PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 90, Number 3, March 1984

FINITE GROUP ACTIONS AND NONSEPARATING 2-SPHERES[1]

STEVEN P. PLOTNICK

ABSTRACT. We extend the splitting theorem of Meeks-Yau for finite group actions on three-manifolds to include manifolds containing nonseparating 2-spheres, and give applications to branched covers of links.

0. Introduction. The purpose of this note is to point that the splitting theorem for finite group actions on three-manifolds of Meeks-Yau [**4**, Theorem 9] can be extended in an appropriate manner to include manifolds with $S^1 \times S^2$ summands. We then give applications to branched covers of links, proving several geometrically "obvious" result.

The splitting theorem basically says that a finite group action on a compact, orientable 3-manifold with no $S^1 \times S^2$ summands splits, modulo permuting homeomorphic summands, as the equivariant connected sum of the actions on the irreducible summands. (As stated in [**4**], homotopy sphere summands are not permitted, but this restriction is no longer necessary. See §1.) This is false when $S^1 \times S^2$ summands are present—even for involutions—as one can see in [**1** and **2**]. In [**2**], however, Kim and Tollefson show that all involutions can be built up from involutions on irreducible 3-manifolds in a simple fashion by finding a suitable collection of 2-spheres along which an involution can be split into simpler actions. This is possible to do by cut and paste methods essentially because intersections arising from an involution are easily visualized. In a similar manner, Gordon and Litherland [**1**] were able to prove a $\mathbf{Z}_2$-equivariant loop theorem. For actions of more general groups, these methods get hopelessly complicated. However, minimal surface methods enables Meeks and Yau to overcome these problems and find invariant collections of spheres and discs. We show below that the presence of nonseparating 2-spheres offers no real problem, and that the results of Kim and Tollefson can be generalized to arbitrary groups.

All manifolds will be oriented, and all actions smooth.

I should like to thank the referee for several useful suggestions.

1. Splitting actions. Let G be a finite group. The most direct way to go from G-actions on irreducible 3-manifolds to G-actions on arbitrary 3-manifolds seems to be the following: Let $X_1, \ldots, X_n$ be orientable, irreducible 3-manifolds, and suppose

Received by the editors November 2, 1981. Presented at the special session on 3-manifolds, 1981 summer meeting of the AMS, Pittsburgh.

1980 *Mathematics Subject Classification*. Primary 57S17; Secondary 57M12.

[1]This material is based upon work partially supported by the National Science Foundation under Grant No. MCS-8002730.

0002-9939/84 $1.00 + $.25 per page

G acts on their disjoint union. Denote the stabilizer of X_i by G_i. In X_i, pick a finite G_i-invariant set $\{x_{i,1},\ldots,x_{i,n_i}\}$ of points, and small balls $B_{i,j}$ about the $x_{i,j}$ so that $\bigcup_{j=1}^{n_i} B_{i,j}$ is also G_i-invariant. Denote the stabilizer of $B_{i,j}$ by $G_{i,j}$. Use the G-action to translate these points and balls to the other X_k in the orbit of X_i. Do this once for each orbit.

Now let $\tau\colon \{x_{i,j}\} \hookleftarrow$ be an equivariant, fixed point free involution, so that $G_{i,j} = G_{\tau(i,j)}$ and the (linear) actions of $G_{i,j}$ on $B_{i,j}$ and $B_{\tau(i,j)}$ are equivalent, where $\tau(x_{i,j}) = x_{\tau(i,j)}$. Remove $\bigcup_{i,j} \mathring{B}_{i,j}$ and glue $\partial B_{i,j}$ to $\partial B_{\tau(i,j)}$ by an equivariant, orientation-reversing diffeomorphism. If, for some i, j, there exists $g \in G$ with $gx_{i,j} = \tau(x_{i,j})$, we also require that $g(\tau(x_{i,j})) = x_{i,j}$ and that the glueing can be chosen to commute with the action of g. (This leads to invariant spheres whose sides are interchanged by certain group elements.) It is clear that we can do the above so that the result, say X, carries a G-action.

Notice that if X contains no nonseparating 2-spheres, this description reduces to that of Meeks and Yau. Keep in mind that some X_i can be S^3. Also, the G-action on X depends strongly on the choice of glueing maps, as is easily seen from elementary examples.

THEOREM. *All finite group actions on compact, orientable* 3*-manifolds arise via the above construction.*

PROOF. Write $X = \#Y_i \# (\#_1^m S^1 \times S^2)$, where each Y_i is irreducible. According to [**4**, p. 480], there exists a collection $\Gamma = \{S_1,\ldots,S_k\}$ of embedded, disjoint 2-spheres in X which generate $\pi_2(X)$ as a $\pi_1(X)$-module, and $G \cdot \Gamma = \Gamma$. According to [**3**], we can also assume that any fake 3-ball in X is split off by one of the S_i.

LEMMA. *Splitting X along Γ disconnects X into a collection of manifolds where every* 2*-sphere separates, i.e. Γ cuts off all handles.*

PROOF. The Hurewicz map $\rho\colon \pi_2(X) \to H_2(X)$ has image $\oplus_1^m \mathbf{Z}$ given by the nonseparating 2-spheres in $\#_1^m S^1 \times S^2$. Since Γ contains a π_1-basis of $\pi_2(X)$, $\rho(\Gamma)$ contains a $\mathbf{Z}$-basis of $H_2(X)$.

With this description of $X - \Gamma$, we see that Assertion 2 of [**4**, p. 480] is still valid. Thus, splitting X open along Γ yields a collection of irreducible manifolds minus open balls, on which G acts. Capping these off yields a collection $X_1,\ldots,X_n$ of irreducible manifolds. The reverse process then determines the involution τ and the glueing data described above.

2. Branched covers of links. Suppose that X is the k-fold cyclic branched cover of a link L in S^3.

PROPOSITION. *If X contains a nonseparating* 2*-sphere, then L is geometrically split.*

PROOF. As we have seen, there exists an invariant family $\mathbf{Z}_k \cdot S^2$ of nonseparating 2-spheres. Splitting X along these yields components $X_1,\ldots,X_n$, with $n < k$. Since the action is semifree, each component is actually $\mathbf{Z}_k$-invariant. (Otherwise, the stabilizer G_i of a component X_i would act freely, and X_i/G_i would live in the quotient space, with nontrivial fundamental group.) We claim that the $\mathbf{Z}_k$ action on

X_i must freely permute the boundary spheres. If not, one of them is either invariant under $\mathbf{Z}_k$ (rotation with two fixed points) or invariant under $\mathbf{Z}_2 \subset \mathbf{Z}_k$, acting freely. In the first case, we can find a simple closed curve in X intersecting this sphere once. The image of this curve in $X/\mathbf{Z}_k = S^3$ intersects the image of this sphere once, a contradiction. The second case leads to a connected sum with $\mathbf{R}P^3$ in the orbit space, again a contradiction. Thus, the boundary spheres are freely permuted, so that $n = 2$. The orbit $\mathbf{Z}_k \cdot S^2$ goes down to a sphere which splits the link, proving the proposition.

Therefore, it suffices to study branched covers with no nonseparating 2-spheres.

PROPOSITION. *Suppose X is the k-fold cyclic branched cover of a link $L \subset S^3$, and X contains no $S^1 \times S^2$ summands. If X splits as a connected sum $\#_1^n X_i$, there is a corresponding splitting of L as $\#_1^n L_i$, with X_i the k-fold cyclic branched cover of L_i.*

PROOF. From Meeks and Yau, we know that the action splits, up to permuting factors. However, since the action is semifree, any permuting of summands must be a free permutation, so that the summand in question goes down homeomorphically to the orbit space, a contradiction. Thus, the action splits, and the result follows.

COROLLARY (see [2, Corollary 4] for the case of involutions). *A nonsplit link in S^3 is prime $\Leftrightarrow$ all cyclic branched covers are prime $\Leftrightarrow$ one cyclic branched cover is prime.*

COROLLARY. $\#_1^{(n-1)(k-1)} S^1 \times S^2$ *arises as the k-fold cyclic branched cover of only the trivial link of n components.*

PROOF. Induction and the Smith Conjecture.

REFERENCES

1. C. McA. Gordon and R. A. Litherland, *Incompressible surfaces in branched coverings*, Sympos. Smith Conjecture, Columbia Univ., 1979 (to appear).

2. P. K. Kim and J. L. Tollefson, *Splitting the PL involutions of non-prime 3-manifolds*, Michigan Math. J. **27** (1980), 259–274.

3. W. H. Meeks III, L. Simon and S. T. Yau, *Embedded minimal surfaces, exotic spheres, and manifolds with positive Ricci curvature*, preprint.

4. W. H. Meeks III and S. T. Yau, *Topology of three dimensional manifolds and the embedding problems in minimal surface theory*, Ann. of Math. (2) **112** (1980), 441–484.

DEPARTMENT OF MATHEMATICS, COLUMBIA UNIVERSITY, NEW YORK, NEW YORK 10027