alpha}} \right].$$

Notice that $d'(\cdot)$ is *strictly concave*, because $h_3\big(g(y)\big) < 0,\ h_4\big(g(y)\big) < 0$ from (8.26), (8.22) and (7.15), (7.16). Furthermore $d'\big(g(y)\big) = 0$ from (8.15).

We cannot have $d'(\cdot) \le 0$ throughout the interval $[f(y), g(y)]$, because that would contradict (A.8). Thus, the only possibilities compatible with the strict concavity of $d'(\cdot)$ and with $d'\big(g(y)\big) = 0$ are: (*i*) $d'(\cdot) > 0$ over $\big(f(y), g(y)\big)$; or (*ii*) $d'(\cdot) < 0$ on $\big(f(y), q\big)$ and $d'(\cdot) > 0$ on $(q, g(y))$, for some q in $\big(f(y), g(y)\big)$. In either case, we have $d(x) < 1$ on $[f(y), g(y))$. □

Proof of (8.32): From (8.27), (8.29) and the proof of Proposition 8.5, we get with $z = f(y)$:

$$\frac{1}{\sqrt{2\alpha}} e^{x\sqrt{2\alpha}} Q_{xy}(x, y) = f'(y)h_1'(z)\left[e^{2x\sqrt{2\alpha}} + e^{2z\sqrt{2\alpha}} \right] = m(z)\left[e^{2x\sqrt{2\alpha}} + e^{2z\sqrt{2\alpha}} \right] < 0.$$

Proof of (8.31): We have $\frac{1}{2}Q_{xx}\big(f(y),y\big) = \big(\alpha Q(x,y) - \lambda x^2\big)\big|_{x=f(y)} = (\alpha\delta - \lambda)f^2(y) > 0$, and $Q_{xx}\big(g(y),y\big) = -Q_{xy}\big(g(y),y\big) > 0$ from (8.32). On the other hand, $\frac{1}{2}(Q_{xx})_{xx} - \alpha Q_{xx} = -2\lambda < 0$ for $f(y) < x < g(y)$, so the maximum principle gives $Q_{xx}(x,y) \geq Q_{xx}\big(f(y),y\big) \wedge Q_{xx}\big(g(y),y\big) > 0$, for $f(y) \leq x \leq g(y)$. $\square$

Proof of (8.30): The equality is satisfied by construction of the function Q, in (8.10). As for the inequality, for fixed $y > 0$, the function $v(x) \triangleq \delta x^2 - Q(x,y),\ f(y) \leq x \leq g(y)$ satisfies $v\big(f(y)\big) = v'\big(f(y)\big) = 0$ by construction of Q, as well as $v''(x) > 0$ because $\frac{1}{2}Q_{xx}(x,y) < \lambda/\alpha < \delta$, thanks to (8.31) and Lemma 8.1. It follows that $v'(\cdot) > 0,\ v(\cdot) > 0$ on $\big(f(y), g(y)\big)$. $\square$

Proof of Theorem 9.1: **(i)** In the region $\{(x,y)\ /\ 0 \leq x \leq f(y),\ 0 \leq y < \infty\}$ we have $Q(x,y) = \delta x^2$, thus also $(Q_x + Q_y)(x,y) = Q_x(x,y) = 2\delta x < 2\delta f_0 < 1$ from (8.23), and from the proof of Lemma 8.1: $\big(\frac{1}{2}Q_{xx} + \lambda x^2 - \alpha Q\big)(x,y) = \delta - (\alpha\delta - \lambda)x^2 \geq \delta - (\alpha\delta - \lambda)f_0^2 > 0$.

(ii) In the region $\{(x,y)\ /\ f(y) < x \leq g(y),\ 0 \leq y < \infty\} \cup \{(x,0)\ /\ x \geq f(0)\}$ we have $\big(\frac{1}{2}Q_{xx} + \lambda x^2 - \alpha Q\big)(x,y) = 0$, and $Q(x,y) < \delta x^2$ from (6.3) and (8.30). On the other hand: $(Q_x + Q_y)(x,y) < 1$ for $f(y) \leq x < g(y),\ 0 < y < \infty$, from (8.34). Finally, $Q\big(f(y)+,y\big) = \delta\big(f(y)\big)^2$ and $Q_x\big(f(y)+,y\big) = 2\delta f(y)$ from (8.11), (8.12), as well as

$$\begin{aligned} Q_y(x,y) &= A'(y)e^{x\sqrt{2\alpha}} + B'(y)e^{-x\sqrt{2\alpha}} = f'(y)\left[h_1'\big(f(y)\big)e^{x\sqrt{2\alpha}} + h_2'\big(f(y)\big)e^{-x\sqrt{2\alpha}}\right] \\ &= f'(y)h_1'\big(f(y)\big)e^{-x\sqrt{2\alpha}}\left(e^{2x\sqrt{2\alpha}} - e^{2f(y)\sqrt{2\alpha}}\right) < 0, \quad f(y) < x \leq g(y) \end{aligned}$$

with $Q_y\big(f(y),y\big) = 0$. In particular, $Q(\cdot,\cdot)$ is of class $\mathcal{C}^1$ across the boundary $\{(x,y)/x = f(y)\}$.

(iii) In the region $\{(x,y)\ /\ g(y) < x < g(0) + y,\ 0 < y < \infty\}$, we have $Q(x,y) = \zeta + Q(\eta,\theta)$ where $\eta = x - \zeta,\ \theta = y - \zeta$ and $\zeta = \zeta(x,y) \in (0,y)$ is defined by (9.2). In particular, we have $\zeta_x = 1/(1 - g'(\theta)) > 0,\ \ \zeta_y = -g'(\theta)/(1 - g'(\theta)) < 0$ and

$$\begin{aligned} Q_x(x,y) &= \zeta_x + (1 - \zeta_x)Q_x(\eta,\theta) + (-\zeta_x)Q_y(\eta,\theta) = Q_x(\eta,\theta) \\ Q_y(x,y) &= \zeta_y + (-\zeta_y)Q_x(\eta,\theta) + (1 - \zeta_y)Q_y(\eta,\theta) = Q_y(\eta,\theta) \\ Q_{xx}(x,y) &= (1 - \zeta_x)Q_{xx}(\eta,\theta) + (-\zeta_x)Q_{xy}(\eta,\theta) = Q_{xx}(\eta,\theta); \end{aligned}$$

similarly $Q_{xy}(x,y) = Q_{xy}(\eta,\theta),\ Q_{yy}(x,y) = Q_{yy}(\eta,\theta)$. We have used here the property $Q_{xx}(\eta,\theta) = -Q_{xy}(\eta,\theta) = Q_{yy}(\eta,\theta)$ for $\eta = g(\theta)$, a consequence of (8.14) and (8.15). It follows that $Q_x + Q_y = 1$ throughout the region, and that the function Q is of class $\mathcal{C}^2$ across the moving boundary $g(\cdot)$. Furthermore,

$$\begin{aligned} &\frac{1}{2}Q_{xx}(x,y) + \lambda x^2 - \alpha Q(x,y) = \frac{1}{2}Q_{xx}(\eta,\theta) + \lambda x^2 - \alpha\big(\zeta + Q(\eta,\theta)\big) \\ &\quad = \left(\frac{1}{2}Q_{xx} - \alpha Q\right)(\eta,\theta) + \lambda x^2 - \alpha\zeta = \lambda[x^2 - (x-\zeta)^2] - \alpha\zeta = \lambda\zeta(2x - \zeta) - \alpha\zeta \\ &\quad = 2\lambda\zeta\left[(x-\zeta) - \frac{\alpha}{2\lambda}\right] + \lambda\zeta^2 = 2\lambda\zeta\left[g(y-\zeta) - \frac{\alpha}{2\lambda}\right] + \lambda\zeta^2 > 0, \end{aligned}$$

in view of (8.26) and (8.23). For the same reason, we also have $Q(x,y) = \zeta + Q(\eta,\theta) < \zeta + \delta\eta^2 = \zeta + \delta(x-\zeta)^2 < \delta x^2$ in this region.

(iv) Similar properties can be established in the region $\{(x,y)/g(0)+y \le x < \infty\}$, where $Q(x,y) = y + Q_0(x-y)$. It can also be seen that the function Q is of class $\mathcal{C}^2$ across the line $\{(x,y)/x = g(0)+y, 0 < y < \infty\}$.

B. APPENDIX: PROOF OF PROPOSITION 8.7

¿From (8.25)-(8.27), (8.18) and (A.2), it is easy to see

$$g'(0) = \mathcal{X}'\big(f(0)\big)\cdot f'(0) = \mathcal{X}'(f_0)\cdot \frac{h_3\big(\mathcal{X}(f_0)\big) - \sqrt{2\alpha}h_1(f_0)}{h_1'(f_0)} = \mathcal{X}'(f_0)\cdot\frac{h_3(g_0)}{h_1'(f_0)}\ , \tag{B.1}$$

$$\mathcal{X}'(f_0) = -\frac{\frac{\partial}{\partial z}q(x;z)}{\frac{\partial}{\partial x}q(x;z)} = \frac{-2(\delta-\frac{\lambda}{\alpha})(1+\sqrt{2\alpha}z)e^{z\sqrt{2\alpha}} + 2\sqrt{2\alpha}e^{2z\sqrt{2\alpha}}h_3(x)}{\sqrt{\frac{2}{\alpha}}(\lambda x - \frac{\alpha}{2})e^{x\sqrt{2\alpha}}(1-e^{2(z-x)\sqrt{2\alpha}})} \tag{B.2}$$

with $z = f_0,\ x = \mathcal{X}(f_0) = g_0$. On the interval $(\frac{\alpha}{2\lambda}, \frac{\alpha}{2\lambda}+\frac{1}{\sqrt{2\alpha}})$, let us define the following function

$$\begin{aligned} M(x;\lambda) \ &\stackrel{\triangle}{=}\ \lambda h_1'(f_0)\cdot\sqrt{\frac{2}{\alpha}}\left(x-\frac{\alpha}{2\lambda}\right)e^{x\sqrt{2\alpha}}(1-e^{2(f_0-x)\sqrt{2\alpha}}) \\ &\qquad -h_3(x)\cdot\left(-2\left(\delta-\frac{\lambda}{\alpha}\right)(1+\sqrt{2\alpha}f_0)e^{f_0\sqrt{2\alpha}} + 2\sqrt{2\alpha}h_3(x)e^{2\sqrt{2\alpha}f_0}\right), \end{aligned} \tag{B.3}$$

for which we have

$$g'(0) \le 1 \quad\Longleftrightarrow\quad M(g_0;\lambda) > 0. \tag{B.4}$$

Let us also define

$$\tilde{M}(u;\lambda) \stackrel{\triangle}{=} \frac{1}{\lambda}M\left(u+\frac{\alpha}{2\lambda};\lambda\right), \qquad \text{for} \qquad u\in(0,1/\sqrt{2\alpha}), \tag{B.5}$$

so that

$$g'(0)\le 1 \quad\Longleftrightarrow\quad \tilde{M}\left(g_0-\frac{\alpha}{2\lambda};\lambda\right) > 0. \tag{B.6}$$

The definition of $\tilde{M}(\cdot\,;\cdot)$ depends on λ, and so do f_0 and g_0. It will be convenient to think of the parameters α and δ as fixed, while λ can vary. We will write $f_0(\lambda)$ (resp., $g_0(\lambda)$) instead of f_0 (resp., g_0), to emphasize that both f_0 and g_0 depend on λ.

B.1 Lemma: *For every fixed* $0 < \lambda \le \lambda^*$, *the function* $u \mapsto \tilde{M}(u;\lambda)$ *is strictly increasing on* $(0,\frac{1}{\sqrt{2\alpha}})$ *with* $\tilde{M}(0;\lambda) < 0$. *For every fixed* $u \in (0,\frac{1}{\sqrt{2\alpha}})$, *the function* $\lambda \mapsto \tilde{M}(u;\lambda)$ *is strictly decreasing on* $(0,\lambda^*]$.

Proof: Suppose $\lambda \in (0,\lambda^*]$ is fixed. Clearly, as u increases, the first term on the right-hand side of (B.3) increases; as for the second term, the function $u \mapsto h_3(u+\frac{\alpha}{2\lambda})$ is increasing (see the proof of

Proposition 8.5). However, we have $h_3(u+\frac{\alpha}{2\lambda})<0$, so $u\mapsto h_3^2(u+\frac{\alpha}{2\lambda})$ decreases. Hence $M(\cdot;\lambda)$ is strictly increasing, and thus so is $\tilde{M}(\cdot;\lambda)$. Furthermore,

$$\text{(B.7)}\quad \lambda\cdot\tilde{M}(0;\lambda)=M(\frac{\alpha}{2\lambda};\lambda)=h_3(\frac{\alpha}{2\lambda})\left[2\left(\delta-\frac{\lambda}{\alpha}\right)(1+\sqrt{2\alpha}f_0)e^{f_0\sqrt{2\alpha}}-2\sqrt{2\alpha}e^{2f_0\sqrt{2\alpha}}h_3(\frac{\alpha}{2\lambda})\right]$$

is negative since $h_3(\frac{\alpha}{2\lambda})<0$ and $\lambda<\alpha\delta$. Now suppose $u\in(0,\frac{1}{\sqrt{2\alpha}})$ is fixed. From (B.3), we have

$$\text{(B.8)}\qquad \begin{aligned}\tilde{M}(u;\lambda)\ =\ &\sqrt{\frac{2}{\alpha}}h_1'(f_0(\lambda))\left(1-e^{2(f_0(\lambda)-u-\frac{\alpha}{2\lambda})\sqrt{2\alpha}}\right)ue^{(u+\frac{\alpha}{2\lambda})\sqrt{2\alpha}}\\ &-\frac{1}{\lambda}h_3\left(u+\frac{\alpha}{2\lambda}\right)\cdot\left[-2\left(\delta-\frac{\lambda}{\alpha}\right)\left(1+f_0(\lambda)\sqrt{2\alpha}\right)e^{f_0(\lambda)\sqrt{2\alpha}}\right.\\ &\left.+2\sqrt{2\alpha}\cdot h_3\left(u+\frac{\alpha}{2\lambda}\right)e^{2f_0(\lambda)\sqrt{2\alpha}}\right]\end{aligned}$$

for the function of (B.5). From (A.1), we observe that the first term

$$\begin{aligned}&\sqrt{\frac{2}{\alpha}}h_1'(f_0(\lambda))\left(1-e^{2(f_0(\lambda)-u-\frac{\alpha}{2\lambda})\sqrt{2\alpha}}\right)ue^{(u+\frac{\alpha}{2\lambda})\sqrt{2\alpha}}\\ =\ &u\left(\delta-\frac{\lambda}{\alpha}\right)\left(\frac{1}{\alpha}+\frac{2f_0(\lambda)}{\sqrt{2\alpha}}\right)\left[e^{\left(u+\frac{\alpha}{2\lambda}-f_0(\lambda)\right)\sqrt{2\alpha}}-e^{-\left(u+\frac{\alpha}{2\lambda}-f_0(\lambda)\right)\sqrt{2\alpha}}\right]\\ =\ &\frac{u}{\alpha^2}\sqrt{\alpha^2\delta^2-\lambda^2}\left[e^{\left(u+\frac{\alpha}{2\lambda}-f_0(\lambda)\right)\sqrt{2\alpha}}-e^{-\left(u+\frac{\alpha}{2\lambda}-f_0(\lambda)\right)\sqrt{2\alpha}}\right]\end{aligned}$$

is clearly decreasing. Recall from (6.9) that $\lambda\mapsto f_0(\lambda)$ is increasing, and observe that we have $\frac{1}{\lambda}h_3(u+\frac{\alpha}{2\lambda})=-\frac{1}{\alpha}(u+\frac{1}{\sqrt{2\alpha}})e^{-(u+\frac{\alpha}{2\lambda})\sqrt{2\alpha}}$, therefore

$$\lambda\mapsto-\frac{1}{\lambda}h_3\left(u+\frac{\alpha}{2\lambda}\right)\cdot2\sqrt{2\alpha}h_3\left(u+\frac{\alpha}{2\lambda}\right)e^{2f_0(\lambda)\sqrt{2\alpha}}=-\frac{2\lambda\sqrt{2\alpha}}{\alpha^2}\left(u+\frac{1}{\sqrt{2\alpha}}\right)^2e^{2(f_0(\lambda)-u-\frac{\alpha}{2\lambda})\sqrt{2\alpha}}$$

is decreasing. It suffices then to show that, in (B.8), the term

$$\lambda\mapsto\frac{1}{\lambda}h_3\left(u+\frac{\alpha}{2\lambda}\right)\cdot2\left(\delta-\frac{\lambda}{\alpha}\right)\left(1+f_0(\lambda)\sqrt{2\alpha}\right)e^{f_0(\lambda)\sqrt{2\alpha}}=-\frac{2\sqrt{\alpha^2\delta^2-\lambda^2}}{\alpha^2}\left(u+\frac{1}{\sqrt{2\alpha}}\right)e^{(f_0(\lambda)-u-\frac{\alpha}{2\lambda})\sqrt{2\alpha}}$$

is decreasing. But this follows from

$$\begin{aligned}\frac{d}{d\lambda}\left(\sqrt{\alpha^2\delta^2-\lambda^2}\,e^{(f_0(\lambda)-\frac{\alpha}{2\lambda})\sqrt{2\alpha}}\right)\ &=\ \left(\frac{-\lambda}{\sqrt{\alpha^2\delta^2-\lambda^2}}+\sqrt{2\alpha}\sqrt{\alpha^2\delta^2-\lambda^2}(\frac{df_0}{d\lambda}+\frac{\alpha}{2\lambda^2})\right)\cdot e^{(f_0(\lambda)-\frac{\alpha}{2\lambda})\sqrt{2\alpha}}\\ &=\ \left(\frac{-\lambda}{\sqrt{\alpha^2\delta^2-\lambda^2}}+\frac{\alpha\delta}{\alpha\delta-\lambda}+\sqrt{\alpha^2\delta^2-\lambda^2}\frac{\alpha\sqrt{2\alpha}}{2\lambda^2}\right)\cdot e^{(f_0(\lambda)-\frac{\alpha}{2\lambda})\sqrt{2\alpha}}>0,\end{aligned}$$

since $\frac{\lambda}{\sqrt{\alpha^2\delta^2-\lambda^2}}<\frac{\lambda}{\alpha\delta-\lambda}<\frac{\alpha\delta}{\alpha\delta-\lambda}$. □

B.2 Lemma: *The function* $\lambda\mapsto g_0(\lambda)-\frac{\alpha}{2\lambda}$ *is decreasing for* $0<\lambda\leq\lambda^*$.

Proof: Note that $g_0 \equiv g_0(\lambda)$ is actually the solution of the equation

$$q(x;f_0) = \sqrt{2\alpha}\big[h_2(f_0) - h_1(f_0)e^{2f_0\sqrt{2\alpha}}\big] + h_3(x)e^{2f_0\sqrt{2\alpha}} - h_4(x) = 0 \tag{B.9}$$

in the notation of (8.22). Since $q(\cdot;f_0)$ is strictly decreasing on the interval $\left(\frac{\alpha}{2\lambda}, \frac{\alpha}{2\lambda}+\frac{1}{\sqrt{2\alpha}}\right)$ for fixed λ (see the proof of Lemma 8.2), we only need show that

$$G(u;\lambda) \triangleq q\left(u+\frac{\alpha}{2\lambda};f_0\right) \quad \text{is a } \textit{decreasing} \text{ function of } \lambda, \tag{B.10}$$

for any fixed $u \in (0, \frac{1}{\sqrt{2\alpha}})$. We compute

(B.11)

$$G(u;\lambda) = -2f_0\left(\delta - \frac{\lambda}{\alpha}\right)e^{f_0\sqrt{2\alpha}} - \frac{\lambda}{\alpha}\left(u+\frac{1}{\sqrt{2\alpha}}\right)e^{(2f_0-u-\frac{\alpha}{2\lambda})\sqrt{2\alpha}} - \frac{\lambda}{\alpha}\left(u-\frac{1}{\sqrt{2\alpha}}\right)e^{(u+\frac{\alpha}{2\lambda})\sqrt{2\alpha}}.$$

Observe from (6.9) that the function $\lambda \mapsto f_0(\lambda)$ is increasing, therefore $\lambda \mapsto \frac{\lambda}{\alpha}\big(u+\frac{1}{\sqrt{2\alpha}}\big)e^{(2f_0(\lambda)-u-\frac{\alpha}{2\lambda})\sqrt{2\alpha}}$ is also increasing. Furthermore, $\lambda \mapsto \lambda e^{\frac{\alpha}{2\lambda}\sqrt{2\alpha}}$ is a decreasing function; this follows from

$$\frac{d}{d\lambda}\left(\lambda e^{\frac{\alpha}{2\lambda}\sqrt{2\alpha}}\right) = \left(1-\frac{\alpha}{2\lambda}\sqrt{2\alpha}\right)\cdot e^{\alpha\sqrt{2\alpha}/(2\lambda)} < 0$$

for $0 < \lambda \le \lambda^*$, since

$$\lambda^* < \frac{\sqrt{2\alpha}}{2}\alpha \quad \Longleftrightarrow \quad \delta < \frac{\sqrt{2\alpha}}{2}\left(1+\frac{\delta/\alpha}{\frac{1}{4\delta}+\frac{1}{\sqrt{2\alpha}}}\right) \quad \Longleftrightarrow \quad \frac{\sqrt{2\alpha}}{2\delta} + \frac{1}{\frac{\sqrt{2\alpha}}{4\delta}+1} > 1$$

and this last inequality is clearly valid. The function $\lambda \mapsto \frac{\lambda}{\alpha}\left(u-\frac{1}{\sqrt{2\alpha}}\right)e^{(u+\frac{\alpha}{2\lambda})\sqrt{2\alpha}}$ is increasing, so it remains to show that $\lambda \mapsto (\alpha\delta-\lambda)f_0(\lambda)e^{f_0(\lambda)\sqrt{2\alpha}}$ is increasing. ¿From (6.9) it develops that $\frac{d}{d\lambda}f_0(\lambda) = \frac{1}{\sqrt{2\alpha}}\frac{\alpha\delta}{(\alpha\delta-\lambda)^2}\sqrt{\frac{\alpha\delta-\lambda}{\alpha\delta+\lambda}}$, hence

$$\begin{aligned}
\frac{d}{d\lambda}\left((\alpha\delta-\lambda)f_0(\lambda)e^{f_0(\lambda)\sqrt{2\alpha}}\right) &= e^{f_0(\lambda)\sqrt{2\alpha}}\left(-f_0(\lambda)+\frac{df_0}{d\lambda}(\lambda)\left(1+f_0(\lambda)\sqrt{2\alpha}\right)(\alpha\delta-\lambda)\right)\\
&= e^{f_0(\lambda)\sqrt{2\alpha}}\left(-f_0(\lambda)+\frac{\alpha\delta(1+f_0(\lambda)\sqrt{2\alpha})}{\sqrt{2\alpha}\sqrt{\alpha^2\delta^2-\lambda^2}}\right)\\
&= \frac{e^{f_0(\lambda)\sqrt{2\alpha}}}{\sqrt{2\alpha}\sqrt{\alpha^2\delta^2-\lambda^2}}\left[\alpha\delta + f_0(\lambda)\sqrt{2\alpha}\left(\alpha\delta-\sqrt{\alpha^2\delta^2-\lambda^2}\right)\right] > 0,
\end{aligned}$$

which completes our proof. □

Let us define

$$\lambda_* \triangleq \sup\left\{0<\lambda\le\lambda^* \;/\; M\big(g_0(\lambda);\lambda\big)\ge 0\right\} \wedge \lambda^*. \tag{B.12}$$

We have then the following claim.

Proposition 8.7: *λ_* is always strictly positive, and $g'(0) \le 1$ if and only if $0<\lambda\le\lambda_*$.*

Proof: To prove $\lambda_* > 0$, we only need to to show $M\big(g_0(\lambda);\lambda\big) \geq 0$ for λ sufficiently small. However, as $\lambda \to 0+$, we have $f_0(\lambda) \to 0$ and $g_0(\lambda) \to +\infty$ (see Lemma B.2), as well as $h_1\big(f_0(\lambda)\big) \to 0$, $h_2\big(f_0(\lambda)\big) \to 0$, $h_3\big(g_0(\lambda)\big) \to 0$. Now equation (8.21) implies $h_4\big(g_0(\lambda)\big) \to 0$, or equivalently, $e^{g_0(\lambda)\sqrt{2\alpha}}\left(\frac{\lambda g_0(\lambda)}{\alpha} - \frac{\lambda}{\alpha\sqrt{2\alpha}} - \frac{1}{2}\right) \to 0$. However, $\lambda e^{g_0(\lambda)\sqrt{2\alpha}} \to +\infty$ as $\lambda \to 0+$, since $g_0(\lambda) > \alpha/2\lambda$. Therefore, $e^{g_0(\lambda)\sqrt{2\alpha}}\big((\lambda x/\alpha) - (1/2)\big) \to +\infty$, as $\lambda \to 0+$. In conjuction with $h_1'(0+) = \delta/\sqrt{2\alpha}$, the equation (B.3) now yields $M\big(g_0(\lambda);\lambda\big) \to +\infty$ as $\lambda \to 0+$.
As for the second part of the claim, it suffices to show

$$M\big(g_0(\lambda);\lambda\big) \geq 0, \quad \forall\, 0 < \lambda \leq \lambda_*, \qquad \text{and} \qquad M\big(g_0(\lambda);\lambda\big) < 0, \quad \forall\, \lambda > \lambda_*.$$

However, $M\big(g_0(\lambda);\lambda\big) \geq 0 \iff \tilde{M}\big(g_0(\lambda) - \frac{\alpha}{2\lambda};\lambda\big) \geq 0$. Now, for every $\lambda \in (0, \lambda_*]$, we observe that

$$\begin{aligned}
\tilde{M}\big(g_0(\lambda) - \tfrac{\alpha}{2\lambda};\lambda\big) &\geq \tilde{M}\big(g_0(\lambda_*) - \tfrac{\alpha}{2\lambda_*};\lambda\big) && \text{(by Lemmata B.1 and B.2)}\\
&\geq \tilde{M}\big(g_0(\lambda_*) - \tfrac{\alpha}{2\lambda_*};\lambda_*\big) && \text{(by Lemma B.1)}\\
&\geq 0, && \text{(since } M\big(g_0(\lambda_*);\lambda_*\big) \geq 0)
\end{aligned}$$

and similarly, $\tilde{M}\big(g_0(\lambda) - \frac{\alpha}{2\lambda};\lambda\big) \leq \tilde{M}\big(g_0(\lambda_*) - \frac{\alpha}{2\lambda_*};\lambda\big) < \tilde{M}\big(g_0(\lambda_*) - \frac{\alpha}{2\lambda_*};\lambda_*\big) \leq 0$, for all $\lambda > \lambda_*$.

C. APPENDIX: PROOFS OF RESULTS IN SECTION 10

The following identities are helpful, along with those of (10.24), and may be verified by substitution. Where needed, the fact $H'(w) = -e^{2h_1^{-1}(w)\sqrt{2\alpha}}$ is used (see Lemma 8.1):

$$\text{(C.1)} \qquad e^{x\sqrt{2\alpha}}\frac{\partial \mathcal{H}_3}{\partial x}(x,y) = e^{-x\sqrt{2\alpha}}\frac{\partial \mathcal{H}_4}{\partial x}(x,y) = \frac{2\lambda y}{\sqrt{2\alpha}}\left[x - \left(\frac{y}{2} + \frac{\alpha}{2\lambda}\right)\right],$$

$$\text{(C.2)} \qquad \frac{\partial L}{\partial x}(x,y) = \frac{\partial \mathcal{H}_3}{\partial x}(x,y)\left[e^{2z\sqrt{2\alpha}} - e^{2x\sqrt{2\alpha}}\right]; \qquad z = h_1^{-1}(\mathcal{H}_3(x,y)).$$

First we establish

$$\text{(C.3)} \qquad \mathcal{H}_3(x,y) \in \left(-\frac{\lambda}{2\alpha^2}\,,\,0\right) \quad \text{for } \; y > 0, \quad x > \left(\frac{\alpha}{2\lambda} + \frac{y}{2} - \frac{1}{\sqrt{2\alpha}}\right)^+.$$

Clearly $\mathcal{H}_3(x,y) = \frac{\lambda y}{\alpha}e^{-x\sqrt{2\alpha}}\left(\frac{\alpha}{2\lambda} + \frac{y}{2} - x - \frac{1}{\sqrt{2\alpha}}\right) < 0$ in the given range. From (C.1), the function $\mathcal{H}_3(\cdot,y)$ has a global minimum at $x = \frac{y}{2} + \frac{\alpha}{2\lambda}$, and $\mathcal{H}_3(\frac{y}{2} + \frac{\alpha}{2\lambda};y) = -\frac{\lambda y}{\alpha\sqrt{2\alpha}}e^{-\sqrt{2\alpha}(y/2+\alpha/2\lambda)} > -\frac{\lambda}{2\alpha^2}$ (use $\sup_{y\geq 0}(ye^{-\beta y}) = \frac{1}{\beta e}$ and $e > 2$).

Proof of (10.29): For $x > \left(\frac{y}{2} + \frac{\alpha}{2} - \frac{1}{\sqrt{2\alpha}}\right)^+$, set $z = z(x,y) \triangleq h_1^{-1}(\mathcal{H}_3(x,y))$; observe $z < f_0$ if $y > 0$, and $z = 0$ if $y = 0$. From (C.1), (10.24), (C.2), we have then

$$\text{(C.4)} \qquad L_x(x,y) = -\frac{2\lambda y}{\sqrt{2\alpha}}\left[x - \left(\frac{y}{2} + \frac{\alpha}{2\lambda}\right)\right]e^{x\sqrt{2\alpha}}\left[1 - e^{2\sqrt{2\alpha}(z-x)}\right],$$

$$\text{(C.5)} \qquad L_y(x,y) = -h_4(x) + \frac{\lambda y}{\alpha} e^{x\sqrt{2\alpha}} + \sqrt{2\alpha} h_2(f_0) e^{y\sqrt{2\alpha}} + e^{2z\sqrt{2\alpha}} \left[h_3(x) + \frac{\lambda y}{\alpha} e^{-x\sqrt{2\alpha}} \right],$$

$$\text{(C.6)} \qquad \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) L(x,y) = \sqrt{2\alpha} \left[\mathcal{H}_4(x,y) - \mathcal{H}_3(x,y) e^{2z\sqrt{2\alpha}} \right] + h_3(x) e^{2z\sqrt{2\alpha}} - h_4(x).$$

Reading (C.4) with $x > \frac{y}{2} + \frac{\alpha}{2\lambda} > f_0 > z$ we see that $L(\cdot, y)$ is strictly decreasing on $(\frac{y}{2} + \frac{\alpha}{2\lambda}, \infty)$, and $\lim_{x\to\infty} L(x,y) = -\infty$. Observe from (10.27), (C.5) that

$$\text{(C.7)} \qquad L(x,0) = 0, \qquad L_y(x,0) = q(x; f_0), \qquad L_x(x,0) = 0.$$

It was shown in Lemma 8.2 that $q(x; f_0) > 0$, for $0 < x < g_0$. Therefore, $\frac{d}{dy} L(y + \frac{\alpha}{2\lambda}; y)\Big|_{y=0} = q(\frac{\alpha}{2\lambda}; f_0) > 0$, and it follows that $Y \stackrel{\triangle}{=} \sup\left\{ y > 0 \ / \ L(y + \frac{\alpha}{2\lambda}; y) > 0 \right\} > 0$. Since $L(\cdot, y)$ is strictly decreasing on $(y + \frac{\alpha}{2\lambda}, \infty)$ and tends to $-\infty$ as $x \to \infty$, the existence of $G(y) \in \left(y + \frac{\alpha}{2\lambda}, \infty\right)$ solving (10.28) is established for $0 < y < Y$. Because $H(\mathcal{H}_3(x,y))$ remains bounded on the domain of L, and because

$$\mathcal{H}_4 \left(y + \frac{\alpha}{2\lambda}; y \right) = \left(-\frac{\lambda}{2\alpha} y^2 + \frac{\lambda}{\alpha\sqrt{2\alpha}} \right) e^{\left(y + \frac{\alpha}{2\lambda}\right)\sqrt{2\alpha}} + h_2(f_0) e^{y\sqrt{2\alpha}},$$

it is clear that $\lim_{y\to\infty} L(y + \frac{\alpha}{2\lambda}; y) = -\infty$. Therefore, $Y < \infty$ and $L(Y + \frac{\alpha}{2\lambda}; Y) = 0$, which shows $G(Y) = Y + \frac{\alpha}{2\lambda}$.

With $G(\cdot)$ so defined, set $F(y) = h_1^{-1}\left(\mathcal{H}_3(G(y), y)\right)$, $0 < y \le Y$. ¿From (C.7) we have $L(x,y) = yq(x; f_0) + R(x,y)$, where $R(x,y)/y \to 0$ as $y \downarrow 0$, uniformly for x in compact sets; it follows that $\lim_{y\downarrow 0} G(y) = g_0$, and the proof of (10.29) is complete.

Proof of (10.30): Just evaluate (C.6) at $x = G(y)$, so that $\mathcal{H}_3(G(y); y) = h_1(F(y))$ and $\mathcal{H}_4(G(y); y) = h_2(F(y))$, to obtain

$$\text{(C.8)} \quad \begin{aligned} (L_x + L_y)(G(y), y) &= \sqrt{2\alpha} \left[h_2(\xi) - h_1(\xi) e^{\xi\sqrt{2\alpha}} \right] + h_3(x) e^{\xi\sqrt{2\alpha}} - h_4(x) \Big|_{\xi = F(y),\ x = G(y)} \\ &= q(G(y); F(y)). \end{aligned}$$

Analysis of $F'(\cdot)$: Differentiation in $h_1(F(y)) = \mathcal{H}_3(G(y), y)$ using (10.32), (C.2) and (10.24), yields

$$\begin{aligned} h_1'(F(y)) \cdot F'(y) &= G'(y) \cdot \frac{\partial \mathcal{H}_3}{\partial x}(G(y), y) + \frac{\partial \mathcal{H}_3}{\partial y}(G(y), y) \\ &= \left[G'(y) - 1 \right] \cdot \frac{\partial \mathcal{H}_3}{\partial x}(G(y), y) + \left(\frac{\partial \mathcal{H}_3}{\partial x} + \frac{\partial \mathcal{H}_3}{\partial y} \right)(G(y), y) \\ &= \frac{q(G(y); F(y))}{e^{2G(y)\sqrt{2\alpha}} - e^{2F(y)\sqrt{2\alpha}}} + h_3(G(y)) - \sqrt{2\alpha} \cdot \mathcal{H}_3(G(y), y). \end{aligned}$$

Letting $y \downarrow 0$ gives $\lim_{y\downarrow 0} F'(y) = \frac{h_3(g_0)}{h_1'(f_0)} = f'(0)$, in conjunction with (8.27) at $y = 0$. For the limit calculation, use $\mathcal{H}_3(g_0, 0) = h_1(f_0) = 0$ and $q(g_0; f_0) = 0$. Further manipulation gives

$$\text{(C.9)} \quad F'(y) h_1'(F(y)) \cdot \left(e^{2G(y)\sqrt{2\alpha}} - e^{2F(y)\sqrt{2\alpha}} \right) = \frac{2\lambda}{\alpha} \left(y + \frac{\alpha}{2\lambda} - G(y) \right) e^{G(y)\sqrt{2\alpha}} + \sqrt{2\alpha} h_2(f_0) e^{y\sqrt{2\alpha}}.$$

The right-hand side is negative, because $h_2(f_0) < 0$ and $G(y) \geq y + \frac{\alpha}{2\lambda}$. Since $h_1'(\cdot) > 0$ and $e^{2G(y)\sqrt{2\alpha}} - e^{2F(y)\sqrt{2\alpha}} > 0$, it follows that $F'(y) < 0$ for $0 \leq y \leq \bar{y}$.

Calculation of $G'(0)$: The function $L(\cdot, \cdot)$ has a saddle point at $(x, y) = (g_0, 0)$, and an analysis by Taylor expansion shows that $G'(0)$ exists and

(C.10)
$$G'(0) = -\frac{L_{yy}}{2L_{xy}}(g_0, 0).$$

To calculate $G'(0)$, it is actually easier to use the equation (10.13) and the linear approximation

$$q(G(y); F(y)) = y\left[q_x(g_0; f_0)G'(0) + q_y(g_0; f_0)f'(0)\right] + o(y).$$

Observe also from (8.25), (C.4) that

(C.11)
$$\frac{\partial L}{\partial x}(G(y); F(y)) = y \cdot q_x(G; F) + \frac{\lambda}{\sqrt{2\alpha}} y^2 e^{G\sqrt{2\alpha}}\left[1 - e^{2\sqrt{2\alpha}(F-G)}\right]\Big|_{F=F(y),\ G=G(y)}.$$

By substituting these in (10.13), recalling (C.4), and taking the limit as $y \downarrow 0$, we obtain

(C.12)
$$G'(0+) \triangleq \lim_{y \downarrow 0} G'(y) = 1 - G'(0) + g'(0),$$

where we have used $g'(0) = -\left(q_y/q_x\right)(g_0; f_0) \cdot f'(0)$ from (B.1), (B.2). From (C.12) we have that $G \in \mathcal{C}^1([0, Y])$, $G'(0+) = G'(0)$ and, solving (C.12) for $G'(0)$, we get $G'(0) = (1 + g'(0))/2$.

Proof of Lemma 10.8: By choice of $F(\cdot)$ and $G(\cdot)$, the function Q and its derivative Q_x are continuous across both boundaries $\{(x, y) \,/\, x = F(y)\}$ and $\{(x, y) \,/\, x = G(y)\}$. Thus, it remains to check the continuity of the derivative Q_y or, equivalently, of the directional derivative

(C.13)
$$U(x, y) \triangleq \left(\frac{\partial Q}{\partial x} + \frac{\partial Q}{\partial y}\right)(x, y).$$

Since $U(x, y) = 2\delta x$ for $x < F(y)$, and $U(x, y) = 1$ for $x > G(y)$, it is necessary to calculate $U(x, y)$ for $F(y) < x < G(y)$ and show continuity at $x = F(y)$ and $x = G(y)$. From (10.5)

$$Q(x, y) = \frac{\lambda}{\alpha}x^2 + \frac{\lambda}{\alpha^2} + \mathcal{H}_3(G(y), y)e^{x\sqrt{2\alpha}} + \mathcal{H}_4(G(y), y)e^{-x\sqrt{2\alpha}}.$$

Now from (10.24) and (C.1), we obtain

$$\begin{aligned}
\frac{d}{dy}\mathcal{H}_3(G(y); y) &= \frac{\partial \mathcal{H}_3}{\partial x}(G(y); y) \cdot G'(y) + \frac{\partial \mathcal{H}_3}{\partial y}(G(y); y) \\
&= \frac{\partial \mathcal{H}_3}{\partial x}(G(y); y) \cdot \big(G'(y) - 1\big) - \sqrt{2\alpha} \cdot \mathcal{H}_3(G(y); y) + h_3(G(y)), \\
\frac{d}{dy}\mathcal{H}_4(G(y); y) &= \frac{\partial \mathcal{H}_4}{\partial x}(G(y); y) \cdot \big(G'(y) - 1\big) + \sqrt{2\alpha} \cdot \mathcal{H}_4(G(y); y) - h_4(G(y)) \\
&= e^{2G(y)\sqrt{2\alpha}}\frac{\partial \mathcal{H}_3}{\partial x}(G(y); y) \cdot \big(G'(y) - 1\big) + \sqrt{2\alpha} \cdot \mathcal{H}_4(G(y); y) - h_4(G(y)).
\end{aligned}$$

Straightforward differentiation and substitution, with help from (10.32) and (C.2), lead to the expression

$$
\text{(C.14)} \qquad \begin{aligned} U(x,y) &= \frac{2\lambda x}{\alpha} + h_3(G(y))e^{x\sqrt{2\alpha}} - h_4(G(y))e^{-x\sqrt{2\alpha}} \\ &\qquad + \big(G'(y)-1\big)\,\frac{\partial \mathcal{H}_3}{\partial x}(G(y);y)e^{x\sqrt{2\alpha}}\left[1 - e^{2\sqrt{2\alpha}(G(y)-x)}\right] \\ &= \frac{2\lambda x}{\alpha} + h_3(G(y))e^{x\sqrt{2\alpha}} - h_4(G(y))e^{-x\sqrt{2\alpha}} \\ &\qquad + q(G(y);F(y))\cdot e^{x\sqrt{2\alpha}}\,\frac{1-e^{2\sqrt{2\alpha}(G(y)-x)}}{e^{2G(y)\sqrt{2\alpha}} - e^{2F(y)\sqrt{2\alpha}}}. \end{aligned}
$$

Direct calculation shows $U(G(y);y) = 1$ by using

$$
h_3(x)e^{x\sqrt{2\alpha}} - h_4(x)e^{-x\sqrt{2\alpha}} = 1 - \frac{2\lambda}{\alpha}x,
$$

and $U(F(y);y) = 2\delta F(y)$ (observe $e^{-F\sqrt{2\alpha}}q(G;F) = \frac{2\lambda F}{\alpha} - 2\delta F + h_3(G)e^{F\sqrt{2\alpha}} - h_4(G)e^{-F\sqrt{2\alpha}}$). Thus U is continuous at both boundaries.

Clearly, (10.34) holds by the construction of Q in (10.5). The proof of (10.33) follows the strategy of the proof of (8.35), writing $U(x,y)$ in place of $d(x)$. As in that proof, it suffices to show that $x \mapsto \frac{\partial U}{\partial x}(x,y)$ is strictly concave on $(F(y), G(y))$, and $\frac{\partial U}{\partial x}(G(y),y) \geq 0$. Now we can write (C.14) in the form

$$
\text{(C.14)}' \qquad U(x,y) = \frac{2\lambda}{\alpha}x + C(y)e^{x\sqrt{2\alpha}} + D(y)e^{-x\sqrt{2\alpha}}.
$$

By collecting terms in (C.14) – or by noticing that U solves $\frac{1}{2}U_{xx} + 2\lambda x = \alpha U$ in $F(y) < x < G(y)$ with boundary conditions $U(F(y),y) = 2\delta F(y)$, $U(G(y),y) = 1$, and then solving for $C(y)$ and $D(y)$ – one finds $C(y) < 0 < D(y)$. As a consequence, $\frac{\partial U}{\partial x}(x,y)$ is strictly concave.

Finally, further calculation using the definition (8.22) of $q(\cdot,\cdot)$, shows

$$
\text{(C.15)} \qquad \left(e^{2G(y)\sqrt{2\alpha}} - e^{2F(y)\sqrt{2\alpha}}\right) U_x(G(y);y) = 2\sqrt{2\alpha}\cdot e^{G(y)\sqrt{2\alpha}}\,q(G(y);F(y)).
$$

Since $q(G(y);F(y)) > 0$ for $0 < y < \bar{y}$, and $q(G(\bar{y});F(\bar{y})) = 0$, we deduce $U_x(G(y);F(y)) \geq 0$ for $0 < y \leq \bar{y}$. $\square$

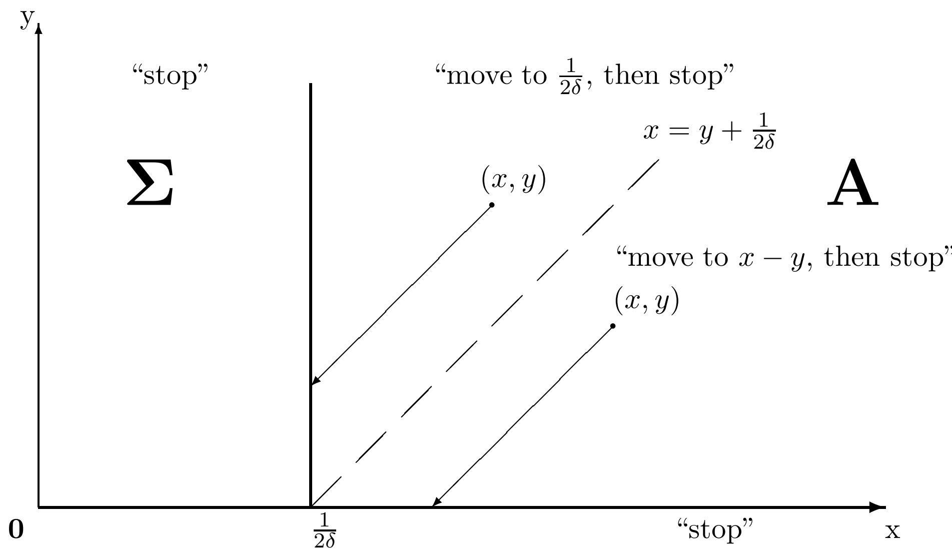

Figure 1: $\lambda \geq \alpha\delta$

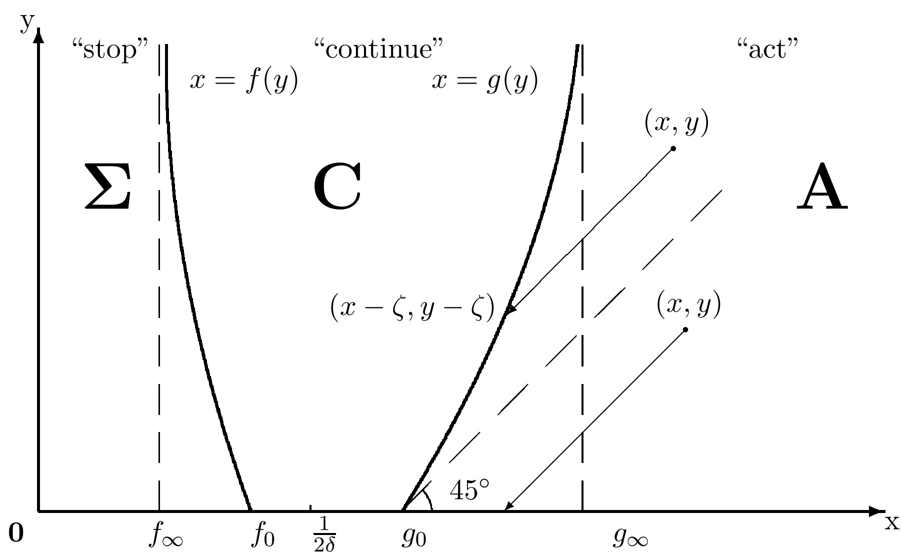

Figure 2: $0 < \lambda \leq \lambda_*$

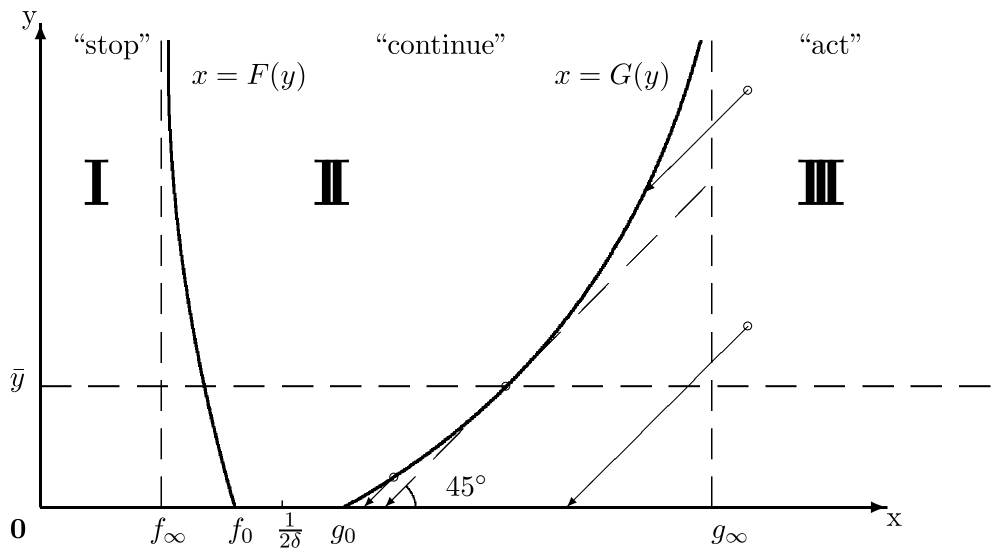

Figure 3: $\lambda_* < \lambda \leq \lambda^*$

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