

Finite, integrable and bounded time embeddings for diffusions

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We solve the Skorokhod embedding problem (SEP) for a general *time-homogeneous diffusion* X : given a distribution ρ , we construct a stopping time τ such that the stopped process X_τ has the distribution ρ . Our solution method makes use of martingale representations (in a similar way to Bass (In *Seminar on Probability XVII. Lecture Notes in Math.* 784 (1983) 221–224 Springer) who solves the SEP for Brownian motion) and draws on law uniqueness of weak solutions of SDEs.

Then we ask if there exist solutions of the SEP which are respectively finite almost surely, integrable or bounded, and when does our proposed construction have these properties. We provide conditions that guarantee existence of *finite time* solutions. Then, we fully characterize the distributions that can be embedded with *integrable* stopping times. Finally, we derive necessary, respectively sufficient, conditions under which there exists a *bounded* embedding.

Keywords: bounded time embedding; Skorokhod's embedding theorem

Introduction

Let X be a one-dimensional time-homogeneous diffusion, and let ρ be a probability measure on \mathbb{R} . The Skorokhod embedding problem (SEP) for ρ in X is to find a stopping time τ such that $X_\tau \sim \rho$. Our main goals in this article are firstly to construct a solution of the Skorokhod embedding problem, and secondly to discuss when does there exist a solution which is finite, integrable or bounded in time, and when does our construction have these properties.

Our construction is based on the observation that we can remove the drift of the time-homogeneous diffusion by changing the space variable via a *scale function*. We can thus simplify the embedding problem to the case where X is a local martingale diffusion. We then consider a random variable that has the distribution we want to embed and that can be represented as a Brownian martingale N on the time interval $[0, 1]$. Further, we set up an ODE that uniquely determines a time-change along every path of X . We then show, by drawing on a result of uniqueness in law for weak solutions of SDEs, that the time-changed diffusion has the same distribution as the martingale N . Thus the time-change provides a solution of the SEP.

Our solution is a generalization of Bass's solution of the SEP for *Brownian motion* (see [3]). Bass also starts with the martingale representation of a random variable with the given distribution. By changing the martingale's clock, he obtains a Brownian motion and an associated embedding stopping time. The time-change is governed by an ODE, a special case of our ODE,

which establishes an analytic link between Brownian paths and the embedding stopping time. This link yields embedding stopping times for *arbitrary* Brownian motions.

Now consider properties of solutions of the SEP. As is well known from the literature, whether a distribution is embeddable into the diffusion X depends on the relation between the support of the distribution and the state space of X and the relation between the initial value X_0 and the first moment of the distribution. We include in our analysis a general discussion of sufficient and necessary conditions for the existence of *finite* embedding stopping times, with particular reference to our proposed construction.

Next, we fully determine the collection of distributions that can be embedded in X with *integrable* stopping times. The associated conditions involve an integrability condition on the target distribution which makes use of a function that also appears in Feller's test for explosions (see, e.g., [10]).

Finally, we address the question of whether a distribution can be embedded in *bounded* time. Recall that the Root solution ([18]) of the SEP has the property that it minimises $\mathbb{E}[(\tau - t)^+]$ uniformly in t . The Root solution τ_R is of the form $\tau_R = \inf\{t : (X_t, t) \in \mathcal{R}_\rho\}$ where \mathcal{R}_ρ is a 'barrier'; in particular $\mathcal{R}_\rho = \{(x, t) : t \geq \beta_\rho(x)\} \subseteq \mathbb{R} \times \mathbb{R}_+$ for some suitably regular function β_ρ depending on the target law. Hence, a necessary and sufficient condition for there to be an embedding τ with $\tau \leq T$ is that $\beta_\rho(\cdot) \leq T$. However, the Root barrier is non-constructive and difficult to analyse (though for some recent progress in this direction see Cox and Wang [7] and Oberhauser and Dos Reis [13]). For this reason, instead of searching for a single set of necessary *and* sufficient conditions we limit ourselves to finding separate sets of necessary conditions and sufficient conditions.

Our original motivation in developing a solution of the SEP for diffusions was to study bounded stopping times with the aim of providing simple sufficient conditions for the existence of a bounded embedding. The boundedness (finiteness) of an embedding is an important property of the embedding used to solve the gambling in contests problem of Seel and Strack [19], and is also relevant in the model-independent pricing of variance swaps, see Carr and Lee [4], Hobson [9] and Cox and Wang [7].

Consider for a moment the case where X is a real-valued Brownian motion, null at 0. Then it is possible to embed any target probability measure ρ in X . Moreover, ρ can be embedded in integrable time if and only if ρ is centred and in L^2 , and then $\mathbb{E}[\tau] = \int x^2 \rho(dx)$. The case of embeddings in bounded time is more subtle. Clearly a necessary condition for there to exist an embedding τ of ρ in X such that $\tau \leq 1$ is that ρ is smaller than μ_G in convex order, where μ_G is the law of a standard Gaussian. But this is not sufficient. Let $\mu_{\pm a}$ be the uniform measure on $\{-a, +a\}$. Then $\mu_{\pm a}$ is smaller than μ_G in convex order if and only if $a \leq \sqrt{2/\pi}$. But any embedding τ of $\mu_{\pm a}$ has $\tau \geq \min\{u : |B_u| \geq a\}$, and thus does not satisfy $\tau \leq T$ for any T . Hence, we would like to find sufficient conditions on ρ such that there exists $\tau \leq T$ with $X_\tau \sim \rho$. The case where X is Brownian motion, possibly with drift, was considered in Ankirchner and Strack [1]. Here we consider general time-homogeneous diffusions.

The paper is organized as follows. In Section 1, we describe our solution method of the SEP for a diffusion *without* drift. In this section, we assume that the initial value of the diffusion coincides with the first moment of the distribution to embed (the centred case). In the following Section 2, we briefly explain how to construct solutions if the first moment does *not* match the initial value (the non-centred case). In Section 3, we collect some general conditions which guarantee that a

distribution can be embedded into X in finite time. We then consider integrable embeddings in Section 4. Distinguishing between the centred and non-centred case, we provide sufficient and necessary conditions for the existence of integrable solutions of the SEP. Section 5 discusses bounded embeddings. In Section 6, we explain how one can reduce the SEP for diffusions with drift to the case without drift. Finally, in Section 7 we illustrate our results with some examples.

1. The martingale case

We will argue in Section 6 below that the problem of interest can be reduced to the case in which the process is a continuous local martingale. In this section, we describe a generalisation of the Bass [3] solution of the SEP. The Bass solution is an embedding of ν in Brownian motion: we consider embeddings in a local martingale diffusion which may be thought of as time-changed Brownian motion.

Consider the time-homogeneous local martingale diffusion M , where M solves

$$dM_s = \eta(M_s) dW_s, \quad \text{with } M_0 = m; \tag{1}$$

here $m \in \mathbb{R}$ and $\eta: \mathbb{R} \rightarrow \mathbb{R}_+$ is Borel-measurable. We assume that the set of points $x \in \mathbb{R}$ with $\eta(x) = 0$ coincides with the set of points where $\frac{1}{\eta}$ is *not* locally square integrable. Then a result by Engelbert and Schmidt implies that the SDE (1) possesses a weak solution that is unique in law (see, e.g., Theorem 5.5.4 in [10]). We define $l = \sup\{x \leq m \mid \eta(x) = 0\}$ and $r = \inf\{x \geq m \mid \eta(x) = 0\}$ so that $-\infty \leq l \leq r \leq \infty$ (to exclude trivialities we assume $l < m < r$) and for $x \in \mathbb{R}$,

$$q(x) = \int_m^x dy \int_m^y \frac{2}{\eta^2(z)} dz. \tag{2}$$

By our assumption on η , q is infinite on $(-\infty, l)$ and (r, ∞) .

Remark 1. By Feller’s test, $\mathbb{P}[\inf_{s \leq t} M_s \leq l] = 0$ for one, and then every, $t > 0$ if and only if $q(l+) = \lim_{x \downarrow l} q(x) = \infty$. Similarly, $\mathbb{P}[\sup_{s \leq t} M_s \geq r] = 0$ if and only if $q(r-) = \infty$ (see, e.g., Theorem 5.5.29 in [10]). Further, by results of Kotani [11], the local martingale M is a martingale provided either $-\infty < l$ or $\int_{-\infty}^l |x| \eta(x)^{-2} dx = \infty$ and either $r < \infty$ or $\int_r^{\infty} x \eta(x)^{-2} dx = \infty$.

Note that our assumption that $\frac{1}{\eta}$ is not locally square integrable at l and r implies that l and r are absorbing boundaries if they can be reached in finite time. Then without loss of generality we may assume that $\eta = 0$ on $(-\infty, l)$ and (r, ∞) and η is positive on (l, r) .

We want to embed a non-Dirac probability measure ν with $\int x d\nu(x) = m$. Let $\underline{\nu} = \inf\{\text{supp}(\nu)\}$ and $\bar{\nu} = \sup\{\text{supp}(\nu)\}$ be the extremes of the support of ν , and let F be the distribution function associated to the target law ν . Moreover, let $\Phi: \mathbb{R} \rightarrow [0, 1]$ be the cumulative distribution function of the normal distribution and $\varphi = \Phi'$ its density. Define the function $h = F^{-1} \circ \Phi$. Let $(\tilde{W}_t)_{t \geq 0}$ be a Brownian motion on a filtration $\tilde{\mathbb{F}} = (\tilde{\mathcal{F}}_t)_{t \geq 0}$. Notice that $h(\tilde{W}_1)$ has the distribution ν . In particular, $h(\tilde{W}_1)$ is integrable and $\mathbb{E}[h(\tilde{W}_1)] = m$.

We define the $\tilde{\mathbb{F}}$ -martingale $N_t = \mathbb{E}[h(\tilde{W}_1) \mid \tilde{\mathcal{F}}_t]$ for $t \in [0, 1]$. Notice that $N_0 = m$, N_1 has distribution ν and $N_t = b(t, \tilde{W}_t)$, where

$$b(t, x) = \int_{\mathbb{R}} h(y)\varphi_{1-t}(x - y) \, dy = (\varphi_{1-t} \star h)(x),$$

and φ_ν is the density of the normal distribution with variance ν .

Since ν is not a Dirac measure we have that h is increasing somewhere, and hence, for all $t \in [0, 1)$, the mapping $x \mapsto b(t, x)$ is strictly increasing. Thus, we can define the inverse function $B : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ implicitly by

$$b(t, B(t, x)) = x, \quad \text{for all } t \in [0, 1), x \in \mathbb{R}; \tag{3}$$

moreover we set $B(1, x) = h^{-1}(x)$. The process N solves the SDE

$$dN_t = b_x(t, B(t, N_t)) \, d\tilde{W}_t, \quad N_0 = \int x \, d\nu(x) = m. \tag{4}$$

Define

$$\lambda(t, y) = \frac{b_x(t, y)}{\eta(b(t, y))}, \quad \Lambda(t, y) = \lambda(t, B(t, y)) = \frac{b_x(t, B(t, y))}{\eta(y)}. \tag{5}$$

The candidate embedding which we want to discuss is $\delta(1)$ where δ solves

$$\delta'(t) = \Lambda(t, M_{\delta(t)})^2 = \frac{b_x(t, B(t, M_{\delta(t)}))^2}{\eta(M_{\delta(t)})^2}, \quad \delta(0) = 0. \tag{6}$$

Note that δ is increasing so that if δ is defined on $[0, 1)$ then we can set $\delta(1) = \lim_{t \uparrow 1} \delta(t)$.

Theorem 1. *If the ODE (6) has a solution on $[0, 1)$ for almost all paths of M , then $\delta(1)$ embeds F into M , that is, the law of $M_{\delta(1)}$ is ν .*

Proof. Let $Y_t = M_{\delta(t)}$ for all $t \in [0, 1)$. By interchanging the time-change and integration, see, for example, Proposition V.1.5 in [15], we get

$$Y_t - m = \int_0^{\delta(t)} \eta(M_s) \, dW_s = \int_0^t \eta(M_{\delta(s)}) \, dW_{\delta(s)} = \int_0^t \eta(Y_s) \, dW_{\delta(s)}.$$

Let $Z_t = \int_0^t \frac{1}{\sqrt{\delta'(s)}} \, dW_{\delta(s)}$, for $t \in [0, 1]$. Notice that $\langle Z, Z \rangle_t = \int_0^t \frac{1}{\delta'(s)} \, d\delta(s) = t$ (Proposition V.1.5 in [15]) and then by Lévy’s characterization theorem, Z is a Brownian motion on $[0, 1]$. Next, observe that

$$\begin{aligned} Y_t - m &= \int_0^t \eta(Y_s) \sqrt{\delta'(s)} \, dZ_s = \int_0^t \eta(Y_s) \Lambda(s, M_{\delta(s)}) \, dZ_s \\ &= \int_0^t b_x(s, B(s, Y_s)) \, dZ_s, \end{aligned} \tag{7}$$

which shows that Y solves the SDE (4) with W replaced by Z ; in other words (Y, Z) is a weak solution of (4).

It follows directly from Lemma 2(a) in Bass [3] that $b_x(t, B(t, x))$ is Lipschitz continuous in x , uniformly in t , on compact subsets of $[0, 1) \times \mathbb{R}$. Therefore, the SDE (4) has at most one strong solution on $[0, 1)$ and hence (4) is pathwise unique, from which it follows (see, e.g., Section 5.3 in [10]) that solutions of (4) are unique in law. Hence, for $t < 1$, $Y_t = M_{\delta(t)}$ has the same distribution as N_t , and in the limit t tends to 1 we have N_1 and hence Y_1 has law ν . \square

Remark 2. Notice that the assumption that $\int x\nu(dx) = m$ is crucial for the conclusion of Theorem 1. Indeed, if $\int x\nu(dx) \neq m$, then Y and N solve the same SDE, but with *different* initial conditions. Hence, one cannot derive that Y_1 has the same distribution as N_1 .

We next aim at showing that $\delta(1)$ is a stopping time with respect to $\mathbb{F}^M = (\mathcal{F}_t^M)_{t \geq 0}$, the smallest filtration containing the filtration generated by the martingale M and satisfying the usual conditions. To this end we consider, as in [3], the ODE satisfied by the inverse of $\delta(t)$. The ODE for the inverse is Lipschitz continuous and hence guarantees that Picard iterations converge to a unique solution.

Lemma 1. *Let M be a path of the solution of (1). Then (6) has a solution on $[0, 1)$ if and only if there exists $a \in \mathbb{R}_+ \cup \{\infty\}$ such that the ODE*

$$\Delta'(s) = \frac{\eta(M_s)^2}{b_x(\Delta(s), B(\Delta(s), M_s))^2} \tag{8}$$

has a solution on $[0, a)$ with $\lim_{s \uparrow a} \Delta(s) = 1$.

Proof. Assume that there exists a solution of (6) on $[0, 1)$. Set $a = \delta(1)$ and define $\Delta(s) = \delta^{-1}(s)$ for all $s \in [0, a)$. Then a straightforward calculation shows that Δ satisfies (8).

The reverse direction can be shown similarly. \square

Remark 3. If $\eta = 1$, then the ODE (8) is the ODE (1) of Bass' paper [3].

Lemma 2. *Suppose the ODE (6) has a solution on $[0, 1)$ for almost all paths of M . Then $\delta(t)$ is an \mathbb{F}^M -stopping time, for all $t \in [0, 1]$.*

Proof. Let C be a compact subset of $[0, 1) \times \mathbb{R}_+$. By Lemma 2 of Bass [3], $b_x(t, x)$ and $B(t, x)$ are Lipschitz continuous on C . Moreover, on C the function b_x is bounded away from zero and bounded from above. This implies that $\frac{1}{b_x(t, B(t, x))^2}$ is Lipschitz continuous on C , too.

Define the mapping $\gamma : (t, y) \mapsto \frac{\eta(M_t)^2}{b_x(y, B(y, M_t))^2}$. Now let D be a compact subset of $\mathbb{R}_+ \times [0, 1)$. Then there exists an $L \in \mathbb{R}_+$ such that for all (t, y) and $(t, \tilde{y}) \in D$ we have

$$|\gamma(t, y) - \gamma(t, \tilde{y})| \leq L\eta(M_t)^2|y - \tilde{y}|,$$

that is, γ is Lipschitz continuous in the second argument.

We define the Picard iterations $\Delta_0(t) = 0$ and for $n \geq 0$,

$$\Delta_{n+1}(t) = 1 \wedge \int_0^t \frac{\eta(M_s)^2}{b_x(\Delta_n(s), B(\Delta_n(s), M_s))^2} ds.$$

We have that $\Delta_n(t) = 1$ after the first time where Δ_n attains 1. The assumptions on η guarantee that $\int_0^s \eta(M_t)^2 dt$ is finite, a.s. for each $s < a$ (see, e.g., Section 5.5 in [10]). By standard arguments, one can now show that $\Delta_n(t)$ converges to a limit $\Delta(t)$ on $[0, a)$, where $\Delta(a) = 1$ for all $t \geq a$. In particular, $\Delta(t)$ is \mathcal{F}_t^M -measurable; moreover $\Delta(t)$ solves the ODE (8) on $[0, a)$.

Now let $t \in [0, 1)$ and $u \in \mathbb{R}_+$. Observe that

$$\{\delta(t) \leq u\} = \{\Delta(u) \geq t\}.$$

The RHS is \mathcal{F}_u^M -measurable, which implies that $\delta(t)$ is an (\mathcal{F}_t^M) -stopping time. The limit $\delta(1) = \lim_{t \uparrow 1} \delta(t)$ is also an (\mathcal{F}_t^M) -stopping time. □

Lemma 3. *There exists a solution of (6) on $[0, 1)$ for almost all paths of M if and only if $\int_0^T \Lambda(t, N_t)^2 dt < \infty$, a.s. for all $T < 1$. In this case, $\delta(1)$ has the same distribution as $\int_0^1 \Lambda(t, N_t)^2 dt$.*

Proof. For all $n \in \mathbb{N}$ let $\xi_n = 1 \wedge \inf\{t \geq 0 \mid \int_0^t \Lambda(s, M_{\delta(s)})^2 ds \geq n\}$ and $\zeta_n = 1 \wedge \inf\{t \geq 0 \mid \int_0^t \Lambda(s, N_s)^2 ds \geq n\}$. By appealing to uniqueness in law of solutions of (4) one can show, as in the proof of Theorem 1, that $(M_{\delta(t)})_{0 \leq t \leq \xi_n}$ and $(N_t)_{0 \leq t \leq \zeta_n}$ have the same distribution. Moreover, ξ_n and ζ_n have the same distribution, and therefore, $\lim_n \mathbb{P}[\xi_n = 1] = 1$ if and only if $\lim_n \mathbb{P}[\zeta_n = 1] = 1$. □

Recall (Monroe [12]) that a solution σ of the SEP for ν in M is *minimal* if whenever τ is a solution of the SEP for ν in M such that $\tau \leq \sigma$ then $\tau = \sigma$ almost surely. The following result shows that $\delta(1)$ is minimal, provided it exists. In particular, the Bass embedding [3] is minimal.

Proposition 1. *Suppose $\int_0^t \Lambda(s, N_s)^2 ds < \infty$ almost surely, for every $t < 1$, or equivalently $\delta(t) < \infty$ almost surely for each $t < 1$. Then $\delta(1)$ is a minimal embedding of ν in M .*

Proof. We have $(N_t)_{0 \leq t \leq 1} = (\mathbb{E}_t[h(\tilde{W}_1)])_{0 \leq t \leq 1}$ is uniformly integrable (UI). Since, by construction $Y \stackrel{\mathcal{L}}{=} N$, it follows that $(Y_t)_{0 \leq t \leq 1}$ is UI. But $Y_t \equiv M_{\delta(t)}$ and $M_s = W_{A_s}$ for some time-change A and some Brownian motion W and hence $(W_{A \circ \delta(s)})_{0 \leq s \leq 1} = (W_{s \wedge (A \circ \delta)(1)})_{s \geq 0}$ is UI. Monroe [12, Theorem 3] proves that in the Brownian case, if τ is an embedding of ν in a Brownian motion W and if $W_0 = \int x\nu(dx)$ then τ is minimal if and only if $(W_{t \wedge \tau})_{t \geq 0}$ is UI. Hence, $A_{\delta(1)}$ is minimal for ν in W . Since A is increasing we can conclude that $\delta(1)$ is minimal for ν in M . □

Theorem 2. *Suppose $\text{supp}(\nu) \subseteq [l, r]$. Recall $M_0 = m$ and suppose $\nu \in L^1$ and $\int x\nu(dx) = m$. Then $\delta(1)$ exists and is a minimal embedding of ν in M .*

Proof. For $t < 1$, $N_t \in (\underline{v}, \bar{v}) \subseteq (\ell, r)$ and since $\frac{1}{\eta}$ is locally square integrable $\int_0^t \Lambda(s, N_s)^2 ds = \int_0^t \frac{b_x(s, B(s, N_s))^2}{\eta(N_s)^2} ds < \infty$ almost surely. Hence, $\delta(t)$ exists and is finite for each $t < 1$ and $M_{\delta(1)}$ has law ν . \square

Suppose ν places mass outside $[\ell, r]$. Then it is clear that it is not possible to embed ν in M using any embedding. To see that this holds true for $\delta(1)$, suppose $\bar{v} > r$. Then for each $t < 1$ we have $b(t, \cdot) : \mathbb{R} \rightarrow (\underline{v}, \bar{v})$ and there exists a continuous function $y(t)$ such that $b(t, y) > r$ for $y > y(t)$. Then, $\int_0^T \lambda(t, \tilde{W}_t)^2 dt = \infty$ for all $T < 1$ such that $\sup_{0 < s < T} \tilde{W}_s - y(s) > 0$. Since the set $\sup_{0 < s < 1} \tilde{W}_s - y(s) > 0$ has positive probability, δ explodes before time 1 with positive probability also.

Standing Assumption 1. Henceforth, we will assume that ν places no mass outside $[\ell, r]$.

Recall that we have assumed that we are given a diffusion with $M_0 = m$, and that the target measure ν satisfies $\nu \in L^1$ and $m = \int x \nu(dx)$. We call this the centred case. In the next section, we consider what happens if we relax this assumption.

In the case where $\nu \in L^1$ but $m \neq \int x \nu(dx)$, we introduce an embedding δ^* which involves running the martingale M until it first hits $\int x \nu(dx)$ and then using the stopping time $\delta(1)$ defined above, but for M started at $\int x \nu(dx)$.

Then in subsequent sections we will ask, when does there exist a finite (respectively {integrable, bounded}) embedding, and when does $\delta(1)$ or more generally δ^* have this property.

2. The non-centred case

In this section, we do not assume that $\nu \in L^1$ and that $m = \int x \nu(dx)$.

When at least one of $\int_{-\infty} |x| \nu(dx)$ and $\int_{-\infty}^{\infty} x \nu(dx)$ is finite we write $\nu^* = \int x \nu(dx) \in \bar{\mathbb{R}}$. Note that we assume that ν has support in the state space of M .

Proposition 2 (Pedersen and Peskir [14], Cox and Hobson [5]). *Suppose $-\infty < l < m < r < \infty$. Then for there to be an embedding of ν in M we must have that $\int x \nu(dx) = m$. In this case M is a uniformly integrable martingale.*

Suppose $-\infty = l < m < r < \infty$. Then there exists an embedding of ν in M if and only if $\nu^ \geq m$. Conversely, if $-\infty < l < m < r = \infty$ there exists an embedding of ν in M if and only if $\nu^* \leq m$.*

Finally, suppose $-\infty = l < m < r = \infty$. Then we can embed any distribution ν in M .

Proof. In the bounded case, the fact that M is a bounded local martingale gives that it is a UI-martingale, and hence $\int x \nu(dx) = \mathbb{E}[M_\tau] = M_0 = m$.

For the second case, the upper bound on the state space means that M is a submartingale so that the condition $m \leq \nu^*$ is necessary. Then provided $\nu \in L^1$ we can run M until it first reaches $\nu^* \in [m, \infty)$. Note that M hits ν^* in finite time by the argument in Karatzas, Shreve [10], Section 5.5 C. Then we can embed ν using the local martingale M started from ν^* (using, for example, the time $\delta(1)$ defined above, or the Azéma–Yor construction as in Pedersen and

Peskir [14]). If ν^* is infinite, then we need a different construction, see, for example, Cox and Hobson [5].

For the final case, any distribution can be embedded in M . If $\nu \in L^1$ then we can run M until it hits ν^* and then consider an embedding for the local martingale started at the mean of the target distribution. If $\nu \notin L^1$, then we can use the construction in [5], but not the one in this paper. \square

Let H_z^M be the first hitting time of z by M , and more generally let H_x^X be the first hitting time of x by a stochastic process X . Suppose $\mu \in L^1$ and let $\delta_{\nu^*}(1)$ be the stopping time $\delta(1)$ constructed in the previous section to embed ν in the time-homogeneous diffusion started at $M_0 = \nu^*$. Then let $\delta^* = H_{\nu^*}^M + \delta_{\nu^*}(1)$. By the results of the proposition, provided $\nu \in L^1$ and both $\nu^* \leq m$ if $r < \infty$ and $\nu^* \geq m$ if $l > -\infty$, then δ^* is an embedding of ν .

3. Finite embeddings

3.1. The centred case

Suppose $\nu \in L^1$ and $m = \int x\nu(dx)$.

Proposition 3.

- (i) If $l > -\infty$, M does not hit l in finite time and $\nu(\{l\}) > 0$ or if $r < \infty$, M does not hit r in finite time and $\nu(\{r\}) > 0$, then any embedding of ν has $\tau = \infty$ with positive probability.
- (ii) Otherwise, either $l = -\infty$, or M does not hit l in finite time and $\nu(\{l\}) = 0$ or M can hit l in finite time and either $r = \infty$, or M does not hit r in finite time and $\nu(\{r\}) = 0$ or M can hit r in finite time. Then if τ is an embedding of ν we have that $\bar{\tau} = \tau \wedge H_\ell^M \wedge H_r^M$ is also an embedding of ν and $\bar{\tau}$ is finite almost surely.

Proof. (i) Suppose τ is any embedding of ν in M . Then $\tau = \infty$ on the set where $M_\tau \in \{l, r\}$. Moreover, this set has positive probability by assumption.

(ii) If τ is an embedding of ν , then $M_{t \wedge \tau}$ converges almost surely, even on the set $\tau = \infty$. However, if $(l = -\infty, r = \infty)$ then by the Rogozin trichotomy (see [17]), $-\infty = \liminf M_t < \limsup M_t = \infty$ and $(M_t)_{t \geq 0}$ does not converge. Hence, we must have $\tau < \infty$.

Otherwise, one or both of $\{l, r\}$ is finite. Then M converges and so if $\tau = \infty$ then either $M_\tau = l$ or $M_\tau = r$.

If l or r is finite but M hits neither l nor r in finite time, then $\tau = \infty$ is excluded outside a set of measure zero by the hypothesis that $\nu(\{l\}) = 0$ and $\nu(\{r\}) = 0$. Hence, $\tau < \infty$ almost surely.

Finally, if M can hit either l or r in finite time then it will do so and $\bar{\tau} = H_\ell^M \wedge H_r^M < \infty$. \square

Corollary 1. *If there exists an embedding τ of ν in M which is finite almost surely then $\delta(1)$ is finite almost surely.*

Proof. If there is a finite embedding, then we must be in case (ii) of the proposition. Then $\delta(1) \wedge H_\ell^M \wedge H_r^M$ is finite almost surely. But also $\delta(1) \leq H_\ell^M \wedge H_r^M$ so that $\delta(1) < \infty$ almost surely. \square

3.2. The non-centred case

Suppose ν and m are such that an embedding exists (recall Proposition 2). Necessarily we must have that at least one of ℓ and r is infinite.

Suppose $\nu \in L^1$ so that ν^* and δ^* are well defined. Then since $H_{\nu^*}^M$ is finite almost surely, we have that δ^* is finite if and only if $\delta_{\nu^*}(1)$ is finite almost surely.

Then the result for the non-centred case is identical to both the proposition and the corollary describing the results in the centred case, modulo the substitution of δ^* for $\delta(1)$ in Corollary 1.

4. Integrable embeddings

4.1. The centred case

Suppose $\nu \in L^1$ and $m = \int x \nu(dx)$.

In this section, we provide an integrability condition on ν that guarantees that (6) has a solution on $[0, 1]$ and that $\delta(1)$ is integrable. Notice that q is twice continuously differentiable on (l, r) . The second derivative

$$q''(x) = \frac{2}{\eta^2(x)}$$

is positive, which means that q is convex. Moreover, q is decreasing on (l, m) and increasing on (m, r) ; in particular $q \geq 0$.

Theorem 3. *If the function q is integrable wrt ν , then the ODE (6) has a solution on $[0, 1]$ for almost all paths of M and $\delta(1)$ is integrable. In this case, $\mathbb{E}[\delta(1)] = \int q(x)\nu(dx)$.*

Proof. Assume first that q is integrable wrt ν . This means that the random variable $q(N_1)$ is integrable. Let

$$\tau_n = 1 \wedge \inf \left\{ t \geq 0 \mid \int_0^t |q'(N_s)b_x(s, B(s, N_s))|^2 ds \geq n \right\}, \tag{9}$$

and observe that $(N_u)_{u \leq s}$ is bounded away from l and r for any $s < 1$, and hence $\tau_n \uparrow 1$, a.s. By Itô's formula, and using $q(N_0) = q(m) = 0$, we get

$$\begin{aligned} q(N_{\tau_n}) &= \int_0^{\tau_n} q'(N_s)b_x(s, B(s, N_s)) d\tilde{W}_s + \frac{1}{2} \int_0^{\tau_n} q''(N_s)b_x(s, B(s, N_s))^2 ds \\ &= \int_0^{\tau_n} q'(N_s)b_x(s, B(s, N_s)) d\tilde{W}_s + \int_0^{\tau_n} \frac{b_x(s, B(s, N_s))^2}{\eta^2(N_s)} ds. \end{aligned}$$

Taking expectations, the martingale part disappears and we obtain

$$\mathbb{E} \left[\int_0^{\tau_n} \frac{b_x(s, B(s, N_s))^2}{\eta^2(N_s)} ds \right] = \mathbb{E}[q(N_{\tau_n})]. \tag{10}$$

Notice that Jensen’s inequality implies that

$$0 \leq q(N_{\tau_n}) \leq \mathbb{E}[q(N_1) \mid \tilde{\mathcal{F}}_{\tau_n}].$$

Since the family $(\mathbb{E}[q(N_1) \mid \tilde{\mathcal{F}}_{\tau_n}])_{n \geq 1}$ is uniformly integrable, also $(q(N_{\tau_n}))_{n \geq 1}$ is uniformly integrable. Therefore we can interchange the expectation operator and the limit $n \rightarrow \infty$ on the RHS of (10). By monotone convergence, we can do so also on the LHS and hence we get

$$\mathbb{E}\left[\int_0^1 \frac{b_x(s, B(s, N_s))^2}{\eta^2(N_s)} ds\right] = \mathbb{E}[q(N_1)] < \infty.$$

Lemma 3 implies that the ODE (6) has a solution on $[0, 1]$ for almost all paths of M and that $\delta(1)$ is integrable. □

The reverse statement of Theorem 3 holds true as-well, that is, if $\delta(1)$ is integrable, then q is integrable wrt ν . Indeed, we next show that the existence of an *arbitrary* integrable solution of the SEP implies that q is integrable wrt ν .

Proposition 4. *Any stopping time τ that solves the SEP satisfies*

$$\mathbb{E}[\tau] \geq \int q(x) d\nu(x). \tag{11}$$

Proof. Recall that $\bar{\tau} = \tau \wedge H_l \wedge H_r$. Since ℓ is absorbing if $H_\ell < \infty$ and similarly if $H_r < \infty$ then r is absorbing, we have that $M_{\bar{\tau}} = M_\tau$ and $\bar{\tau}$ is also an embedding of ν .

Let τ be an stopping time with $M_\tau \sim \nu$. Suppose that τ is integrable; else the statement is trivial. Let

$$\sigma_n = n \wedge \inf\left\{t \geq 0 \mid \int_0^t |q'(M_s)|^2 \eta(M_s)^2 ds \geq n\right\}.$$

Observe that $\sigma_n \uparrow H_l \wedge H_r$, a.s. Using Itô’s formula, we obtain

$$\begin{aligned} \mathbb{E}[q(M_{\tau \wedge \sigma_n})] &= q(M_0) + \mathbb{E}\left[\frac{1}{2} \int_0^{\tau \wedge \sigma_n} q''(M_s) d\langle M, M \rangle_s\right] \\ &= \mathbb{E}[\tau \wedge \sigma_n]. \end{aligned} \tag{12}$$

Then Fatou’s lemma implies

$$\mathbb{E}[q(M_\tau)] = \mathbb{E}[q(M_{\bar{\tau}})] \leq \liminf_n \mathbb{E}[q(M_{\tau \wedge \sigma_n})] \leq \mathbb{E}[\bar{\tau}] \leq \mathbb{E}[\tau],$$

and hence (11). □

Remark 4. Notice that if M attains the boundary l with positive probability in finite time, then the function q is finite at l . In this case ν can have mass on l . If M does not attain the boundary l in finite time, then obviously a distribution ν with mass in l can not be embedded with an integrable stopping time. Similar considerations apply at the right boundary r .

Theorems 3 and 4 imply the following corollaries.

Corollary 2. *Suppose $v \in L^1$ and $m = \int xv(dx)$. There exists an integrable solution τ of the SEP if and only if q is integrable wrt v . In this case, τ satisfies (11).*

Corollary 3. *Suppose $v \in L^1$ and $m = \int xv(dx)$. Whenever there exists an integrable solution of the SEP, then $\delta(1)$ is also an integrable solution.*

4.2. The non-centred case

Suppose we are given a local martingale diffusion M started at $M_0 = m$ and a measure $v \in L^1$ with $v^* \neq m$.

Recall the definition of q in (2). To emphasise the role of the initial point, write q_m for this expression. More generally, for $n \in (l, r)$ define

$$q_n(x) = \int_n^x dy \int_n^y dz \frac{2}{\eta(z)^2}. \tag{13}$$

Then $q_m(z) = q_n(z) + q_m(n) + q'_m(n)(z - n)$. As $q = q_m$, in particular

$$q(z) = q_{v^*}(z) + q(v^*) + q'(v^*)(z - v^*)$$

and $\int q(z)v(dz) = \int q_{v^*}(z)v(dz) + q(v^*)$. Hence, $\int q(z)v(dz)$ is finite if and only if $\int q_{v^*}(z)v(dz)$ is finite.

We state the following theorem in the case $m > v^*$ which necessitates $r = \infty$, and then $\ell \in [-\infty, m)$. There is a corresponding result for $m < v^*$ in which the condition $\lim_{n \uparrow \infty} q(n)/n < \infty$ is replaced by $\lim_{n \uparrow \infty} q(-n)/n < \infty$. Note that the limit $\lim_n q(n)/n$ is well defined because q is convex.

Theorem 4. *Suppose $m > v^*$.*

Suppose $\int q(z)v(dz) < \infty$ and $\lim q(n)/n < \infty$. Then δ^ is an integrable embedding of v .*

Conversely, suppose there exists an integrable embedding τ of v in M . Then $\int q(z)v(dz) < \infty$ and $\lim q(n)/n < \infty$.

Proof. Consider the first part of the theorem. By the comments before the theorem, we may assume that $\int q_{v^*}(z)v(dz) < \infty$ and hence, for M started at v^* , $\mathbb{E}[\delta_{v^*}(1)] < \infty$. Then, it is sufficient to show that $\mathbb{E}[H_{v^*}^M] < \infty$. But

$$\begin{aligned} \mathbb{E}[H_{v^*}^M] &= \lim_{n \uparrow \infty} \mathbb{E}[H_{v^*}^M \wedge H_n^M] = \lim_{n \uparrow \infty} \mathbb{E}[q(M_{H_{v^*}^M \wedge H_n^M})] \\ &= q(v^*) \lim_{n \uparrow \infty} \frac{n - m}{n - v^*} + \lim_{n \uparrow \infty} q(n) \frac{m - v^*}{n - v^*} \\ &= q(v^*) + (m - v^*) \lim_{n \uparrow \infty} \frac{q(n)}{n}, \end{aligned}$$

which is finite under the assumption that $\lim q(n)/n < \infty$.

For the converse, suppose that τ is an integrable embedding. Without loss of generality, we may assume that τ is minimal; if not we may replace it with a smaller embedding which is also integrable. Then

$$\begin{aligned} \int q(x)\nu(dx) &= \mathbb{E}\left[\liminf_{n \rightarrow \infty} q(M_{\tau \wedge H_{-n}^M \wedge H_n^M})\right] \leq \liminf_{n \rightarrow \infty} \mathbb{E}[q(M_{\tau \wedge H_{-n}^M \wedge H_n^M})] \\ &= \lim_{n \rightarrow \infty} \mathbb{E}[\tau \wedge H_{-n}^M \wedge H_n^M] \\ &= \mathbb{E}[\tau] < \infty. \end{aligned}$$

It remains to show that $\mathbb{E}[H_{\nu^*}^M] < \infty$. Recall that this is equivalent to the condition $\lim_{n \uparrow \infty} q(n)/n < \infty$.

Recall that by the Dubins–Schwarz theorem (Rogers and Williams [16], page 64) we can write $M_t = \hat{W}_{C_t}$ for a $\mathbb{G} = (\mathcal{G}_t)_{t \geq 0}$ -Brownian motion \hat{W} where $\mathcal{G}_s = \mathcal{F}_{C_s}^{-1}$. Let $\sigma = C_\tau$. Then $M_\tau = \hat{W}_{C_\tau} = \hat{W}_\sigma$ and σ embeds ν in \hat{W} .

Since σ is a minimal embedding of ν in \hat{W} , by Theorem 5 of Cox and Hobson [6]

$$\lim_n n\mathbb{P}[\sigma > H_{-n}^{\hat{W}}] = 0. \tag{14}$$

Moreover, by arguments in the proof of Lemma 11 of Cox and Hobson [6], for any stopping time $\tilde{\sigma} \leq \sigma$

$$\mathbb{E}[|\hat{W}_{\tilde{\sigma}}|] \leq \mathbb{E}[|\hat{W}_\sigma|] = \int |z|\nu(dz).$$

Hence, $(\hat{W}_{t \wedge \sigma})_{t \geq 0}$ is bounded in L^1 , and then by Theorem 1 of Azéma *et al.* [2], $(\hat{W}_{t \wedge \sigma})_{t \geq 0}$ is uniformly integrable if and only if $\lim_n n\mathbb{P}[\sigma > H_{-n}^{\hat{W}} \wedge H_n^{\hat{W}}] = 0$. Since ν is not centred and $(\hat{W}_{t \wedge \sigma})_{t \geq 0}$ is not UI, it follows from (14) that $\lim_n n\mathbb{P}[\sigma > H_n^{\hat{W}}] > 0$. But $(\sigma > H_n^{\hat{W}}) \equiv (\tau > H_n^M)$ so $\limsup_n n\mathbb{P}[\tau > H_n^M] > 0$.

Then

$$\begin{aligned} \mathbb{E}[\tau] &= \lim_n \mathbb{E}[q(M_{\tau \wedge H_n^M})] \\ &\geq \lim_n \mathbb{E}[q(n); \tau > H_n^M] \\ &= \lim_n \left(\frac{q(n)}{n} n\mathbb{P}[\tau > H_n^M] \right) \\ &\geq \lim \frac{q(n)}{n} \cdot \limsup n\mathbb{P}[\tau > H_n^M]. \end{aligned}$$

Then, if $\mathbb{E}[\tau] < \infty$ it follows that $\lim \frac{q(n)}{n} < \infty$ and $\mathbb{E}[H_{\nu^*}^M] < \infty$. □

Finally, we consider the case where $\nu \notin L^1$.

Lemma 4. *Suppose $\nu \notin L^1$. If τ is an embedding of ν , then τ is not integrable.*

Proof. Observe that $q(x) \geq 0$ and that if $\nu \notin L^1$ then since q is convex we must have $\int q(x)\nu(dx) = \infty$. Then if τ is an embedding of ν

$$\begin{aligned} \mathbb{E}[\tau] &= \lim_{n \uparrow \infty} \mathbb{E}[\tau \wedge H_n^{q(M)} \wedge H_{-n}^{q(M)}] = \lim_{n \uparrow \infty} \mathbb{E}[q(M_{\tau \wedge H_n^{q(M)} \wedge H_{-n}^{q(M)}})] \\ &\geq \mathbb{E}\left[\liminf_{n \uparrow \infty} q(M_{\tau \wedge H_n^{q(M)} \wedge H_{-n}^{q(M)}})\right] = \mathbb{E}[q(M_\tau)] = \infty. \end{aligned} \quad \square$$

5. Bounded time embedding

5.1. The centred case

In this section, we analyze the question under which conditions we can guarantee the stopping time $\delta(1)$ to be bounded, that is, $\delta(1) \leq T \in \mathbb{R}_+$. Let us first state a necessary condition which places a lower bound on how little mass must be embedded in each a neighbourhood of a point x .

Theorem 5. *Suppose that η is locally bounded and denote by η^* its upper semicontinuous envelope. If a distribution with distribution function F can be embedded before time $T > 0$, then for all $x \in \mathbb{R}$ with $0 < F(x) < 1$ it must hold that*

$$\limsup_{\varepsilon \downarrow 0} -\varepsilon^2 \ln(F(x + \varepsilon) - F(x - \varepsilon)) \leq \frac{\pi^2}{8} T \eta^*(x)^2. \quad (15)$$

Proof. Fix x and suppose t' is such that $M_{t'} = x$.

For $\varepsilon > 0$ define $B_\varepsilon(x) = \{y \mid |y - x| < \varepsilon\}$ and $\tilde{\eta}(x, \varepsilon) = \max\{\eta^*(z) \mid z \in \bar{B}_\varepsilon(x)\}$. Note that on $t \geq t'$ the process \tilde{M} which solves the SDE $d\tilde{M}_t = \tilde{\eta}(M_t) dW_t$ where

$$\tilde{\eta}(m) = (1_{\{m \in B_\varepsilon(x)\}} \eta(m) + 1_{\{m \notin B_\varepsilon(x)\}} \tilde{\eta}(m, \varepsilon))$$

subject to $\tilde{M}_{t'} = M_{t'} = x$, coincides with M up to the first leaving time of $B_\varepsilon(x)$. Moreover, there exists a Brownian motion \tilde{W} such that on $t \geq t'$, $\tilde{M}_t = \tilde{W}_{\Gamma(t)}$, where $\Gamma(t) = \int_{t'}^t \tilde{\eta}(M_s)^2 ds \leq \tilde{\eta}(x, \varepsilon)^2 t$. Then

$$\begin{aligned} \mathbb{P}\left[\sup_{t' \leq t \leq T} |M_t - M_{t'}| < \varepsilon\right] &= \mathbb{P}\left[\sup_{t' \leq t \leq T} |\tilde{M}_t - \tilde{M}_{t'}| < \varepsilon\right] \\ &= \mathbb{P}\left[\sup_{t' \leq t \leq T} |\tilde{W}_{\Gamma(t)} - \tilde{W}_{\Gamma(t')}| < \varepsilon\right] \\ &\geq \mathbb{P}\left[\sup_{0 \leq s \leq \tilde{\eta}(x, \varepsilon)^2 T} |W_s| < \varepsilon\right]. \end{aligned}$$

The probability for the absolute value of the Brownian motion W to stay within the ball $B_\varepsilon(0)$ up to time $KT \geq 0$ is given by (see Section 5, Chapter X in Feller [8])

$$\begin{aligned} \mathbb{P}\left[\sup_{s \in [0, K^2 T]} |W_s| < \varepsilon\right] &= \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{1}{2n+1} e^{-(2n+1)^2 \pi^2 / (8\varepsilon^2) KT} (-1)^n \\ &\geq \frac{4}{\pi} e^{-\pi^2 / (8\varepsilon^2) KT} - \frac{4}{3\pi} e^{-9\pi^2 / (8\varepsilon^2) KT} \geq \frac{8}{3\pi} e^{-\pi^2 / (8\varepsilon^2) KT}. \end{aligned}$$

Assume that there exists a stopping time τ such that M_τ has the distribution F . Denote by $\zeta = \inf\{t \geq 0 : M_t = x\}$ the first time the process M hits x . Since $F(x) \notin \{0, 1\}$, the event $A = \{\zeta < \tau\}$ occurs with positive probability.

Let \mathcal{F}_ζ be the σ -field generated by M up to time ζ and observe that $A \in \mathcal{F}_\zeta$. Note further that the process $Z = (Z_h)_{h \geq 0}$ given by $Z_h = M_{h+\zeta} - M_\zeta$ is independent of \mathcal{F}_ζ .

Now suppose τ is bounded by T . The mass of F on the ball $B_\varepsilon(x)$ has to be at least as large as the probability that A occurs and that X stays within the ball $B_\varepsilon(x)$ between ζ and T . Therefore,

$$\begin{aligned} F(x + \varepsilon) - F(x - \varepsilon) &\geq \mathbb{P}\left[A \cap \left\{\sup_{\zeta \leq s \leq T} |M_s - M_\zeta| < \varepsilon\right\}\right] \\ &= \mathbb{P}[A] \mathbb{P}\left[\sup_{\zeta \leq s \leq T} |M_s - M_\zeta| < \varepsilon\right] \\ &\geq \mathbb{P}[A] \frac{8}{3\pi} e^{-\pi^2 / (8\varepsilon^2) \bar{\eta}(x, \varepsilon)^2 T}. \end{aligned}$$

Hence, we have

$$-\varepsilon^2 \ln(F(x + \varepsilon) - F(x - \varepsilon)) \leq \varepsilon^2 \ln \frac{3\pi}{8\mathbb{P}[A]} + \frac{\pi^2}{8} \bar{\eta}(x, \varepsilon)^2 T,$$

which implies the result. □

Now we turn to the converse, and sufficient conditions for these to exist an embedding of ν in bounded time. Suppose again that $\nu \in L^1$ and $M_0 = m = \int x \nu(dx)$.

Recall the definition of r in (5). The first result is an immediate corollary of Theorem 1.

Corollary 4. *If $\lambda(t, y)^2$ is bounded by $T \in \mathbb{R}_+$, for all $y \in \mathbb{R}$ and $t \in [0, 1]$, then the stopping time $\delta(1)$ is also bounded by T .*

Proposition 5. *Assume that F is absolutely continuous and has compact support. Suppose F has density f . If η and f are bounded away from zero, then the stopping time $\delta(1)$ is bounded.*

Proof. Note that $h' = \frac{\varphi}{f \circ F^{-1} \circ \Phi}$ and thus it follows from f bounded away from zero that h' is bounded. Hence, b_x is bounded and thus $\lambda(t, y)$ is bounded. □

Lemma 5. *Suppose that η is concave on (l, r) . Let F be an absolutely continuous distribution with $\text{supp}(F) \subseteq [l, r]$ and suppose that $\sup_{x \in [l, r]} \frac{h'(x)}{\eta(h(x))} \leq \sqrt{T} < \infty$. Then F is embeddable in bounded time, and there exists an embedding τ with $\tau \leq T$.*

Proof. We have $b(t, x) = (\varphi_{1-t} \star h)(x)$ and

$$b_x(t, x) = (\varphi_{1-t} \star h')(x) \leq \sqrt{T}(\varphi_{1-t} \star (\eta \circ h))(x) \leq \sqrt{T}\eta \circ (\varphi_{1-t} \star h)(x) = \sqrt{T}\eta(b(t, x))$$

and then $\lambda(t, x)^2 \leq T$ and the result follows from Corollary 4. \square

Remark 5. More generally for the existence of a bounded embedding it is sufficient that there is a concave function ξ and ε in $(0, 1)$ for which $\varepsilon\xi \leq \eta \leq \varepsilon^{-1}\xi$. Then if $\sup_{x \in [l, r]} \frac{h'(x)}{\eta(h(x))} \leq \sqrt{T}$

$$b_x(t, x) \leq \sqrt{T}\varepsilon^{-1}(\varphi_{1-t} \star (\xi \circ h))(x) \leq \sqrt{T}\varepsilon^{-1}\xi \circ (\varphi_{1-t} \star h)(x) \leq \sqrt{T}\varepsilon^{-2}\eta(b(t, x)),$$

and $\lambda(t, x)^2 \leq T\varepsilon^{-4}$.

Remark 6. The sufficient condition from Lemma 5 implies a stronger version of the necessary condition of Theorem 5. Indeed, it implies that the limit superior of the left hand side of equation (15) is equal to zero. To show this, let η be locally bounded and assume that F satisfies the assumptions of Lemma 5. Then for all $z \in (\underline{v}, \bar{v})$, the interior of $\text{supp}(F)$, we have

$$f(z) \geq \frac{1}{\sqrt{T}} \frac{\varphi \circ \Phi^{-1} \circ F(z)}{\eta(z)}.$$

Let $x \in (\underline{v}, \bar{v})$. Since η is locally bounded there exists $B \in \mathbb{R}_+$ such that $\eta(z) \leq B$ for z close enough to x . Then, for ε small we have

$$\begin{aligned} \ln(F(x + \varepsilon) - F(x - \varepsilon)) &= \ln\left(\int_{x-\varepsilon}^{x+\varepsilon} f(z) \, dz\right) \\ &\geq \ln\left(\frac{1}{\sqrt{T}B} \int_{x-\varepsilon}^{x+\varepsilon} \varphi \circ \Phi^{-1} \circ F(z) \, dz\right), \end{aligned}$$

and applying Jensen's inequality we obtain

$$\ln(F(x + \varepsilon) - F(x - \varepsilon)) \geq \ln \frac{2\varepsilon}{\sqrt{T}B} - \frac{1}{4\varepsilon} \int_{x-\varepsilon}^{x+\varepsilon} [(\Phi^{-1} \circ F(z))^2 + \ln 2\pi] \, dz.$$

Consequently $\limsup_{\varepsilon \downarrow 0} -\varepsilon^2 \ln(F(x + \varepsilon) - F(x - \varepsilon)) = 0$.

5.2. The non-centred case

If M is a martingale and if τ is bounded by L , then $M_{t \wedge \tau}$ is uniformly integrable and $\mathbb{E}[M_\tau] = m$. Hence, there are no embeddings of ν in M if $\nu \notin L^1$ or $\nu^* \neq m$.

If M is a local martingale but not a martingale, then we may have $\tau \leq L$ and $\mathbb{E}[M_\tau] \neq M_0 = m$. However, δ^* is not bounded since $\delta^* > H_{\nu^*}^M$ which is not bounded.

For example, let $m = 1$ and $\eta(x) = x^2$ so that $dM_t = M_t^2 dB_t$, and M is the reciprocal of a 3-dimensional Bessel process. Let $\nu = \mathcal{L}(M_1)$. Then $\nu \in L^1$ and $\nu^* \leq 1 = m$. Then, trivially, $\tau \equiv 1$ is a bounded embedding of 1.

6. General diffusions

Let $(X_t)_{t \geq 0}$ be a solution to

$$dX_t = \beta(X_t) dt + \alpha(X_t) dW_t, \quad \text{with } X_0 = x_0,$$

where $x_0 \in \mathbb{R}$, $\beta : \mathbb{R} \rightarrow \mathbb{R}$ and $\alpha : \mathbb{R} \rightarrow \mathbb{R}$ are Borel-measurable. We assume that X takes values only in an interval $[l, r]$ with $-\infty \leq l < x_0 < r \leq \infty$. Moreover, we assume that $\alpha(x) \neq 0$ for all $x \in (l, r)$ and that $\frac{1+|\beta|}{\alpha^2}$ is locally integrable on (l, r) .

Suppose we want to embed ρ in X with a stopping time τ .

By changing the space scale one can transform the diffusion X into a continuous local martingale. To this end, we define the *scale function* s (cf. [15], Chapter VII, §3) via

$$s(x) = \int_{x_0}^x \exp\left(-\int_{x_0}^y \frac{2\beta(z)}{\alpha(z)^2} dz\right) dy, \quad x \in (l, r).$$

Note that we are always free to choose the scale function such that $M_0 = s(x_0) = 0$, and we have done so.

Then s solves $\beta(x)s'(x) + \frac{1}{2}\alpha^2(x)s''(x) = 0$. Note that the scale function s is strictly increasing and continuously differentiable. Itô's formula implies that $M_t = s(X_t)$ is a local martingale with integral representation

$$M_t = \int_0^t s'(X_s)\alpha(X_s) dW_s.$$

Thus $dM_t = \eta(M_t) dW_t$ where $\eta \equiv (s'\alpha) \circ s^{-1}$.

Note that

$$\int_{\chi-\varepsilon}^{\chi+\varepsilon} \frac{1}{((s'\alpha) \circ s^{-1})^2(z)} dz = \int_{s^{-1}(\chi-\varepsilon)}^{s^{-1}(\chi+\varepsilon)} \frac{1}{\alpha^2(z)s'(z)} dz.$$

Since s is continuous, $\frac{1}{\eta}$ is locally square integrable provided

$$\frac{1}{\alpha(y)\sqrt{s'(y)}} = \frac{1}{\alpha(y)} \exp\left(-\int_{x_0}^y \frac{2\beta(z)}{\alpha(z)^2} dz\right)^{-1/2} = \frac{1}{\alpha(y)} \exp\left(\int_{x_0}^y \frac{\beta(z)}{\alpha(z)^2} dz\right)$$

is locally square integrable which follows from our assumptions on the pair (α, β) .

Let F_ρ be the distribution function of ρ . If $\nu = \rho \circ s^{-1}$ so that $F(x) = F_\rho(s^{-1}(x))$, then $X_\tau \sim \rho$ is equivalent to $M_\tau \sim \nu$. Then ν has mean zero if and only if

$$\int_{\mathbb{R}} s(x)\rho(dx) = 0. \tag{16}$$

Clearly the requirement that τ is finite, integrable or bounded is invariant under the change of scale. However, in the case of bounded embeddings we can give a simple sufficient condition in terms of data relating to the general diffusion X .

Define $g = F_\rho^{-1} \circ \Phi$ and $h = s \circ g = F^{-1} \circ \Phi$.

Theorem 6. *If $x \mapsto -\frac{2\beta(x)}{\alpha(x)} + \alpha'(x)$ is non-increasing and $\frac{g'}{\alpha \circ g}$ is bounded by \sqrt{T} , then ρ can be embedded in X in bounded time. In particular, there exists an embedding τ with $\tau \leq T$.*

Proof. We prove in the first step that $\eta \equiv (s'\alpha) \circ s^{-1}$ is concave. We have

$$\eta' = ((s'\alpha) \circ s^{-1})' = \frac{(s''\alpha + s'\alpha') \circ s^{-1}}{s' \circ s^{-1}} = \left(-\frac{2\beta}{\alpha} + \alpha' \right) \circ s^{-1}$$

where we have used the fact that s solves $\alpha s'' = -2\beta s'/\alpha$. As s^{-1} is monotone increasing, under the first hypothesis of the theorem we have that $((s'\alpha) \circ s^{-1})'$ is non-increasing and hence η is concave.

We have $h = s \circ g$ and hence, again by hypothesis,

$$\frac{h'}{\eta \circ h} = \frac{(s' \circ g)g'}{(s'\alpha) \circ g} = \frac{g'}{\alpha \circ g} \leq \sqrt{T}.$$

Lemma 5 implies that ν can be embedded in M with a stopping time τ satisfying $\tau \leq T$, and the same stopping time embeds ρ in X . □

7. Examples

7.1. Brownian motion with drift

Let X be a Brownian motion with drift, that is,

$$X_t = x_0 + \gamma t + \theta W_t,$$

where $\gamma \in \mathbb{R}$, $\theta > 0$ and $x_0 = 0$. The scale function equals

$$s(x) = \begin{cases} \frac{1}{\kappa}(1 - \exp(-\kappa x)) & \text{for } \kappa \neq 0, \\ x & \text{for } \kappa = 0, \end{cases}$$

with $\kappa = \frac{2\gamma}{\theta^2}$. If $\kappa > 0$ then $s(\mathbb{R}) = (-\infty, 1/\kappa)$, whereas if $\kappa < 0$ then $s(\mathbb{R}) = (1/\kappa, \infty)$. Then, if $M = s(X)$ we have $dM_t = \theta(1 - \kappa M_t) dW_t$, and M is a martingale.

Suppose the aim is to embed ρ . Let F_ρ be the distribution function of ρ and write $\nu = \rho \circ s^{-1}$. Since ρ is a measure on \mathbb{R} , ν is a measure on (l, r) and any embedding τ is finite. Note that

$$\nu^* = \int s(x)\rho(dx) = \frac{1}{\kappa} \left(1 - \int_{\mathbb{R}} e^{-\kappa x} \rho(dx) \right).$$

Then, by Proposition 2 there is an embedding of ν if and only if one of the following conditions is satisfied

1. $\nu^* \geq 0$ and $\kappa > 0$.
2. $\nu^* \leq 0$ and $\kappa < 0$.
3. $\kappa = 0$.

Condition 1 and 2 simplify to $0 \leq \nu^*\kappa = 1 - \int_{\mathbb{R}} e^{-\kappa x} \rho(dx)$ and hence $\int_{\mathbb{R}} e^{-\kappa x} \rho(dx) \leq 1$ is necessary for the existence of an embedding if $\kappa \neq 0$.

7.1.1. *The centred case*

Suppose $\int e^{-\kappa x} \rho(dx) = 1$. Then ν has zero mean.

Proposition 6. *For $\kappa \neq 0$ ($\kappa = 0$) there exists an integrable stopping time embedding ρ into X if and only if x (x^2) is integrable with respect to ρ . In this case, any minimal and integrable stopping time τ satisfies*

$$\mathbb{E}[\tau] = \begin{cases} \frac{1}{\gamma} \int x\rho(dx) & \text{for } \kappa \neq 0, \\ \frac{1}{\nu^2} \int x^2\rho(dx) & \text{for } \kappa = 0. \end{cases}$$

Proof. Note that q is given by

$$q(x) = \begin{cases} -\frac{2}{\kappa\theta^2} \left(\frac{1}{\kappa} \ln(1 - \kappa x) + x \right) & \text{for } \kappa \neq 0, \\ \frac{x^2}{\theta^2} & \text{for } \kappa = 0. \end{cases}$$

Moreover,

$$\int q(x)\nu(dx) = \int q(s(x))\rho(dx) = \begin{cases} \frac{1}{\gamma} \int x\rho(dx) & \text{for } \kappa \neq 0, \\ \frac{1}{\nu^2} \int x^2\rho(dx) & \text{for } \kappa = 0. \end{cases}$$

The result follows now from Theorem 3 and Proposition 4. □

Finally, we consider sufficient conditions for there to exist a bounded embedding. It turns out that the embedding stopping time $\delta(1)$ is bounded if $h = F^{-1} \circ \Phi = s \circ F_\rho^{-1} \circ \Phi$ is Lipschitz continuous with parameter L . We can thus recover the sufficient condition from Section 3.2 in [1].

Proposition 7. *Suppose that $F_\rho^{-1} \circ \Phi$ is Lipschitz continuous with Lipschitz constant $L \in \mathbb{R}_+$. Then there exists an embedding τ of ρ in X such that $\tau \leq \frac{L^2}{\theta^2}$.*

Proof. For this example, $x \mapsto -2\beta/\alpha + \alpha'$ is the constant map. Hence, the result follows from Theorem 6. □

7.1.2. *The non-centred case*

Suppose $\int e^{-\kappa x} \rho(dx) < 1$. It is clear that δ^* is finite almost surely, but the arguments of Section 5.2 show that there can be no embedding of ρ which is bounded. Further, if $\int e^{-\kappa x} \rho(dx) < 1$ then it follows that $v \in L^1$ and that $v^* \in (0, 1/\kappa)$ (or $-1/\kappa, 0$).

Now consider integrable embeddings. By Theorem 4, there exists an integrable embedding if and only if $\mathbb{E}[H_{v^*}^M] < \infty$ and $\int q(x)v(dx) < \infty$. But $\mathbb{E}[H_{v^*}^M] = \mathbb{E}[H_{s^{-1}(v^*)}^X]$ and, since X is drifting Brownian motion, provided $\text{sgn}(z) = \text{sgn}(\kappa) = \text{sgn}(\gamma)$, X hits z in finite mean time. Hence, $\mathbb{E}[H_{v^*}^M] < \infty$. Further

$$\begin{aligned} \int q(x)v(dx) &= \int q(s(x))\rho(dx) \\ &= -\frac{2}{\kappa\theta^2} \int \left[\left(\frac{1}{\kappa} \ln(1 - \kappa s(x)) + s(x) \right) \right] \rho(dx) \\ &= -\frac{2}{\kappa\theta^2} \int \left[-x + \frac{1}{\kappa}(1 - e^{-\kappa x}) \right] \rho(dx) \\ &= \int \frac{x}{\gamma} \rho(dx) - \frac{1}{\gamma} \int \frac{1}{\kappa}(1 - e^{-\kappa x}) \rho(dx) \\ &= \frac{1}{\gamma} \left(\int x\rho(dx) - v^* \right). \end{aligned}$$

Hence, there is an integrable embedding if $\int x\rho(dx) < \infty$ and δ^* is integrable.

7.2. Bessel process

Let R be the radial part of 3-dimensional Brownian motion so that R solves $dR_t = dB_t + R_t^{-1} dt$ and suppose that $R_0 = 1$. Then the scale function is given by $s(r) = 1 - r^{-1}$, and we can embed any distribution ρ on \mathbb{R}_+ in R provided $\int r^{-1}\rho(dr) \leq 1$ (see Proposition 2).

7.2.1. *The centred case*

Suppose that $\int r^{-1}\rho(dr) = 1$. Then v has zero mean.

Proposition 8. *There exists an integrable stopping time that embeds ρ into R if and only if $\int r^2\rho(dr) < \infty$. In this case, any minimal and integrable stopping time τ satisfies $\mathbb{E}[\tau] = -\frac{1}{3} + \int \frac{1}{3}r^2\rho(dr)$.*

Proof. Note that $\eta(x) = (s'\alpha) \circ s^{-1}(x) = (1 - x)^2$. Moreover,

$$q(x) = -\frac{2}{3}x + \frac{1}{3} \frac{1}{(1-x)^2} - \frac{1}{3}.$$

Notice that $\int q(x)(\rho \circ s^{-1})(dx) = \int (\frac{1}{3}r^2 + \frac{2}{3}r^{-1} - 1)\rho(dr)$, and hence the result follows from Theorem 3 and Proposition 4. □

By Remark 1, we have that $M_t = 1 - R_t^{-1}$ is not a martingale (this is the Johnson–Helms example of a strict local martingale). Further, the map $r \mapsto -2\beta(r)/\alpha(r) + \alpha'(r) = -2/r$ is increasing.

However, suppose we want to embed a target law ρ in R in bounded time, where the support of ρ is bounded away from both 0 and ∞ by \hat{l} and \hat{r} , respectively. Let $\bar{l} = s(\hat{l})$ and $\bar{r} = s(\hat{r})$. Let \hat{R} be the stopped Bessel process $\hat{R}_t = R_{t \wedge H_{\bar{l}} \wedge H_{\bar{r}}}$ and let $\bar{M} = s(\hat{R})$. Then \bar{M} is a martingale, which is absorbed at both \bar{l} and \bar{r} . Then a necessary condition for there to exist an embedding of ν in \bar{M} in bounded time is that ν has support in $[\bar{l}, \bar{r}]$ and $\int_{\bar{l}}^{\bar{r}} x\nu(dx) = 0$. Hence, a necessary condition for it to be possible to embed ρ in R in bounded time is that $\int_{\hat{l}}^{\hat{r}} r^{-1}\rho(dr) = 1$. By Remark 5, a sufficient condition is that $\int_{\hat{l}}^{\hat{r}} r^{-1}\rho(dr) = 1$ and $\log F_{\rho}^{-1} \circ \Phi$ is Lipschitz continuous.

7.2.2. *The non-centred case*

Suppose $\int r^{-1}\rho(dr) < 1$. It is clear that δ^* is finite almost surely, but the arguments of Section 5.2 show that there can be no embedding of ρ which is bounded.

Consider integrable embeddings. By Theorem 4, there exists an integrable embedding if and only if $\lim_{n \rightarrow \infty} \frac{q(n)}{n} < \infty$ and $\int q(x)\nu(dx) < \infty$. For the first part, we have that

$$\lim_{n \rightarrow \infty} \frac{q(n)}{n} = \lim_{n \rightarrow \infty} -\frac{2}{3} + \frac{1}{3} \frac{1}{(1-n)^2n} - \frac{1}{3n} = -\frac{2}{3} < \infty.$$

Furthermore,

$$\begin{aligned} \int q(x)\nu(dx) &= \int q(s(x))\rho(dx) = \int_{\mathbb{R}_+} \left(\frac{1}{3}r^2 + \frac{2}{3}r^{-1} - 1\right)\rho(dx) \\ &= \int_{\mathbb{R}_+} \left(\frac{1}{3}r^2 - \frac{1}{3} - \frac{2}{3}s(x)\right)\rho(dx) \\ &= \frac{1}{3} \left(\int_{\mathbb{R}_+} r^2\rho(dx) - 1 - 2\nu^*\right). \end{aligned}$$

Hence, there exists an integrable stopping time if r^2 is integrable with respect to ρ .

7.3. Ornstein–Uhlenbeck process

Let X be an Ornstein–Uhlenbeck solving the SDE

$$dX_t = \xi X_t dt + \sigma dW_t,$$

where $\xi \in \mathbb{R}$, $\sigma > 0$ and $X_0 = 0$. The scale function is given by $s(x) = \int_0^x e^{(-\xi/\sigma^2)y^2} dy$.

The centred case

Let ρ be a distribution with $\int s(x)\rho(dx) = 0$. Then ν has zero mean.

We next give sufficient conditions for ρ to be embeddable in bounded time. We need to distinguish between a positive and negative mean reversion speed ξ .

Suppose first that $\xi > 0$, and the process is mean repelling. Then the scale function is bounded. In this case, $-\frac{2\beta(x)}{\alpha(x)} + \alpha'(x) = -\frac{2\xi}{\sigma}x$ is decreasing. Moreover, for $g = F_\rho^{-1} \circ \Phi$, $\frac{g'}{\alpha \circ g} = \frac{1}{\sigma}g'$. Therefore, by Theorem 6, if g is Lipschitz continuous with Lipschitz constant L , then there exists an embedding that is bounded by $\frac{L^2}{\sigma^2}$.

Suppose next that $\xi < 0$. Then the derivative of the scale function satisfies $s'(x) \geq 1$, $x \in \mathbb{R}$. Moreover, $\eta(x) = (s' \circ \alpha) \circ s^{-1}(x) \geq \sigma$ and the intensity of the time change satisfies $r^2(t, x) \leq \frac{1}{\sigma^2}b_x^2(t, x)$. Therefore, if $h = s \circ F_\rho^{-1} \circ \Phi$ is Lipschitz continuous with Lipschitz constant L , then there exists an embedding that is bounded by $\frac{L^2}{\sigma^2}$. (Note that $h' = (s' \circ g)g' \geq g'$ so that the requirement that h is Lipschitz is stronger than the requirement that g is Lipschitz.)

Finally, suppose that $\xi = 0$. Then the scale function is the identity function, and the arguments from each of the last two paragraphs apply and yield the same sufficient condition.

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