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FINITE JET DETERMINATION OF LOCAL ANALYTIC CR AUTOMORPHISMS AND THEIR PARAMETRIZATION BY 2-JETS IN THE FINITE TYPE CASE

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Abstract

We show that germs of local real-analytic CR automorphisms of a real-analytic hypersurface M in $\mathbb{C}^2$ at a point $p \in M$ are uniquely determined by their jets of some finite order at p if and only if M is not Levi-flat near p. This seems to be the first necessary and sufficient result on finite jet determination and the first result of this kind in the infinite type case.

If M is of finite type at p, we prove a stronger assertion: the local real-analytic CR automorphisms of M fixing p are analytically parametrized (and hence uniquely determined) by their 2-jets at p. This result is optimal since the automorphisms of the unit sphere are not determined by their 1-jets at a point of the sphere. The finite type condition is necessary since otherwise the needed jet order can be arbitrarily high [Kow1,2], [Z2]. Moreover, we show, by an example, that determination by 2-jets fails for finite type hypersurfaces already in $\mathbb{C}^3$.

We also give an application to the dynamics of germs of local biholomorphisms of $\mathbb{C}^2$.

1 Introduction

By H. Cartan's classical uniqueness theorem [Ca], a biholomorphic automorphism of a bounded domain $D \subset \mathbb{C}^n$ is uniquely determined by its value and its first order derivatives (that is, by its 1-jet) at any given point $p \in D$. The example of the unit ball D shows that if p is taken on the boundary ∂D the same uniqueness phenomenon does not hold for 1-jets

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but rather for 2-jets at p. (In this case, all automorphisms extend holomorphically across the boundary and, hence, any jet at a boundary point is well defined.). More generally, results of E. Cartan [C1,2], N. Tanaka [T] and S.-S. Chern–J.K. Moser [ChM] (see also H. Jacobowitz [J]) show that unique determination by 2-jets at a point p holds for (germs at p of) local biholomorphisms of $\mathbb{C}^n$ sending a (germ at p of an) open piece $M \subset \partial D$ into itself provided M is a *Levi-nondegenerate* real-analytic hypersurface.

The case of a *degenerate* real-analytic hypersurface M is much less understood, even in $\mathbb{C}^2$. It was previously known that unique determination, as above, by jets at p of some finite order k holds if M is of *finite type* at p, due to a recent result of the first author jointly with M.S. Baouendi and L.P. Rothschild [BER5, Corollary 2.7]. (We remark here that both notions of finite type, i.e. that in the sense of [Ko], [BlG] and that of [D], coincide in $\mathbb{C}^2$.) On the other hand, the only known situation where the germs of local biholomorphisms of $\mathbb{C}^2$ sending M into itself are not uniquely determined by their k-jets at p, for any k, is that where M is *Levi flat*. The first main result of this paper fills the gap between these two situations by showing that the Levi-flat case is indeed the only exception:

Theorem 1.1. *Let $M \subset \mathbb{C}^2$ be a real-analytic hypersurface and $p \in M$. The following are equivalent:*

(i) *M is not Levi-flat near $p \in M$.*
(ii) *There is an integer k such that if H^1 and H^2 are germs at p of local biholomorphisms of $\mathbb{C}^2$ sending M into itself with $j_p^k H^1 = j_p^k H^2$, then $H^1 \equiv H^2$.*

As mentioned above, the implication (ii)⇒(i) is well known. Thus, we shall only be concerned with the opposite implication. We also mention that the situation in higher dimensions is more complicated. The implication (i)⇒(ii) is clearly false as can be easily seen by the example $M = M_0 \times \mathbb{C}^{n-2} \subset \mathbb{C}^n$ where $M_0 \subset \mathbb{C}^2$ is any real-analytic Levi-nonflat hypersurface. The reader is referred to the survey paper [Z2] for a discussion of the higher dimensional case.

We remark that this and all the following results in this paper about local biholomorphisms sending M into itself also hold for local biholomorphisms sending M into another real-analytic hypersurface $M' \subset \mathbb{C}^2$. Indeed, any fixed local biholomorphism f_0 sending M into M' defines a one-to-one correspondence between the set of germs preserving M and that sending M to M' (as well as those of their jets) via $g \mapsto f_0 \circ g$.

Since germs of local biholomorphisms of $\mathbb{C}^2$ sending M into itself are in one-to-one correspondence with germs of real-analytic CR automorphisms of M by a theorem of Tomassini [To], we obtain the following immediate consequence of Theorem 1.1:

COROLLARY 1.2. *Let $M \subset \mathbb{C}^2$ be a real-analytic hypersurface which is not Levi-flat near a point $p \in M$. Then there exists an integer k such that if h^1 and h^2 are germs at p of real-analytic CR automorphisms of M with $j_p^k h^1 = j_p^k h^2$, then $h^1 \equiv h^2$.*

The proof of Theorem 1.1 relies on the parametrization of local biholomorphisms along the zero Segre variety given in §3, the method of singular complete systems recently developed by the first author [E2] and on a finite determination result for solutions of singular differential equations given in §5.

We should point out that for a general real-analytic Levi-nonflat hypersurface M, the integer k needed to make the statement of Theorem 1.1 (and of Corollary 1.2) true, may be arbitrarily large. In fact, examples given by Kowalski [Kow1,2] and the third author [Z2] show that, for any integer $k \geq 2$, there exists a real-analytic Levi-nonflat hypersurface $M_k \in \mathbb{C}^2$ and two local biholomorphisms H^1, H^2 near $p \in M_k$ sending M_k into itself with $j_p^k H^1 = j_p^k H^2$, but $j_p^{k+1} H^1 \neq j_p^{k+1} H^2$.

The above mentioned examples, where a high order jet is needed to distinguish between biholomorphic selfmaps, are all of *infinite type* at the point p. This fact is explained by the second main result of this paper, which improves the finite determination results above in the *finite type case* in two different directions. First we show that, just as in the Levi-nondegenerate case, the 2-jets are always sufficient for unique determination of local CR automorphisms of hypersurfaces in $\mathbb{C}^2$ of finite type. This conclusion contrasts strongly with most known results for hypersurfaces of finite type, where one usually has to take at least as many derivatives as the type (i.e. the minimal length of commutators of vector fields on M in the complex tangent direction required to span the full tangent space) as well as with the situation in higher dimension as is illustrated by the following example.

EXAMPLE 1.3. For $\ell \geq 2$, let $M \subset \mathbb{C}^3$ be the real algebraic hypersurface defined by

$$\operatorname{Im} w = |z_1|^2 + \operatorname{Re} z_1^\ell \bar{z}_2 ,$$

where the coordinates of $\mathbb{C}^3$ are (z_1, z_2, w). Observe that M is of finite type (indeed, type 2) at 0 and, for any $a \in \mathbb{R}$, the polynomial automorphism

$$H_a(z_1, z_2, w) = (z_1, z_2 + iaz_1^\ell, w) ,$$

sends $(M,0)$ into itself and its $(\ell-1)$-jet at 0 coincides with that of the identity.

Thus, for a hypersurface M of finite type at $p \in M$ in dimension higher than 2, there is not even a number k depending on the type of M at p such that biholomorphic selfmaps are uniquely determined by their k-jets at p. In the general setting of Theorem 1.1 in any dimension, upper estimates on how large the number k must be chosen, were previously only known for *finitely nondegenerate* manifolds M (see e.g. [BER3] for this notion), due to results in [BER1] (see also [Z1] and [BER2]) and the estimate for k in this case was at least twice the type of M minus two. We would like to point out that finite nondegeneracy is a strictly stronger notion than finite type; e.g. the hypersurface

$$M := \{(z,w) \in \mathbb{C}^2 : \mathsf{Im}\, w = |z|^4\}$$

is of finite type but is not finitely nondegenerate at 0. Since, however, the hypersurfaces in Example 1.3 are finitely nondegenerate, it is shown that, in dimension higher than 2, a uniform estimate for k is not possible even in this more restrictive class. In this paper, we show that, in $\mathbb{C}^2$, 2-jets are sufficient for unique determination as in Theorem 1.1 regardless of the type of M at p.

The second improvement in the finite type case is the stronger conclusion (than that of Theorem 1.1) that local biholomorphisms are not only uniquely determined but are analytically parametrized by their 2-jets. We denote by $G_p^2(\mathbb{C}^2)$ the group of all 2-jets at p of local biholomorphisms $H\colon (\mathbb{C}^2,p) \to (\mathbb{C}^2,p')$.

Theorem 1.4. *Let $M \subset \mathbb{C}^2$ be a real-analytic hypersurface of finite type at a point $p \in M$. Then there exist an open subset $\Omega \subset \mathbb{C}^2 \times G_p^2(\mathbb{C}^2)$ and a real-analytic map $\Psi(Z,\Lambda)\colon \Omega \to \mathbb{C}^2$, which in addition is holomorphic in Z, such that the following holds. For every local biholomorphism H of $\mathbb{C}^2$ sending (M,p) into itself, the point $(p, j_p^2 H)$ belongs to Ω and the identity*

$$H(Z) \equiv \Psi(Z, j_p^2 H)$$

holds for all $Z \in \mathbb{C}^2$ near p.

The conclusion of Theorem 1.4 was previously known for Levi-nondegenerate hypersurfaces in $\mathbb{C}^N$ due to the classical results mentioned above and for Levi-nondegenerate CR submanifolds of higher codimension due to a more recent result of V.K. Beloshapka [Be]. The existence of a k-jet parametrization, for some k, was known in the more general case (than that of Levi nondegenerate hypersurfaces) where M is finitely nondegenerate at p (see [BER4] and previous results in [BER1] and [Z1]). For other results on finite jet determination and finite jet parametrization of local CR automorphisms (in the finite type case), the reader is referred to the papers [H], [BER2,4], [Ha], [L], [BMR], [E1], [K], [KZ]. Theorem 1.4 will be a consequence of Theorem 4.1 which will be proved in §4.

We conclude this introduction by giving some applications of our main results. In view of a regularity result of X. Huang [Hu], a unique determination result can be formulated for *continuous* CR homeomorphisms as follows. Recall that a continuous mapping $h\colon M \to \mathbb{C}^2$ is called CR if it is annihilated, in the sense of distributions, by all CR vector fields on M (i.e. by the traces of $(0,1)$-vector fields in $\mathbb{C}^2$ tangent to M). A homeomorphism h between two hypersurfaces M and M' is called CR if both h and h^{-1} are CR mappings. Huang showed in the above mentioned paper that every continuous CR mapping between two real-analytic hypersurfaces $M, M' \subset \mathbb{C}^2$ of finite type extends holomorphically to a neighborhood of M in $\mathbb{C}^2$. Thus Theorem 1.4 implies:

COROLLARY 1.5. *Let $M, M' \subset \mathbb{C}^2$ be real-analytic hypersurfaces of finite type. Then, for any $p \in M$, if h_1 and h_2 are germs at p of local CR homeomorphisms between M and M' such that*

$$h_1(x) - h_2(x) = o\big(\|x\|^2\big)\,, \quad x \to 0\,, \tag{1}$$

then $h_1 \equiv h_2$, where $x = (x_1, x_2, x_3)$ are any local coordinates on M vanishing at p.

Observe that, since a homeomorphism need not preserve the vanishing order, a uniqueness statement in the spirit of Corollary 1.5 may not be reduced to the case $M = M'$ in general. In our case, however, the reduction is possible due to the above mentioned theorem of Huang.

Our next application of Theorem 1.4 is a structure result for the group $\mathsf{Aut}(M,p)$ of all local biholomorphisms $H\colon (\mathbb{C}^2,p) \to (\mathbb{C}^2,p)$ sending M (with $p \in M$) into itself. A fundamental problem, usually referred to as the *local biholomorphic equivalence problem*, is to determine for which pairs of (germs of) real submanifolds (M,p) and (M',p') there exist local biholomorphisms sending (M,p) into (M',p') or, formulated in a slightly different way, to describe, for a given germ of a manifold (M,p), its equivalence class under local biholomorphic transformations. There is, of course, no loss of generality in assuming that $p = p'$. The action of the group $\mathcal{E}_p$ of all local biholomorphisms $H\colon (\mathbb{C}^2,p) \to (\mathbb{C}^2,p)$ will take us from any germ (M,p) to any other germ (M',p) in the same equivalence class. However, $\mathcal{E}_p$ does not in general act freely on the equivalence class of (M,p). To understand the structure of the equivalence classes one is therefore led to study the structure of the isotropy group $\mathsf{Aut}(M,p)$. Observe that $\mathsf{Aut}(M,p)$ is a topological group equipped with a natural direct limit topology which it inherits as a subgroup of $\mathcal{E}_p$. A sequence of germs H^j is convergent if all germs H^j extend holomorphically to a common neighborhood of p on which they converge uniformly (cf. e.g. [BER1]). By standard techniques (see e.g. [BER1] and [BER4]), Theorem 1.4 implies the following:

Corollary 1.6. *Let (M,p) be a germ of a real-analytic hypersurface in $\mathbb{C}^2$ of finite type. Then the jet evaluation homomorphism*

$$j_p^2\colon \mathsf{Aut}(M,p) \to G_p^2(\mathbb{C}^2) \tag{2}$$

is a homeomorphism onto a closed Lie subgroup of $G_p^2(\mathbb{C}^2)$ and hence defines a Lie group structure on $\mathsf{Aut}(M,p)$.

We conclude with an application to the dynamics of germs of local biholomorphisms of $\mathbb{C}^2$:

Theorem 1.7. *Let $H\colon (\mathbb{C}^2,0) \to (\mathbb{C}^2,0)$ be a local biholomorphism tangent to the identity at 0, i.e. of the form $H(Z) = Z + O(|Z|^2)$. Suppose that H preserves a germ of a Levi-nonflat real-analytic hypersurface at 0. Then H fixes each point of a complex hypersurface through 0.*

Theorem 1.7 is a consequence of Theorem 3.1, whose statement and proof are given in §3.

2 Preliminaries

Recall that a real-analytic hypersurface $M \subset \mathbb{C}^N$ ($N \geq 2$) is called *Levi-flat* if its Levi form

$$L(\xi) := \sum_{k,j} \frac{\partial^2 \rho}{\partial z_k \partial \bar{z}_j} \xi_k \bar{\xi}_j$$

vanishes identically on the complex tangent subspace $T_p^c M := T_p M \cap i T_p M$ for any $p \in M$, where M is locally given by $\{\rho = 0\}$ with $d\rho \neq 0$. It is well known (and not difficult to see) that M is Levi-flat if and only if, at any point $p \in M$, there are local holomorphic coordinates $(z,w) \in \mathbb{C}^{N-1} \times \mathbb{C}$ in which M has the form $\{\mathsf{Im}\, w = 0\}$.

In general, let $M \subset \mathbb{C}^N$ be a real-analytic hypersurface with $p \in M$. We may choose local coordinates $(z,w) \in \mathbb{C}^{N-1} \times \mathbb{C}$, vanishing at p, so that M is defined locally near $p = (0,0)$ by an equation of the form

$$M : \mathsf{Im}\, w = \varphi(z, \bar{z}, \mathsf{Re}\, w)\,, \tag{3}$$

where $\varphi(z,\bar{z},s)$ is a real-valued, real-analytic function satisfying

$$\varphi(z,0,s) \equiv \varphi(0,\chi,s) \equiv 0\,. \tag{4}$$

Such coordinates are called *normal coordinates* for M at p; the reader is referred to e.g. [BER3] for the existence of such coordinates and related basic material concerning real submanifolds in complex space. We mention also that the hypersurface M is of finite type at $p = (0,0)$ if and only if, in normal coordinates, $\varphi(z,\chi,0) \not\equiv 0$.

3 Parametrization of Jets Along the Zero Segre Variety

Let $M \subset \mathbb{C}^N$ be a real-analytic hypersurface with $p \in M$. We shall choose normal coordinates $(z,w) \in \mathbb{C}^{N-1} \times \mathbb{C}$ for M at p; i.e. (z,w) vanishes at p and M is defined locally near $p = (0,0)$ by (3) where $\varphi(z,\bar{z},s)$ is a real-valued, real-analytic function satisfying (4). Denote by $J^k_{0,0}(\mathbb{C}^N)$ the space of all k-jets at 0 of holomorphic mappings $H\colon (\mathbb{C}^N,0) \to (\mathbb{C}^N,0)$, and by $\mathsf{T}(\mathbb{C}^N) \subset J^1_{0,0}(\mathbb{C}^N)$ the group of invertible upper triangular matrices. Given coordinates Z and Z' near the origins of the source and target copy of $\mathbb{C}^N$, respectively, we obtain associated coordinates $\Lambda = (\Lambda_i^\alpha)_{1\le i\le N, 1\le|\alpha|\le k} \in J^k_{0,0}(\mathbb{C}^N)$ (where $i \in \mathbb{Z}_+$ and $\alpha \in \mathbb{Z}_+^N$), in which a jet $j_0^k H$ is given by $\Lambda_i^\alpha = \partial_Z^\alpha H_i(0)$. In this paper, we shall be concerned with the situation $N = 2$ where we have coordinates (z,w) near 0 in the source copy of $\mathbb{C}^2$ and (z',w') near 0 in the target $\mathbb{C}^2$. A map $H\colon (\mathbb{C}^2,0) \to (\mathbb{C}^2,0)$ is then given in coordinates by $H(z,w) = (F(z,w), G(z,w))$. We shall use, for a given k, the notation $\Lambda = (\lambda^{ij}, \mu^{ij})_{1\le i+j\le k}$, for the associated coordinates on $J^k_{0,0}(\mathbb{C}^2)$, where $\lambda^{ij} = F_{z^iw^j}(0)$ and $\mu^{ij} = G_{z^iw^j}(0)$, and we use the notation $F_{z^iw^j} = \partial_z^i\partial_w^j F$ etc. for partial derivatives. In this section, we shall prove the following result.

Theorem 3.1. *Let $M, M' \subset \mathbb{C}^2$ be real-analytic hypersurfaces that are not Levi-flat, and let $(z,w) \in \mathbb{C}^2$ and $(z',w') \in \mathbb{C}^2$ be normal coordinates for M and M' vanishing at $p \in M$ and $p' \in M'$, respectively. Then, for any integer $k \ge 0$, the identity*

$$H_{w^k}(z,0) \equiv \Phi^k\big(z, H'(0), \overline{H'(0)}, j_0^{k+1}H, \overline{j_0^{k+1}H}\big) \tag{5}$$

holds for any local biholomorphism $H\colon (\mathbb{C}^2,0) \to (\mathbb{C}^2,0)$ sending M into M', where $\Phi^k(z,\Lambda_1,\widetilde{\Lambda}_1,\Lambda_2,\widetilde{\Lambda}_2)$ is a polynomial in $(\Lambda_2,\widetilde{\Lambda}_2) \in J^{k+1}_{0,0}(\mathbb{C}^2)\times\overline{J^{k+1}_{0,0}(\mathbb{C}^2)}$ with coefficients that are holomorphic in $(z,\Lambda_1,\widetilde{\Lambda}_1) \in \mathbb{C}\times J^1_{0,0}(\mathbb{C}^2)\times\overline{J^1_{0,0}(\mathbb{C}^2)}$ in a neighborhood of $\{0\}\times\mathsf{T}(\mathbb{C}^2)\times\overline{\mathsf{T}(\mathbb{C}^2)}$. Moreover, the $\mathbb{C}^2$-valued functions Φ^k depend only on (M,p) and (M',p').

Observe, that in normal coordinates, the line $\{(z,0) : z \in \mathbb{C}\}$ is the zero Segre variety and hence we see the conclusion of Theorem 3.1 as a parametrization of jets along the zero Segre variety. We first show how Theorem 3.1 implies Theorem 1.7.

Proof of Theorem 1.7. Let H be a local biholomorphism tangent to the identity at 0, as in Theorem 1.7, and $(M,0)$ the germ of a Levi-nonflat real-analytic hypersurface which is preserved by H. By assumption, we have $j_0^1 H = j_0^1 \mathsf{id}$. Thus, by Theorem 3.1, with $k=0$, applied to the local biholomorphisms H and id, both sending M into itself, we conclude that $H(z,0)\equiv z$. Hence each point of the complex hypersurface $\{z=0\}\subset \mathbb{C}^2$ is a fixed point of H. This completes the proof of Theorem 1.7. □

Before entering the proof of Theorem 3.1, we shall introduce some notation. The equation (3) can be written in complex form

$$M : w = Q(z,\bar z,\bar w)\,, \tag{6}$$

where $Q(z,\chi,\tau)$ is a holomorphic function satisfying

$$Q(z,0,\tau)\equiv Q(0,\chi,\tau)\equiv \tau\,. \tag{7}$$

Equation (6) defines a real hypersurface if and only if (see [BER3])

$$Q\big(z,\chi,\bar Q(\chi,z,w)\big)\equiv w\,, \tag{8}$$

where we use the notation $\bar h(\zeta) := \overline{h(\bar\zeta)}$.

We shall study the derivatives of $Q(z,\chi,\tau)$ with respect to (z,τ) denoted as follows:

$$q_{\alpha\mu}(\chi) := Q_{z^\alpha\tau^\mu}(0,\chi,0)\,,\quad q_{\alpha\mu}(0)=0\,,\quad \alpha\ge 1\,. \tag{9}$$

Note that we do not use the function $q_{\alpha\mu}(\chi)$ for $\alpha=0$, since the corresponding derivatives can be computed directly by (7):

$$Q_{\tau^\mu}(0,\chi,0)\equiv \begin{cases} 1 & \text{for } \mu=1\\ 0 & \text{for } \mu>1\,. \end{cases} \tag{10}$$

We now use the functions $q_{\alpha\mu}(\tau)$ to define biholomorphic invariants of (M,p) as follows. Let m_0 be the positive integer (or ∞) given by

$$m_0 := \min\big\{m\in\mathbb{Z}_+ : q_{\alpha\mu}(\chi)\not\equiv 0\,,\ \alpha+\mu=m\big\}\,, \tag{11}$$

where we set $m_0=\infty$ if $q_{\alpha\mu}(\chi)\equiv 0$ for all (α,μ). Thus, $m_0=\infty$ is equivalent to M being Levi-flat. In what follows, we shall assume that M is not Levi-flat, i.e. $m_0<\infty$. We also define

$$\mu_0 := \min\big\{\mu\in\mathbb{Z}_+ : q_{\alpha\mu}(\chi)\not\equiv 0\,,\ \alpha+\mu=m_0\big\}\,, \tag{12}$$

and set $\alpha_0 := m_0-\mu_0$.

Let $M'\subset\mathbb{C}^2$ be another real-analytic hypersurface, and let $(z',w')\in\mathbb{C}^2$ be normal coordinates for M' at some point p'. In what follows, we shall use a $'$ to denote an object associated to M' corresponding to one defined previously for M. Let furthermore $H=(F,G)$ be a local mapping

$(\mathbb{C}^2, p) \to (\mathbb{C}^2, p')$. Then H sends (a neighborhood of p in) M into M' if and only if it satisfies the identity

$$G\big(z, Q(z,\chi,\tau)\big) \equiv Q'\big(F(z,Q(z,\chi,\tau)), \bar{F}(\chi,\tau), \bar{G}(\chi,\tau)\big)\,, \tag{13}$$

for $(z,\chi,\tau) \in \mathbb{C}^3$. In particular, setting $\chi = \tau = 0$, we deduce that

$$G(z,0) \equiv 0\,. \tag{14}$$

It follows from (14) that, in normal coordinates, the 2×2 matrix $H'(0)$ is triangular and therefore H is a local biholomorphism if and only if

$$F_z(0)\, G_w(0) \neq 0\,. \tag{15}$$

We have the following property.

Proposition 3.2. *Let $M \subset \mathbb{C}^2$ be a real-analytic hypersurface. Then, the integers m_0, α_0, and μ_0 defined above are biholomorphic invariants.*

Proof. We have to show that, if (M,p) and (M',p') are locally biholomorphic, then $\mu_0 = \mu_0'$ and $\alpha_0 = \alpha_0'$. We introduce the following ordering of the pairs $(\alpha,\mu) \in \mathbb{Z}_+^2$. We write $(\alpha,\mu) \prec (\beta,\nu)$ if either $\alpha+\mu < \beta+\nu$, or if $\alpha+\mu = \beta+\nu$ and $\mu < \nu$ (or, equivalently, $\alpha > \beta$). We prove the statement by contradiction. Suppose $(\alpha_0,\mu_0) \prec (\alpha_0',\mu_0')$. We differentiate (13) by the chain rule α_0 times in z and μ_0 times in τ, evaluate the result at $(z,\tau)=0$ and use the identities (7), (10) and (14). On the right-hand side we obtain a sum of terms each of which has a factor of $Q'_{z^\alpha\chi^\beta\tau^\mu}(0,\bar{F}(\chi,0),0)$ with $(\alpha,\mu) \preceq (\alpha_0,\mu_0)$. By the assumption $(\alpha_0,\mu_0) \prec (\alpha_0',\mu_0')$, all these derivatives are zero. Similarly, using the fact that $Q_{z^\alpha\tau^\mu}(0,\chi,0) \equiv 0$ for $(\alpha,\mu) \prec (\alpha_0,\mu_0)$ on the left-hand side, we conclude that

$$G_{z^{\alpha_0}w^{\mu_0}}(0) + G_w(0)\, q_{\alpha_0\mu_0}(\chi) \equiv 0\,. \tag{16}$$

Since $G_w(0) \neq 0$ and $q_{\alpha_0\mu_0}(\chi) \not\equiv \text{const}$, we reach the desired contradiction. We must then have $(\alpha_0,\mu_0) \succeq (\alpha_0',\mu_0')$. The opposite inequality follows by reversing the roles of M and M' by considering the inverse mapping H^{-1}. Hence, $(\alpha_0,\mu_0) = (\alpha_0',\mu_0')$ as claimed. □

We are now ready to prove Theorem 3.1.

Proof of Theorem 3.1. As in the proof of Proposition 3.2, we differentiate (13) by the chain rule α_0 times in z and μ_0 times in τ, evaluate the result at $(z,\tau)=0$ and use the identities (7), (10) and (14):

$$\begin{aligned} &G_{z^{\alpha_0}w^{\mu_0}}(0) + G_w(0)\, q_{\alpha_0\mu_0}(\chi) \\ &\qquad \equiv q'_{\alpha_0\mu_0}\big(\bar{F}(\chi,0)\big)\big(F_z(0) + F_w(0) q_{10}(\chi)\big)^{\alpha_0}\overline{G_w}(\chi,0)^{\mu_0}. \end{aligned} \tag{17}$$

By putting $\chi = 0$ and using (9) we obtain $G_{z^{\alpha_0}w^{\mu_0}}(0) = 0$. Furthermore, the derivative $\overline{G_w}(\chi,0)$ on the right-hand side of (17) can be computed by differentiating (13) in τ at $(z,\tau)=0$:

$$\overline{G_w}(\chi,0) \equiv G_w(0) - q'_{10}(\bar{F}(\chi,0))F_w(0)\,. \tag{18}$$

Note that for $\chi=0$, we obtain $\overline{G_w}(0)=G_w(0)$, i.e. $G_w(0)$ is real. After these observations, (17) can be rewritten as

$$\begin{aligned} G_w(0)\,q_{\alpha_0\mu_0}(\chi) \equiv q'_{\alpha_0\mu_0}\big(\bar{F}(\chi,0)\big)\big(F_z(0) \\ + q_{10}(\chi)F_w(0)\big)^{\alpha_0}\big(G_w(0) - q'_{10}(\bar{F}(\chi,0))F_w(0)\big)^{\mu_0}. \end{aligned} \tag{19}$$

We point out that $q_{10}(\chi)\not\equiv 0$ (and similarly $q'_{10}(\chi')\not\equiv 0$) if and only if $m_0=1$ (in this case, $(\alpha_0,\mu_0)=(1,0)$). It can be shown that $m_0=1$ if and only if M is *finitely nondegenerate* at p (see e.g. [BER3]). We will not use this fact in the present paper.

It follows, that for $m_0\ge 2$, (19) reduces to the form

$$q_{\alpha_0\mu_0}(\chi) \equiv q'_{\alpha_0\mu_0}\big(\bar{F}(\chi,0)\big)\,F_z(0)^{\alpha_0}G_w(0)^{\mu_0-1}. \tag{20}$$

If $m_0=1$, in which case $(\alpha_0,\mu_0)=(1,0)$, we instead have

$$G_w(0)\,q_{10}(\chi) \equiv q'_{10}(\bar{F}(\chi,0))\big(F_z(0)+F_w(0)q_{10}(\chi)\big). \tag{21}$$

Define $l\ge 1$ to be the order of vanishing at 0 of the function $q_{\alpha_0\mu_0}(\chi)\not\equiv 0$. (It is not difficult to verify from (8) that $l\ge\alpha_0$. We will not use this property.) Together with (15), it follows from (20) and (21), respectively, that $l=l'$. Let us write

$$q_{\alpha_0\mu_0}(\chi)\equiv (q(\chi))^l, \quad q'_{\alpha_0\mu_0}(\chi')\equiv (q'(\chi'))^l, \tag{22}$$

where q and q' are local biholomorphisms of $\mathbb{C}$ at 0. Taking lth roots from both sides of (20) and (21) respectively we obtain

$$q'\big(\bar{F}(\chi,0)\big) \equiv E(\chi)q(\chi)\,, \tag{23}$$

where

$$E(\chi) := \begin{cases} \big(G_w(0)(F_z(0)+F_w(0)q_{10}(\chi))^{-1}\big)^{1/l}, & m_0=1\,,\\ \big(F_z(0)^{-\alpha_0}G_w(0)^{1-\mu_0}\big)^{1/l}, & m_0\ge 2\,, \end{cases}$$

and the branch of the lth root has to be chosen appropriately. We want to write E as a holomorphic function in χ, $F_z(0)$, $\overline{F_z}(0)$ and $F_w(0)$. For this, observe that there exists a constant $c\ne 0$ such that

$$E(\chi) \equiv \begin{cases} c\big(1+F_w(0)F_z(0)^{-1}q_{10}(\chi)\big)^{-1/l}, & m_0=1\,,\\ c\,, & m_0\ge 2\,, \end{cases} \tag{24}$$

where we used the principal branch of the lth root near 1 (for χ small). We substitute the expressions (24) for $E(\chi)$ in (23) and differentiate once in χ at $\chi=0$ to obtain

$$q'_{\chi'}(0)\overline{F_z}(0) \equiv c q_\chi(0) \tag{25}$$

in both cases $m_0 = 1$ and $m_0 \geq 2$. Solving (25) for c and using (23) and (24) we obtain

$$\bar{F}(\chi, 0) \equiv \chi \Psi\big(\chi, H'(0), \overline{H'(0)}\big) , \tag{26}$$

where Ψ is a holomorphic function in its arguments, defined in a neighborhood of the subset $\{0\} \times \mathsf{T}(\mathbb{C}^2) \times \overline{\mathsf{T}(\mathbb{C}^2)} \subset \mathbb{C} \times J^1_{0,0}(\mathbb{C}^2) \times \overline{J^1_{0,0}(\mathbb{C}^2)}$ such that

$$\Psi(0, \Lambda, \widetilde{\Lambda}) \neq 0 \tag{27}$$

for any $(\Lambda, \widetilde{\Lambda}) \in \mathsf{T}(\mathbb{C}^2) \times \overline{\mathsf{T}(\mathbb{C}^2)}$. In particular, we see from (26) that the mapping H along the Segre variety $\{w = 0\}$ is completely determined by its first jet at 0. In fact $H(z, 0)$ depends only on the derivatives $F_w(0)$, $F_z(0)$, $\overline{F_z}(0)$.

We shall now determine derivatives of H with respect to w along the Segre variety $\{w = 0\}$ in terms of jets of H at the origin. We begin with the derivatives $F_{w^k}(z, 0)$ (or $\overline{F_{w^k}}(\chi, 0)$), $k \geq 1$. To get an expression involving them we use the same strategy as above, but now we differentiate (13) α_0 times in z and $\mu_0 + k$ times in τ and then set $(z, \tau) = 0$. We shall use the notation $(F_{z^i w^j})_{i+j \leq k+1}$ etc. to denote strings of partial derivatives of *positive order*. We obtain

$$\begin{aligned} &G_{z^{\alpha_0} w^{\mu_0 + k}}(0) + \Psi_1(\chi, j_0^{k+1} G) \\ &\quad \equiv \partial_{\bar z} q'_{\alpha_0 \mu_0}\big(\bar{F}(\chi, 0)\big)\big(F_z(0) + F_w(0) q_{10}(\bar z)\big)^{\alpha_0} \overline{F_{w^k}}(\chi, 0) \overline{G_w}(\chi, 0)^{\mu_0} \\ &\qquad + \Psi_2\big(\chi, \bar{F}(\chi, 0), j_0^{k+1} F, (\overline{F_{w^r}}(\chi, 0))_{r \leq k-1}, (\overline{G_{w^s}}(\chi, 0))_{s \leq k+1}\big) , \end{aligned} \tag{28}$$

where the functions $\Psi_1(\chi, \Lambda)$ and $\Psi_2(\chi, \chi', \Lambda_1, \Lambda_2, \Lambda_3)$ are polynomials in Λ and $(\Lambda_1, \Lambda_2, \Lambda_3)$ respectively with holomorphic coefficients in (χ, χ'). (More precisely, the coefficients are polynomials in $(q_{\alpha\mu}(\chi))$ and $(q'_{\alpha\mu}(\chi'))$ and in their derivatives.) We claim, however, that no term involving $\overline{G_{w^{k+1}}}(\chi, 0)$ occurs with a nontrivial coefficient in Ψ_2 when $m_0 = 1$. Indeed, in this case $\alpha_0 = 1$ and $\mu_0 = 0$. Thus, to obtain the expression (28) we differentiate once in z and k times in τ. Consequently, no term of the form $\overline{G_{w^{k+1}}}(\chi, 0)$ can appear, as claimed.

We observe that from the identity (28) with $\chi = 0$ we obtain a polynomial expression for the derivative $G_{z^{\alpha_0} w^{\mu_0 + k}}(0)$ in terms of $j_0^{k+1} H$ and $\overline{j_0^{k+1} H}$. (Recall that $H = (F, G)$.) After substituting in (28) this expression for $G_{z^{\alpha_0} w^{\mu_0 + k}}(0)$, the right-hand side of (18) for $\overline{G_w}(\chi, 0)$ and the right-hand side of (26) for $\bar{F}(\chi, 0)$, we obtain

$$\begin{aligned} \partial_{\chi'} q'_{\alpha_0 \mu_0}\big(\chi \Psi(\chi, H'(0), \overline{H'(0)}\big) \overline{F_{w^k}}(\chi, 0) \equiv \Psi_3\big(\chi, H'(0), \overline{H'(0)}, j_0^{k+1} H, \\ \overline{j_0^{k+1} H}, (\overline{F_{w^r}}(\chi, 0))_{r \leq k-1}, (\overline{G_{w^s}}(\chi, 0))_{s \leq k+1}\big) , \end{aligned} \tag{29}$$

where $\Psi_3(\chi, \Lambda_1, \widetilde{\Lambda}_1, \Lambda_2, \widetilde{\Lambda}_2, \Lambda_3, \Lambda_4)$ is a polynomial in $(\Lambda_2, \widetilde{\Lambda}_2, \Lambda_3, \Lambda_4)$ with holomorphic coefficients in $(\chi, \Lambda_1, \widetilde{\Lambda}_1)$. We also observe that the coefficient of $\overline{F_{w^k}}(\chi, 0)$ on the left-hand side does not vanish identically by the choice of (α_0, μ_0).

As before, we get an expression for the derivatives $\overline{G_{w^s}}(\chi, 0)$ that occur on the right-hand side of (29) by differentiating (13) in τ at $(z, \tau) = 0$, this time $s \geq 2$ times:

$$\begin{aligned} G_{w^s}(0) \equiv\ & q'_{10}\big(\bar{F}(\chi, 0)\big) F_{w^s}(0) \\ & + \partial_{\chi'} q'_{10}\big(\bar{F}(\chi, 0)\big) F_w(0) \overline{F_{w^{s-1}}}(\chi, 0) + \overline{G_{w^s}}(\chi, 0) \\ & + \Psi_4\big(\bar{F}(\chi, 0), (F_{w^i}(0))_{i \leq s-1}, (\overline{F_{w^r}}(\chi, 0))_{r \leq s-2}, (\overline{G_{w^j}}(\chi, 0))_{j \leq s-1}\big), \end{aligned} \tag{30}$$

where $\Psi_4(\chi', \Lambda_1, \Lambda_2, \Lambda_3)$ is a polynomial in $(\Lambda_1, \Lambda_2, \Lambda_3)$ with holomorphic coefficients in χ'. We solve (30) for $\overline{G_{w^s}}(\bar{z}, 0)$ in terms of $\overline{G_{w^t}}(\bar{z}, 0)$ with $t < s$. Then, by induction on s and by (18), we obtain for any $s \geq 1$ an identity of the form

$$\begin{aligned} \overline{G_{w^s}}(\chi,0) \equiv\ & G_{w^s}(0) - q'_{10}\big(\bar{F}(\chi,0)\big) F_{w^s}(0) - \partial_{\chi'} q'_{10}(\bar{F}(\chi,0)) F_w(0) \overline{F_{w^{s-1}}}(\chi,0) \\ & + \Psi_5\big(\bar{F}(\chi, 0), (H_{w^i}(0))_{i \leq s-1}, (\overline{F_{w^r}}(\chi, 0))_{r \leq s-2}\big), \end{aligned} \tag{31}$$

where $\Psi_5(\chi', \Lambda_1, \Lambda_2)$ is a polynomial in (Λ_1, Λ_2) with holomorphic coefficients in χ'. Note that the term $\overline{F_{w^{s-1}}}(\chi, 0)$ only occurs in (31) when $m_0 = 1$.

We now substitute the right-hand side of (30) for $\overline{G_{w^s}}(\chi, 0)$ and the right-hand side of (26) for $\bar{F}(\chi, 0)$ in the identity (29) to obtain

$$\begin{aligned} & \partial_{\chi'} q'_{\alpha_0 \mu_0}\big(\chi \Psi(\chi, H'(0), \overline{H'(0)}\big) \overline{F_{w^k}}(\chi, 0) \\ & \qquad \equiv \Psi^k\big(\chi, H'(0), \overline{H'(0)}, j_0^{k+1} H, \overline{j_0^{k+1} H}, (\overline{F_{w^r}}(\chi, 0))_{r \leq k-1}\big), \end{aligned} \tag{32}$$

where $\Psi^k(\chi, \Lambda_1, \widetilde{\Lambda}_1, \Lambda_2, \widetilde{\Lambda}_2, \Lambda_3)$ is a polynomial in $(\Lambda_2, \widetilde{\Lambda}_2, \Lambda_3)$ with holomorphic coefficients in $(\chi, \Lambda_1, \widetilde{\Lambda}_1)$. We claim that an identity of the form

$$\bar{F}_{w^k}(\chi, 0) \equiv \widetilde{\Psi}^k\big(\chi, H'(0), \overline{H'(0)}, j_0^{k+1} H, \overline{j_0^{k+1} H}\big) \tag{33}$$

holds, where $\widetilde{\Psi}^k(\chi, \Lambda_1, \widetilde{\Lambda}_1, \Lambda_2, \widetilde{\Lambda}_2)$ is a polynomial in $(\Lambda_2, \widetilde{\Lambda}_2)$ with holomorphic coefficients in $(\chi, \Lambda_1, \widetilde{\Lambda}_1)$. We prove the claim by induction on k. For $k = 1$, there is no occurrence of $\Lambda_3 = \overline{F_{w^r}}(\chi, 0)$ on the right-hand side of (32). We now would like to divide both sides of (32) by the first factor $\Gamma := \partial_{\chi'} q'_{\alpha_0 \mu_0}(\chi \Psi(\chi, H'(0), \overline{H'(0)})$ on the left-hand side. We know from (27) that this factor does not vanish identically. However, it may happen that it vanishes for $\chi = 0$. In order to obtain a holomorphic function on the right-hand side after this division, we have to make sure that the vanishing order of the right-hand side with respect to χ at the origin is not smaller that the vanishing order of Γ. Of course, by (32), this is true for those jet values of $(\Lambda_1, \widetilde{\Lambda}_1, \Lambda_2, \widetilde{\Lambda}_2)$ that come from a local biholomorphism H sending M into M'. On the other hand, we have no information about the vanishing order of $\Psi^1(\chi, \Lambda_1, \widetilde{\Lambda}_1, \Lambda_2, \widetilde{\Lambda}_2)$ for other values of $(\Lambda_1, \widetilde{\Lambda}_1, \Lambda_2, \widetilde{\Lambda}_2)$. Hence we may not be able to divide. The idea for solving this problem is to extract the "higher order part" of Ψ^1 that is divisible by Γ.

Let $0 \leq \nu < \infty$ be the vanishing order of the function $\partial_{\chi'} q'_{\alpha_0\mu_0}(\chi')$ at $\chi' = 0$. Then the vanishing order of the left-hand side of (32) is at least ν. By truncating the power series expansion in χ of the coefficients of the polynomial Ψ^1 on the right-hand side of (32), we can write it in a unique way as a sum $\Psi^1 \equiv \Psi^1_1 + \Psi^1_2$ such that both Ψ^1_1 and Ψ^1_2 are polynomials in $(\Lambda_2, \widetilde{\Lambda}_2)$ with holomorphic coefficients in $(\chi, \Lambda_1, \widetilde{\Lambda}_1)$, each coefficient of Ψ^1_1 is a polynomial in χ of order at most $\nu - 1$ with holomorphic coefficients in $(\Lambda_1, \widetilde{\Lambda}_1)$ and each coefficient of Ψ^1_2 is of vanishing order at least ν with respect to χ. We now remark that, whenever we set $(\Lambda_1, \widetilde{\Lambda}_1, \Lambda_2, \widetilde{\Lambda}_2) = (H'(0), \overline{H'(0)}, j_0^{k+1}H, \overline{j_0^{k+1}H})$ for a mapping H satisfying (32), the first polynomial Ψ^1_1 must vanish identically in χ. Therefore, the identity (32) will still hold after we replace Ψ^1 by Ψ^1_2. Now it follows from (27) that Ψ^1_2 is divisible by Γ and, hence, we have proved the claim for $k = 1$. The induction step for $k > 1$ is essentially a repetition of the above argument. Once (33) is shown for $k < k_0$, we substitute it for $\overline{F_{w^r}}(\chi, 0)$ in the right-hand side of (32). Then we obtain a polynomial without Λ_3 and the above argument can be used to obtain (33) for $k = k_0$. The claim is proved.

A formula for $\bar{G}_{w^k}(\chi, 0)$ similar to (33) is obtained by substituting the result for F in (31) (and also using (26)). The proof of Theorem 3.1 is complete. □

4 Parametrization of Biholomorphisms in the Finite Type Case

In this section, we will prove the following theorem, from which Theorem 1.4 is a direct consequence and Corollary 1.6 follows by standard techniques (see [BER1] and [BER4]). We keep the setup and notation introduced in previous sections. We also denote by $\mathsf{T}^2(\mathbb{C}^2)$ the subspace of 2-jets in $J^2_{0,0}(\mathbb{C}^2)$ whose first derivative matrix is upper triangular.

Theorem 4.1. *Let $M, M' \subset \mathbb{C}^2$ be real-analytic hypersurfaces of finite type, and let $(z, w) \in \mathbb{C}^2$ and $(z', w') \in \mathbb{C}^2$ be normal coordinates for M*

and M' at $p \in M$ and $p' \in M'$, respectively. Then an identity of the form

$$H(z,w) \equiv \Theta\big(z,w,j_0^2H,\overline{j_0^2H}\big),$$

holds for any local biholomorphism $H\colon(\mathbb{C}^2,p)\to(\mathbb{C}^2,p')$ sending M into M', where $\Theta(z,w,\Lambda,\widetilde{\Lambda})$ is a holomorphic function in a neighborhood of the subset $\{(0,0)\}\times\mathsf{T}^2(\mathbb{C}^2)\times\overline{\mathsf{T}^2(\mathbb{C}^2)}$ in $\mathbb{C}^2\times J^2_{0,0}(\mathbb{C}^2)\times\overline{J^2_{0,0}(\mathbb{C}^2)}$, depending only on (M,p) and (M',p').

Proof. We use the expansion of the function $Q(z,\chi,\tau)$ (and similarly for the function $Q'(z',\chi',\tau')$, using a $'$ to denote corresponding objects associated to M') as follows

$$Q(z,\chi,\tau)=\tau+\sum_{\beta\ge 1} r_\beta(\chi,\tau)z^\beta,\quad r_\beta(0,0)=0\,. \tag{34}$$

Recall that M is of finite type at 0 if and only if $Q(z,\chi,0)\not\equiv 0$, i.e. if $r_\beta(\chi,0)\not\equiv 0$ for some β. We define a positive integer associated to (M,p) by

$$\beta_0 := \min\big\{\beta\colon r_\beta(\chi,0)\not\equiv 0\big\} \tag{35}$$

and an integer β_0' associated to (M',p') by the analogous formula. It follows easily, by setting $\tau=0$ in the equation (13), differentiating in z and then setting $z=0$ as in the proof of Proposition 3.2, that $\beta_0=\beta_0'$ (i.e. β_0 is a biholomorphic invariant). Indeed, by using also the fact that $G(z,0)\equiv 0$, we obtain in this way the identity

$$r_{\beta_0}(\chi,0)G_w(0)\equiv r'_{\beta_0}\big(\bar F(\chi,0),0\big)\big(F_z(0)+F_w(0)r_1(\chi,0)\big)\,. \tag{36}$$

By differentiating (13) with respect to z and setting $\tau=\bar Q(\chi,z,0)$, we also obtain

$$\begin{aligned} Q_z\big(z,\chi,\bar Q(\chi,z,0)\big)G_w(z,0)\equiv Q'_z\big(F(z,0),\bar F(\chi,\bar Q(\chi,z,0)),\bar G(\chi,\bar Q(\chi,z,0))\big)\\ \cdot\big(F_z(z,0)+Q_z(z,\chi,\bar Q(\chi,z,0))F_w(z,0)\big)\,, \end{aligned}\tag{37}$$

where we have used (8) and the fact that $G(z,0)\equiv 0$. By using the conjugate of (13) to substitute for $\bar G(\chi,\bar Q(\chi,z,0))$, we obtain

$$\begin{aligned} Q_z\big(z,\chi,\bar Q\big)G_w(z,0)\equiv Q'_z\big(F(z,0),\bar F(\chi,\bar Q),\bar Q'(\bar F(\chi,\bar Q),F(z,0),0)\big)\\ \cdot\big(F_z(z,0)+Q_z(z,\chi,\bar Q)F_w(z,0)\big)\,, \end{aligned}\tag{38}$$

where we have used the notation $\bar Q := \bar Q(\chi,z,0)$. Observe, by differentiating (8) with respect to z and setting $w=0$, that

$$Q_z\big(z,\chi,\bar Q(\chi,z,0)\big)\equiv -Q_\tau\big(z,\chi,\bar Q(\chi,z,0)\big)\bar Q_z(\chi,z,0)\,. \tag{39}$$

Since $Q_\tau(z,0,0)\equiv 1$ by (7), we conclude from (34) and (39) that

$$Q_z\big(z,\chi,\bar Q(\chi,z,0)\big)=\partial_z\overline{r_{\beta_0}}(z,0)\chi^{\beta_0}+O(\chi^{\beta_0+1})\,. \tag{40}$$

Also, observe that $\partial_z\overline{r_{\beta_0}}(z,0)\not\equiv 0$ by the choice of β_0. It follows that $\partial_z\overline{r_{\beta_0}}(z,0)\ne 0$ for all z in any sufficiently small punctured disc centered at 0. In such a punctured disc there exist β_0 locally defined holomorphic functions $\psi(z,\chi)$, differing by multiplication by a β_0th root of unity, such that

$$Q_z\big(z,\chi,\bar{Q}(\chi,z,0)\big) \equiv \psi(z,\chi)^{\beta_0} \tag{41}$$

and

$$\psi(z,0)\equiv 0, \quad \psi_\chi(z,0)\neq 0 \text{ for } z\neq 0 \text{ near } 0 \tag{42}$$

for each choice of ψ. Moreover, each local function ψ extends as a multiple-valued holomorphic function (with at most β_0 branches) to a neighborhood of $(0,0)$ in $\mathbb{C}^2\setminus\{z=0\}$. If we choose z in a sufficiently small punctured disc centered at 0, then also $\partial_{z'}\overline{r'_{\beta_0}}(F(z,0),0)\neq 0$ and the equation (38) can be written

$$\psi(z,\chi)^{\beta_0}G_w(z,0)\big(F_z(z,0)+Q_z(z,\chi,\bar{Q})F_w(z,0)\big)^{-1} \\ \equiv \psi'\big(F(z,0),\bar{F}(\chi,\bar{Q}(\chi,z,0))\big)^{\beta_0}. \tag{43}$$

$G_w(z_0,0)\neq 0$. Taking β_0th roots from both sides of (43) and substituting the conjugate of (26) for $F(z,0)$ we obtain

$$\psi'\big(z\bar{\Psi}(z,\overline{H'(0)},H'(0)),\bar{F}(\chi,\bar{Q}(\chi,z,0))\big)\equiv E(z,\chi)\psi(z,\chi) \tag{44}$$

with

$$E(z,\chi):=\big(G_w(z,0)(F_z(z,0)+\psi(z,\chi)^{\beta_0}F_w(z,0))^{-1}\big)^{1/\beta_0},$$

and the branch of the β_0th root has to be appropriately chosen. Analogously to the proof of Theorem 3.1, we write

$$E(z,\chi)\equiv c\bigg(\frac{G_w(z,0)}{G_w(0)}\Big(\frac{F_z(z,0)}{F_z(0)}+Q_z(z,\chi,\bar{Q})\frac{F_w(z,0)}{F_z(0)}\Big)^{-1}\bigg)^{1/\beta_0}, \tag{45}$$

where c is a constant and where we used the principal branch of the β_0th root near 1. We now use Theorem 3.1 to rewrite (45) as

$$E(z,\chi)\equiv c\,\Phi\big(z,\chi,H'(0),\overline{H'(0)},j_0^2H,\overline{j_0^2H}\big)^{1/\beta_0}, \tag{46}$$

where $\Phi(z,\chi,\Lambda_1,\widetilde{\Lambda}_1,\Lambda_2,\widetilde{\Lambda}_2)$ is a holomorphic function as in Theorem 3.1. Here the problem arises that the values of $\Phi(0,0,\Lambda_1,\widetilde{\Lambda}_1,\Lambda_2,\widetilde{\Lambda}_2)$ may differ from 1 and thus the values of the β_0th root in (46) are not uniquely determined. To solve this problem we use a trick to replace the function $\Phi(z,\chi,\Lambda_1,\widetilde{\Lambda}_1,\Lambda_2,\widetilde{\Lambda}_2)$ by

$$\Phi(z,\chi,\Lambda_1,\widetilde{\Lambda}_1,\Lambda_2,\widetilde{\Lambda}_2)-\Phi(0,0,\Lambda_1,\widetilde{\Lambda}_1,\Lambda_2,\widetilde{\Lambda}_2)+1\,.$$

The new function (denote it by Φ instead of the old one) takes the right value 1 for $(z,\chi)=0$, so that the principal branch of its β_0th root is defined, and the identity (46) still holds for any local biholomorphism $H=(F,G)$ of $\mathbb{C}^2$ sending (M,p) into (M',p').

We substitute the expression (46) for $E(z,\chi)$ in (44) and differentiate once in χ at $\chi = 0$ to obtain

$$\psi'_{\chi'}\big(z\bar\Psi(z,\overline{H'(0)},H'(0)),0\big) \equiv c\,\Phi\big(z,0,H'(0),\overline{H'(0)},j_0^2H,\overline{j_0^2H}\big)^{1/\beta_0}\psi_\chi(z,0)\,. \tag{47}$$

By (41), the coefficient on the right-hand side does not vanish identically. Hence we can solve c from (47) as a *multiple-valued holomorphic function*

$$c = c\big(z,H'(0),\overline{H'(0)},j_0^2H,\overline{j_0^2H}\big) \tag{48}$$

and thus ignore the fact that c is constant. We know from (46), however, that, whenever the arguments of Ψ and Φ are jets of a local biholomorphism, the "function" c is actually (locally) constant. The different values of c are due to the choice of different branches of ψ and ψ'. Recall that the values of ψ and ψ' may only differ by β_0th roots of unity. If ψ and ψ' are multiplied by the roots of unity ϵ and ϵ' respectively, then c is multiplied by $\epsilon'\epsilon^{-1}$.

We claim that, for every fixed jet

$$(\Lambda_1^0,\widetilde\Lambda_1^0,\Lambda_2^0,\widetilde\Lambda_2^0)\in \mathsf{T}(\mathbb{C}^2)\times\overline{\mathsf{T}(\mathbb{C}^2)}\times J^2_{0,0}(\mathbb{C}^2)\times\overline{J^2_{0,0}(\mathbb{C}^2)}\,,$$

the (multiple-valued) function $c(z,\Lambda_1,\widetilde\Lambda_1,\Lambda_2,\widetilde\Lambda_2)$ is uniformly bounded for $(z,\Lambda_1,\widetilde\Lambda_1,\Lambda_2,\widetilde\Lambda_2)$ in some neighborhood of $(0,\Lambda_1^0,\widetilde\Lambda_1^0,\Lambda_2^0,\widetilde\Lambda_2^0)$. Indeed, it follows from the construction that the derivative $\psi_\chi(z,0)$ equals the β_0th root of $\partial_z\overline{r_{\beta_0}}(z,0)$. Furthermore, by equation (36), the functions $\partial_z\overline{r_{\beta_0}}(z,0)$ and $\partial_z\overline{r'_{\beta_0}}(F(z,0),0)$ have the same vanishing order at $z=0$. Then, taking β_0th powers of both sides in (47) we obtain (single-valued) holomorphic functions of the same vanishing order at $z=0$ (recall that both functions Ψ and Φ do not vanish at $z=0$). Hence c^{β_0} is bounded as a ratio of two holomorphic functions having the same vanishing order. The claim is proved.

We now substitute (48) for c into (46) and then use (44) to obtain

$$\begin{aligned}&\psi'\big(z\bar\Psi(z,\overline{H'(0)},H'(0)),\bar F(\chi,\bar Q(\chi,z,0))\big)\\ &\quad\equiv c\big(z,H'(0),\overline{H'(0)},j_0^2H,\overline{j_0^2H}\big)\,\Phi\big(z,\chi,H'(0),\overline{H'(0)},j_0^2H,\overline{j_0^2H}\big)^{1/\beta_0}\psi(z,\chi)\end{aligned} \tag{49}$$

Since $\psi'_\chi(F(z,0),0)\neq 0$ for $z\neq 0$ near 0, we may apply the implicit function theorem and solve for $\bar F(\chi,\bar Q(\chi,z,0))$ in (49) to conclude that

$$\bar F\big(\chi,\bar Q(\chi,z,0)\big)\equiv \Phi_1(z,\chi,j_0^2H,\overline{j_0^2H})\,, \tag{50}$$

where $\Phi_1(z,\chi,\Lambda,\widetilde\Lambda)$ is an (a priori multiple-valued) holomorphic function defined in a domain

$$D\subset\mathbb{C}\times\mathbb{C}\times J^2_{0,0}(\mathbb{C}^2)\times\overline{J^2_{0,0}(\mathbb{C}^2)} \tag{51}$$

that contains all points $(z_0, 0, \Lambda_0, \overline{\widetilde{\Lambda}_0})$ with $z_0 \neq 0$ that are sufficiently close to the set $\{0\} \times \{0\} \times \mathsf{T}^2(\mathbb{C}^2) \times \overline{\mathsf{T}^2(\mathbb{C}^2)}$. A value of Φ_1 depends on the values of ψ, ψ' and c. We have seen that, if ψ and ψ' are multiplied by ϵ and ϵ' respectively, then c is multiplied by $\epsilon'\epsilon^{-1}$. We now observe, that in fact the identity (49) is invariant under this change. Hence we obtain *exactly the same value* for Φ_1 for *all possible values* of ψ and ψ'. We conclude that Φ_1 is single-valued.

A similar expression for $\bar{G}$ is obtained by substituting (50) and (26) into (13). In summary, we obtain

$$\bar{H}(\chi, \bar{Q}(\chi, z, 0)) \equiv \Xi\big(z, \chi, j^2 H(0), \overline{j^2 H(0)}\big)\,, \tag{52}$$

where $\Xi(z, \chi, \Lambda, \widetilde{\Lambda})$ is a ($\mathbb{C}^2$-valued) holomorphic function in a domain D as in (51). Here we conjugated and switched the variables.

To complete the proof of Theorem 4.1, we shall proceed as in [BER1]. We consider the equation

$$\tau = \bar{Q}(\chi, z, 0)\,, \tag{53}$$

and try to solve it for z as a function of (χ, τ) in a neighborhood of a point $z_0 \neq 0$ close to 0. (We cannot apply the method of [BER1] at 0, since the function Ξ in (52) may not be defined in a neighborhood of $\{0\} \times \{0\} \times \mathsf{T}^2(\mathbb{C}^2) \times \overline{\mathsf{T}^2(\mathbb{C}^2)}$.) We expand $\bar{Q}(\chi, z, 0)$ in powers of $z - z_0$ near $z = z_0$ as follows

$$\bar{Q}(\chi, z, 0) \equiv p_0(\chi; z_0) + \sum_{\gamma \geq 1} p_\gamma(\chi; z_0)(z - z_0)^\gamma \tag{54}$$

with $p_\gamma(0; z_0) \equiv 0$ for all γ, since $\bar{Q}(0, z, 0) \equiv 0$. Let γ_0 be the smallest integer $\gamma \geq 1$ such that $p_\gamma(\chi; z_0) \not\equiv 0$ with z_0 fixed. The existence of $\gamma_0 < \infty$ is guaranteed by the finite type condition. Moreover, if $z_0 \neq 0$ is sufficiently small, γ_0 does not depend on z_0. After dividing (53) by $p_{\gamma_0}(\chi; z_0)^{\gamma_0+1}$ and using (54) we obtain

$$\frac{\tau - p_0(\chi; z_0)}{p_{\gamma_0}(\chi; z_0)^{\gamma_0+1}} = \Big(\frac{z - z_0}{p_{\gamma_0}(\chi; z_0)}\Big)^{\gamma_0} + \sum_{\gamma \geq \gamma_0+1} C_\gamma(\chi; z_0)\Big(\frac{z - z_0}{p_{\gamma_0}(\chi; z_0)}\Big)^{\gamma}, \tag{55}$$

where $C_\gamma(\chi; z_0) := p_\gamma(\chi; z_0) p_{\gamma_0}(\chi; z_0)^{\gamma - \gamma_0 - 1}$. Then, by the implicit function theorem, the equation

$$\eta = t^{\gamma_0} + \sum_{\gamma \geq \gamma_0+1} C_\gamma(\chi; z_0) t^\gamma$$

has γ_0 solutions of the form $t = g(\chi, \eta^{1/\gamma_0}; z_0)$, where $g(\chi, \zeta; z_0)$ is a holomorphic function in a neighborhood of $\{0\} \times \{0\} \times (\Delta_\delta \setminus \{0\})$ with $g(0, 0; z_0) \equiv 0$, where $\Delta_\delta := \{z_0 \in \mathbb{C} : |z_0| < \delta\}$ is a sufficiently small disc. Hence the equation (53) can be solved for z in the form

$$z = z_0 + p_{\gamma_0}(\chi; z_0) g\bigg(\chi, \bigg(\frac{\tau - p_0(\chi; z_0)}{p_{\gamma_0}(\chi; z_0)^{\gamma_0+1}}\bigg)^{1/\gamma_0}; z_0\bigg) \tag{56}$$

for $\chi \neq 0$ and $(\tau - p_0(\bar z; z_0))/p_{\gamma_0}(\bar z; z_0)^{\gamma_0+1}$ both sufficiently small. We now substitute (56) for z in the identity (52) to obtain

$$\bar H(\chi,\tau) \equiv \widetilde{\Xi}\bigg(\bigg(\frac{\tau - p_0(\chi; z_0)}{p_{\gamma_0}(\chi; z_0)^{\gamma_0+1}}\bigg)^{1/\gamma_0}, \chi, j_0^2 H, \overline{j_0^2 H}; z_0\bigg), \tag{57}$$

where $\widetilde{\Xi}(\zeta, \chi, \Lambda, \widetilde{\Lambda}; z_0)$ is a holomorphic function defined for all $(\zeta, \chi, \Lambda, \widetilde{\Lambda}; z_0)$ with $(\chi, \zeta; z_0)$ in a neighborhood of $\{0\} \times \{0\} \times (\Delta_\delta \setminus \{0\})$ and

$$\big(z_0 + p_{\gamma_0}(\chi; z_0)^{\gamma_0+1} g(\chi, \zeta; z_0), \chi, \Lambda, \widetilde{\Lambda}\big) \in D\,,$$

where D is the domain of definition of the function Ξ in (52). It follows from the above description of D that $(0, 0, \Lambda_0, \widetilde{\Lambda}_0; z_0)$ is in the domain of definition of $\widetilde{\Xi}$ whenever $z_0 \neq 0$ and $(z_0, 0, \Lambda_0, \widetilde{\Lambda}_0)$ is in a sufficiently small neighborhood of $\{0\} \times \{0\} \times \mathsf{T}^2(\mathbb{C}^2) \times \overline{\mathsf{T}^2(\mathbb{C}^2)}$.

Let us expand the function $\widetilde{\Xi}(\zeta, \chi, \Lambda, \widetilde{\Lambda}; z_0)$ in ζ

$$\widetilde{\Xi}(\zeta, \chi, \Lambda, \widetilde{\Lambda}; z_0) \equiv \sum_{k \geq 0} A_k(\chi, \Lambda, \widetilde{\Lambda}; z_0) \zeta^k, \tag{58}$$

and decompose it as $\widetilde{\Xi} = \widetilde{\Xi}_1 + \widetilde{\Xi}_2$, where

$$\begin{aligned} \widetilde{\Xi}_1(\zeta, \chi, \Lambda, \widetilde{\Lambda}; z_0) &:= \sum_{j \geq 0} A_{j\gamma_0}(\chi, \Lambda, \widetilde{\Lambda}; z_0) \zeta^{j\gamma_0} \\ \widetilde{\Xi}_2(\zeta, \chi, \Lambda, \widetilde{\Lambda}; z_0) &:= \sum_{k \notin \gamma_0 \mathbb{Z}_+} A_k(\chi, \Lambda, \widetilde{\Lambda}; z_0) \zeta^k. \end{aligned} \tag{59}$$

Since $\bar H(\chi, \tau)$ is holomorphic in a neighborhood of 0 in $\mathbb{C}^2$, the function of $(\chi, \tau; z_0)$ on the right-hand side of (57) is independent of the value of the γ_0th root, independent of $z_0 \neq 0$ and extends holomorphically to a neighborhood of 0 in $\mathbb{C}^3$. Let us denote by $\Lambda_0 \in \mathsf{T}^2(\mathbb{C}^2)$ the value of $j_0^2 H$. Since the function $\widetilde{\Xi}_1(\zeta, \chi, \Lambda_0, \bar\Lambda_0; z_0)$, in which we substitute

$$\zeta = \bigg(\frac{\bar w - p_0(\chi; z_0)}{p_{\gamma_0}(\chi; z_0)^{\gamma_0+1}}\bigg)^{1/\gamma_0}, \tag{60}$$

is single valued on $|\bar w| = \varepsilon$, for $\varepsilon > 0$ sufficiently small (depending on χ with $p_{\gamma_0}(\chi; z_0) \neq 0$), we conclude that the function $(\zeta, \chi; z_0) \mapsto \widetilde{\Xi}_2(\zeta, \chi, \Lambda_0, \bar\Lambda_0; z_0)$ is identically 0. Hence, we must have

$$\bar H(\chi, \tau) \equiv \Xi_1\left(\frac{\bar w - p_0(\chi; z_0)}{p_{\gamma_0}(\chi; z_0)^{\gamma_0+1}}, \chi, j_0^2 H, \overline{j_0^2 H}; z_0\right), \tag{61}$$

where $\Xi_1(\eta, \chi, \Lambda, \widetilde{\Lambda}; z_0) := \sum_{j \geq 0} A_{j\gamma_0}(\chi, \Lambda, \widetilde{\Lambda}; z_0) \eta^j$.

Next, we decompose each $A_{j\gamma_0}(\chi,\Lambda,\widetilde{\Lambda};z_0)$ uniquely as follows

$$A_{j\gamma_0}(\chi,\Lambda,\widetilde{\Lambda};z_0) \equiv B_j(\chi,\Lambda,\widetilde{\Lambda};z_0)p_{\gamma_0}(\chi;z_0)^{j(\gamma_0+1)} + \sum_{0\le l\le Kj-1} R_{jl}(\Lambda,\widetilde{\Lambda};z_0)\chi^l, \tag{62}$$

where K denotes the order of vanishing of $p_{\gamma_0}(z;z_0)^{\gamma_0+1}$ at 0. It is not difficult to see that we have

$$\sup_{|\chi|\le\delta} \left\| B_j(\chi,\Lambda,\widetilde{\Lambda};z_0)\right\| \le C^j \sup_{|\chi|\le\delta} \left\| A_{j\gamma_0}(\chi,\Lambda,\widetilde{\Lambda};z_0)\right\|, \tag{63}$$

for some small $\delta > 0$ and constant C, where $\|v\|$ denotes the maximum of $|v_1|$ and $|v_2|$ for $v \in \mathbb{C}^2$. Hence, the power series

$$\Gamma(\kappa,\chi,\Lambda,\widetilde{\Lambda};z_0) := \sum_{j\ge 0} B_j(\chi,\Lambda,\widetilde{\Lambda};z_0)\kappa^j \tag{64}$$

defines a holomorphic function whose domain of definition contains any point $(0,0,\Lambda_0,\widetilde{\Lambda}_0;z_0)$ with $z_0 \ne 0$ and $(z_0,0,\Lambda_0,\widetilde{\Lambda}_0)$ in a sufficiently small neighborhood of $\{0\}\times\{0\}\times \mathsf{T}(\mathbb{C}^2)\times\overline{\mathsf{T}(\mathbb{C}^2)}$.

We now wish to show that Ξ_1 in (61) can be replaced by Γ with $\kappa = \tau - p_0(\chi;z_0)$. For this, we decompose the function Ξ_1 uniquely as $\Xi_1 = \Xi_2 + \Xi_3$, where

$$\begin{aligned} \Xi_2(\eta,\chi,\Lambda,\widetilde{\Lambda};z_0) &:= \sum_{j\ge 0} B_j(\chi,\Lambda,\widetilde{\Lambda};z_0)p_{\gamma_0}(\chi;z_0)^{j(\gamma_0+1)}\eta^j \\ \Xi_3(\eta,\chi,\Lambda,\widetilde{\Lambda};z_0) &:= \sum_{j\ge 0} R_j(\chi,\Lambda,\widetilde{\Lambda};z_0)\eta^j, \end{aligned} \tag{65}$$

where $R_j(\chi,\Lambda,\widetilde{\Lambda};z_0) := \sum_{0\le l\le Kj-1} R_{jl}(\Lambda,\widetilde{\Lambda};z_0)\chi^l$ is the remainder polynomial in the division (62). Now, observe that $\Xi_2(\eta,\chi,\Lambda,\widetilde{\Lambda};z_0)$, with $\eta = \kappa/p_{\gamma_0}(\chi;z_0)^{\gamma_0+1}$, coincides with the function $\Gamma(\kappa,\chi,\Lambda,\widetilde{\Lambda};z_0)$. Since the right-hand side of (61) is holomorphic in (χ,τ) near 0, it is not difficult to see that $(z,\zeta)\mapsto \Xi_3(\eta,\chi,\Lambda_0,\bar{\Lambda}_0;z_0)$ must be identically 0, and that

$$\bar{H}(\chi,\tau) \equiv \Gamma\big(\tau - p_0(\chi;z_0),\chi, j_0^2H, \overline{j_0^2H}; z_0\big). \tag{66}$$

It remains to remark that the right-hand side of (66) is holomorphic in $(\chi,\tau;z_0)$ near $(0,0,\widetilde{z_0})$ with any $\widetilde{z_0}\ne 0$ sufficiently small and is independent of z_0. The proof of Theorem 4.1 is complete. □

Proof of Corollary 1.6. The proof can be obtained by repeating the arguments from [BER1]. □

5 Finite Jet Determination for Solutions of Singular ODEs

The proof of Theorem 1.1 in the infinite type case is based on Theorem 3.1, on the first author's results in [E2] (see Theorem 6.1 below) and on the following property of solutions of singular ordinary differential equations which we prove in this section, and which may be of independent interest.

Theorem 5.1. *Consider a singular differential equation for an* $\mathbb{R}^n$*-valued function* $y(x,\theta)$*, where* $x \in \mathbb{R}$*,* $\theta \in \mathbb{R}^m$*, of the form*

$$x^{\gamma+1}\partial_x y(x,\theta) = \frac{p(x,y(x,\theta),\theta)}{q(x,y(x,\theta),\theta)}\,, \tag{67}$$

where $\gamma \geq 0$ *is an integer,* $p(x,y,\theta)$ *and* $q(x,y,\theta)$ *are real-analytic functions (valued in* $\mathbb{R}^n$ *and* $\mathbb{R}$*, respectively) defined in a neighborhood of* 0 *in* $\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^m$ *with* $q(0,0,\theta)\not\equiv 0$*. Let* $\hat{y}(x,\theta)$ *be a real-analytic solution of* (67) *near* 0 *with* $\hat{y}(0,\theta)\equiv 0$*. Then there exists an integer* $k \geq 0$ *such that, if* $y(x,\theta)$ *is another solution near the origin with* $\partial_x^l y(0,\theta) \equiv \partial_x^l \hat{y}(0,\theta)$ *for* $0 \leq l \leq k$*, then* $y(x,\theta)\equiv\hat{y}(x,\theta)$*.*

Proof. We write $f(x,y,\theta)$ for the right-hand side of (67). For $\gamma = 0$, the proof is rather simple. Expand both sides of the equation (67) in powers of x and identify the coefficients of x^k. It is not difficult to see that one may solve the resulting equation for the coefficients $a_k = a_k(\theta)$ of $y(x,\theta)$ in terms of $a_l = a_l(\theta)$, with $l \leq k-1$, unless k is an eigenvalue of $f_y(0,0,\theta)$. The last possibility does not happen if k is sufficiently large and θ is outside a countable union Γ of proper real-analytic subvarieties. For every $\theta \notin \Gamma$ and $y, \hat{y}$ satisfying the hypotheses in the theorem, we conclude that $y(x,\theta)\equiv\hat{y}(x,\theta)$. The required statement follows by continuity and the fact that the complement of Γ is dense in a neighborhood of 0 in $\mathbb{R}^m$. The details are left to the reader.

For the rest of the proof we assume $\gamma \geq 1$. We shall write $y^{(s)} \in \mathbb{R}^n$ for the sth derivative of $y(x,\theta)$ in x evaluated at $(0,\theta)$ and $y^{(s+1,\dots,s+l)} \in \mathbb{R}^{ln}$ for the column of the l derivatives $y^{(s+1)},\dots,y^{(s+l)}$.

We shall differentiate (67) in x at $(0,\theta)$ and apply the chain rule. Since $y(0,\theta)\equiv 0$, the derivatives of the right-hand side $f(x,y,\theta)$ will be always evaluated at $(0,0,\theta)$. Hence they will be ratios of real-analytic functions where the denominator is some power of $q(0,0,\theta)$. We consider the ring of all ratios of this kind and all polynomials (and rational functions) below will be understood over this ring (i.e. a polynomial below will be a polynomial with coefficients in this ring).

By taking the sth derivative ($s \geq \gamma+1$) of the identity (67) in x, evaluating at $(0,\theta)$, and using the chain rule, we obtain

$$c_s y^{(s-\gamma)} = P_l(y^{(1,\dots,l-1)},\theta)y^{(s-l+1,\dots,s)} + R_{l,s}(y^{(1,\dots,s-l)},\theta) \tag{68}$$

for any $1 \leq l \leq s/2$, where $c_s = \binom{s}{\gamma+1}$, P_l is a $\mathbb{R}^{n\times nl}$-valued matrix polynomial in $y^{(1,\dots,l-1)}$ depending only on l and $R_{l,s}$ is a $\mathbb{R}^n$-valued polynomial in $y^{(1,\dots,s-l)}$ depending on both l and s. In fact, P_l can be written as $P_l = (P_l^1, \dots, P_l^l)$ with each P_l^i being an $n \times n$ matrix given by the formula

$$P_l^i(y^{(1,\dots,l-1)},\theta) = (d/dx)^{l-i}\big(f_y(x,y(x),\theta)\big)\big|_{x=0}\,, \quad i=1,\dots,l\,.$$

We further fix integers t and r satisfying $t+1 \le (r+1/\gamma)/2$, collect the identities (68) in blocks for $l=(t+1)\gamma$ and $s = r\gamma+1,\dots,(r+1)\gamma$ and write them in the form

$$C_r y^{((r-1)\gamma+1,\dots,r\gamma)} = \sum_{0\le j\le t} Q^j(y^{(1,\dots,(j+1)\gamma-1)},\theta)y^{((r-j)\gamma+1,\dots,(r-j+1)\gamma)} + S_{r,t}(y^{(1,\dots,(r-t)\gamma)},\theta)\,, \quad (69)$$

where C_r is the diagonal $\gamma n \times \gamma n$ matrix with eigenvalues $c_{r\gamma+1},\dots,c_{(r+1)\gamma}$, each of multiplicity n, Q^j are $\mathbb{R}^{\gamma n\times\gamma n}$-valued matrix polynomials in $y^{(1,\dots,(j+1)\gamma-1)}$ and $S_{t,r}$ is a $\mathbb{R}^{\gamma n}$-valued polynomial in $y^{(1,\dots,(r-t)\gamma)}$ depending on both t and r. Here we put all the terms containing $y^{(i)}$ with $i \le (r-t)\gamma$ into $S_{t,r}$.

We first try to solve the system (69) with respect to the γ highest derivatives $y^{(r\gamma+1)},\dots,y^{((r+1)\gamma)}$. We can do this provided the coefficient matrix $Q^0(y^{(1,\dots,\gamma-1)},\theta)$ is invertible. In general, a solution can be obtained only modulo the kernel of $Q^0(y^{(1,\dots,\gamma-1)},\theta)$. Here the dimension of the kernel may change as $(y^{(1,\dots,\gamma-1)},\theta)$ changes. To avoid this problem we consider only solutions $y(x,\theta)$ of (67) with $y^{(1,\dots,\gamma-1)} = \hat{y}^{(1,\dots,\gamma-1)}$ (as we may by *a priori* assuming $k \ge \gamma-1$). For these solutions, we obtain from (69) an identity

$$y^{(r\gamma+1,\dots,(r+1)\gamma)} = T_{r,t}(y^{(1,\dots,r\gamma)},\theta) \mod \ker Q^0 \quad (70)$$

where $T_{r,t}$ is a $\mathbb{R}^{\gamma n}$-valued polynomial in $y^{(1,\dots,r\gamma)}$ and $Q^0{:=}Q^0(\hat{y}^{(1,\dots,\gamma-1)},\theta)$. Here two cases are possible. If the kernel of Q^0 is trivial for some θ, it is trivial for θ outside a proper real-analytic subvariety. Then for such values of θ, (70) can be iterated to determine all derivatives $y^{(s)}$, for $s \ge \gamma+1$, in terms of $y^{(1,\dots,\gamma)}$. The proof is complete by continuity.

If the kernel of Q^0 is nontrivial for all θ, it has a constant dimension for θ outside a proper real-analytic subvariety. Then we consider the system (69) with r replaced by $r+1$. Here the γ-tuple of new unknown derivatives $y^{((r+1)\gamma+1,\dots,(r+2)\gamma)}$ with the coefficient matrix Q^0 is involved. However, we can still extract some information when the image $\operatorname{im} Q^0$ is a proper subspace of $\mathbb{R}^{\gamma n}$ (which happens precisely when $\ker Q^0 \ne \{0\}$), namely

$$(C_{r+1} - Q^1(y^{(1,\dots,2\gamma-1)},\theta))y^{(r\gamma+1,\dots,(r+1)\gamma)} =$$

$$\sum_{2\leq j\leq t} Q^j(y^{(1,\dots,(j+1)\gamma-1)},\theta)y^{((r-j+1)\gamma+1,\dots,(r-j+2)\gamma)}$$
$$+ S_{r+1,t}(y^{(1,\dots,(r-t+1)\gamma)},\theta) + Q^0 y^{((r+1)\gamma+1,\dots,(r+2)\gamma)}. \quad (71)$$

We now use the explicit form of C_{r+1} to conclude that, for every r sufficiently large, the matrix

$$C_{r+1} - Q^1(y^{(1,\dots,2\gamma-1)},\theta) \quad (72)$$

on the left-hand side is invertible for $y^{(1,\dots,2\gamma-1)} = \hat{y}^{(1,\dots,2\gamma-1)}$ and for θ outside a proper subvariety. By taking the union of these subvarieties for different r we see that, for each θ outside a countable union of proper subvarieties, the matrices (72) are invertible for all r. We write

$$A^1_{r+1}(y^{(1,\dots,2\gamma-1)},\theta) := \big(C_{r+1} - Q^1(y^{(1,\dots,2\gamma-1)},\theta)\big)^{-1}.$$

Thus A^1_{r+1} is a rational function in $y^{(1,\dots,2\gamma-1)}$. By applying $A^1_{r+1}(y^{(1,\dots,2\gamma-1)},\theta)$ to both sides of (71), we obtain a rational expression for $y^{(r\gamma+1,\dots,(r+1)\gamma)}$ in terms of lower order derivatives modulo the linear subspace

$$V^1_r(y^{(1,\dots,2\gamma-1},\theta) := A^1_{r+1}(y^{(1,\dots,2\gamma-1)},\theta)\ \mathsf{im}\, Q^0 \subset \mathbb{R}^{\gamma n}. \quad (73)$$

Previously we fixed the first derivatives $y^{(1,\dots,\gamma-1)}$. In this step, we further assume that $y^{(1,\dots,2\gamma-1)} = \hat{y}^{(1,\dots,2\gamma-1)}$. We can then drop the dependence of Q^1, A^1_{r+1} and V^1_r on the derivatives as we did for Q^0. Taking also (70) into account, we conclude that $y^{(r\gamma+1,\dots,(r+1)\gamma)}$ is determined modulo

$$\mathsf{ker}\, Q^0 \cap V^1_r\,. \quad (74)$$

If this space is zero-dimensional (for some θ), the two equations (70) and (71) determine $y^{(r\gamma+1,\dots,(r+1)\gamma)}$ completely and yield a polynomial expression for these derivatives in terms of lower derivatives (for this value of θ).

We now observe that we may write $A^1_{r+1} = c^{-1}_{(r+2)\gamma}B^1_{r+1}$, where the matrix B^1_{r+1} tends to the identity (for θ fixed) as $r\to\infty$. Moreover the linear operator $B^1(\epsilon) := B^1_{r+1}$, where $\epsilon := 1/r$ (and the integers r in B^1_{r+1} are replaced by a continuous variable r in the obvious way), is a ratio of analytic functions for (ϵ,θ) in a neighborhood of 0. By Cramer's rule, there exist real-analytic functions $v_1(\theta),\dots,v_\mu(\theta)$ and $u_{\mu+1}(\theta),\dots,u_{n\gamma}(\theta)$ that represent bases for $\mathsf{ker}\, Q^0$ and for $\mathsf{im}\, Q^0$ respectively for every θ outside a proper subvariety. If we write $\Delta(\epsilon,\theta)$ for the determinant of the matrix of $n\gamma$-vectors

$$v_1,\dots,v_\mu, B^1u_{\mu+1},\dots,B^1u_{n\gamma}\,, \quad (75)$$

then the intersection (74) is positive dimensional if and only if $\Delta(\epsilon,\theta)$ vanishes at $(1/r,\theta)$. Since $\Delta(\epsilon,\theta)$ is real-analytic near 0, we conclude that either $\Delta(\epsilon,\theta)$ is identically 0 or there exists r_0 such that $\Delta^1(1/r,\theta)\neq 0$

for $r \geq r_0$ and for θ outside a proper subvariety Γ^1_r. In the second case the intersection (74) is zero-dimensional for $r \geq r_0$ and $\theta \notin \Gamma^1_r$ and we obtain $y^{(r\gamma+1,\dots,(r+1)\gamma)}$ as a polynomial expression in lower derivatives. By a simple inductive argument and the analyticity of y and $\hat{y}$, it follows that $y(x,\theta) = \hat{y}(x,\theta)$ for all x and all θ outside the union of Γ^1_r, $r \geq r_0$. The desired statement follows by continuity, since the union of Γ^1_r's is nowhere dense.

Thus, we may assume that $\Delta(\epsilon,\theta) \equiv 0$. In this case, we consider (69) with r replaced by $r+2$. By solving (71), with $r+1$ in place of r, for $y^{((r+1)\gamma+1,\dots,(r+2)\gamma)}$ (modulo $\operatorname{im} Q^0(y^{1,\dots,\gamma-1},\theta)$), we conclude that

$$\begin{aligned} &y^{((r+1)\gamma+1,\dots,(r+2)\gamma)} \\ &\quad = A^1_{r+2}\Big(\sum_{2\leq j\leq t} Q^j(y^{(1,\dots,(j+1)\gamma-1)},\theta)y^{((r-j+2)\gamma+1,\dots,(r-j+3)\gamma)} \\ &\qquad + S_{r+2,t}(y^{(1,\dots,(r-t+1)\gamma)},\theta)\Big) + A^1_{r+2}Q^0y^{((r+2)\gamma+1,\dots,(r+3)\gamma)}, \end{aligned} \tag{76}$$

where A^1_{r+2} and $S_{r+2,t}$ are defined as above. By applying Q^0 to both sides and substituting the right-hand side for $Q^0y^{((r+1)\gamma+1,\dots,(r+2)\gamma)}$ in (71), we deduce that

$$D^2_{r+1}(y^{(1,\dots,3\gamma-1)},\theta)y^{(r\gamma+1,\dots,(r+1)\gamma)} = R^2_{r+1}(y^{(1,\dots,r\gamma)},\theta) \mod Q^0(V^1_{r+1}), \tag{77}$$

where V^1_{r+1} is as defined above and D^2_{r+1} is the invertible (for r large enough and θ outside a proper subvariety Γ^2_r, provided that we have $y^{(1,\dots,3\gamma-1)} = \hat{y}^{(1,\dots,3\gamma-1)}$) matrix

$$D^2_{r+1}(y^{(1,\dots,3\gamma-1)}) := C_{r+1} - Q^1 - Q^0A^1_{r+2}Q^2(y^{(1,\dots,3\gamma-1)},\theta). \tag{78}$$

As before we assume in this step that $y^{(1,\dots,3\gamma-1)} = \hat{y}^{(1,\dots,3\gamma-1)}$ and drop the dependence on these derivatives.

Now observe that the assumption that $\Delta(\epsilon,\theta) \equiv 0$ (which is equivalent to the intersections (74) being nontrivial for all r sufficiently large) implies that the space $Q^0(V^1_{r+1})$ is of strictly lower dimension than V^1_{r+1}. This is a crucial observation. Let us write A^2_{r+1} for the inverse of D^2_{r+1}. It follows that we can solve (77) for $y^{(r\gamma+1,\dots,(r+1)\gamma)}$ modulo the subspace

$$V^2_r := A^2_{r+1}Q^0(V^1_{r+1}). \tag{79}$$

If the intersection

$$\operatorname{ker} Q^0 \cap V^2_r \tag{80}$$

is zero-dimensional, we can find a polynomial expression for $y^{(r\gamma+1,\dots,(r+1)\gamma)}$ in terms of lower derivatives by using equation (70). We claim again that this intersection will be either zero-dimensional for all sufficiently large r

and θ outside a proper subvariety Γ_r^3 or positive-dimensional for all sufficiently large r and all θ. The argument is as before. We can detect positive dimensionality of the intersection (80) by the vanishing of suitable determinants formed by the vectors in (75) with B^1 replaced by $B^2 = B^2(\epsilon)$, where $B^2(\epsilon)$ (which also of course depends on θ) is defined as follows. Let us factor the scalar $c^{-1}_{(r+2)\gamma}$ in A^2_{r+1}, writing $A^2_{r+1} = c^{-1}_{(r+2)\gamma} B^2_{r+1}$. Then we define $B^2(\epsilon) := B^2_{r+1} Q^0 B^1_{r+2}$, where as before $\epsilon := 1/r$. It is not difficult to see that B^2 is analytic in (ϵ, θ) near 0 with $B^2(0,\theta)$ equal to the identity. Since any determinant formed by the vectors in (75) with B^1 replaced by B^2 will be analytic, the claim now follows as above.

As mentioned above, if the intersection (80) is trivial for all sufficiently large r, we are done. If not, we must go iterate the procedure above, and start with the equation (69) with r replaced by $r+3$. In this way we will obtain a subspace $V_r^3(y^{(1,\dots,4\gamma-1)},\theta)$ (in a way analogous to that yielding $V_r^2(y^{(1,\dots,3\gamma-1)},\theta)$). By the same argument as above, the fact that we are forced to go to the next iteration (i.e. the intersection (80) is nontrivial for all large r) implies that V_r^3 has strictly lower dimension than V_r^2. The crucial observation is that, if we are forced to make another iteration, the dimension of the subspaces $V_r^j := V_r^j(\hat{y}^{(1,\dots,(j+1)\gamma-1)},\theta)$ drops. Hence, the process will terminate after at most $n\gamma$ steps. The details of the iterations are left to the reader.

Summarizing, we obtain a linear system for $y^{(r\gamma+1,\dots,(r+1)\gamma)}$ (providing r is large enough) in terms of lower order derivatives and the matrix coefficient of $y^{(r\gamma+1,\dots,(r+1)\gamma)}$ is polynomial in $y^{(1,\dots,(n\gamma+1)\gamma)}$ and is invertible for θ outside a countable union of proper subvarieties. Then the proof is completed by the analyticity of $y(x,\theta)$ and $\hat{y}(x,\theta)$ and by continuity as before. □

6 Finite Jet Determination in the Infinite Type Case; Proof of Theorem 1.1

In this section we complete the proof of Theorem 1.1. We shall only prove the implication (i)⇒(ii). The opposite implication is well known.

If M is of finite type at p, the statement is a special case of Theorem 1.4. Hence, to complete the proof of Theorem 1.1, we may assume that the hypersurface $M \subset \mathbb{C}^2$ is of *infinite type* at p. This is equivalent to the property that the Segre variety E of p is contained in M. As before we denote the same objects associated to another real-analytic hypersurface

$M' \subset \mathbb{C}^2$ by $'$. Given M', H^1 and H^2 as in Theorem 1.1, we set $p' := H^1(p)$. It follows that M' is also of infinite type at p'. The proof of Theorem 1.1 in this case is based on Theorem 3.1 and on the following result.

Theorem 6.1. *Let h^0 be a C^∞-smooth CR-diffeomorphism between real-analytic hypersurfaces M and M' in $\mathbb{C}^2$. Suppose that M is of infinite type at a point $p \in M$ and set $p' := h^0(p) \in M'$. Choose local coordinates $y = (x, s) \in \mathbb{R}^2 \times \mathbb{R}$ on M and $y' = (x', s') \in \mathbb{R}^2 \times \mathbb{R}$ on M' vanishing at p and p', respectively, such that the Segre varieties $E \subset M$ and $E' \subset M'$ at p and p' are locally given by $s = 0$ and $s' = 0$, respectively. Then there exists an integer $m \geq 1$ such that, if h is a C^∞-smooth CR-diffeomorphism between open neighborhoods of p and p' in M and M' respectively with $h(p)$ sufficiently close to p' and we set $g(y) := s'(h(y))$, then there is a (unique) C^∞-smooth function $v_h(y)$ on M near p satisfying*

$$s^m \partial_s g(y) \equiv v_h(y) g(y)^m. \tag{81}$$

For any such h we write

$$u_h(y) := \big((\partial_{x_i} h(y))_{1 \leq i \leq 2}, s^m \partial_s f(y), v_h(y)\big) \in \mathbb{R}^9,$$

where $f(y) := x'(h(y))$, and set

$$u_h^\alpha(y) := (s^m \partial_s)^{\alpha_0} \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} u_h(y) \tag{82}$$

for any multi-index $\alpha = (\alpha_0, \alpha_1, \alpha_2) \in \mathbb{Z}_+^3$. Then there exist real-analytic functions $q(y)$ on M near p and $q'(y')$ on M' near p' with $q(x,0)$ and $q'(x',0)$ both not identically zero, an open neighborhood $\Omega \subset J^2(M, \mathbb{R}^9)_p \times M' \times M$ of $((u_{h^0}^\beta(p))_{0 \leq |\beta| \leq 2}, p', p)$, and, for every multi-index $\alpha \in \mathbb{Z}_+^3$ with $|\alpha| = 3$, real-analytic functions $r^\alpha(\Lambda, y', y)$ on Ω such that, for any h as above with

$$\big((u_h^\beta(p))_{0 \leq |\beta| \leq 2}, h(p), p\big) \in \Omega\,, \tag{83}$$

the equation

$$u_h^\alpha(y) \equiv \frac{r^\alpha((u_h^\beta(y))_{0 \leq |\beta| \leq 2}, h(y), y)}{q(y) q'(h(y))} \tag{84}$$

holds at every $y \in M$ near p for which the denominator does not vanish.

At points $p \in M$ and $p' \in M'$ where $q(p) \neq 0$ and $q'(p') \neq 0$ (the latter of which is, in fact, a consequence of the former), Theorem 6.1 is a reformulation of Theorem 2.1 in [E2]. Theorem 6.1, as stated above, follows by repeating the proof of Theorem 2.1 in [E2] at a general point p, accepting the presence of a denominator which may vanish at p. The result is the conclusion of Theorem 6.1. We shall omit the details and refer the reader to the proof in [E2] for inspection.

Proof of Theorem 1.1. Let H^1, H^2 be as in Theorem 1.1 with $j_p^k H^1 = j_p^k H^2$, for some $k \geq 2$ (which will be specified later). Observe that, for any local biholomorphism $H\colon (\mathbb{C}^2, p) \to \mathbb{C}^2$ sending M into itself, the restriction $h := H|_M$ is a local CR-diffeomorphism between open neighborhoods of p and $p' := H(p)$ in M. We shall write $h^j := H^j|_M$, $j = 1, 2$. If we take $h^0 := h^1$, $h := h^2$ and $M' = M$ in Theorem 6.1, then h satisfies (83) and hence the equation (84). For an h as in Theorem 6.1, we set

$$U_h(y) := \big((u_h^\beta(y))_{0\le|\beta|\le 2}, h(y)\big)\,.$$

Equations (81) and (84) then imply that

$$s^m \partial_s U_h(y) \equiv \frac{R(U_h, y)}{q(y)q'(h(y))} \tag{85}$$

for some real-analytic function $R(U, y)$ defined in a neighborhood of $(U_{h^0}, 0)$.

The assumption $j_p^k H^1 \equiv j_p^k H^2$ and Theorem 3.1 imply that

$$j_Z^{k-1} H^1 \equiv j_Z^{k-1} H^2\,, \quad \forall Z \in E\,, \tag{86}$$

near p. Let us write, for $j = 1, 2$,

$$U_{h^j}(x, s) = \sum_{k=0}^{\infty} U_k^j(x) s^k. \tag{87}$$

We conclude, by the construction of U_h and (86), that $U_l^1(x) \equiv U_l^2(x)$ for $l \le k-4$. To complete the proof of Theorem 1.1, consider the functions $\tilde U_{h^j}(x, s) := U_{h^j}(x, s) - U_{h^j}(x, 0)$, for $j = 1, 2$, which satisfy $\tilde U_{h^j}(x, 0) = 0$. For $k \ge 4$, we have $U_{h^1}(x, 0) = U_{h^2}(x, 0)$ and hence both $\tilde U_{h^1}$ and $\tilde U_{h^2}$ satisfy the same system of differential equations

$$s^m \partial_s \tilde U_h(y) = \frac{\tilde R(y, \tilde U_h)}{q(y)q'(h(y))}\,, \tag{88}$$

where $\tilde R(y, \tilde U) := R(y, \tilde U + U^1(x, 0))$, as does any other $\tilde U_h$ arising from a CR diffeomorphism h with $U_h(x, 0) = U^1(x, 0)$. Recall that $h(y)$ is one of the components of U_h. The existence of the integer k such that $H^1 \equiv H^2$ (which is equivalent to $h^1 \equiv h^2$) if $j_p^k H^1 = j_p^k H^2$ now follows from Theorem 5.1, although the choice of the integer k appears to depend on the mapping H^1. However, we shall show that one can find a k that works for every H^1. Let k be the integer obtained by the above procedure applied to H^1, H^2 where $H^1(Z)$ is the identity mapping id. We conclude that if $H\colon (\mathbb{C}^2, p) \to (\mathbb{C}^2, p)$ sends M into itself and $j_p^k H = \mathsf{id}$, then $H \equiv \mathsf{id}$. We claim that the same number k satisfies the conclusion of Theorem 1.1 for any H^1, H^2. Indeed, for any H^1, H^2 as in Theorem 1.1, the mapping $H := (H^1)^{-1} \circ H^2$ sends (M, p) into itself and satisfies $j_p^k H = \mathsf{id}$. Hence, by the construction of k, we must have $(H^1)^{-1} \circ H^2 \equiv \mathsf{id}$ which proves the claim. This completes the proof of Theorem 1.1. □

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