Tôhoku Math. Journ.
35 (1983), 415–424.

FINITE GROUPS OF POLYNOMIAL AUTOMORPHISMS IN C^n

MIKIO FURUSHIMA

(Received August 9, 1982)

Introduction. Let C^n be an n-dimensional complex Euclidean space. A biholomorphic transformation $g: C^n \to C^n$ of C^n onto C^n is called a polynomial automorphism if g and the inverse g^{-1} are given by n polynomials in n variables. We shall denote by $\mathrm{Aut}(C^n)$ the group of all polynomial automorphisms in C^n. Let X be a projective algebraic compactification of C^n, let $\iota: C^n \to X$ be an inclusion and put $A = X - \iota(C^n)$. Then A is a closed subvariety of X. For simplicity, we shall denote this compactification by $(C^n, \iota, X; A)$. Let us denote by $\mathrm{Aut}(X)$ the group of all birational and biregular automorphisms of X, and define a subgroup $\mathrm{Aut}(X; A)$ of $\mathrm{Aut}(X)$ by $\mathrm{Aut}(X; A) = \{\hat{g} \in \mathrm{Aut}(X); \hat{g}(A) = A\}$. Then we have the following theorem.

THEOREM 1. *Let G be a finite subgroup of* $\mathrm{Aut}(C^n)$. *Then there exist a non-singular projective algebraic compactification* $(C^n, \iota, X; A)$ *and a finite subgroup $\hat{G}$ of* $\mathrm{Aut}(X; A)$ *such that $\iota^{-1} \circ \hat{G} \circ \iota = G$, namely $\{\iota^{-1} \circ \hat{g} \circ \iota; \hat{g} \in \hat{G}\} = G$ on C^n.*

Applying Theorem 1 and Morrow's classification of the minimal normal compactifications of C^2 [13], we shall give an elementary proof of the following theorem which was obtained by Gizatullin-Danilov [4], Miyanishi [12] and Kambayashi [10], independently (see also [3]).

THEOREM 2 ([4], [12], [10]). *Let G be a finite subgroup of* $\mathrm{Aut}(C^2)$. *Then G is conjugate in* $\mathrm{Aut}(C^2)$ *with a finite subgroup of $GL(2, C)$, namely, there exists a polynomial automorphism $\alpha \in \mathrm{Aut}(C^2)$ such that $\alpha \circ G \circ \alpha^{-1}$ is a finite subgroup of $GL(2, C)$.*

REMARK 1. For $n = 2$, Theorem 1 is a special case of the theorem of Gizatullin-Danilov [4, § 6]. For $n \geqq 3$, it seems to be effective in answering the following general question (see § 3).

QUESTION. Let G be a finite subgroup of $\mathrm{Aut}(C^n)$. Then is G conjugate in $\mathrm{Aut}(C^n)$ with a finite subgroup of $GL(n, C)$?

1. Proof of Theorem 1. Let G be a finite subgroup of $\mathrm{Aut}(C^n)$ $(n \geqq 2)$. Let C^n/G be the quotient space of C^n by the group G, and

$\pi: C^n \to C^n/G$ the projection. Since G is a finite group of polynomial automorphisms in C^n, by Cartan [2], C^n/G is a normal affine algebraic variety of dimension n and the projection π is a proper finite regular mapping. Let Y be the normalization of the algebraic closure of C^n/G in some complex projective spece P^N, where $N > 0$ is a sufficiently large integer. Then Y is a normal projective algebraic variety of dimension n. Let $\tau: C^n/G \to Y$ be the natural inclusion and put $B_0 = Y - \tau(C^n/G)$. The triple $R = (C^n, \pi, C^n/G)$ is a branched algebraic covering over C^n/G. Let B_1 be the algebraic closure in Y of the branch locus in C^n/G and put $B = B_0 \cup B_1$. Then B is a closed subvariety of Y. Then the triple $\mathfrak{R}' = (C^n - \pi^{-1}(B), \pi, C^n/G - B)$ is an unbranched covering over $Y - B$ $(= C^n/G - B)$. By Stein [16, Satz 1], there exists a topologically branched finite covering $\mathfrak{R}_0 = (X_0, \pi_0, Y)$ over Y with the following properties:

(i) the branch locus is contained in the set B,

(ii) X_0 contains C^n as an open subset, and

(iii) $\pi_0 | C^n = \pi$.

Further, such a covering $\mathfrak{R}_0$ is uniquely determined up to topological isomorphisms. Since π_0 is a proper finite mapping and Y is compact, X_0 is also compact. Since Y is a normal complex space, by the well-known theorem of Grauert-Remmert [6], we can introduce a normal complex structure on X_0 and the projection π_0 is holomorphic with respect to this complex structure. Since Y is projective algebraic and π_0 is proper finite holomorphic, by Grauert-Remmert [5] (see also Remmert-Stein [15, Satz 8]), so is X_0. Thus, π_0 is a proper finite regular mapping. Let $\iota_0: C^n \to X_0$ be the natural inclusion and put $A_0 = X_0 - \iota_0(C^n)$. Then A_0 is a closed subvariety of X_0.

Let g be an arbitrary element of G. Since $\pi_0 \circ g$ $(= \pi \circ g = \pi)$: $X_0 - A_0 \to Y$ is continued to the regular mapping $\pi_0: X_0 \to Y$ of X_0 into Y, by Stein [16, Hilfssatz 2], g can be uniquely extended to a continuous mapping $g_0: X_0 \to X_0$. By the Riemann extension theorem, g_0 is a holomorphic (therefore regular) mapping of X_0 onto X_0. Similarly, the inverse g^{-1} can be uniquely extended to a regular mapping $g_0^{-1}: X_0 \to X_0$ of X_0 onto X_0, and we have $g_0 \circ g_0^{-1} = \mathrm{id}_{X_0}$. Since $g(C^n) = C^n$, we have $g_0(A_0) = A_0$, namely, $g_0 \in \mathrm{Aut}(X_0; A_0)$, and further we have $\iota_0^{-1} \circ g_0 \circ \iota_0 = g$ on C^n. Thus we have the following:

PROPOSITION 1. *Let G be a finite subgroup of* Aut (C^n). *Then there exist a (not necessarily non-singular) projective algebraic compactification* $(C^n, \iota_0, X_0; A_0)$ *and a finite subgroup G_0 of* Aut $(X_0: A_0)$ *such that* $\iota_0^{-1} \circ G_0 \circ \iota_0 = G$ *on* C^n.

By Hironaka's equivariant resolution theorem [8, §7], there exists a non-singular model $\phi\colon X \to X_0$ of X_0 such that any automorphism $g_0 \in \mathrm{Aut}(X_0)$ can be uniquely extended to an automorphism $\hat{g} \in \mathrm{Aut}(X)$ and satisfies $\phi \circ \hat{g} = g_0 \circ \phi$.

From this theorem and the facts that the singularities of X_0 do not lie on C^n and that $g_0(C^n) = C^n$ for every $g_0 \in G_0$, there exists a finite subgroup $\hat{G}$ of $\mathrm{Aut}(X; A)$, where $A = \phi^{-1}(A_0)$, such that $\phi \circ \hat{G} = G_0 \circ \phi$, that is, for any $g_0 \in G_0$, there exists a unique element $\hat{g} \in \hat{G}$ such that $\phi \circ \hat{g} = g_0 \circ \phi$. Putting $\iota = \phi^{-1} \circ \iota_0\colon C^n \to X$, the proof of Theorem 1 is completed.

2. Proof of Theorem 2. Let G be a finite subgroup of $\mathrm{Aut}(C^2)$. By Theorem 1, there exist a non-singular projective algebraic compactification $(C^2, \iota, X; A)$ and a subgroup $\hat{G}$ of $\mathrm{Aut}(X; A)$ such that $\iota^{-1} \circ \hat{G} \circ \iota = G$. We put $A = \bigcup_{i=1}^{l} A_i$, where each A_i is an irreducible algebraic curve. We need the following two elementary lemmas.

LEMMA 1. *Let M be a two-dimensional complex manifold and $e = \{x_1, \cdots, x_k\}$ a set of finitely many points in M. Let $f\colon M \to M$ be a biholomorphic transformation with $f(e) = e$. Let $Q_e(M)$ be the quadratic transformation of M at the set e, and $\phi\colon Q_e(M) \to M$ the projection. Put $\phi^{-1}(e) = E = \bigcup_{i=1}^{k} E_i$, where $E_i = \phi^{-1}(x_i)$ is an exceptional curve of the first kind. Then there exists a unique biholomorphic transformation $\hat{f}\colon Q_e(M) \to Q_e(M)$ with $\hat{f}(E) = E$ such that $\phi \circ \hat{f} = f \circ \phi$.*

LEMMA 2. *Let $\hat{M}$ be a two-dimensional complex manifold and $E = \bigcup_{i=1}^{k} E_i$ a disjoint union of exceptional curves of the first kind. Let $\hat{g}\colon \hat{M} \to \hat{M}$ be a biholomorphic transformation with $\hat{g}(E) = E$. Let $M = \hat{M}/E$ be the contraction of E, $\psi\colon \hat{M} \to M$ the projection and put $\psi(E) = e = \{x_1, \cdots, x_k\}$. Then there exists a unique biholomorphic transformation $g\colon M \to M$ with $g(e) = e$ such that $\psi \circ g = g \circ \psi$.*

The proof of Lemma 1 is contained in that of the Lemma of Hopf [9] and Lemma 2 follows from the Riemann extension theorem.

Since the singularities of the (reducible) curve A is $\hat{G}$-invariant, blowing up such singularities and using Lemma 1, we may assume that each A_i is non-singular and A_i's cross each other normally if they intersect. Further, we may assume that $(C^2, \iota, X; A)$ is a minimal normal compactification (see Morrow [13]). Indeed, taking account of Morrow's classification of the minimal normal compactifications of C^2 (see also Figure), we see that the irreducible components A_i $(1 \leqq i \leqq k)$ of A with the following properties (i) and (ii) are $\hat{G}$-invariant.

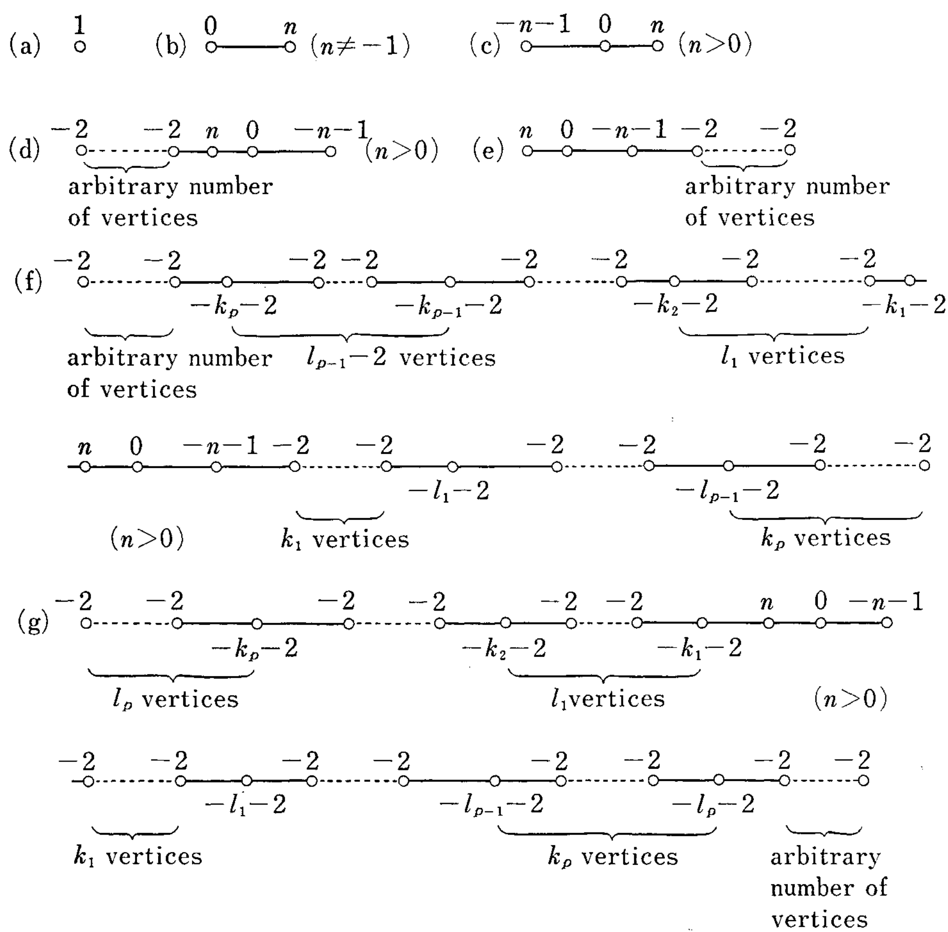
(i) A_i is an exceptional curve of the first kind, and

(ii) the number of irreducible components of A, different from A_i, which intersect A_i is at most two.

Blowing down such irreducible components A_i $(1 \leqq i \leqq k)$ to points, and using Lemma 2 at each step, the above assertion is finally proved. Thus we have the following:

PROPOSITION 2. *Let G be a finite subgroup of* Aut(C^2). *Then there exist a minimal compactification* $(C^2, \iota, X; A)$ *of* C^2 *and a finite subgroup* $\hat{G}$ *of* Aut$(X; A)$ *such that* $\iota^{-1} \circ \hat{G} \circ \iota = G$ *on* C^2.

REMARK 2. We can also prove Proposition 2 without using Hironaka's



FIGURE

equivariant resolution theorem. Indeed, by Proposition 1 and the uniqueness of the minimal resolution of singularities of a two-dimensional complex analytic space (cf. Laufer [11]), we can easily see that there exist a non-singular projective algebraic compactification $(\boldsymbol{C}^2, \iota_0, X_0; A_0)$ of $\boldsymbol{C}^2$ and a finite subgroup $\hat{G}$ of $\mathrm{Aut}\,(X_0; A_0)$ such that $\iota_0^{-1} \circ \hat{G} \circ \iota_0 = G$ on $\boldsymbol{C}^2$. Using Lemmas 1 and 2 repeatedly, we have finally Proposition 2.

Now, by Morrow [13], the types of the graph $\Gamma(A)$ of A $(= \bigcup_{i=1}^{l} A_i)$ are the following, where each vertex of the graph represents a non-singular rational curve A_i, adjacent to which we write the self-intersection number (A_i^2) of A_i. Two vertices are joined by a segment if and only if the two corresponding rational curves intersect each other (see Figure).

(CASE 1). The type of $\Gamma(A)$ is (a). In this case, X is a complex projective plane $\boldsymbol{P}^2$ and $A = X - \iota(\boldsymbol{C}^2)$ is a line L in $\boldsymbol{P}^2$. More precisely, let $(X_i)_{0 \leqq i \leqq 2}$ be homogeneous coordinates in $\boldsymbol{P}^2$. Then $A = X - \iota(\boldsymbol{C}^2) = V(X_0)$.

(CASE 2). The type of $\Gamma(A)$ is (b). In this case, X is a rational ruled surface $\boldsymbol{F}_n$ with the minimal section s_0 whose self-intersection number is $(s_0^2) = -n$ $(n \geqq 0)$. Let s_∞ be a section with $(s_\infty^2) = n$ and l a fiber. Then we have $A = \boldsymbol{F}_n - \iota(\boldsymbol{C}^2) = s_\infty \cup l$.

(CASE 3). The type of $\Gamma(A)$ is one of $(c) \sim (g)$. Let A_0 (resp. A_1, A_2) be the irreducible component of A with $(A_0^2) = 0$ (resp. $(A_1^2) = n$, $(A_2^2) = -n - 1$). Since the self-intersection number is invariant under an automorphism of X, we have $\hat{g}(A_i) = A_i$ $(i = 0, 1)$ for every $\hat{g}$ of $\mathrm{Aut}\,(X; A)$. Since A_0 and A_1 are $\hat{g}$-invariant, so is A_2. Blowing up the intersection point of A_0 and A_1, and blowing down the proper transform of A_0 to a point, we have a new minimal normal compactification $(\boldsymbol{C}^2, \iota_1, X_1; B)$ of $\boldsymbol{C}^2$. It is easily seen that the type of the graph $\Gamma(B)$ of B is the same as that of $\Gamma(A)$ with n replaced by $n - 1$, provided $n \geqq 2$. If $n = 1$, the type changes as follow:

$$(c) \rightarrow (b)\,, \qquad (d) \rightarrow (e)\,, \qquad (e) \rightarrow (b) \quad \text{and} \quad (f) \leftrightarrow (g)\,.$$

Thus repeating this process finitely many times and using Lemmas 1 and 2 at each step, we see finally that every element of $\mathrm{Aut}\,(X; A)$ induces a unique element of $\mathrm{Aut}\,(\boldsymbol{F}_n, s_\infty \cup l)$. More precisely, let $\psi: X \rightarrow \boldsymbol{F}_n$ be the birational mapping obtained by the above process. By the construction, the restriction $\psi | \iota(\boldsymbol{C}^2)$ of ψ to $\iota(\boldsymbol{C}^2) = X - A$ is a one-to-one regular mapping and the mapping $\psi \circ \iota: \boldsymbol{C}^2 \rightarrow \boldsymbol{F}_n$ gives an inclusion. We put $s_\infty \cup l = \boldsymbol{F}_n - \psi \circ \iota(\boldsymbol{C}^2)$. Then there exists a finite subgroup $\hat{G}$ of $\mathrm{Aut}\,(\boldsymbol{F}_n, s_\infty \cup l)$ such that $(\psi \circ \iota)^{-1} \circ \hat{G} \circ (\psi \circ \iota) = G$. Thus we have the following:

PROPOSITION 3. *Let G be a finite subgroup of* $\mathrm{Aut}\,(\boldsymbol{C}^2)$. *Then the* *following two cases arise:*

(1) *There exists a finite subgroup* $\hat{G}$ *of* $\mathrm{Aut}(\boldsymbol{P}^2, L)$ *such that* $\iota^{-1} \circ \hat{G} \circ \iota = G$, *where* $\iota: \boldsymbol{C}^2 \to \boldsymbol{P}^2$ *is an inclusion and* $L = \boldsymbol{P}^2 - \iota(\boldsymbol{C}^2)$ *is a line.*

(2) *There exists a finite subgroup* $\tilde{G}$ *of* $\mathrm{Aut}(\boldsymbol{F}_n, s_\infty \cup l)$ *such that* $\tau^{-1} \circ \tilde{G} \circ \tau = G$, *where* $\tau: \boldsymbol{C}^2 \to \boldsymbol{F}_n$ *is an inclusion,* s_∞ *is a section with the self-intersection number* $(s_\infty^2) = n$ $(n \geqq 0)$ *and* l *is a fiber of* $\boldsymbol{F}_n$.

Now, since $\iota^{-1} \circ \hat{G} \circ \iota = G$ (resp. $\tau^{-1} \circ \tilde{G} \circ \tau = G$), we have $\iota \circ G \circ \iota^{-1} = \hat{G}|\boldsymbol{C}^2$ (resp. $\tau \circ G \circ \tau^{-1} = \tilde{G}|\boldsymbol{C}^2$), where $\hat{G}|\boldsymbol{C}^2$ (resp. $\tilde{G}|\boldsymbol{C}^2$) means the restriction of the group $\hat{G}$ (resp. $\tilde{G}$) to $\iota(\boldsymbol{C}^2)$ (resp. $\tau(\boldsymbol{C}^2)$). For simplicity, we identify $\iota(\boldsymbol{C}^2)$ and $\tau(\boldsymbol{C}^2)$ with $\boldsymbol{C}^2$. On the other hand, $\mathrm{Aut}(\boldsymbol{P}^2)$ and $\mathrm{Aut}(\boldsymbol{F}_n)$ are well-known, and we can write down every element of $\mathrm{Aut}(\boldsymbol{P}^2, L)$ or $\mathrm{Aut}(\boldsymbol{F}_n, s_\infty \cup l)$ (see [4]). In fact, choosing suitable coordinates x and y in $\boldsymbol{C}^2$, we find that for every element $\hat{g}$ of $\mathrm{Aut}(\boldsymbol{P}^2, L)$ (resp. $\tilde{g}$ of $\mathrm{Aut}(\boldsymbol{F}_n, s_\infty \cup l)$) the restriction $\hat{g}|\boldsymbol{C}^2$ (resp. $\tilde{g}|\boldsymbol{C}^2$) has the following form:

$$\begin{cases} x' = ax + by + \lambda \\ y' = cx + dy + \mu, \quad \text{where} \quad ad - bc \neq 0 \text{ and } \lambda, \mu \in \boldsymbol{C} \end{cases}$$

$$\left(\text{resp.} \quad \begin{cases} x' = ax + \lambda \\ y' = dy + \nu(x), \quad \text{where} \quad ad \neq 0 \text{ and } \nu(x) \in \boldsymbol{C}[x] \end{cases}\right).$$

Since ι (resp. τ) is a regular mapping of $\boldsymbol{C}^2$ into $\boldsymbol{P}^2$ (resp. $\boldsymbol{F}_n$), ι and τ can be regarded as elements of $\mathrm{Aut}(\boldsymbol{C}^2)$. Consequently we have the following:

PROPOSITION 4. *Let* G *be a finite subgroup of* $\mathrm{Aut}(\boldsymbol{C}^2)$. *Then there exists a polynomial automorphism* β *in* $\boldsymbol{C}^2$ *such that for every* g *of* G, *we have*

$$\beta \circ g \circ \beta^{-1}: \begin{cases} x' = a_g x + b_g y + \lambda_g \\ y' = c_g x + d_g y + \mu_g \end{cases}$$

or

$$\beta \circ g \circ \beta^{-1}: \begin{cases} x' = l_g x + \lambda'_g \\ y' = m_g y + \nu_g(x), \end{cases}$$

where $a_g, b_g, c_g, d_g, l_g, m_g, \lambda_g, \lambda'_g, \mu_g \in \boldsymbol{C}$, $a_g d_g - b_g c_g \neq 0$, $l_g m_g \neq 0$ *and* $\nu_g(x) \in \boldsymbol{C}[x]$.

Finally, put

$$\gamma_1 = 1/|G| \cdot \sum_{g \in G} \begin{pmatrix} a_g & b_g \\ c_g & d_g \end{pmatrix}^{-1} \circ (\beta \circ g \circ \beta^{-1})$$

or

$$\gamma_2 = 1/|G| \cdot \sum_{g \in G} \begin{pmatrix} l_g & 0 \\ 0 & m_g \end{pmatrix}^{-1} \circ (\beta \circ g \circ \beta^{-1}).$$

We can easily see that

$$\gamma_1: \begin{cases} x' = x + 1/|G| \cdot \sum_{g \in G} (\lambda_g d_g - b_g \mu_g)/(a_g d_g - b_g c_g) \\ y' = y + 1/|G| \cdot \sum_{g \in G} (a_g \mu_g - \lambda_g c_g)/(a_g d_g - b_g c_g) \end{cases}$$

and

$$\gamma_2: \begin{cases} x' = x + 1/|G| \cdot \sum_{g \in G} \gamma'_g/l_g \\ y' = y + 1/|G| \cdot \sum_{g \in G} \nu_g(x)/m_g \,. \end{cases}$$

Thus γ_1 and γ_2 are polynomial automorphisms in C^2. For any element h of G, we have

$$\begin{aligned} &\gamma_1 \circ (\beta \circ h \circ \beta^{-1}) \\ &= 1/|G| \cdot \sum_{g \in G} \begin{pmatrix} a_g & b_g \\ c_g & d_g \end{pmatrix}^{-1} \circ (\beta \circ g \circ \beta^{-1}) \circ (\beta \circ h \circ \beta^{-1}) \\ &= 1/|G| \cdot \sum_{g \in G} \begin{pmatrix} a_h & b_h \\ c_h & d_h \end{pmatrix} \circ \left\{ \begin{pmatrix} a_g & b_g \\ c_g & d_g \end{pmatrix} \circ \begin{pmatrix} a_h & b_h \\ c_h & d_h \end{pmatrix} \right\}^{-1} \circ (\beta \circ g \circ h \circ \beta^{-1}) \\ &= \begin{pmatrix} a_h & b_h \\ c_h & d_h \end{pmatrix} \circ 1/|G| \cdot \sum_{g \circ h \in G} \left\{ \begin{pmatrix} a_g & b_g \\ c_g & d_g \end{pmatrix} \circ \begin{pmatrix} a_h & b_h \\ c_h & d_h \end{pmatrix} \right\}^{-1} \circ \beta \circ (g \circ h) \circ \beta^{-1} \\ &= \begin{pmatrix} a_h & b_h \\ c_h & d_h \end{pmatrix} \circ \gamma_1 \,. \end{aligned}$$

Similarly, we have

$$\gamma_2 \circ (\beta \circ h \circ \beta^{-1}) = \begin{pmatrix} l_g & 0 \\ 0 & m_g \end{pmatrix} \circ \gamma_2 \,.$$

Therefore, for every element g of G, we have

$$\gamma_1 \circ (\beta \circ g \circ \beta^{-1}) \circ \gamma_1 = \begin{pmatrix} a_g & b_g \\ c_g & d_g \end{pmatrix} \in GL(2, C)$$

or

$$\gamma_2 \circ (\beta \circ g \circ \beta^{-1}) \circ \gamma_2 = \begin{pmatrix} l_g & 0 \\ 0 & m_g \end{pmatrix} \in GL(2, C) \,.$$

We have only to let $\alpha = \gamma_1 \circ \beta$ or $\alpha = \gamma_2 \circ \beta$. Thus the proof of Theorem 2 is completed.

3. Example. Let G be a finite subgroup of Aut (C^3). By Theorem 1, there exists a non-singular projective algebraic compactification $(C^3, \iota, X; A)$ and a finite subgroup $\hat{G}$ of Aut $(X; A)$ such that $\iota^{-1} \circ \hat{G} \circ \iota = G$. Here, if we can choose the complex projective space P^3 or a non-singular quadric hypersurface Q^3 in P^4 as such a compactification X, there exists an element α of $\mathrm{Aut}(C^3)$ such that $\alpha \circ G \circ \alpha^{-1}$ is a finite subgroup of $GL(3, C)$. Indeed, if $X = P^3$, then it is obvious. Suppose that $X = Q^3 \hookrightarrow P^4$. Let $(X_i)_{0 \leqq i \leqq 4}$ (resp. $(Y_i)_{1 \leqq i \leqq 4}$) be the homogeneous coordinates of P^4 (resp. P^3). We may assume that

$$X \cong V(X_0X_1 + X_2^2 + X_3^2 + X_1^2)\,,$$
$$A \cong V(X_0) \cap X \cong V(Y_2^2 + Y_3^2 + Y_4^2) \hookrightarrow P^3\,.$$

In fact, we shall first consider the following standard sequence:

$$\to H_c^i(C^3, Z) \to H^i(X, Z) \to H^i(A, Z) \to H_c^{i+1}(C^3, Z) \to .$$

Since $H_i^c(C^3, Z) = 0$ for $1 \leqq i \leqq 4$, we have

$$H^i(X, Z) \cong H^i(A, Z) \quad \text{for} \quad 1 \leqq i \leqq 4\,.$$

By the Lefschetz hyperplane section theorem, we have $H^2(X, Z) \cong H^2(P^4, Z) \cong Z$. We can see that the line bundle $[A]$ is ample on X, and the first Chern class $C_1([A])$ of $[A]$ generates the cohomology ring $H^2(X, Z)$ $(\cong Z)$. By the adjunction formula, we have $K_X \cong [A]^{-3}$ (cf. Brenton-Morrow [1]). Since A is a hyperplane section and $H^2(A, Z) \cong Z$, A is an irreducible quadric hypersurface in $V(X_0) \cong P^3$ with an isolated singularity. By elementary arguments, we see that the minimal resolution of A is the rational ruled surface F_2. Thus we may assume that A is isomorphic to the variety $V(Y_2^2 + Y_3^2 + Y_4^2) \hookrightarrow P^3$, and that X is isomorphic to the variety $V(X_0X_1 + X_2^2 + X_3^2 + X_4^2)$ (see Griffiths-Harris [7]). It is easy to verify that such a (X, A) is a non-singular compactification of C^3.

Now, we put $x = (1:0:0:0) \in X$. Then x is a singular point of A. Let $p_1\colon Q_x(X) \to X$ be the quadratic transformation of X at the point x with $p_1^{-1}(x) = E \cong P^2$. We define the projection $p_2\colon Q_x(X) \to P^3$ of $Q_x(X)$ onto P^3 by

$$p_2^{-1}(y) = \begin{cases} \text{(i)} & \text{the point with} \quad X_0 = -\sum_{i=2}^{4} y_i^2/y_1\,, \quad X_i = y_i \quad (1 \leqq i \leqq 4) \\ & \text{if} \quad y_1 \neq 0\,, \\ \text{(ii)} & \text{the point with} \quad X_0 = 1\,, \quad X_i = 0 \quad (1 \leqq i \leqq 4) \\ & \text{if} \quad y_1 = 0 \quad \text{and} \quad \sum_{i=2}^{4} y_i^2 \neq 0\,, \\ \text{(iii)} & \text{any of the line of points with} \quad X_0 = t\,, \quad x_i = sy_i \\ & (1 \leqq i \leqq 4) \quad \text{if} \quad y_1 = \sum_{i=2}^{4} y_i^2 = 0\,. \end{cases}$$

Thus we have the following diagram

$$\begin{array}{ccc} & Q_x(X) & \\ {}^{p_1}\swarrow & & \searrow^{p_2} \\ X & \cdots\cdots\cdots & \boldsymbol{P}^3 \end{array} .$$

Let $\bar{A}$ be the proper transform of A in $Q_x(X)$. Then we have $p_2(p_1^{-1}(A)) = V(Y_1) \hookrightarrow \boldsymbol{P}^3$ and $p_2(\bar{A})$ is a conic $\gamma: \{Y_1 = Y_2^2 + Y_3^2 + Y_4^2 = 0\} \hookrightarrow V(Y_1)$ (see Mumford [14]).

Let g be an arbitrary element of $\mathrm{Aut}(X; A)$. Then $g(x) = x$, since the point x is the only singular point of A. Therefore, for the same reason as in Lemma 1, there exists a unique automorphism $\hat{g}$ of $\mathrm{Aut}(Q_x(X); p_1^{-1}(A))$ such that $p_1 \circ \hat{g} = g \circ p_1$. Further by the Riemann extension theorem, there exists a unique automorphism $\tilde{g}$ of $\mathrm{Aut}(\boldsymbol{P}^3; V(Y_1))$ such that $p_2 \circ \tilde{g} = \hat{g} \circ p_2$. We put $\alpha = p_2 \circ p_1^{-1}$. Then α is a one-to-one regular mapping of $\boldsymbol{C}^3$ into $\boldsymbol{P}^3$ with $V(Y_1) = \boldsymbol{P}^3 - \alpha(\boldsymbol{C}^3)$ and $\alpha \circ g = \tilde{g} \circ \alpha$, namely, $\alpha \circ g \circ \alpha^{-1} = \tilde{g} | \boldsymbol{C}^3$. Since $\tilde{g} \in \mathrm{Aut}(\boldsymbol{P}^3; V(Y_1))$, $\tilde{g} | \boldsymbol{C}^3$ is a linear transformation. Therefore G is conjugate in $\mathrm{Aut}(\boldsymbol{C}^3)$ with a finite subgroup of $GL(3, \boldsymbol{C})$.

References

[1] L. Brenton and J. Morrow, Compactifications of $\boldsymbol{C}^n$, Trans. Amer. Math. Soc., 246 (1978), 139-153.

[2] H. Cartan, Quotient d'un espace analytique par un groupe d'automorphismes, Algebraic Geometry and Topology, A symposium in honor of S. Lefschetz, Princeton Univ. Press (1957), 90-102.

[3] M. Furushima, Finite groups of polynomial automorphisms in the complex affine plane (I), Mem. Fac. Sci. Kyushu Univ., 36 (1982), 85-105.

[4] M. H. Gizatullin and V. I. Danilov, Automorphisms of affine surfaces I, Math. USSR. Izv., 9 (1975), 493-534.

[5] H. Grauert and R. Remmert, Bilder und Urbilder analytischer Garben, Ann. Math., 68 (1958), 393-443.

[6] H. Grauert and R. Remmert, Komplexe Räume, Math. Ann., 136 (1958), 245-318.

[7] P. Griffiths and J. Harris, Principles of Algebraic Geometry, John Willey & Sons, 1978.

[8] H. Hironaka, Bimeromorphic smoothing of a complex analytic space, Math. Inst. Warwick Univ., England (1971).

[9] H. Hopf, Schlichte Abbildungen und lokale Modifikationen komplexer Mannigfaltigkeiten, Comm. Math. Helv., 29 (1955), 132-156.

[10] T. Kambayashi, Automorphism group of a polynomial ring and algbraic group action on an affine space, J. of Algebra, 60 (1979), 439-451.

[11] H. B. Laufer, Normal two-dimensional singularities, Ann. Math. Studies, 59 (1973).

[12] M. Miyanishi, Lectures on curves on rational and unirational surface, Tata Inst. Fundamental Research, Bombay, Springer-Verlag, Berlin-Heidelberg-New York, 1978.

[13] J. Morrow, Minimal normal compactification of $\boldsymbol{C}^2$, Proceedings of the Conference of Complex Analysis, Rice Univ. Studies 59 (1973), 97-112.

[14] D. MUMFORD, Algebraic Geometry I: Complex Projective varieties, Grundlehren Math. Wissenschaften 221, Springer-Verlag, Berlin-Heidelberg-New York, 1976.
[15] R. REMMERT AND K. STEIN, Eigentliche holomorphe Abbildungen, Math. Z., 73 (1960), 159-189.
[16] K. STEIN, Analytische Zerlegungen komplexer Räume, Math. Ann., 132 (1956), 63-93.

DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
KYUSHU UNIVERSITY
FUKUOKA, 812
JAPAN

Current Address:
KUMAMOTO RADIO TECHNICAL COLLEGE
NISHI-GÔSHI-MACHI
KUMAMOTO, 861-11
JAPAN