Research Article

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Finite groups whose intersection power graphs are toroidal and projective-planar

Abstract: The intersection power graph of a finite group G is the graph whose vertex set is G, and two distinct vertices x and y are adjacent if either one of x and y is the identity element of G, or $\langle x \rangle \cap \langle y \rangle$ is non-trivial. In this paper, we completely classify all finite groups whose intersection power graphs are toroidal and projective-planar.

Keywords: intersection power graph, finite group, genus

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1 Introduction

A *simple graph* is an undirected graph without loops and multiple edges. A graph is called *finite* if its vertex set is finite. All graphs considered in this paper are finite and simple. Let Γ be a graph. Denote by $V(\Gamma)$ and $E(\Gamma)$ the vertex set and the edge set of Γ, respectively. An *embedding* of a graph into a surface is a drawing of the graph on the surface in such a way that its edges may intersect only at their endpoints. A graph is called *planar* if it can be embedded in the plane. A non-planar graph can be embedded in some surface obtained from the sphere by attaching some handles or crosscaps. We denote by $\mathbb{S}_k$ a sphere with $k \geq 0$ handles and by $\mathbb{N}_k$ a sphere with k crosscaps. Note that $\mathbb{S}_0 = \mathbb{N}_0$ is the sphere, while $\mathbb{S}_1$ and $\mathbb{N}_1$ are a torus and a projective plane, respectively. The smallest non-negative integer k such that a graph Γ can be embedded on $\mathbb{S}_k$ is called the *orientable genus* or *genus* of Γ and is denoted by $\gamma(\Gamma)$. The *non-orientable genus* of Γ, denoted by $\bar{\gamma}(\Gamma)$, is instead the smallest integer k such that Γ can be embedded on $\mathbb{N}_k$. A graph Γ is called *toroidal* if $\gamma(\Gamma) = 1$ and Γ is called *projective-planar* if $\bar{\gamma}(\Gamma) = 1$. Note that Γ is planar if and only if $\gamma(\Gamma) = 0$, if and only if, $\bar{\gamma}(\Gamma) = 0$. We follow the book [1] for undefined notation and terminology.

Graphs associated with groups and other algebraic structures have been actively investigated in the literature, because they have valuable applications [2] and are related to automata theory [3,4]. For example, the Cayley graph of a group, which has a long history. The *undirected power graph* of a group G, denoted $\mathcal{P}(G)$, is a simple graph whose vertex set is G and two distinct vertices are adjacent if one is a power of the other. Kelarev and Quinn [5] first introduced the concept of power graph of a group, as a directed graph. The concept of the undirected power graph of a group was introduced first by Chakrabarty et al. [6]. In the past two decades, the study of power graphs of groups has been growing. See, for example, [7–15] and the survey paper [16] with many results and open questions on power graphs.

In 2018, Bera [17] defined the intersection power graph $\mathcal{P}_I(G)$ of a group G, where the vertex set of $\mathcal{P}_I(G)$ is G, and distinct vertices x and y are adjacent if either one of x and y is the identity element of G or $|\langle x \rangle \cap \langle y \rangle| > 1$. In [17], the author studied some properties of $\mathcal{P}_I(G)$ and determined the full automorphism

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group of the intersection power graph of a cyclic group. A *book* is a collection of half-planes having the same line as their boundary. A planar embedding of a graph into a book is called the *book embedding*. The smallest possible number of half-planes for any book embedding of a graph is called the *book thickness* of the graph. In [18], the authors determined the finite groups whose intersection power graphs have book thickness at most two.

All groups considered in this paper are finite. Let G be a group. The set of orders of elements of G is denoted by $\pi_e(G)$. $\mathbb{Z}_n$ denotes the cyclic group of order n. Also, we use Ψ to denote the set of all groups G such that the following two conditions hold:

(i) $\{4\} \subseteq \pi_e(G) \subseteq \{1, 2, 3, 4\}$;
(ii) G has exactly two cyclic subgroups of order 4 intersecting at a subgroup of order 2.

Note that $\mathbb{Z}_2 \times \mathbb{Z}_4 \in \Psi$. Also, we identify the relevant groups using their unique identifiers in the SmallGroups library [19], which is distributed with GAP [20]. The mth group of order n in the SmallGroups library is identified as SmallGroup (n, m).

One can easily see that, for any group G, $\mathcal{P}(G)$ is a spanning subgraph of $\mathcal{P}_I(G)$. In [21], the authors classified all finite groups whose power graphs have (non)orientable genus one. In [22], the authors classified all abelian groups whose intersection power graphs have (non)orientable genus one. In this paper, we completely classify all groups whose intersection power graphs have (non)orientable genus one. Our main results are the following theorems, where D_{2n} denotes the dihedral group of order $2n$.

Theorem 1.1. *Let G be a group. Then $\mathcal{P}_I(G)$ is toroidal if and only if*

$$G \in \Psi \cup \{\mathbb{Z}_5, \mathbb{Z}_6, \mathbb{Z}_7, D_{10}, D_{12}, D_{14}, SmallGroup(20, 3), SmallGroup(21, 1)\}.$$

Theorem 1.2. *Let G be a group. Then $\mathcal{P}_I(G)$ is projective-planar if and only if*

$$G \in \Psi \cup \{\mathbb{Z}_5, \mathbb{Z}_6, D_{10}, D_{12}, SmallGroup(20, 3)\}.$$

2 Preliminaries

In this section, we briefly recall some notations, terminologies, and basic results and prove a lemma, which we need in the sequel.

G always denotes a group and its identity element is e. Let $g \in G$. The *order* of g, denoted $o(g)$, is the cardinality of the cyclic subgroup $\langle g \rangle$ generated by g. The *exponent* of G, denoted by $\exp(G)$, is defined as the least common multiple of the orders of all elements of the group. Let $a, b \in G$. Then $[a, b] = a^{-1}b^{-1}ab$ is called the *commutator* of a and b. Note that $[a, b] = e$ if and only if a and b commute.

Recall that D_{2n} is the dihedral group of order $2n$, and a presentation of D_{2n} is given by

$$D_{2n} = \langle a, b : a^n = b^2 = e, bab = a^{-1} \rangle.$$

Observe that D_{2n} is abelian if and only if $n \in \{1, 2\}$. Moreover, we have

$$D_{2n} = \langle a \rangle \cup \{b, ab, a^2b, \ldots, a^{n-1}b\}, \quad o(a^ib) = 2 \quad \text{for any } 1 \le i \le n. \tag{1}$$

Note that $\mathbb{Z}_2 \times D_8 \in \Psi$.

For $n \ge 2$, Johnson [23, pp. 44–45] defined the generalized quaternion group, which is denoted by Q_{4n} and has the presentation

$$Q_{4n} = \langle x, y : x^n = y^2, x^{2n} = e, y^{-1}xy = x^{-1} \rangle.$$

It is known that Q_{4n} has order $4n$ and its unique involution is x^n. Moreover, $o(x^iy) = 4$ for any $1 \le i \le 2n$, and $Q_{4n} = \langle x \rangle \cup \{x^iy : 1 \le i \le 2n\}$. If $n = 2$, then Q_8 is the quaternion group of order 8. Note that $Q_8 \notin \Psi$.

Recall now the following elementary results.

Lemma 2.1. [24, Theorem 5.4.10(ii)] *Let p be a prime. Then a p-group having a unique subgroup of order p is either cyclic or generalized quaternion.*

Lemma 2.2. [25, Lemma 3.1] *There is no group that has precisely two cyclic subgroups of order 6.*

A graph is called *complete* if every two distinct vertices of this graph are adjacent. The complete graph of order n is denoted by K_n. A complete bipartite graph Γ is a graph whose vertex set can be partitioned into two non-empty partite sets V_1 and V_2 such that for every two distinct vertices $u \in V_i$ and $v \in V_j$ where $i, j \in \{1, 2\}$, u and v are adjacent if and only if $i \neq j$. If $|V_1| = m$ and $|V_2| = n$, then the complete bipartite graph Γ is denoted by $K_{m,n}$.

Theorem 2.3. [1] *Let $n \geq 3$ and $m \geq 2$ be two integers. Then*
(a) $\gamma(K_n) = \left\lceil \frac{1}{12}(n-3)(n-4) \right\rceil$.
(b) *For $n \neq 7$, $\bar{\gamma}(K_n) = \left\lceil \frac{1}{6}(n-3)(n-4) \right\rceil$. Moreover, $\bar{\gamma}(K_7) = 3$.*
(c) $\gamma(K_{m,n}) = \left\lceil \frac{1}{4}(m-2)(n-2) \right\rceil$.
(d) $\bar{\gamma}(K_{m,n}) = \left\lceil \frac{1}{2}(m-2)(n-2) \right\rceil$.

The following corollary is an immediate consequence of Theorem 2.3.

Corollary 2.4.
(i) $\gamma(K_n) = 1$ *if and only if $n \in \{5, 6, 7\}$.*
(ii) $\bar{\gamma}(K_n) = 1$ *if and only if $n \in \{5, 6\}$.*
(iii) $\gamma(K_{4,6}) = 2$ *and $\bar{\gamma}(K_{4,6}) = 4$.*

A *block* of a graph Γ is a subgraph B of Γ maximal with respect to the property that removing any single vertex of B does not disconnect B. It is easy to see that the intersection of two distinct blocks has at most one vertex (see (1) of [26]). Given a graph Γ, as stated in [26], there exists a unique finite collection $\mathcal{B}$ of blocks B of Γ such that

$$\Gamma = \bigcup_{B \in \mathcal{B}} B.$$
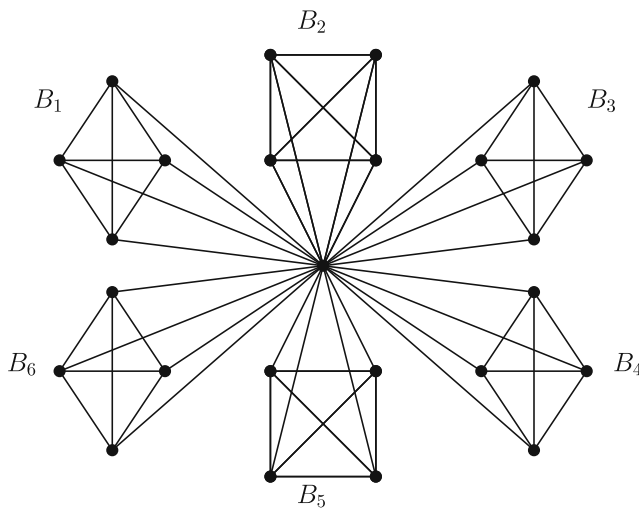
Figure 1: $\mathcal{P}_I(\mathbb{Z}_5 \times \mathbb{Z}_5)$.

The collection $\mathcal{B}$ is called the *block decomposition* of Γ. For more information on the block decomposition of a graph, the readers are referred to [26] and [27].

We use the following example to illustrate the block decomposition of a graph.

Example 2.5. Let $G = \mathbb{Z}_5 \times \mathbb{Z}_5$. Then $\mathcal{P}_I(G)$ is a union of six complete graphs of order 5 that share the identity element of G, as displayed in Figure 1. Thus, the block decomposition of $\mathcal{P}_I(G)$ is $\{B_1, B_2, \ldots, B_6\}$, where $B_i \cong K_5$ for each $1 \leq i \leq 6$.

The following result tells us how to compute the (non)orientable genus of a graph using its blocks.

Theorem 2.6. [26, Theorem 1], [28, Corollary 3] *Let Γ be a connected graph with block decomposition* $\{B_1, \ldots, B_n\}$. *Then*

(I)

$$\gamma(\Gamma) = \sum_{i=1}^{n} \gamma(B_i).$$

(II) *If $\bar{\gamma}(B_i) = 2\gamma(B_i) + 1$ for each i, then*

$$\bar{\gamma}(\Gamma) = 1 - n + \sum_{i=1}^{n} \bar{\gamma}(B_i).$$

Otherwise,

$$\bar{\gamma}(\Gamma) = 2n - \sum_{i=1}^{n} \mu(B_i),$$

where $\mu(B_i) = \max\{2 - 2\gamma(B_i), 2 - \bar{\gamma}(B_i)\}$.

The following result is an easy observation and will be frequently used in the sequel sometimes without explicit reference.

Observation 2.7.
  (i) Let Γ be a graph. Then Γ is planar if and only if for any subgraph Ω of Γ, Ω is planar.
 (ii) If Ω is a subgraph of a graph Γ, then $\gamma(\Omega) \leq \gamma(\Gamma)$ and $\bar{\gamma}(\Omega) \leq \bar{\gamma}(\Gamma)$.
(iii) If H is a subgroup of G, then $\mathcal{P}_I(H)$ is an induced subgraph of $\mathcal{P}_I(G)$. In particular, $\gamma(\mathcal{P}_I(H)) \leq \gamma(\mathcal{P}_I(G))$ and $\bar{\gamma}(\mathcal{P}_I(H)) \leq \bar{\gamma}(\mathcal{P}_I(G))$.
(iv) Every generator of $\mathbb{Z}_n$ is adjacent to every other vertex in $\mathcal{P}_I(\mathbb{Z}_n)$. Moreover, $\mathbb{Z}_n$ has $\varphi(n)$ generators, where φ is Euler's totient function.

Lemma 2.8. [17, Theorem 3.1] *$\mathcal{P}_I(G)$ is complete if and only if G is either a cyclic p-group or a generalized quaternion 2-group.*

Lemma 2.9. [12, Theorem 2] *$\mathcal{P}(G)$ is planar if and only if $\pi_e(G) \subseteq \{1, 2, 3, 4\}$.*

Lemma 2.10. *$\mathcal{P}_I(G)$ is planar if and only if G satisfies the following two conditions*:
 (I) $\pi_e(G) \subseteq \{1, 2, 3, 4\}$;
(II) *If G has two distinct cyclic subgroups $\langle x \rangle$ and $\langle y \rangle$ of order 4, then $|\langle x \rangle \cap \langle y \rangle| = 1$.*

Proof. If G satisfies (I) and (II), then it is easy to see that $\mathcal{P}_I(G) = \mathcal{P}(G)$, and so $\mathcal{P}_I(G)$ is planar by Lemma 2.9. Conversely, suppose that $\mathcal{P}_I(G)$ is planar. Then $\mathcal{P}(G)$ also is planar. It follows from Lemma 2.9 that (I) holds. Now suppose, by contradiction, that G has two distinct cyclic subgroups of order 4, say, $\langle x \rangle$ and $\langle y \rangle$, such that $|\langle x \rangle \cap \langle y \rangle| = 2$. Then the subgraph induced by $\langle x \rangle \cup \langle y \rangle$ is isomorphic to K_6, against the fact that K_6 is non-planar. □
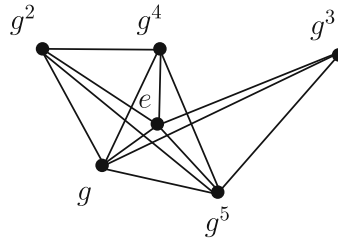
Figure 2: $\mathcal{P}_I(\mathbb{Z}_6)$.

Lemma 2.10 implies that $\gamma(\mathcal{P}_I(\mathbb{Z}_n)) = 0$ if and only if $n \in \{1, 2, 3, 4\}$, if and only if $\overline{\gamma}(\mathcal{P}_I(\mathbb{Z}_n)) = 0$. In the following, we classify all cyclic groups whose intersection power graphs have (non)orientable genus one.

Lemma 2.11.
(i) $\gamma(\mathcal{P}_I(\mathbb{Z}_n)) = 1$ *if and only if* $n \in \{5, 6, 7\}$.
(ii) $\overline{\gamma}(\mathcal{P}_I(\mathbb{Z}_n)) = 1$ *if and only if* $n \in \{5, 6\}$.

Proof. (i) Clearly, we have that $\mathcal{P}_I(\mathbb{Z}_5) \cong K_5$ and $\mathcal{P}_I(\mathbb{Z}_7) \cong K_7$ by Lemma 2.8. Thus, $\gamma(\mathcal{P}_I(\mathbb{Z}_5)) = \gamma(\mathcal{P}_I(\mathbb{Z}_7)) = 1$ by Corollary 2.4. Moreover, it is easy to see that $\mathcal{P}_I(\mathbb{Z}_6) = \mathcal{P}(\mathbb{Z}_6)$, as displayed in Figure 2, where $\mathbb{Z}_6 = \langle g \rangle$. Thus, it follows from [21, Theorem 3.2] that $\gamma(\mathcal{P}(\mathbb{Z}_6)) = 1$, and so $\gamma(\mathcal{P}_I(\mathbb{Z}_6)) = 1$.

Conversely, suppose that $\gamma(\mathcal{P}_I(\mathbb{Z}_n)) = 1$. It follows from Lemma 2.10 that $n \geq 5$. Suppose, by contradiction that $n \geq 10$. Then $\varphi(n) \geq 4$, and so $\mathbb{Z}_n$ has at least four generators. Since every generator of $\mathbb{Z}_n$ is adjacent to every other vertex in $\mathcal{P}_I(\mathbb{Z}_n)$, we have that $\mathcal{P}_I(\mathbb{Z}_n)$ has a subgraph isomorphic to $K_{4,6}$. Since $\gamma(K_{4,6}) = 2$ by Corollary 2.4, we also have $\gamma(\mathcal{P}_I(\mathbb{Z}_n)) \geq 2$, a contradiction. We conclude that $n \leq 9$. Moreover, since $\gamma(\mathcal{P}_I(\mathbb{Z}_8)) = 2$ and $\gamma(\mathcal{P}_I(\mathbb{Z}_9)) = 3$ by Lemma 2.8 and Theorem 2.3, we deduce $n \leq 7$, as desired.

(ii) By Lemma 2.8 and Corollary 2.4, we have $\overline{\gamma}(\mathcal{P}_I(\mathbb{Z}_5)) = \overline{\gamma}(K_5) = 1$. Moreover, since $\overline{\gamma}(K_5) \leq \overline{\gamma}(\mathcal{P}_I(\mathbb{Z}_6)) \leq \overline{\gamma}(K_6)$ and $\overline{\gamma}(K_6) = 1$, we have $\overline{\gamma}(\mathcal{P}_I(\mathbb{Z}_6)) = 1$. Conversely, suppose that $\overline{\gamma}(\mathcal{P}_I(\mathbb{Z}_n)) = 1$. It follows from Lemma 2.10 that $n \geq 5$. Similar to the proof of (i), if $n \geq 10$, then $\mathcal{P}_I(\mathbb{Z}_n)$ has a subgraph isomorphic to $K_{4,6}$, which is impossible as $\overline{\gamma}(K_{4,6}) = 4$ by Corollary 2.4. Hence, $n \leq 9$. Moreover, since $\overline{\gamma}(\mathcal{P}_I(\mathbb{Z}_7)) = 3$, $\overline{\gamma}(\mathcal{P}_I(\mathbb{Z}_8)) = 4$ and $\overline{\gamma}(\mathcal{P}_I(\mathbb{Z}_9)) = 5$ by Lemma 2.8 and Theorem 2.3, we deduce $n \leq 6$, as desired. □

Lemma 2.12. *Let* $G \in \Psi$. *Then* $\gamma(\mathcal{P}_I(G)) = 1$ *and* $\overline{\gamma}(\mathcal{P}_I(G)) = 1$.

Proof. By definition, G has exactly one pair of cyclic subgroups $\langle x \rangle$ and $\langle y \rangle$ of order 4 such that $|\langle x \rangle \cap \langle y \rangle| = 2$. Then the subgraph induced by $\langle x \rangle \cup \langle y \rangle$ is isomorphic to K_6. Moreover, by the definition of an intersection power graph, we have that $\mathcal{P}_I(G)$ is a union of some complete graphs of order at most 4 and a complete graph of order 6 that share the identity element of G, as displayed in Figure 3. It follows that $\mathcal{P}_I(G)$ has a block
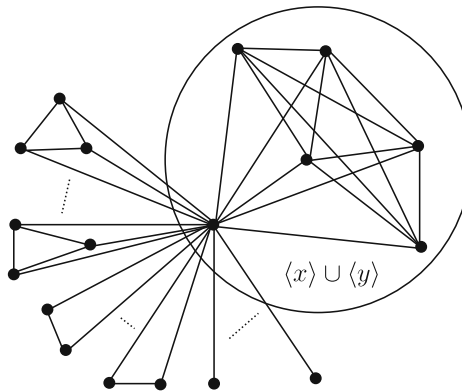

Figure 3: $\mathcal{P}_I(G)$ with $G \in \Psi$.

decomposition $\mathcal{B}$ such that every block $B \in \mathcal{B}$ is isomorphic to K_2, K_3, K_4, or K_6. Since $\mathcal{P}_I(G)$ has precisely one block isomorphic to K_6, we have that $\gamma(\mathcal{P}_I(G)) = 1$ and $\overline{\gamma}(\mathcal{P}_I(G)) = 1$ by Theorem 2.6. $\qquad\square$

3 Preliminary results

In this section, we prove some lemmas needed to give the proofs of our main theorems.

Lemma 3.1. *Let G be a group with $\pi_e(G) \subseteq \{1, 2, 3, 4\}$. The following statements are equivalent*:
(a) $\gamma(\mathcal{P}_I(G)) = 1$;
(b) $\overline{\gamma}(\mathcal{P}_I(G)) = 1$;
(c) $G \in \Psi$.

Proof. We first prove that (a) and (c) are equivalent. It follows from Lemma 2.12 that (c) implies (a). Suppose now that $\gamma(\mathcal{P}_I(G)) = 1$. If G has no cyclic subgroup of order 4 or has precisely one cyclic subgroup of order 4, then Lemma 2.10 implies that $\mathcal{P}_I(G)$ is planar, a contradiction. Thus, $\{4\} \subseteq \pi_e(G) \subseteq \{1, 2, 3, 4\}$ and there exist at least two cyclic subgroups $\langle x \rangle$ and $\langle y \rangle$ of order 4 such that $|\langle x \rangle \cap \langle y \rangle| = 2$, we want to show that they are unique. If there exists $z \in G\backslash(\langle x \rangle \cup \langle y \rangle)$ such that $o(z) = 4$ and $|\langle x \rangle \cap \langle z \rangle| = 2$, then the subgraph induced by $\langle x \rangle \cup \langle y \rangle \cup \langle z \rangle$ is isomorphic to K_8, which has genus 2 by Theorem 2.3, a contradiction. We conclude that if there exists an element $z \in G\backslash(\langle x \rangle \cup \langle y \rangle)$ of order 4, then $|\langle z \rangle \cap \langle x \rangle| = |\langle z \rangle \cap \langle y \rangle| = 1$. Suppose, by contradiction, that there exist two distinct cyclic subgroups $\langle x' \rangle$ and $\langle y' \rangle$ of order 4 such that $|\langle x' \rangle \cap \langle y' \rangle| = 2$ and $x', y' \in G\backslash(\langle x \rangle \cup \langle y \rangle)$. By the previous argument we know that $|\langle x' \rangle \cap \langle x \rangle| = |\langle x' \rangle \cap \langle y \rangle| = 1$. Thus, the subgraph Ω induced by $\langle x \rangle \cup \langle y \rangle \cup \langle x' \rangle \cup \langle y' \rangle$ is a union of two complete graphs of order 6 that share the identity vertex, as displayed in Figure 4. Thus, Ω has two blocks isomorphic to K_6, and so $\gamma(\Omega) = 2$ by Theorems 2.3 and 2.6. It follows that $\gamma(\mathcal{P}_I(G)) \geq \gamma(\Omega) = 2$, contrary to $\gamma(\mathcal{P}_I(G)) = 1$. We conclude that G has exactly one pair of cyclic subgroups of order 4 such that their intersection has size 2, and so $G \in \Psi$.

Note next that $\overline{\gamma}(K_8) = 4$ and $\overline{\gamma}(\Omega) = 2$ by Theorems 2.3 and 2.6. Then similar to the above proof, we get that $\overline{\gamma}(\mathcal{P}_I(G)) = 1$ if and only if $G \in \Psi$. $\qquad\square$

Lemma 3.2. *Let G be a group with $\{5\} \subseteq \pi_e(G) \subseteq \{1, 2, 3, 4, 5\}$. Then the following facts are equivalent*:
(a) $\gamma(\mathcal{P}_I(G)) = 1$;
(b) *G is isomorphic to one of the groups*

$$\mathbb{Z}_5, D_{10}, SmallGroup(20, 3) \cong \langle g, w : g^5 = w^4 = e, wgw^{-1} = g^2 \rangle; \qquad (2)$$

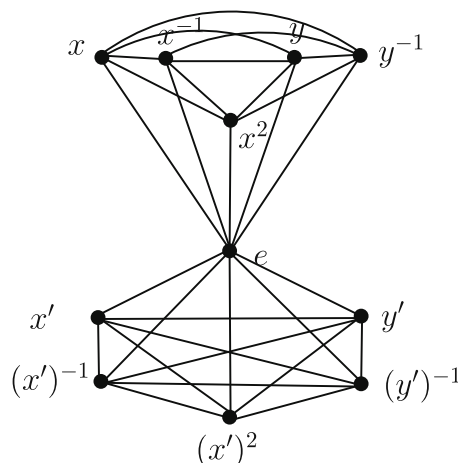(c) $\overline{\gamma}(\mathcal{P}_I(G)) = 1$.

Figure 4: The subgraph Ω.

Proof. We first prove that if G is isomorphic to one group in (2), then $\gamma(\mathcal{P}_I(G)) = 1$. By Lemma 2.11, we have $\gamma(\mathcal{P}_I(\mathbb{Z}_5)) = 1$. Moreover, by (1), it is easy to see that $\mathcal{P}_I(D_{10})$ has exactly one block isomorphic to K_5, and every other block is isomorphic to K_2. It follows that $\gamma(\mathcal{P}_I(D_{10})) = 1$ by Theorems 2.3 and 2.6. Now let $G = \text{SmallGroup}(20, 3)$. Using GAP [20], we deduce that G has a unique subgroup of order 5 and has five distinct cyclic subgroups of order 4 such that the intersection of each two of them has size 1. It follows that $\mathcal{P}_I(G)$ is a union of a complete graph of order 5 and four complete graphs of order 4 that share the identity vertex. Consequently, every block of $\mathcal{P}_I(G)$ is isomorphic to K_4 or K_5. Since $\mathcal{P}_I(G)$ has precisely one block isomorphic to K_5, we infer that $\gamma(\mathcal{P}_I(G)) = 1$ by Theorems 2.3 and 2.6, as desired.

We next prove that if $\gamma(\mathcal{P}_I(G)) = 1$, then G is isomorphic to one group in (2). Suppose, by contradiction, that G has two distinct subgroups of order 5. Then the subgraph Δ induced by the two distinct subgroups of order 5 is a union of two complete graphs of order 5 that share the identity vertex. Thus, Δ has two distinct blocks isomorphic to K_5, and so $\gamma(\Delta) = 2$ by Theorems 2.3 and 2.6. This implies a contradiction as $\gamma(\mathcal{P}_I(G)) \geq \gamma(\Delta) = 2$. As a consequence, G has precisely one subgroup $\langle g \rangle$ of order 5, and so $\langle g \rangle$ is normal in G. If G has an element x of order 3, then $\langle g, x \rangle = \langle g \rangle \langle x \rangle$ has order 15 and hence is cyclic, against the fact that G has no element of order 15. Thus, $\pi_e(G) \subseteq \{1, 2, 4, 5\}$. If $\pi_e(G) = \{1, 5\}$, since G has precisely one subgroup $\langle g \rangle$ of order 5, we have $G = \langle g \rangle \cong \mathbb{Z}_5$ by Lemma 2.1. Therefore, in the following we may assume that G has some involutions.

Case 1. G has no elements of order 4.

Then $\pi_e(G) = \{1, 2, 5\}$. Let a be an involution of G. Since every group of order 10 is isomorphic to D_{10} or $\mathbb{Z}_{10}$, we have $\langle a, g \rangle \cong D_{10}$ by Lemma 2.11. Suppose, by contradiction, that $G \neq \langle a, g \rangle$. Then there exists an involution $b \in G \backslash \langle a, g \rangle$. Using again Lemma 2.11 we have $\langle b, g \rangle \cong D_{10}$. If $[a, b] = e$, since $aga = bgb = g^{-1}$, we have $(gab)^2 = gb(aga)b = g(bg^{-1}b) = g^2$ and so $(gab)^5 = g^4gab = ab$, which implies $o((gab)^2) = 5$ and $o((gab)^5) = 2$, it follows that $o(gab) = 10$, a contradiction. We conclude that $[a, b] \neq e$. Recall that two distinct involutions generate a dihedral group. Since $\pi_e(G) = \{1, 2, 5\}$, we deduce that $\langle a, b \rangle \cong D_{10}$. Consequently, $g \in \langle a, b \rangle$, and so $\langle a, g \rangle \subseteq \langle a, b \rangle$. Since $\langle a, g \rangle \cong D_{10}$, we then have $\langle a, b \rangle = \langle a, g \rangle$, which implies $b \in \langle a, g \rangle$, a contradiction. We conclude that $G = \langle a, g \rangle \cong D_{10}$, as desired.

Case 2. G has an element w of order 4.

Then $\pi_e(G) = \{1, 2, 4, 5\}$. Let $a = w^2$. By Lemma 2.11, we necessarily have $\langle a, g \rangle \cong D_{10}$. We first claim that every involution of G belongs to $\langle a, g \rangle$. Suppose, by contradiction, that G has an involution $b \in G \backslash \langle a, g \rangle$. Then, as in the proof of Case 1, we deduce that $[a, b] \neq e$. It follows that $\langle a, b \rangle \cong D_8$ or D_{10}. If $\langle a, b \rangle \cong D_8$, since $o(ab) = 4$ and $aga = bgb = g^{-1}$, we have

$$
\begin{aligned}
(gab)^4 &= (gab)^2(gabgab) \\
&= (gab)^2(ga(bgb)bab) \\
&= (gab)^2(g(ag^{-1}a)abab) \\
&= (gab)^2 g^2(ab)^2 \\
&= g^2(ab)^2 g^2(ab)^2 \\
&= g^2(aba)(bg^2b)(babab) \\
&= g^2(ab)(ag^{-2}a)(ababab) \\
&= g^2(a)(bg^2b)(bababab) \\
&= g^2(ag^{-2}a)(abababab) \\
&= g^4(abababab) \\
&= g^4
\end{aligned}
$$

and so $(gab)^5 = ab$, which implies $o((gab)^4) = 5$ and $o((gab)^5) = 4$, it follows that $o(gab) = 20$, a contradiction. Hence, $\langle a, b \rangle \cong D_{10}$. Therefore, we have $b \in \langle a, b \rangle = \langle a, g \rangle$, a contradiction.

Let P be a Sylow 2-subgroup of G with $w \in P$. Since G has a unique subgroup of order 5 by Lemma 2.1, we have that $\langle g \rangle$ is the unique Sylow 5-subgroup of G. Since $\pi_e(G) = \{1, 2, 4, 5\}$, we have $G = P\langle g \rangle$. Note that G has precisely five involutions a, ga, g^2a, g^3a, g^4a by the above claim. If P has an involution u with $u \neq a$, then $u = g^ia$ for some $1 \leq i \leq 4$, and so $g^iaa^{-1} = g^i \in P$, a contradiction as $o(g^i) = 5$. We conclude that P has a unique involution. Note that $\pi_e(G) = \{1, 2, 4, 5\}$. By Lemma 2.1, we know that P is isomorphic to either $\mathbb{Z}_4$ or Q_8. If $P \cong Q_8$, then the subgraph induced by P would be isomorphic to K_8 by Lemma 2.8, which is impossible as $\gamma(K_8) = 2$. Therefore, $P \cong \mathbb{Z}_4$, and so $G \cong \mathbb{Z}_4 \ltimes \mathbb{Z}_5$. By GAP [20], there are five groups of order 20 up to isomorphism, and every group H of order 20 has an element of order 10 if $H \not\cong \mathrm{SmallGroup}(20, 3)$. Thus, $G \cong \mathrm{SmallGroup}(20, 3) \cong \langle g, w : g^5 = w^4 = e, wgw^{-1} = g^2 \rangle$, as desired. Thus, (a) and (b) are equivalent.

Note next that $\overline{\gamma}(K_8) = 4$ and $\overline{\gamma}(\Delta) = 2$ by Theorems 2.3 and 2.6. Then similar to the above proof, we get that (b) and (c) are equivalent. □

Lemma 3.3. *Let G be a group with $\{7\} \subseteq \pi_e(G) \subseteq \{1, 2, 3, 4, 7\}$. Then $\gamma(\mathcal{P}_I(G)) = 1$ if and only if G is isomorphic to one of the following groups*:

$$\mathbb{Z}_7, D_{14}, \mathrm{SmallGroup}(21, 1) \cong \langle g, w : g^7 = w^3 = e, w^{-1}gw = g^4 \rangle.$$

Proof. By Lemma 2.11, $\gamma(\mathcal{P}_I(\mathbb{Z}_7)) = 1$. Moreover, by (1), it is easy to see that $\mathcal{P}_I(D_{14})$ has exactly one block isomorphic to K_7, and every other block is isomorphic to K_2. As a consequence, $\gamma(\mathcal{P}_I(D_{14})) = 1$ by Theorem 2.6. Now let $G = \mathrm{SmallGroup}(21, 1)$. By GAP [20], we check that G has a unique subgroup of order 7, and has seven distinct subgroups of order 3. Thus, every block of $\mathcal{P}_I(G)$ is isomorphic to K_7 or K_3. Note that $\mathcal{P}_I(G)$ has precisely one block isomorphic to K_7. It follows that $\gamma(\mathcal{P}_I(G)) = 1$ by Theorem 2.6, as desired.

Conversely, suppose that $\gamma(\mathcal{P}_I(G)) = 1$. Suppose, by contradiction, that G has two distinct subgroups of order 7. Then the subgraph Ω induced by the two distinct subgroups of order 7 is a union of two complete graphs of order 7 that share the identity vertex. Thus, Ω has two distinct blocks isomorphic to K_7, and so $\gamma(\mathcal{P}_I(G)) \geq \gamma(\Omega) = 2$ by Theorems 2.3 and 2.6, a contradiction. As a consequence, G has precisely one subgroup $\langle g \rangle$ of order 7, and so $\langle g \rangle$ is normal in G. If there exists an element $x \in G$ such that $o(x) = 4$, then the subgroup $\langle x, g \rangle$ has order 28, which is impossible since every group of order 28 has an element of order 14 by GAP [20]. Consequently, we have $\pi_e(G) \subseteq \{1, 2, 3, 7\}$.

Assume that $\pi_e(G) = \{1, 7\}$. Since G has precisely one subgroup $\langle g \rangle$ of order 7, by Lemma 2.1, we have that $G = \langle g \rangle \cong \mathbb{Z}_7$, as desired. In the following, we assume that $\{1, 7\} \subset \pi_e(G)$.

Case 1. G has an involution a.

Then $\langle g, a \rangle$ has order 14. Since G has no elements of order 14, we have

$$\langle g, a \rangle = \langle g, a : g^7 = a^2 = e, aga = g^{-1} \rangle \cong D_{14}.$$

Suppose, by contradiction, that there exists an involution $b \in G$ such that $b \notin \langle g, a \rangle$. As before $\langle g, b \rangle \cong D_{14}$ and $bgb = g^{-1}$. If $[a, b] = e$, then $gabgab = ga(bgb)a = gag^{-1}a = g^2$, and so $(gab)^7 = ab$, which implies $o(gab) = 14$, which is impossible. We conclude that $[a, b] \neq e$. It follows that $\langle a, b \rangle \cong D_6$ or D_{14} by $\pi_e(G) \subseteq \{1, 2, 3, 7\}$. If $\langle a, b \rangle \cong D_{14}$, then $g, a \in \langle a, b \rangle$, and so $\langle a, b \rangle = \langle g, a \rangle$, contrary to $b \notin \langle g, a \rangle$. As a consequence, we have $\langle a, b \rangle \cong D_6$, and so $(ab)^3 = e$. Therefore, we have

$$(gab)^3 = gab(gab)^2 = (gab)(ga(bgb)bab) = (gab)(g(ag^{-1}a)abab) = (gab)g^2(ab)^2 = g^3(ab)^3 = g^3$$

and so $(gab)^7 = ab$, which implies $o((gab)^3) = 7$ and $o((gab)^7) = 3$, it follows that $o(gab) = 21$, a contradiction. We conclude that every involution of G belongs to $\langle g, a \rangle$. It follows that G has precisely seven involutions $a, ga, g^2a, \ldots, g^6a$. Suppose now, by contradiction, that there exists an element c of order 3. Then $\langle g, c \rangle = \langle g \rangle \langle c \rangle$ has order 21. By GAP [20], there are two groups of order 21 up to isomorphism, that is, $\mathbb{Z}_{21}$ and $\mathrm{SmallGroup}(21, 1)$. Since G has no elements of order 21, we have $\langle g, c \rangle = \langle g, c : g^7 = c^3 = e, c^{-1}gc = g^4 \rangle \cong \mathrm{SmallGroup}(21, 1)$. Note that cac^{-1} is an involution. We may assume that $cac^{-1} = g^ta$ for some $0 \leq t \leq 6$. Note that $aga = g^{-1}$ and $cgc^{-1} = g^2$. We have

$$c(g^{-t}a) = ca(ag^{-t}a)$$
$$= ca(g^t)$$
$$= (g^t ac)g^t$$
$$= (g^t a)(cg^t)$$
$$= (g^t a)(g^{2t}c)$$
$$= (g^t)(ag^{2t}a)(ac)$$
$$= (g^t)(g^{-2t})(ac)$$
$$= (g^{-t}a)c.$$

It follows that $g^{-t}ac$ is an element of order 6, a contradiction. Thus, in this case, $\pi_e(G) = \{1, 2, 7\}$. Since every involution of G belongs to $\langle g, a \rangle$, and G has precisely one subgroup $\langle g \rangle$ of order 7, we have that $G = \langle g, a \rangle \cong D_{14}$, as desired.

Case 2. *G* has no involutions.

Then $\pi_e(G) = \{1, 3, 7\}$. Let $w \in G$ with $o(w) = 3$. Note that a group of order 21 is isomorphic to either $\mathbb{Z}_{21}$ or SmallGroup(21, 1). Hence, by Lemma 2.11, we have that

$$\langle g, w \rangle \cong \text{SmallGroup}(21, 1) \cong \langle g, w : g^7 = w^3 = e, w^{-1}gw = g^4 \rangle.$$

Now let P be a Sylow 3-subgroup of G. Since $\pi_e(G) = \{1, 3, 7\}$, we have $G = P\langle g \rangle$. Suppose, by contradiction, that $|P| > 3$. Then every non-trivial element of P has order 3. Note that the fact that a non-trivial p-group has non-trivial center. We have that there exist $u, v \in P$ such that $o(u) = o(v) = 3$, $\langle u \rangle \neq \langle v \rangle$, and $[u, v] = e$. Note that $\langle g, u \rangle \cong \langle g, v \rangle \cong \text{SmallGroup}(21, 1)$. We have $u^{-1}gu = v^{-1}gv = g^4$. It follows that

$$(u^2 vg)^3 = (u^2 v(gu)uvg)(u^2 vg)$$
$$= (u^2 v(ug^4)uvg)(u^2 vg)$$
$$= (vg^4 uvg)(u^2 vg)$$
$$= (gvuvg)(u^2 vg)$$
$$= (guv^{-1}g)(u^2 vg)$$
$$= (ug^4 g^4 v^{-1})(u^2 vg)$$
$$= (ugv^{-1})(u^2 vg)$$
$$= ug(u^{-1}g)$$
$$= ugg^4 u^{-1}$$
$$= ug^5 u^{-1},$$

and so $o((u^2 vg)^3) = o(g^5) = 7$. Moreover, we have

$$(u^2 vg)^7 = (ug^5 u^{-1})(ug^5 u^{-1})(u^2 vg)$$
$$= (ug^3 u^{-1})(u^2 vg)$$
$$= (ug^3 u^{-1}v)(u^{-1}g)$$
$$= (ug^3 u^{-1})(vg^4)u^{-1}$$
$$= (ug^3 u^{-1})(gv)u^{-1}$$
$$= ug^3(u^{-1}g)vu^{-1}$$
$$= ug^3(g^4 u^{-1})vu^{-1}$$
$$= u^2 v,$$

and so $o((u^2 vg)^7) = 3$. It follows that $u^2 vg$ has order 21, a contradiction. We conclude that $|P| = 3$, and thus G has order 21 so have that $G = \langle g, w \rangle \cong \text{SmallGroup}(21, 1)$, as desired. $\square$

Lemma 3.4. *Let G be a group with $\{6\} \subseteq \pi_e(G) \subseteq \{1, 2, 3, 4, 6\}$. Then the following facts are equivalent*:
(a) $\gamma(\mathcal{P}_I(G)) = 1$;
(b) $G \cong \mathbb{Z}_6$ *or* D_{12};
(c) $\bar{\gamma}(\mathcal{P}_I(G)) = 1$.

Proof. We first prove that (a) and (b) are equivalent. By Lemma 2.11, we have $\gamma(\mathcal{P}_I(\mathbb{Z}_6)) = 1$. Moreover, by (1), it is easy to see that $\mathcal{P}_I(D_{12})$ has exactly one block isomorphic to K_6, and every other block is isomorphic to K_2. As a consequence, $\gamma(\mathcal{P}_I(D_{12})) = 1$ by Theorem 2.6. Thus, we have that (b) implies (a).

Conversely, suppose that $\gamma(\mathcal{P}_I(G)) = 1$. Let $g \in G$ with $o(g) = 6$. We first claim that G has a unique cyclic subgroup of order 6. By Lemma 2.2, we have that the number of cyclic subgroups of order 6 is not equal to 2. Suppose, by contradiction, that G has at least three distinct cyclic subgroups $\langle x \rangle$, $\langle y \rangle$, and $\langle z \rangle$ of order 6. Assume that $\langle x \rangle$, and $\langle y \rangle$ satisfy $|\langle x \rangle \cap \langle y \rangle| = 1$ or 2. Let $U = \{u \in \langle x \rangle \cup \langle y \rangle : o(u) \neq 2\}$. Then the subgraph Δ induced by U is a union of two complete graphs of order 5 that share the identity vertex, as displayed in Figure 5. Thus, Δ has two distinct blocks isomorphic to K_5. It follows that $\gamma(\mathcal{P}_I(G)) \geq \gamma(\Delta) \geq 2$ by Theorem 2.6, a contradiction. The same argument holds for $\langle x \rangle$ and $\langle z \rangle$ or $\langle y \rangle$ and $\langle z \rangle$. We conclude that $|\langle x \rangle \cap \langle y \rangle \cap \langle z \rangle| = 3$, which implies that the subgraph induced by $(\langle x \rangle \cup \langle y \rangle \cup \langle z \rangle) \setminus \{x^3, y^3, z^3\}$ is isomorphic to K_9, a contradiction as $\gamma(K_9) = 3$. Consequently, G has a unique cyclic subgroup $\langle g \rangle$ of order 6 and has exactly two elements of order 6, that is, g and g^{-1}. In particular, $\langle g \rangle$ is normal in G.

If G has an element a of order 3 such that $a \notin \langle g \rangle$, then $\langle g, a \rangle = \langle g \rangle \langle a \rangle$ has order 18. But by GAP [20], there is no group A of order 18 such that $\pi_e(A) \subseteq \{1, 2, 3, 4, 6\}$ and A has precisely two elements of order 6. We conclude that G has exactly two elements of order 3, that is, g^2 and g^4.

Case 1. G has a unique involution.

If G has no elements of order 4, then by the above discussion, $G \cong \mathbb{Z}_6$, as desired. Suppose next, by contradiction, that G has an element of order 4. If G has two distinct cyclic subgroups $\langle b \rangle$ and $\langle c \rangle$ of order 4, since G has a unique involution, we have $|\langle b \rangle \cap \langle c \rangle| = 2$. Note that $|\langle b \rangle \cap \langle c \rangle \cap \langle g \rangle| = 2$. Then the subgraph induced by $\langle b \rangle \cup \langle c \rangle \cup \langle g \rangle$ has a subgraph Λ is a union of a complete graph of order 5 and a complete graph of order 6 that share the identity vertex, as displayed in Figure 6. Thus, Λ has two distinct blocks isomorphic to K_5 and K_6. Consequently, $\gamma(\mathcal{P}_I(G)) \geq \gamma(\Lambda) \geq 2$ by Theorem 2.6, a contradiction. We conclude that G has a unique cyclic subgroup of order 4. Note now that G has a unique involution, two distinct elements of order 4, two distinct elements of order 6, and two distinct elements of order 3. Since $\pi_e(G) \subseteq \{1, 2, 3, 4, 6\}$, it follows that G has order 8, which is impossible since a group of order 8 has no elements of order 6.

Case 2. G has at least two involutions.

We first prove that G has no elements of order 4. Suppose, by contradiction, that G has an element h of order 4. Then $\langle g, h \rangle = \langle g \rangle \langle h \rangle$ has order 24 or 12. By GAP [20], there is no group B of order 24 such that $\pi_e(B) \subseteq \{1, 2, 3, 4, 6\}$ and B has precisely two elements of order 6, we have that $\langle g, h \rangle$ has order 12. It follows that the intersection of $\langle g \rangle$ and $\langle h \rangle$ has size 2. This implies that if $\langle k \rangle \subseteq G$ is of order 4 and different from $\langle h \rangle$, then $|\langle k \rangle \cap \langle h \rangle| = 2$. Since the intersection of two distinct cyclic subgroups of order 4 has size 1 by the proof of Case 1, we have that G has exactly one cyclic subgroup $\langle h \rangle$ of order 4. Note that G has exactly two elements g^2, g^4 of order 3. It follows that both $\langle h \rangle$ and $\langle g^2 \rangle$ are normal in G, and so $h^{-1}g^{-2}hg^2 \in \langle h \rangle \cap \langle g^2 \rangle = \{e\}$. Thus, h and g^2 commute, and so $o(hg^2) = 12$, a contradiction since $\pi_e(G) \subseteq \{1, 2, 3, 4, 6\}$. We conclude that G has no elements of order 4.

Thus, $\pi_e(G) = \{1, 2, 3, 6\}$. Let u be an involution of G with $u \notin \langle g \rangle$. Then $\langle g, u \rangle = \langle g \rangle \langle u \rangle$ has order 12. Note that G has a unique cyclic subgroup of order 6. We deduce that $\langle g, u \rangle \cong D_{12}$ by GAP [20]. In order to
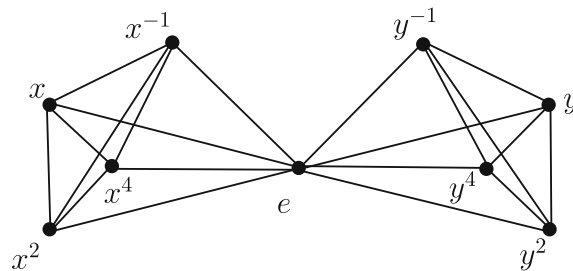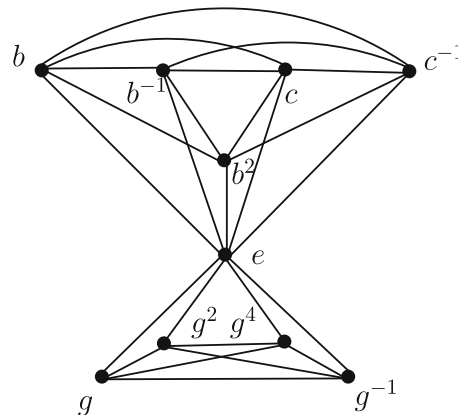

Figure 5: The subgraph Δ.

Figure 6: The subgraph Λ.

prove $G = \langle g, u \rangle$, it suffices to show that every involution of G belongs to $\langle g, u \rangle$. Suppose, by contradiction, that G has an involution $v \notin \langle g, u \rangle$. Then $\langle g, v \rangle \cong D_{12}$, and so $vgv = g^{-1}$.

Assume that $[u, v] = e$. Then $(guv)^2 = gu(vgv)u = g^2$, and so $(guv)^3 = g^3uv \neq e$ since $v \notin \langle g, u \rangle$. Note that $(g^3uv)^2 = g^3u(vg^3v)u = g^6 = e$. It follows that $o((guv)^2) = 3$ and $o((guv)^3) = 2$, and so guv has order 6. Now it is easy to see that $guv \notin \{g, g^5\}$, hence G has at least three distinct elements of order 6, a contradiction.

Assume next that $[u, v] \neq e$. Note that $\langle u, v \rangle$ is a dihedral group. It follows from $\pi_e(G) = \{1, 2, 3, 6\}$ that $\langle u, v \rangle \cong D_6$ or D_{12}. If $\langle u, v \rangle \cong D_{12}$, then $g \in \langle u, v \rangle$, and so $v \in \langle u, v \rangle = \langle g, u \rangle$, a contradiction. As a consequence, we have $\langle u, v \rangle \cong D_6$, which implies $o(uv) = 3$. Note that G has exactly two elements g^2, g^4 of order 3. Thus, we have $uv = g^i$ where $i = 2$ or 4. It follows that $v = ug^i \in \langle g, u \rangle$, a contradiction. Now we have that (a) and (b) are equivalent.

Note next that $\bar{\gamma}(\Delta) = 2$ and $\bar{\gamma}(\Lambda) = 2$ by Theorems 2.3 and 2.6. Then similar to the above proof, we get that (b) and (c) are equivalent. □

4 Proofs of Theorems 1.1 and 1.2

We are now ready to prove our main theorems.

Proof of Theorem 1.1. The sufficiency follows from Lemmas 3.1, 3.2, 3.3, and 3.4. Suppose next that $\gamma(\mathcal{P}_I(G)) = 1$. Then Lemma 2.11 implies $\pi_e(G) \subseteq \{1, 2, 3, 4, 5, 6, 7\}$. If $\pi_e(G) \subseteq \{1, 2, 3, 4\}$, then Lemma 3.1 implies the desired result. Thus, we may assume there that G has an element of order 5, 6, or 7. Suppose that G has an element a of order 5. If G has an element b of order 6 or 7, then the subgraph induced by $\langle a \rangle \cup \langle b \rangle$ has a subgraph Δ, which is isomorphic to a union of two complete graphs of order 5 that share the identity element of G. Thus, Δ has precisely two blocks isomorphic to K_5, and so $\gamma(\Delta) = 2$ by Theorems 2.3 and 2.6, a contradiction. Thus, $\{5\} \subseteq \pi_e(G) \subseteq \{1, 2, 3, 4, 5\}$, and Lemma 3.2 implies the desired result. Similarly, we have the desired result if G contains an element of order 6 or 7 using Lemmas 3.3 and 3.4. □

The proof of Theorem 1.1 can be modified to prove Theorem 1.2. Thus, we omit the proof of Theorem 1.2 for the sake of brevity.

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