Open Mathematics

Research Article

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Finite groups with star-free noncyclic graphs

https://doi.org/10.1515/math-2019-0071
Received October 7, 2018; accepted June 19, 2019

Abstract: For a finite noncyclic group G, let $\mathrm{Cyc}(G)$ be the set of elements a of G such that $\langle a, b\rangle$ is cyclic for each b of G. The noncyclic graph of G is a graph with the vertex set $G \setminus \mathrm{Cyc}(G)$, having an edge between two distinct vertices x and y if $\langle x, y\rangle$ is not cyclic. In this paper, we classify all finite noncyclic groups whose noncyclic graphs are $K_{1,n}$-free, where $K_{1,n}$ is a star and $3 \le n \le 6$.

Keywords: Noncyclic graph, finite group, star-free

MSC: 05C25, 20B05

1 Introduction

All groups considered in this paper are finite. Let G be a noncyclic group. The *cyclicizer* $\mathrm{Cyc}(G)$ of G is the set

$$\{a \in G : \langle a, b\rangle \text{ is cyclic for each } b \in G\},$$

which is a normal cyclic subgroup of G (see [1]). Graphs associated with groups and other algebraic structures have been actively investigated, since they have valuable applications (cf. [2–5]) and are related to automata theory (cf. [5, 6]).

The *noncyclic graph* Γ_G of G is the graph whose vertex set is $G \setminus \mathrm{Cyc}(G)$, and two distinct vertices are adjacent if they do not generate a cyclic subgroup. In 2007, Abdollahi and Hassanabadi [7] introduced the concept of a noncyclic graph and established some basic graph theoretical properties of noncyclic graphs. In [8], Abdollahi and Hassanabadi investigated the clique number of a noncyclic graph. Recently, Costa *et. al* [9] studied the Eulerian properties of noncyclic graphs of finite groups. Aalipour *et. al* [10] studied the relationship between the complement graph of a noncyclic graph and two well-studied graphs–power graphs [11–17] and commuting graphs [18]. Finite groups whose noncyclic graphs have genus one were classified by Selvakumar and Subajini [19] and, independently, by Ma [20]. Moreover, the full automorphism group of a noncyclic graph was determined in [21].

A graph is said to be *Γ-free* if it has no induced subgraphs isomorphic to Γ. Forbidden graph characterization appears in many contexts; for instance, forbidden subgraph problem (Turán-type problem), or extremal graph theory where lower and upper bounds can be obtained for various numerical invariants of the corresponding graphs. Some graphs obtained from groups with small forbidden induced subgraphs have been studied in the literature. For example, Doostabadi *et al.* [22] studied the finite groups with $K_{1,3}$-free power graphs. Akhlaghi and Tong-Viet [23] studied the finite groups with K_4-free prime graphs, where K_4 is the complete graph of order 4. Kayacan [24] classified the finite groups with $K_{3,3}$-free intersection graphs of

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subgroups, where $K_{3,3}$ is the complete bipartite graph with each partition of size 3. In [25], Das and Nongsiang classified K_3-free commuting graphs of finite non-abelian groups.

In this paper, we study noncyclic graphs of finite groups. In Sect. 2, we classify all finite groups G with a unique involution and $\pi_e(G) = \{2, 3, 4, 6\}$, where $\pi_e(G)$ is the set of natural numbers consisting of orders of non-identity elements of G. In Sect. 3, we classify all finite noncyclic groups whose noncyclic graphs are $K_{1,n}$-free, where $3 \le n \le 6$.

2 A result on finite groups

An element of order 2 in a group is called an *involution*. The *exponent* of G is the least common multiple of the orders of the elements of G. We denote the cyclic group of order n and the quaternion group of order 8 by $\mathbb{Z}_n$ and Q_8, respectively. Also $\mathbb{Z}_n^m$ is used for the m-fold direct product of the cyclic group $\mathbb{Z}_n$ with itself.

In this section we prove the following a result about finite groups, which will be used to classify finite groups with $K_{1,5}$-free noncyclic graphs.

Theorem 2.1. *Let G be a finite group having a unique involution and that $\pi_e(G) = \{2, 3, 4, 6\}$. Then, either $G \cong SL(2, 3)$ or $G \cong \mathbb{Z}_3^n \rtimes \mathbb{Z}_4$, where $\mathbb{Z}_4$ acts on $\mathbb{Z}_3^n$ by inversion.*

Let G be a finite group and p a prime number dividing $|G|$. Denote by $\mathrm{Syl}_p(G)$ and $O_p(G)$ the set of all Sylow p-subgroups of G and the largest normal p-subgroup of G, respectively. Note that $O_p(G) = \bigcap_{P \in \mathrm{Syl}_p(G)} P$. Let $n_p = |\mathrm{Syl}_p(G)|$ and $P \in \mathrm{Syl}_p(G)$. Recall that $n_p = |G : N_G(P)| \equiv 1 \pmod p$ and n_p is a divisor of $|G : P|$.

Lemma 2.2. *Let G be a finite group and suppose that $n_p = p + 1$ for some prime number p. Then for any two distinct $P_i, P_j \in \mathrm{Syl}_p(G)$, $P_i \cap P_j = O_p(G)$.*

Proof. Let $m = n_p$ and $L = \mathrm{Syl}_p(G) = \{P_1, P_2, \dots, P_m\}$. Now in order to prove the required result, it suffices to prove the equality $P_1 \cap P_2 = O_p(G)$.

Let $R = P_1 \cap P_2$ and let R act on L by conjugation. Note that for all i, we have that $(R \cap N_G(P_i))P_i = P_i(R \cap N_G(P_i))$ and $(R \cap N_G(P_i))P_i$ is a p-subgroup of $N_G(P_i)$, so $R \cap N_G(P_i) \subseteq P_i$. It follows that $R_{P_i} = R \cap N_G(P_i) = R \cap P_i$, where R_{P_i} is the stabilizer of P_i in R. Also, since $R = P_1 \cap P_2$, we deduce that $|\mathrm{Orbit}_R(P_1)| = |\mathrm{Orbit}_R(P_2)| = 1$, where $\mathrm{Orbit}_R(P_i)$ is the R-orbit containing P_i. Note that every R-orbit has length 1 or p. Since $|L| = p + 1$, we have that every R-orbit has length 1. This implies that $R = R_{P_i}$ for all i. It follows that for each $i \ge 3$, $P_1 \cap P_2 = P_1 \cap P_2 \cap P_i$, that is, $P_1 \cap P_2 \subseteq P_i$. Thus, $P_1 \cap P_2 \subseteq \bigcap_{i=3}^m P_i$ and so $P_1 \cap P_2 = \bigcap_{P \in \mathrm{Syl}_p(G)} P = O_p(G)$, as desired. ☐

Note that for any prime number p, a p-group with a unique subgroup of order p is either a cyclic group or a generalized quaternion group (see [26, Theorem 5.4.10 (ii)]).

Proof of Theorem 2.1. Suppose that $|G| = 2^t \cdot 3^n$ for some $t \ge 1, n \ge 1$. Let Q and P be a Sylow 2-subgroup and a Sylow 3-subgroup of G, respectively. Since G has a unique involution and $8 \notin \pi_e(G)$, we know that $Q \in \{Q_8, \mathbb{Z}_4\}$. Since G has no elements of order 9, we deduce that P has exponent 3. Denote by x the unique involution of G. Then $x \in Z(G)$, the center of G.

Case 1. $Q = Q_8$.

Let $\langle a \rangle$, $\langle b \rangle$, and $\langle c \rangle$ be the three cyclic subgroups of Q of order 4, and that $ab = c$. Then $a^2 = b^2 = c^2 = x$. Since in this case $|G| = 8 \cdot 3^n$, we have that n_3 is a divisor of 8. This implies that $n_3 = 1$ or 4.

Suppose that $O_3(G) \neq 1$. Then let a, b, c act on $O_3(G)$ by conjugation. Neither of them can fix any non-identity elements of $O_3(G)$, since G has no elements of order 12. Thus, a, b, c act as fixed-point-free automorphisms of $O_3(G)$. Since $a^2 = b^2 = c^2 \in Z(G)$, we have that a, b, c act as fixed-point-free automorphisms of order 2. Now by Burnside's result (see [26, Theorem 1.4, page 336] or [27]), we know that

$O_3(G)$ is abelian and for each non-trivial element $g \in O_3(G)$, we have that $g^a = g^{-1} = g^b = g^c$. It then follows that $g^c = g^{ab} = g$, and hence $|abg| = 12$, a contradiction. Therefore, we conclude $O_3(G) = 1$.

Now we know that $n_3 = 4$. By Lemma 2.2 we have that for any two distinct $P_i, P_j \in \mathrm{Syl}_3(G)$, $P_i \cap P_j = 1$. It follows that the number of elements of order 3 is $4(3^n - 1)$. Also, since every element of order 3 and x can generate a cyclic subgroup of order 6, the number of elements of order 6 is $4(3^n - 1)$. Now all that remains is to count the number of elements of order 4.

Let w be an element of order 4 in G. Then, there is a $Q_1 \in \mathrm{Syl}_2(G)$ so that $w \in Q_1$. Note that $Q_1 \cong Q_8$. It follows that $Q_1 \subseteq N_G(\langle w\rangle)$. If there exists an element y of order 3 such that $\langle w\rangle^y = \langle w\rangle$, then $\langle w\rangle$ is normal in $\langle w\rangle\langle y\rangle$ and $|\langle w\rangle\langle y\rangle| = 12$, and so by the N/C lemma we have $C_G(\langle w\rangle) = N_G(\langle w\rangle)$, which implies that $\langle w\rangle\langle y\rangle \cong \mathbb{Z}_{12}$, a contradiction. It follows that $Q_1 = N_G(\langle w\rangle)$. Thus, every element of order 4 is contained in a unique Sylow 2-subgroup of G. It means that the number of elements of order 4 is $6n_2$.

Suppose that $N_G(Q) = Q$. Then $n_2 = 3^n$. Counting all the elements of G gives that

$$8 \cdot 3^n = 6 \cdot 3^n + 8(3^n - 1) + 2.$$

This implies that $3^n = 1$, contrary to the order of G. Thus, we have $Q \subset N_G(Q)$.

Suppose that $|P \cap N_G(Q)| \geq 9$. Then there exist $w_1, w_2 \in P \cap N_G(Q)$ so that $\langle w_1, w_2\rangle$ is an abelian group of order 9. Now both w_1 and w_2 act on Q by conjugation. We conclude that both w_1 and w_2 act as 3-cycles on $\{\langle a\rangle, \langle b\rangle, \langle c\rangle\}$ because otherwise $12 \in \pi_e(G)$. But then there is an element u of order 3 that fixes some cyclic subgroup $\langle v\rangle$ of order 4, where $u = w_1 w_2^i$ for some integer i. It follows that there exists an element of order 12 in $\langle u\rangle\langle v\rangle$, a contradiction. Thus, we get $|P \cap N_G(Q)| \leq 3$.

Note by the modular law that $N_G(Q) = G \cap N_G(Q) = Q(P \cap N_G(Q))$. Since $Q \subset N_G(Q)$, we have that $P \cap N_G(Q)$ is a subgroup of order 3 and $|N_G(Q)| = 24$. This forces that $n_2 = 3^{n-1}$. Now as above we get that

$$8 \cdot 3^n = 6 \cdot 3^{n-1} + 8(3^n - 1) + 2,$$

which implies $n = 1$ and so $|G| = 24$. Note that in this case Q is normal in G. It is easy to see that $G \cong SL(2, 3)$.

Case 2. $Q = \mathbb{Z}_4$.

Let $Q = \langle y\rangle$. Since $\langle x\rangle P \subseteq N_G(P)$, we deduce $|G : N_G(P)| \neq 4$. Note that n_3 is a divisor of 4. Then $n_3 = 1$ and so P is normal in G. Now as above y acts as a fixed-point-free automorphism of order 2 on P by conjugation. By Burnside's result, P is abelian and so $P \cong \mathbb{Z}_3^n$ for some n, and for all $w \in P$ we have $w^y = w^{-1}$. It follows that $G \cong \mathbb{Z}_3^n \rtimes \mathbb{Z}_4$, as desired. □

3 Main results

In this section we classify all finite groups with $K_{1,n}$-free noncyclic graphs, where $3 \leq n \leq 6$.

In the remainder of this paper, we always use G to denote a finite noncyclic group with the identity element e. Euler's totient function is denoted by ϕ. A proper cyclic subgroup $\langle x\rangle$ is said to be *maximal* in G if $\langle x\rangle \subseteq \langle y\rangle$ implies that $\langle x\rangle = \langle y\rangle$, where y is an element of G. We first begin with the following two lemmas which will be used frequently in the sequel.

Lemma 3.1. *Suppose that $\langle g\rangle$ is a maximal cyclic subgroup of G. Then Γ_G has an induced subgraph isomorphic to $K_{1,\phi(|g|)}$.*

Proof. Let $n = \phi(|g|)$ and let $\{g_1, g_2, \ldots, g_n\}$ be all generators of $\langle g\rangle$. Note that G is noncyclic. Pick an element a in $G \setminus \langle g\rangle$. Since $\langle g\rangle$ is maximal cyclic, we deduce that $\langle a, g_i\rangle$ is not cyclic for each $i \in \{1, 2, \ldots, n\}$. This implies that $\{g_1, g_2, \ldots, g_n, a\}$ induces a subgraph isomorphic to $K_{1,n}$. □

For a graph Γ, we denote the sets of the vertices and the edges of Γ by $V(\Gamma)$ and $E(\Gamma)$, respectively. An *independent set* of Γ is is a subset of the vertices such that no two vertices in the subset represent an edge of Γ. The *independence number* of a graph Γ is the cardinality of the largest independent set and is denoted by $\alpha(\Gamma)$. The following result follows from [7, Proposition 4.6].

Lemma 3.2. $\alpha(\Gamma_G) = \max\{|g| : g \in G\} - |\mathrm{Cyc}(G)|$.

Γ_G is complete if and only if G is an elementary abelian 2-group (see [7, Proposition 3.1]). So, as Γ_G is connected (see [7, Proposition 3.2]), we first note that Γ_G is $K_{1,2}$-free if and only if G is an elementary abelian 2-group.

A *claw* is another name for the complete bipartite graph $K_{1,3}$. We first classify the finite groups whose noncyclic graphs are claw-free.

Theorem 3.3. *Γ_G is claw-free if and only if G is isomorphic to one of the following groups:*
(*a*) Q_8;
(*b*) $\mathbb{Z}_2^n$, $n \geq 2$;
(*c*) *A noncyclic* 3*-group of exponent* 3;
(*d*) *A noncyclic group G with* $\pi_e(G) = \{2, 3\}$.

Proof. Since $\Gamma_{Q_8} \cong K_{2,2,2}$, we have that Γ_{Q_8} is claw-free. Also, by Lemma 3.2 we see that the independence number of the noncyclic graph of every group in (*b*)–(*d*) is at most 2, and so each of the noncyclic graphs is claw-free.

Now we suppose that Γ_G is claw-free. It follows from Lemma 3.1 that for every maximal cyclic subgroup $\langle g\rangle$ of G, $\phi(|g|) \leq 2$. This implies that every cyclic subgroup of G has at most two generators. Thus, $\pi_e(G) \subseteq \{2, 3, 4, 6\}$.

Suppose that G has an element a of order 6. Note that G is noncyclic. Pick an element x in $G \setminus \langle a\rangle$. If $|x| = 2$, since $\langle x, a^3\rangle$ is noncyclic, we have that $\{x, a, a^3, a^5\}$ induces a subgraph isomorphic to $K_{1,3}$, which is impossible. If the order of x is 3 or 4, since $\langle x, a^2\rangle$ is noncyclic, it follows that $\{x, a, a^2, a^5\}$ induces a subgraph isomorphic to $K_{1,3}$, which is also impossible. We conclude $|x| = 6$. Namely, every element of $G \setminus \langle a\rangle$ has order 6. However, in this case we have that both x^2 and x^3 belong to $\langle a\rangle$. It follows that $x \in \langle a\rangle$, a contradiction.

Thus, we conclude that $\pi_e(G) \subseteq \{2, 3, 4\}$. Suppose that there exists an element g of order 4 in G. If there is $x \in G \setminus \langle g\rangle$ with $|x| = 2$ or 3, then $\langle x, g^2\rangle$ is noncyclic, and so $\{x, g, g^2, g^3\}$ induces a subgraph isomorphic to $K_{1,3}$, a contradiction. Consequently, in this case G has a unique involution and $\pi_e(G) = \{2, 4\}$. By [26, Theorem 5.4.10 (ii)], we see that G is isomorphic to Q_8.

We now assume $\pi_e(G) \subseteq \{2, 3\}$. If $\pi_e(G) = \{2\}$, then G is an elementary abelian 2-group, as desired. If $\pi_e(G) = \{3\}$, then G is a 3-group of exponent 3, as desired. □

Theorem 3.4. *Γ_G is $K_{1,4}$-free if and only if G is isomorphic to one of the following groups:*
(*a*) *A noncyclic group G with* $\pi_e(G) \subseteq \{2, 3, 4\}$;
(*b*) $\mathbb{Z}_6 \times \mathbb{Z}_2^m$, $m \geq 1$.

Proof. If G is isomorphic to a noncyclic group with $\pi_e(G) \subseteq \{2, 3, 4\}$, then $\alpha(\Gamma_G) \leq 3$ by Lemma 3.2, and so Γ_G is $K_{1,4}$-free, as desired. If $G \cong \mathbb{Z}_6 \times \mathbb{Z}_2^m$ for some $m \geq 1$, then $|\mathrm{Cyc}(G)| = 3$ and we can obtain that Γ_G is a complete multipartite graph whose each partite set has size 3, which implies that in this case Γ_G is also $K_{1,4}$-free.

For the converse, suppose that Γ_G is $K_{1,4}$-free. Note that $\phi(n)$ is even for any integer $n \geq 3$. By Lemma 3.1 we see that each cyclic subgroup of G has at most two generators. It follows that $\pi_e(G) \subseteq \{2, 3, 4, 6\}$. In order to get the desired result, we now suppose that G has an element g of order 6. Clearly, $\langle g\rangle$ is maximal cyclic. If there exists an element a in $G \setminus \langle g\rangle$ such that $|a| = 3$ or 4, then $\{a, g, g^2, g^4, g^5\}$ induces a subgraph isomorphic to $K_{1,4}$, a contradiction. This means that G has a unique subgroup of order 3 and $\pi_e(G) = \{2, 3, 6\}$.

Let P and Q be a Sylow 2-subgroup and a Sylow 3-subgroup, respectively. Then $G = P \ltimes Q$, P is an elementary abelian 2-group of order great than 2 and $Q \cong \mathbb{Z}_3$. Pick an involution u in G. If $\langle u, g^2\rangle$ is noncyclic, then $\{u, g, g^2, g^4, g^5\}$ induces a subgraph isomorphic to $K_{1,4}$, a contradiction. Thus, we have that every element of P and every element of Q commute. It follows that

$$G = P \times Q \cong \mathbb{Z}_6 \times \mathbb{Z}_2^m, \quad m \geq 1,$$

as required. □

Theorem 3.5. *Γ_G is $K_{1,5}$-free if and only if G is isomorphic to one of the following groups:*
(*a*) *A noncyclic group G with $\pi_e(G) \subseteq \{2, 3, 4, 5\}$;*
(*b*) $\mathbb{Z}_6 \times \mathbb{Z}_2^m, \quad m \geq 1$;
(*c*) $\mathbb{Z}_2 \times Q$, *where Q is a noncyclic 3-group of exponent 3;*
(*d*) *The special linear group $SL(2, 3)$;*
(*e*) $\mathbb{Z}_3^n \rtimes \mathbb{Z}_4$, *where $\mathbb{Z}_4$ acts on $\mathbb{Z}_3^n$ by inversion and $n \geq 1$.*

Proof. If G is isomorphic to a group in (a), then $\alpha(\Gamma_G) \leq 4$ by Lemma 3.2, and so Γ_G is $K_{1,5}$-free. Moreover, by Theorem 3.4, $\Gamma_{\mathbb{Z}_6 \times \mathbb{Z}_2^m}$ is $K_{1,5}$-free, where $m \geq 1$. If $G \cong \mathbb{Z}_2 \times Q$ for some noncyclic 3-group Q of exponent 3, then $|\mathrm{Cyc}(G)| = 2$ and we may check that Γ_G is a complete multipartite graph whose each partite set has size 4, which implies that in this case Γ_G is also $K_{1,5}$-free. Furthermore, if G is isomorphic to $SL(2, 3)$ or a group in (e), then it is easy to see that Γ_G is a complete multipartite graph whose maximal partite set has size 4. Thus, Γ_G is $K_{1,5}$-free if G is one group of (d) and (e).

Conversely, suppose that Γ_G is $K_{1,5}$-free. It follows from Lemma 3.1 that $\pi_e(G) \subseteq \{2, 3, 4, 5, 6, 8, 10, 12\}$.

Suppose that $g \in G$ with $|g| = 12$. If there exists an element a with order 5 or 8, then $\{a, g, g^2, g^5, g^7, g^{11}\}$ induces a subgraph isomorphic to $K_{1,5}$, a contradiction. This implies that $\pi_e(G) \subseteq \{2, 3, 4, 6, 12\}$. If G has an element b of $G \setminus \langle g \rangle$ with $|b| < 12$, then $\{b, g, g^5, g^7, g^{11}, g^t\}$ induces a subgraph isomorphic to $K_{1,5}$, where $|g^t| = |b|$. Thus, in this case one has $G \cong \mathbb{Z}_{12}$, a contradiction. This means that G has no elements of order 12. Similarly, we can get $10 \notin \pi_e(G)$. If G has an element of order 8, then a similar argument implies that G is a 2-group and it has a unique involution and a unique cyclic subgroup of order 4, which implies that G is a generalized quaternion group having precisely two elements of order 4, a contradiction. Thus, we conclude $\pi_e(G) \subseteq \{2, 3, 4, 5, 6\}$.

In order to get the desired result, we suppose that G has an element h of order 6. Then it is easy to see that $5 \notin \pi_e(G)$.

Case 1. G has two distinct cyclic subgroups of order 6.

Assume that $H_1 = \langle h \rangle$ and $H_2 = \langle h_2 \rangle$ are two distinct cyclic subgroups of order 6. In order to avoid that $\{h_2, h, h^2, h^3, h^4, h^5\}$ induces $K_{1,5}$, we may assume that $|H_1 \cap H_2| \geq 2$. In fact, any two distinct cyclic subgroups of order 6 have nontrivial intersection.

Subcase 1.1. There exist two distinct cyclic subgroups of order 6 such that their intersection has order 3.

Without loss of generality, we may assume that $|H_1 \cap H_2| = 3$. Suppose that G has an element x of order 4. If $\langle h^3, x \rangle$ is not cyclic, then $\{x, h, h^2, h^3, h^4, h^5\}$ induces a subgraph isomorphic to $K_{1,5}$, a contradiction. We conclude that $\langle h^3, x \rangle$ is cyclic, and so $h^3 = x^2$. Similarly, we also can obtain $h_2^3 = x^2$. It follows that $x^2 \in H_1 \cap H_2$, which is impossible as $|H_1 \cap H_2| = 3$. Hence, in this subcase $\pi_e(G) \subseteq \{2, 3, 6\}$. Since every generator of any maximal cyclic subgroup of order 2 or 3 is adjacent to each of $\langle h \rangle \setminus \{e\}$, every cyclic subgroup of order 2 or 3 is not maximal. If $\langle y \rangle \neq \langle h^2 \rangle$ is a subgroup of order 3, and let $\langle y \rangle \subseteq \langle h_3 \rangle$ with $|h_3| = 6$, then $|\langle h_3 \rangle \cap H_i| = 2$ for $i = 1, 2$ and so $h_3^3 = h^3 = h_2^3$, which is impossible as $H_1 \neq H_2$. This implies that G has a unique subgroup of order 3. Now we know that $G \cong \mathbb{Z}_2^m \ltimes \mathbb{Z}_3$ for some integer $m \geq 2$. Pick an involution u in G. Then since $\langle u \rangle$ is not maximal, there exists an element h' of order 6 such that $\langle u \rangle \subseteq \langle h' \rangle$. By the uniqueness of the subgroup of order 3, we see that $\langle h' \rangle \cap \langle h \rangle = \langle h^2 \rangle$. It follows that $\langle u, h^2 \rangle$ is cyclic. Namely, every involution of G and h^2 commute. This implies that $G \cong \mathbb{Z}_2^m \times \mathbb{Z}_3$ for some integer $m \geq 2$, as desired.

Subcase 1.2. The intersection of each two distinct cyclic subgroups of order 6 has order 2.

In this case we first claim that G has a unique involution. Assume, to the contrary, that u is an involution of G such that $u \neq h^3$. Then $\langle u, h^2 \rangle$ is not cyclic, since there are no two cyclic subgroups of order 6 such that their intersection has order 3. This implies that $\{u, h, h^2, h^3, h^4, h^5\}$ induces a subgraph isomorphic to $K_{1,5}$, a contradiction. Thus, our claim is valid.

Now note that $\pi_e(G) \subseteq \{2, 3, 4, 6\}$. If $4 \notin \pi_e(G)$, then $G \cong \mathbb{Z}_2 \times Q$, where Q is a noncyclic 3-group of exponent 3. Thus, we may assume that $\pi_e(G) = \{2, 3, 4, 6\}$. Note that $\mathbb{Z}_3 \rtimes \mathbb{Z}_4$ has a unique cyclic subgroup of order 6, where $\mathbb{Z}_4$ acts on $\mathbb{Z}_3$ by inversion. By Theorem 2.1 we see that that $G \cong SL(2, 3)$ or $G \cong \mathbb{Z}_3^n \rtimes \mathbb{Z}_4$, where $\mathbb{Z}_4$ acts on $\mathbb{Z}_3^n$ by inversion, and $n \geq 2$, as required.

Case 2. G has a unique cyclic subgroup of order 6.

We first see that $\langle h \rangle$ is a normal subgroup of G. Note that $\pi_e(G) \subseteq \{2, 3, 4, 6\}$. If G has an element x in $G \setminus \langle h \rangle$ such that $x \in C_G(h)$, the centralizer of h in G, then G has a subgroup isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_6$ or $\mathbb{Z}_3 \times \mathbb{Z}_6$, which contradicts the fact that G has precisely two elements of order 6. This implies that $C_G(h) = \langle h \rangle$. So $G/\langle h \rangle$ is isomorphic to a subgroup of $\mathbb{Z}_2$, and hence $|G| = 6$ or 12. It follows that $G \cong \mathbb{Z}_3 \rtimes \mathbb{Z}_4$, where $\mathbb{Z}_4$ acts on $\mathbb{Z}_3$ by inversion, as desired. □

Theorem 3.6. *Γ_G is $K_{1,6}$-free if and only if G is isomorphic to one of the following groups:*
(a) A noncyclic group G with $\pi_e(G) \subseteq \{2, 3, 4, 5, 6\}$;
(b) $\mathbb{Z}_{10} \times \mathbb{Z}_2^m$, $m \geq 1$.

Proof. If G is isomorphic to a group in (a), then it follows from Lemma 3.2 that $\alpha(\Gamma_G) \leq 5$, and so Γ_G is $K_{1,6}$-free. If $G \cong \mathbb{Z}_{10} \times \mathbb{Z}_2^m$ for some $m \geq 1$, then $|\mathrm{Cyc}(G)| = 5$ and we may check that Γ_G is a complete multipartite graph whose each partite set has size 5, which implies Γ_G is $K_{1,6}$-free.

For the converse, suppose that Γ_G is $K_{1,6}$-free. Since $\phi(n)$ is even for $n \geq 3$, by Lemma 3.1, we have that $\phi(|g|) \leq 4$ for any $g \in G$. It follows that $\pi_e(G) \subseteq \{2, 3, 4, 5, 6, 8, 10, 12\}$. An argument similar to the one used in the third paragraph of the proof of Theorem 3.5 shows that $8, 12 \notin \pi_e(G)$. Consequently, we have $\pi_e(G) \subseteq \{2, 3, 4, 5, 6, 10\}$.

In order to get the desired result, we suppose that G has an element h of order 10. Then it is easy to see that $\pi_e(G) = \{2, 5, 10\}$ and G has a unique subgroup of order 5. Thus, we may assume that $G \cong P \ltimes \mathbb{Z}_5$, where P is an elementary abelian 2-group of order at least 4. Pick any involution u in P. If $\langle u, h^2 \rangle$ is not cyclic, then $\{u, h^2, h^4, h^6, h^8, h^5, h\}$ induces a subgraph isomorphic to $K_{1,6}$, a contradiction. Thus, every element in P and h^2 commute. It follows that $G \cong P \times \mathbb{Z}_5$, that is, $G \cong \mathbb{Z}_{10} \times \mathbb{Z}_2^m$ for some $m \geq 1$, as desired. □

Acknowledgements: We are grateful to the referee for many useful suggestions and comments.

Ma was supported by the National Natural Science Foundation of China (Grant No. 11801441), the Scientific Research Program funded by Shaanxi Provincial Education Department (Program No. 18JK0623), and the Young Talent fund of University Association for Science and Technology in Shaanxi, China (Program No. 20190507). Wang was supported by the National Natural Science Foundation of China (Grant No. 11671043).

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