

# Pacific Journal of Mathematics

**FINITE GROUP ACTION AND EQUIVARIANT BORDISM**

S. S. KHARE

# FINITE GROUP ACTION AND EQUIVARIANT BORDISM

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Conner and Floyd proved that if  $\mathbf{Z}_2^k$  acts on a closed manifold  $M$  differentiably and without any fixed point, then  $M$  is a boundary. Stong gave a stronger result proving that if  $(M, \theta)$  is a closed  $\mathbf{Z}_2^k$ -differential manifold with no stationary point, then  $(M, \theta)$  is a  $\mathbf{Z}_2^k$ -boundary. In the present note, we discuss this problem for a finite group in detail. Let  $G$  be a finite group. By the 2-central component  $G_2(C)$  of  $G$ , we will mean the subgroup of  $G$  consisting of the identity element and all the elements of order 2 in the center of  $G$ . We prove in this note that the fixed data of the 2-central component  $G_2(C)$  of  $G$  determines  $G$ -bordism.

**1. Preliminaries.** Throughout the note we will take  $G$  to be a finite group. By a  $G$ -manifold we will mean a differential compact manifold with a differential action of  $G$  on it. A family  $\mathcal{F}$  in  $G$  is a collection of subgroups of  $G$  such that if  $H \in \mathcal{F}$ , then all the subgroups of  $H$  and all the conjugates of  $H$  are in  $\mathcal{F}$ . Let  $\mathcal{F}' \subset \mathcal{F}$  be families in  $G$  such that  $\exists$  a central element  $a$  in  $G$  of order 2 such that

- (i)  $a \notin H, \forall H \in \mathcal{F} - \mathcal{F}'$
- (ii)  $H \in \mathcal{F}' \Rightarrow [H \cup \{a\}] \in \mathcal{F}'$

(iii) The intersection  $S$  of all members of  $\mathcal{F} - \mathcal{F}'$  is in  $\mathcal{F} - \mathcal{F}'$ . We call such a pair  $(\mathcal{F}, \mathcal{F}')$  of families an admissible pair of families in  $G$  with respect to  $a \in G$ .

**EXAMPLE 2.1.** Let  $G$  be a finite group. We can write the 2-central component  $G_2(C)$  as  $\mathbf{Z}_2^r = [t_1, \dots, t_r]$ , where  $t_1, \dots, t_r$  are generators of  $\mathbf{Z}_2^r$  with  $t_i^2 =$  the identity element and  $t_i t_j = t_j t_i$ . Let  $\mathcal{F}_k$  be the family of all subgroups of  $G$  not containing  $\mathbf{Z}_2^k$ ,  $0 < k \leq r$ , where  $\mathbf{Z}_2^k$  denotes the subgroup of  $G$  generated by the first  $k$  generators  $t_1, \dots, t_k$ . Then  $(\mathcal{F}_{k+1}, \mathcal{F}_k)$  is an admissible pair with respect to  $t_{k+1}$ ,  $0 < k < r$ .

**2. Stationary point free action of  $G_2(C)$  and  $G$ -bordism.** The object of this section is to show that if  $(M, \theta)$  is a  $G$ -manifold with the stationary point free action of  $G_2(C)$  then  $(M, \theta)$  is  $G$ -boundary. Following the notation of Stong [2], let  $\mathfrak{N}_*(G; \mathcal{F}, \mathcal{F}')$  denote the  $(\mathcal{F}, \mathcal{F}')$ -free  $G$ -bordism group for a pair  $(\mathcal{F}, \mathcal{F}')$  of families in  $G$ . For a given family  $\mathcal{F}$

in  $G$  and an element  $g$  in  $G$ , let  $\mathcal{F}_g$  denote the smallest family in  $G$  consisting of all subgroups  $[H \cup \{g\}]$ ,  $H \in \mathcal{F}$ .

**THEOREM 3.1.** *If  $(\mathcal{F}, \mathcal{F}')$  is an admissible pair of families in  $G$  with respect to  $a$  in  $G$ , then an  $(\mathcal{F}, \mathcal{F}')$ -free element in  $\mathfrak{N}_*(G, \mathcal{F}, \mathcal{F}')$  is zero in  $\mathfrak{N}_*(G; \mathcal{F}_a, \mathcal{F}'_a)$ .*

*Proof.* Let  $[M, \theta]$  be in  $\mathfrak{N}_*(G, \mathcal{F}, \mathcal{F}')$ . Let  $F$  denote the fixed points set of  $S$  in  $M$ ,  $S$  being the intersection of all the members of  $\mathcal{F} - \mathcal{F}'$ . Since  $\mathcal{F} - \mathcal{F}'$  is invariant under conjugation,  $S$  is normal in  $G$  and hence the action  $\theta$  on  $M$  induces an action on  $F$  which we denote once again by  $\theta$ . Let  $\nu$  be the normal bundle of the imbedding of  $F$  in the interior of  $M$  and  $D(\nu)$  be its disc bundle with the action  $\theta^*$  of  $G$  on  $D(\nu)$  induced by the real vector bundle maps covering the action  $\theta$  on  $F$ . Since  $F$  is fixed point set of  $S$ ,  $a \notin H$ ,  $\forall H \in \mathcal{F} - \mathcal{F}'$  and no point of  $F$  is fixed by the subgroup  $[S \cup \{a\}]$  generated by  $S \cup \{a\}$ ,  $a$  will act freely on  $F$  and hence on  $D(\nu)$ . Let  $F' = F/[a]$  and  $D'(\nu) = D(\nu)/[a]$ . Since  $a$  is central the actions  $\theta$  and  $\theta^*$  on  $F$  and  $D(\nu)$  induce actions  $\theta'$  and  $\theta'^*$  on  $F'$  and  $D'(\nu)$  respectively. Let  $C_1$  and  $C_2$  be the mapping cylinders of the equivariant double covers  $q_1: F \rightarrow F'$  and  $q_2: D(\nu) \rightarrow D'(\nu)$  respectively and  $\psi_1$  and  $\psi_2$  be the induced actions on  $C_1$  and  $C_2$  respectively. We have the following commutative diagram

$$\begin{array}{ccc} C_2 & \rightarrow & D'(\nu) \\ \downarrow \alpha & & \downarrow \nu' \\ C_1 & \rightarrow & F' \end{array},$$

where  $\alpha: C_2 \rightarrow C_1$  is the map induced from  $\nu': D'(\nu) \rightarrow F'$  by going to mapping cylinders. Clearly  $\partial C_1$  is homeomorphic to  $F$ ,  $\alpha^{-1}(\partial C_1)$  is homeomorphic to  $D(\nu)$  and the action  $\psi_1$  on  $\alpha^{-1}(\partial C_1)$  is isomorphic to the action  $\theta^*$  on  $D(\nu)$ . Consider

$$W = (M \times [0, 1]) \cup C_2 / \sim,$$

where  $\sim$  is the equivalence relation in  $W$  obtained by identifying  $D(\nu) \times \{1\}$  with  $\alpha^{-1}(\partial C_1)$ . Let the action  $\phi$  of  $G$  on  $W$  be given by  $\phi|_{M \times [0, 1]} = \theta \times 1$  and  $\phi|_{C_2} = \psi_1$ . Take  $V$  to be  $(\partial M \times [0, 1]) \cup (M \times \{1\} - (D(\nu) \times \{1\})^\circ) \cup (\partial C_2 - (\alpha^{-1}(\partial C_1))^\circ)$ , where  $^\circ$  denotes the interior operator. Since  $S$  is the intersection of all the members of  $\mathcal{F} - \mathcal{F}'$ ,  $V$  will be  $(\mathcal{F}'_a, \mathcal{F}'_a)$ -free. Also  $W$  is  $(\mathcal{F}'_a, \mathcal{F}'_a)$ -free and  $\partial W$  is homeomorphic to  $M \cup V$  by identifying  $\partial V$  with  $\partial M$ . This shows that  $[M, \theta]$  is zero in  $\mathfrak{N}_*(G; \mathcal{F}_a, \mathcal{F}'_a)$ .  $\square$

Let  $\mathfrak{A}$  denote the family of all subgroups of  $G$  and  $\mathcal{F}_0$  denote the empty family. Then following the notations of Example 2.1 and using the above Theorem, one immediately gets the following.

**COROLLARY 3.2.** *For every  $k$ ,  $0 \leq k < r$ , the homomorphism  $\mathfrak{N}_*(G; \mathcal{F}_{k+1}, \mathcal{F}_k) \rightarrow \mathfrak{N}_*(G; \mathfrak{A}, \mathcal{F}_k)$  induced from the inclusion map  $(\mathcal{F}_{k+1}, \mathcal{F}_k) \rightarrow (\mathfrak{A}, \mathcal{F}_k)$  is zero.*

*Proof.* Since  $(\mathcal{F}_{k+1}, \mathcal{F}_k)$  is admissible pair of families with respect to  $t_{k+1}$  for  $0 \leq k < r$  and no point of the submanifold  $V$  in the above construction is fixed by  $\mathbf{Z}_2^k$ . Theorem 3.1 gives the Corollary immediately.  $\square$

**COROLLARY 3.3.** *Let  $\mathbf{P}$  be the family of all subgroups of  $G$  which do not contain  $G_2(C)$ . Then the homomorphism  $\mathfrak{N}_*(G; \mathbf{P}) \rightarrow \mathfrak{N}_*(G; \mathfrak{A})$  induced from the inclusion map  $\mathbf{P} \rightarrow \mathfrak{A}$  is the zero homomorphism.*

*Proof.* By Corollary 3.2, one gets that

$$\mathfrak{N}_*(G; \mathcal{F}_{k+1}, \mathcal{F}_k) \xrightarrow{i_*} \mathfrak{N}_*(G; \mathfrak{A}, \mathcal{F}_k)$$

is the zero homomorphism,  $0 \leq k < r$ . Consider the exact bordism sequence for the triple

$$\begin{aligned} (\mathfrak{A}, \mathcal{F}_{k+1}, \mathcal{F}_k) &\rightarrow \cdots \rightarrow \mathfrak{N}_*(G; \mathcal{F}_{k+1}, \mathcal{F}_k) \xrightarrow{i_*} \mathfrak{N}_*(G; \mathfrak{A}, \mathcal{F}_k) \\ &\xrightarrow{i_*} \mathfrak{N}_*(G; \mathfrak{A}, \mathcal{F}_{k+1}) \rightarrow \cdots \end{aligned}$$

where  $j_*$  is the homomorphism induced from the inclusion  $j: (\mathfrak{A}, \mathcal{F}_k) \rightarrow (\mathfrak{A}, \mathcal{F}_{k+1})$ . Since  $i_*$  is the zero homomorphism,  $j_*$  will be a monomorphism. Therefore the composite

$$\mathfrak{N}_*(G; \mathfrak{A}, \mathcal{F}_0) \rightarrow \mathfrak{N}_*(G; \mathfrak{A}, \mathcal{F}_1) \rightarrow \cdots \rightarrow \mathfrak{N}_*(G; \mathfrak{A}, \mathcal{F}_r)$$

is a monomorphism and hence by the exact bordism sequence of the triple  $(\mathfrak{A}, \mathcal{F}_r, \mathcal{F}_0)$ , one get that  $\mathfrak{N}_*(G; \mathcal{F}_r, \mathcal{F}_0) \rightarrow \mathfrak{N}_*(G; \mathfrak{A}, \mathcal{F}_0)$  is the zero homomorphism. This completes the proof since  $\mathcal{F}_r = \mathbf{P}$  and  $\mathcal{F}_0 = \emptyset$ .  $\square$

**COROLLARY 3.4.** *If  $G_2(C)$  acts on  $M$  under  $\theta$  without any stationary point then  $(M, \theta)$  is a  $G$ -boundary.*

**3. The stationary points set  $F_{G_2(C)}$  and the normal bundle.** In the last section we dealt with the case when  $F_{G_2(C)}$  is empty. In this section we consider the case when  $F_{G_2(C)} \neq \emptyset$ . For this we introduce the concept of

equivariant trivial normal bundle and use this concept to settle the case  $F_{G_2(C)} \neq \emptyset$  in the form of Theorem 4.2.

Let  $(M^n, \theta)$  be a closed  $G$ -manifold. Consider the decomposition of  $F = F_{G_2(C)}(M^n)$  as  $F = \bigcup_{l=0}^n F^l$ , where  $F^l$  denotes the  $l$ -dimensional component of  $F$ . Let  $\mathcal{D}(\nu_l)$  be the normal disc bundle of  $F^l$  in  $M^n$  with the induced action  $\theta_l$  of  $G$  on  $\mathcal{D}(\nu_l)$ .

**DEFINITION 4.1.**  $F$  is said to have an equivariant trivial normal bundle in  $M^n$ , if  $G/G_2(C)$  acts trivially on  $F$  and  $\exists$  some positive dimensional  $G$ -representations  $(W_l, \phi_l)$ ,  $0 \leq l \leq n$ , such that in  $\mathfrak{R}_*(G; \mathfrak{A}, \mathbf{P})$

$$[D(\nu_l), \theta_l] = [F^l][D(W_l), \phi_l],$$

$D(W_l)$  being the unit disc of  $W_l$ .

Let  $\{V_k, \psi_k\}_{1 \leq k \leq m}$  be the finite set of all irreducible representations of  $G$ . Let  $\mathbf{Z}^+$  be the set of all non-negative integers. Then any  $G$ -representation can be written as  $(V(f), \psi(f))$  for some map  $f: \{1, \dots, m\} \rightarrow \mathbf{Z}^+$  where  $V(f) = \bigoplus_{k=1}^m (V_k, \psi_k)^{f(k)}$ ,  $(V_k, \psi_k)^{f(k)}$  being the direct sum of  $f(k)$  copies of  $(V_k, \psi_k)$ . Let us denote the unit disc and the unit sphere of  $V(f)$  by  $D(f)$  and  $S(f)$ .

**THEOREM 4.2.** *If  $F$  has an equivariant trivial normal bundle in  $M^n$ , then  $F$  is a boundary and  $(M^n, \theta)$  is a  $G$ -boundary.*

*Proof.* Since  $F$  has an equivariant trivial normal bundle in  $M^n$ , we have

$$[\mathcal{D}(\nu_l), \theta_l] = [F^l][D(W_l), \theta_l]$$

for some positive dimensional  $G$ -representations  $(W_l, \phi_l)$ ,  $0 \leq l \leq n$ . Also  $(W_l, \phi_l) = (V(f_l), \psi(f_l))$  for some map  $f_l: \{1, \dots, m\} \rightarrow \mathbf{Z}^+$ . Therefore

$$[\mathcal{D}(\nu_l), \theta_l] = [F^l][D(f_l), \psi(f_l)].$$

Let  $i_*: \mathfrak{R}_*(G; \mathfrak{A}) \rightarrow \mathfrak{R}_*(G; \mathfrak{A}, \mathbf{P})$  be the homomorphism induced by the inclusion map  $i: (\mathfrak{A}, \phi) \rightarrow (\mathfrak{A}, \mathbf{P})$ . Then

$$i_*[M^n, \theta] = \sum_{l=0}^n [\mathcal{D}(\nu_l), \theta_l] = \sum_{l=0}^n [F^l][D(f_l), \psi(f_l)].$$

Therefore

$$\partial_* i_*[M^n, \theta] = \sum_{l=0}^n [F^l][S(f_l), \psi(f_l)] = 0$$

in  $\mathfrak{N}_*(G; \mathbf{P})$ , where  $\partial_*: \mathfrak{N}_*(G; \mathfrak{A}, \mathbf{P}) \rightarrow \mathfrak{N}_*(G; \mathbf{P})$  is the boundary homomorphism. Therefore  $\exists$  a  $\mathbf{P}$ -free  $G$ -manifold  $(D, \eta)$  such that

$$(1) \quad (\partial D, \eta) = \bigcup_{l=0}^n (F^l \times (S(f_l), \psi(f_l))).$$

Since  $(W_l, \phi_l)$  is positive dimensional  $G$ -representation,  $\forall l, \exists$  a member  $k(l)$  in the set  $\{1, \dots, m\}$  such that  $f_l(k(l)) \neq 0$ . Consider the irreducible  $G$ -representation  $(V_{k(l)}, \psi_{k(l)})$ . Let  $(\tilde{V}_{k(l)}, \tilde{\psi}_{k(l)})$  be an irreducible component of the  $G_2(C)$ -representation induced by the  $G$ -representation  $(V_{k(l)}, \psi_{k(l)})$ . Then  $\exists$  a subgroup  $H_{k(l)}$  of  $G$  isomorphic to  $\mathbf{Z}_2^{r-1}$  which fixes  $\tilde{V}_{k(l)}$ ,  $G_2(C)$  being  $\mathbf{Z}_2^r$ . Let us fix some  $\beta, 0 \leq \beta \leq n$ .

From the equation (1), we get

$$F_{H_{k(\beta)}}(\partial D, \eta) = F_{H_{k(\beta)}} \left( \bigcup_{l=0}^n (F^l \times (S(f_l), \psi(f_l))) \right).$$

Let  $F_{H_{k(\beta)}}(D) = F^*$  and  $\mathbf{Z}_{2,\beta} \approx \mathbf{Z}_2$  be the complement of  $H_{k(\beta)}$  in  $G_2(C) = \mathbf{Z}_2^r$ . Then one gets

$$(\partial F^*, \eta |_{\mathbf{Z}_{2,\beta}}) = \bigcup_{l=0}^n (F^l \times (S^{\Delta(l,\beta)-1}, a)),$$

where  $a$  is the antipodal involution and the integer  $\Delta(l, \beta)$  is the nonnegative integer depending on  $l$  and  $\beta$ . Since  $H_{k(\beta)}$  fixed  $\tilde{V}_{k(\beta)}$  and  $f_\beta(k(\beta)) \neq 0$ , one infers that  $\Delta(\beta, \beta) \geq 1$ . Since  $D$  is  $\mathbf{P}$ -free,  $\mathbf{Z}_{2,\beta}$  will act freely on  $F^*$  and therefore  $[\partial F^*, \eta |_{\mathbf{Z}_{2,\beta}}]$  is zero in  $\mathfrak{N}_*(\mathbf{Z}_{2,\beta}; \mathcal{F}_1)$ ,  $\mathcal{F}_1$  being the family consisting of only trivial subgroup of  $\mathbf{Z}_{2,\beta}$ . This gives

$$\sum_{l=0}^n [F^l] [S^{\Delta(l,\beta)-1}, a] = 0$$

in  $\mathfrak{N}_*(\mathbf{Z}_{2,\beta}; \mathcal{F}_1)$ . But  $\mathfrak{N}_*(\mathbf{Z}_{2,\beta}; \mathcal{F}_1)$  is free  $\mathfrak{N}_*$ -module with a set  $\{[S^n, a], n \in \mathbf{Z}^+\}$  of generators. This together with the fact that  $\Delta(\beta, \beta) \geq 1$  gives  $[F^\beta] = 0$  in  $\mathfrak{N}_*$ . By varying  $\beta$ , one gets  $[F^\beta] = 0, \forall \beta = 0, \dots, n$ . Hence  $[F] = 0$  in  $\mathfrak{N}_*$ . Therefore

$$i_*[M^n, \theta] = \sum_{l=0}^n [F^l] [D(f_l), \psi(f_l)] = 0 \quad \text{in } \mathfrak{N}_*(G; \mathfrak{A}, \mathbf{P}).$$

But from Corollary 3.3, one infers that  $i_*: \mathfrak{N}_*(G, \mathfrak{A}) \rightarrow \mathfrak{N}_*(G; \mathfrak{A}, \mathbf{P})$  is an injection. Therefore  $[M^n, \theta]$  is zero in  $\mathfrak{N}_*(G; \mathfrak{A})$ .  $\square$

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