FINITE LARMOR RADIUS APPROXIMATION FOR COLLISIONAL MAGNETIC CONFINEMENT. PART I: THE LINEAR BOLTZMANN EQUATION

By

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Abstract. The subject matter of this paper concerns the derivation of the finite Larmor radius approximation, when collisions are taken into account. Several studies are performed, corresponding to different collision kernels : the relaxation and the Fokker-Planck operators. Gyroaveraging the relaxation operator leads to a position-velocity integral operator, whereas gyroaveraging the linear Fokker-Planck operator leads to diffusion in velocity but also with respect to the perpendicular position coordinates.

1. Introduction.

Many studies in plasma physics concern the energy production through thermonuclear fusion. In particular this reaction can be achieved by magnetic confinement *i.e.,* a tokamak plasma is controlled by applying a strong magnetic field. Large magnetic fields induce high cyclotronic frequencies corresponding to the fast particle dynamics around the magnetic lines. We concentrate on the linear problem by neglecting the self-consistent electro-magnetic field. The external electro-magnetic field is supposed to be a given smooth field

$$E = -\nabla_x \phi, \quad B^\varepsilon = \frac{B(x)}{\varepsilon}\, b(x), \quad |b| = 1$$

when $\varepsilon > 0$ is a small parameter, destined to converge to 0, in order to describe strong magnetic fields. The scalar function ϕ stands for the electric potential, $B(x) > 0$ is the rescaled magnitude of the magnetic field and $b(x)$ denotes its direction. As usual, we appeal to the kinetic description for studying the evolution of the plasma. The notation $f^\varepsilon = f^\varepsilon(t, x, v) \geq 0$ stands for the presence density of a population of charged particles

with mass m and charge q. This density satisfies

$$\partial_t f^\varepsilon + v \cdot \nabla_x f^\varepsilon + \frac{q}{m}(E + v \wedge B^\varepsilon) \cdot \nabla_v f^\varepsilon = Q(f^\varepsilon), \quad (t, x, v) \in \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3 \quad (1.1)$$

$$f^\varepsilon(0, x, v) = f^{\text{in}}(x, v), \quad (x, v) \in \mathbb{R}^3 \times \mathbb{R}^3 \quad (1.2)$$

where Q denotes a collision kernel. The interpretation of the density f^ε is straightforward : the number of charged particles contained at time t inside the infinitesimal volume $dxdv$ around the point (x, v) of the position-velocity phase space is given by $f^\varepsilon(t, x, v)dxdv$. The equation (1.1) accounts for the fluctuation of the density f^ε due to the transport but also to the collisions. We analyze here the linear relaxation operator. The bilinear Fokker-Planck-Landau operator will be studied in [6] following similar lines.

When neglecting the collisions, the limit model as $\varepsilon \searrow 0$ comes by averaging with respect to the fast cyclotronic motion [1–4, 10, 15, 19]. The problem reduces to homogenization analysis and can be solved using the notion of two-scale convergence [11–13].

We point out that a linearized and gyroaveraged collision operator has been written in [20], but the implementation of this operator seems very hard. We refer to [8, 9] for a general guiding-center bilinear Fokker-Planck collision operator. Another difficulty lies in the relaxation of the distribution function towards a local Maxwellian equilibrium. Most of the available model operators, in particular those which are linearized near a Maxwellian, are missing this property. Very recently a set of model collision operators has been obtained in [14], based on entropy variational principles [7].

We study here the finite Larmor radius scaling; *i.e.,* the typical perpendicular spatial length is of the same order as the Larmor radius, and the parallel spatial length is much larger. We assume that the magnetic field is homogeneous and stationary

$$B^\varepsilon = \left(0, 0, \frac{B}{\varepsilon}\right)$$

for some constant $B > 0$, and therefore (1.1) becomes

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon}(v_1 \partial_{x_1} f^\varepsilon + v_2 \partial_{x_2} f^\varepsilon) + v_3 \partial_{x_3} f^\varepsilon + \frac{q}{m} E \cdot \nabla_v f^\varepsilon + \frac{\omega_c}{\varepsilon}(v_2 \partial_{v_1} f^\varepsilon - v_1 \partial_{v_2} f^\varepsilon) = Q(f^\varepsilon)$$
$$(1.3)$$

where $\omega_c = qB/m$ stands for the rescaled cyclotronic frequency. The density f^ε is decomposed into a dominant density f and fluctuations of orders $\varepsilon, \varepsilon^2, ...$

$$f^\varepsilon = f + \varepsilon f^1 + \varepsilon^2 f^2 + ... \quad (1.4)$$

Combining (1.3), (1.4) yields, with the notations $\overline{x} = (x_1, x_2)$, $\overline{v} = (v_1, v_2)$, $^\perp \overline{v} = (v_2, -v_1)$,

$$\mathcal{T}f := \overline{v} \cdot \nabla_{\overline{x}} f + \omega_c \, ^\perp \overline{v} \cdot \nabla_{\overline{v}} f = 0 \quad (1.5)$$

$$\partial_t f + v_3 \partial_{x_3} f + \frac{q}{m} E \cdot \nabla_v f + \mathcal{T}f^1 = Q(f) \quad (1.6)$$

$$\vdots$$

The equation (1.5) appears as a divergence constraint

$$\text{div}_{x,v}\{f(\overline{v}, 0, \omega_c \, ^\perp \overline{v}, 0)\} = 0.$$

Equivalently, (1.5) says that at any time t, the density $f(t, \cdot, \cdot)$ remains constant along the flow associated to $\overline{v} \cdot \nabla_{\overline{x}} + \omega_c \, {}^{\perp}\overline{v} \cdot \nabla_{\overline{v}}$

$$\frac{\mathrm{d}\overline{X}}{\mathrm{d}s} = \overline{V}(s), \quad \frac{\mathrm{d}X_3}{\mathrm{d}s} = 0, \quad \frac{\mathrm{d}\overline{V}}{\mathrm{d}s} = \omega_c \, {}^{\perp}\overline{V}(s), \quad \frac{\mathrm{d}V_3}{\mathrm{d}s} = 0 \tag{1.7}$$

and therefore, at any time t, the density $f(t, \cdot, \cdot)$ depends only on the invariants of (1.7)

$$f(t, x, v) = g\left(t, x_1 + \frac{v_2}{\omega_c}, x_2 - \frac{v_1}{\omega_c}, x_3, r = |\overline{v}|, v_3\right).$$

The time evolution for f comes by eliminating f^1 in (1.6). For doing that, we project onto the kernel of $\mathcal{T}$, which is orthogonal to the range of $\mathcal{T}$. In order to get an explicit model for f we need a simpler representation for the orthogonal projection on $\ker \mathcal{T}$. Actually this projection appears as the average along the characteristic flow (1.7). Denoting this projection by $\langle \cdot \rangle$, we obtain

$$\left\langle \partial_t f + v_3 \partial_{x_3} f + \frac{q}{m} E \cdot \nabla_v f \right\rangle = \langle Q(f) \rangle, \quad (t, x, v) \in \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3. \tag{1.8}$$

On one hand, averaging $\partial_t + v_3 \partial_{x_3} + \frac{q}{m} E \cdot \nabla_v$ leads to another transport operator

$$\left\langle \partial_t f + v_3 \partial_{x_3} f + \frac{q}{m} E \cdot \nabla_v f \right\rangle = \partial_t f + \frac{\langle {}^{\perp}\overline{E} \rangle}{B} \cdot \nabla_{\overline{x}} f + v_3 \partial_{x_3} f + \frac{q}{m} \langle E_3 \rangle \partial_{v_3} f.$$

The key point here is to choose as new coordinates the invariants of (1.7) and to observe that the partial derivatives with respect to these invariants commute with the average operator. More generally, for any smooth vector field $\xi = (\xi_x, \xi_v)$, we obtain the following commutation formula between the divergence and average operators, cf. Proposition 3.3:

$$\begin{aligned}
\langle \mathrm{div}_{x,v} \xi \rangle &= \mathrm{div}_{\overline{x}} \left\{ \left\langle \xi_{\overline{x}} + \frac{{}^{\perp}\xi_{\overline{v}}}{\omega_c} \right\rangle + \left\langle \xi_{\overline{v}} \cdot \frac{{}^{\perp}\overline{v}}{|\overline{v}|} \right\rangle \frac{\overline{v}}{\omega_c |\overline{v}|} - \left\langle \xi_{\overline{v}} \cdot \frac{\overline{v}}{|\overline{v}|} \right\rangle \frac{{}^{\perp}\overline{v}}{\omega_c |\overline{v}|} \right\} + \partial_{x_3} \langle \xi_{x_3} \rangle \\
&\quad + \mathrm{div}_{\overline{v}} \left\{ \left\langle \xi_{\overline{v}} \cdot \frac{{}^{\perp}\overline{v}}{|\overline{v}|} \right\rangle \frac{{}^{\perp}\overline{v}}{|\overline{v}|} + \left\langle \xi_{\overline{v}} \cdot \frac{\overline{v}}{|\overline{v}|} \right\rangle \frac{\overline{v}}{|\overline{v}|} \right\} + \partial_{v_3} \langle \xi_{v_3} \rangle.
\end{aligned}$$

On the other hand, we need to compute the average of the collision kernel Q which is a more complicated task. We focus on the relaxation Boltzmann operator [16]

$$Q_B(f(t, x, \cdot))(v) = \frac{1}{\tau} \int_{\mathbb{R}^3} s(v, v') \{ M(v) f(t, x, v') - M(v') f(t, x, v) \} \, \mathrm{d}v'$$

where $\tau > 0$ is the relaxation time, $s(v, v')$ is the scattering cross section and M is the Maxwellian equilibrium with temperature θ

$$M(v) = \frac{1}{(2\pi\theta/m)^{3/2}} e^{-\frac{m|v|^2}{2\theta}}, \quad v \in \mathbb{R}^3.$$

We need to average functions like $(x, v) \to \int_{\mathbb{R}^3} C(v, v') f(x, v') \, \mathrm{d}v'$, where $C(v, v')$ is a given function. Since the invariants of the flow $(\overline{X}, \overline{V})$ combine $\overline{x}$ and $\overline{v}$, we get a position-velocity integral operator cf. Proposition 4.2

$$\left\langle \int_{\mathbb{R}^3} C(v, v') f(x, v') \, \mathrm{d}v' \right\rangle (x, v) = \omega_c^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \mathcal{C}(|\overline{v}|, v_3, |\overline{v}'|, v_3', z) f(\overline{x}', x_3, v') \, \mathrm{d}v' \mathrm{d}x_1' \mathrm{d}x_2'$$

with $z = \omega_c \overline{x} + {}^\perp \overline{v} - (\omega_c \overline{x'} + {}^\perp \overline{v'})$. We prove that averaging Q_B will lead to a position-velocity integral operator of the same form

$$\langle Q_B \rangle f(x,v) := \langle Q_B(f) \rangle (x,v)$$
$$= \frac{\omega_c^2}{\tau} \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \mathcal{S}(|\overline{v}|, v_3, |\overline{v'}|, v_3', z)\{M(v)f(\overline{x'}, x_3, v') - M(v')f(x,v)\} \, \mathrm{d}v' \mathrm{d}x_1' \mathrm{d}x_2'$$

(see Theorem 1.1 for the definition of $\mathcal{S}$). Observe that $\langle Q_B \rangle$ is global in $(\overline{x}, v)$ but remains local in x_3. In particular it satisfies only a global mass balance, which comes easily by the Fubini theorem and the symmetry of $\mathcal{S}$ cf. Remark 4.6

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \langle Q_B \rangle f(x,v) \, \mathrm{d}v \mathrm{d}x = 0.$$

In the case of the relaxation operator Q_B, we obtain the limit model

THEOREM 1.1. Assume that the scattering cross section satisfies (4.2), (4.7) and that $E(x) = -\nabla_x \phi(x), \phi \in W^{2,\infty}(\mathbb{R}^3)$. Let us consider $f^{\mathrm{in}} \geq 0, f^{\mathrm{in}} \in L^1(\mathbb{R}^3 \times \mathbb{R}^3) \cap L^2(M^{-1}\mathrm{d}x\mathrm{d}v)$ and denote by f^ε the weak solution of (1.3), (1.2) with $Q = Q_B$ for any $\varepsilon > 0$. We assume that $(f^\varepsilon)_{\varepsilon > 0}$ is bounded in $L^\infty(\mathbb{R}_+, L^2(M^{-1}\mathrm{d}x\mathrm{d}v))$. Then the family $(f^\varepsilon)_{\varepsilon > 0}$ converges weakly $\star$ in $L^\infty(\mathbb{R}_+, L^2(M^{-1}\mathrm{d}x\mathrm{d}v))$ to the weak solution of

$$\partial_t f + \frac{\langle {}^\perp \overline{E} \rangle}{B} \cdot \nabla_{\overline{x}} f + v_3 \partial_{x_3} f + \frac{q}{m} \langle E_3 \rangle \partial_{v_3} f = \langle Q_B \rangle f, \quad (t,x,v) \in \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3 \quad (1.9)$$

$$f(0,x,v) = \langle f^{\mathrm{in}} \rangle (x,v), \quad (x,v) \in \mathbb{R}^3 \times \mathbb{R}^3 \quad (1.10)$$

where the averaged relaxation operator is given by

$$\langle Q_B \rangle f(x,v) = \frac{\omega_c^2}{\tau} \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \mathcal{S}(|\overline{v}|, v_3, |\overline{v'}|, v_3', z)\{M(v)f(\overline{x'}, x_3, v') - M(v')f(x,v)\} \, \mathrm{d}v' \mathrm{d}x_1' \mathrm{d}x_2'$$

with $z = \omega_c \overline{x} + {}^\perp \overline{v} - (\omega_c \overline{x'} + {}^\perp \overline{v'})$ and the averaged scattering cross section writes as

$$\mathcal{S}(r, v_3, r', v_3', z) = \sigma(\sqrt{|z|^2 + (v_3 - v_3')^2}) \, \chi(r, r', z)$$

with

$$\chi(r, r', z) = \frac{\mathbf{1}_{\{|r-r'|<|z|<r+r'\}}}{\pi^2 \sqrt{|z|^2 - (r-r')^2} \sqrt{(r+r')^2 - |z|^2}}, \quad r, r' \in \mathbb{R}_+, v_3, v_3' \in \mathbb{R}, \ z \in \mathbb{R}^2.$$

The averaging technique allows us to treat many different collision operators, such as the Fokker-Planck kernel, for example (see Appendix A for details)

$$Q_{FP}(f) = \frac{\theta}{m\tau} \mathrm{div}_v \left(\nabla_v f + \frac{m}{\theta} v f \right) = \frac{\theta}{m\tau} \mathrm{div}_v \left\{ M \nabla_v \left(\frac{f}{M} \right) \right\}.$$

THEOREM 1.2. The limit model when $\varepsilon \searrow 0$ of (1.3), (1.2) with $Q = Q_{FP}$ is given by

$$\partial_t f + \frac{\langle {}^\perp \overline{E} \rangle}{B} \cdot \nabla_{\overline{x}} f + v_3 \partial_{x_3} f + \frac{q}{m} \langle E_3 \rangle \partial_{v_3} f = \langle Q_{FP} \rangle f, \quad (t,x,v) \in \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3 \quad (1.11)$$

$$f(0,x,v) = \langle f^{\mathrm{in}} \rangle (x,v), \quad (x,v) \in \mathbb{R}^3 \times \mathbb{R}^3 \quad (1.12)$$

where the averaged Fokker-Planck operator and the diffusion matrix $\mathcal{L}$ write as

$$\langle Q_{FP} \rangle f(x,v) = \frac{\theta}{m\tau} \mathrm{div}_{\omega_c x, v} \left\{ M \mathcal{L} \nabla_{\omega_c x, v} \left(\frac{f}{M} \right) \right\}$$

$$\mathcal{L} = \begin{pmatrix} 2(I_3 - e_3 \otimes e_3) & -E \\ E & I_3 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Notice that the averaged Fokker-Planck operator contains no derivatives with respect to x_3 since the diffusion matrix $\mathcal{L}$ has only zero entries on the third line and column; averaging the Fokker-Planck operator leads to diffusion in velocity but also with respect to the perpendicular position coordinates.

Our paper is organized as follows. In Section 2 we introduce the average operator along a characteristic flow. Section 3 is devoted to the commutation properties between average and first-order differential operators. The average of the linear Boltzmann kernel is computed in Section 4. We establish its main properties and we prove the convergence result stated in Theorem 1.1.

2. Average operator.

We recall briefly the definition and properties of the average operator corresponding to the transport operator $\mathcal{T}$, whose definition in the $L^2(\mathbb{R}^3 \times \mathbb{R}^3)$ setting is

$$\mathcal{T}u = \mathrm{div}_{x,v}(u\,b), \quad b = (\overline{v}, 0, \omega_c\,{}^{\perp}\overline{v}, 0), \quad \omega_c = \frac{qB}{m}$$

for any function u in the domain

$$\mathrm{D}(\mathcal{T}) = \{u(x,v) \in L^2(\mathbb{R}^3 \times \mathbb{R}^3) \ : \ \mathrm{div}_{x,v}(u\,b) \in L^2(\mathbb{R}^3 \times \mathbb{R}^3)\}.$$

We denote by $\|\cdot\|$ the standard norm of $L^2(\mathbb{R}^3 \times \mathbb{R}^3)$. The characteristics $(X,V)(s;x,v)$ associated to $\overline{v} \cdot \nabla_{\overline{x}} + \omega_c\,{}^{\perp}\overline{v} \cdot \nabla_{\overline{v}}$, see (1.7), satisfy

$$\frac{\mathrm{d}}{\mathrm{d}s}\left\{ \overline{X} + \frac{{}^{\perp}\overline{V}}{\omega_c} \right\} = 0, \quad \frac{\mathrm{d}\overline{V}}{\mathrm{d}s} = \omega_c\,{}^{\perp}\overline{V}, \quad \frac{\mathrm{d}X_3}{\mathrm{d}s} = 0, \quad \frac{\mathrm{d}V_3}{\mathrm{d}s} = 0$$

implying that

$$\overline{V}(s) = R(-\omega_c s)\overline{v}, \quad \overline{X}(s) = \overline{x} + \frac{{}^{\perp}\overline{v}}{\omega_c} - \frac{{}^{\perp}\overline{V}(s)}{\omega_c}, \quad X_3(s) = x_3, \quad V_3(s) = v_3$$

where $R(\alpha)$ stands for the rotation of angle α

$$R(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}.$$

All the trajectories are $T_c = 2\pi/\omega_c$ periodic and we introduce the average operator, see [2], for any function $u \in L^2(\mathbb{R}^3 \times \mathbb{R}^3)$

$$\begin{aligned} \langle u \rangle (x,v) &= \frac{1}{T_c} \int_0^{T_c} u(X(s;x,v), V(s;x,v)) \, \mathrm{d}s \\ &= \frac{1}{2\pi} \int_0^{2\pi} u\left(\overline{x} + \frac{{}^{\perp}\overline{v}}{\omega_c} - \frac{{}^{\perp}\{R(\alpha)\overline{v}\}}{\omega_c}, x_3, R(\alpha)\overline{v}, v_3 \right) \, \mathrm{d}\alpha. \end{aligned}$$

It is convenient to introduce the notation $e^{i\varphi}$ for the $\mathbb{R}^2$ vector $(\cos\varphi, \sin\varphi)$. Assume that the vector $\overline{v}$ writes as $\overline{v} = |\overline{v}|e^{i\varphi}$. Then $R(\alpha)\overline{v} = |\overline{v}|e^{i(\alpha+\varphi)}$ and the expression for

$\langle u \rangle$ becomes

$$
\begin{aligned}
\langle u \rangle (x, v) &= \frac{1}{2\pi} \int_0^{2\pi} u \left(\overline{x} + \frac{^\perp \overline{v}}{\omega_c} - \frac{^\perp \{ |\overline{v}| e^{i(\alpha + \varphi)} \}}{\omega_c}, x_3, |\overline{v}| e^{i(\alpha + \varphi)}, v_3 \right) \, d\alpha \\
&= \frac{1}{2\pi} \int_0^{2\pi} u \left(\overline{x} + \frac{^\perp \overline{v}}{\omega_c} - \frac{^\perp \{ |\overline{v}| e^{i\alpha} \}}{\omega_c}, x_3, |\overline{v}| e^{i\alpha}, v_3 \right) \, d\alpha.
\end{aligned} \tag{2.1}
$$

Notice that $\langle u \rangle$ depends only on the invariants $\overline{x} + \frac{^\perp \overline{v}}{\omega_c}, |\overline{v}|, x_3, v_3$ and therefore belongs to $\ker \mathcal{T}$. The following two results are justified in [3], Propositions 2.1, 2.2. The first one states that averaging reduces to orthogonal projection onto the kernel of $\mathcal{T}$. The second one concerns the invertibility of $\mathcal{T}$ on the subspace of zero average functions and establishes a Poincaré inequality.

PROPOSITION 2.1. *The average operator is linear continuous. Moreover it coincides with the orthogonal projection on the kernel of $\mathcal{T}$, i.e.,*

$$
\langle u \rangle \in \ker \mathcal{T} \text{ and } \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (u - \langle u \rangle) \varphi \, dv dx = 0, \ \forall \, \varphi \in \ker \mathcal{T}. \tag{2.2}
$$

REMARK 2.1. Notice that $(\overline{X}, \overline{V})$ depends only on s and $(\overline{x}, \overline{v})$ and thus the variational characterization in (2.2) holds true at any fixed $(x_3, v_3) \in \mathbb{R}^2$. Indeed, for any $\varphi \in \ker \mathcal{T}$, $(x_3, v_3) \in \mathbb{R}^2$ we have

$$
\begin{aligned}
\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} (u\varphi)(x, v) \, d\overline{v} d\overline{x} &= \frac{1}{T_c} \int_0^{T_c} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} u(x, v) \varphi(\overline{X}(-s; x, v), x_3, \overline{V}(-s; x, v), v_3) \, d\overline{v} d\overline{x} ds \\
&= \frac{1}{T_c} \int_0^{T_c} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} u(\overline{X}(s; x, v), x_3, \overline{V}(s; x, v), v_3) \varphi(x, v) \, d\overline{v} d\overline{x} ds \\
&= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \langle u \rangle (x, v) \varphi(x, v) \, d\overline{v} d\overline{x}.
\end{aligned}
$$

We have the orthogonal decomposition of $L^2(\mathbb{R}^3 \times \mathbb{R}^3)$ into invariant functions along the characteristics (1.7) and zero average functions

$$
u = \langle u \rangle + (u - \langle u \rangle), \ \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (u - \langle u \rangle) \langle u \rangle \, dv dx = 0.
$$

Notice that $\mathcal{T}^\star = -\mathcal{T}$ and thus the equality $\langle \cdot \rangle = \mathrm{Proj}_{\ker \mathcal{T}}$ implies

$$
\ker \langle \cdot \rangle = (\ker \mathcal{T})^\perp = (\ker \mathcal{T}^\star)^\perp = \overline{\mathrm{Range} \, \mathcal{T}}.
$$

In particular $\mathrm{Range} \, \mathcal{T} \subset \ker \langle \cdot \rangle$. Actually we show that $\mathrm{Range} \, \mathcal{T}$ is closed, which will give a solvability condition for $\mathcal{T}u = w$ (cf. [3], Propositions 2.2).

PROPOSITION 2.2. *The restriction of $\mathcal{T}$ to $\ker \langle \cdot \rangle$ is a one-to-one map onto $\ker \langle \cdot \rangle$. Its inverse belongs to $\mathcal{L}(\ker \langle \cdot \rangle, \ker \langle \cdot \rangle)$ and we have the Poincaré inequality*

$$
\|u\| \leq \frac{2\pi}{|\omega_c|} \|\mathcal{T}u\|, \ \omega_c = \frac{qB}{m} \neq 0
$$

for any $u \in \mathrm{D}(\mathcal{T}) \cap \ker \langle \cdot \rangle$.

The natural space when dealing with the linear Boltzmann kernel Q_B is $L^2\left(M^{-1}\mathrm{d}x\mathrm{d}v\right)$ rather than $L^2(\mathrm{d}x\mathrm{d}v)$. Motivated by that, we introduce the operator $\mathcal{T}_M : \mathrm{D}(\mathcal{T}_M) \subset L^2\left(M^{-1}\mathrm{d}x\mathrm{d}v\right) \to L^2\left(M^{-1}\mathrm{d}x\mathrm{d}v\right)$ given by $\mathcal{T}_M u = \mathrm{div}_{x,v}(ub)$ for any function u in the domain

$$\mathrm{D}(\mathcal{T}_M) = \{u(x,v) \in L^2\left(M^{-1}\mathrm{d}x\mathrm{d}v\right) \ : \ \mathrm{div}_{x,v}(ub) \in L^2\left(M^{-1}\mathrm{d}x\mathrm{d}v\right)\}.$$

Straightforward arguments show that $u \in \mathrm{D}(\mathcal{T}_M)$ iff $u/\sqrt{M} \in \mathrm{D}(\mathcal{T})$ and $\mathcal{T}_M(u) = \sqrt{M}\mathcal{T}(u/\sqrt{M})$ for any $u \in \mathcal{D}(\mathcal{T}_M)$. In particular we have $\ker \mathcal{T}_M = \sqrt{M}\ker \mathcal{T}$. Notice that formula (2.1) still defines a linear bounded operator on $L^2\left(M^{-1}\mathrm{d}x\mathrm{d}v\right)$, denoted by $\langle \cdot \rangle_M$, which coincides with the orthogonal projection on the kernel of $\mathcal{T}_M$ with respect to the scalar product of $L^2\left(M^{-1}\mathrm{d}x\mathrm{d}v\right)$. Indeed, taking into account that $M(v)$ is constant along the characteristic flow of (1.7), we have for any $u \in L^2\left(M^{-1}\mathrm{d}x\mathrm{d}v\right)$

$$\langle u \rangle_M = \sqrt{M}\left\langle \frac{u}{\sqrt{M}} \right\rangle \in \sqrt{M}\ker \mathcal{T} = \ker \mathcal{T}_M$$

and for any $\varphi \in \ker \mathcal{T}_M$

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} (u - \langle u \rangle_M)\,\varphi(x,v)\,\frac{\mathrm{d}x\mathrm{d}v}{M(v)} = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\left(\frac{u}{\sqrt{M}} - \left\langle \frac{u}{\sqrt{M}} \right\rangle\right)\frac{\varphi}{\sqrt{M}}\,\mathrm{d}v\mathrm{d}x = 0.$$

The Poincaré inequality also holds true, with the same constant, since for any $u \in \mathrm{D}(\mathcal{T}_M) \cap \ker \langle \cdot \rangle_M$ we can write

$$\|u\|_{L^2(M^{-1})} = \left\|\frac{u}{\sqrt{M}}\right\| \leq \frac{2\pi}{|\omega_c|}\left\|\mathcal{T}\left(\frac{u}{\sqrt{M}}\right)\right\| = \frac{2\pi}{|\omega_c|}\left\|\frac{\mathcal{T}_M u}{\sqrt{M}}\right\| = \frac{2\pi}{|\omega_c|}\|\mathcal{T}_M u\|_{L^2(M^{-1})}.$$

From now on, for the sake of simplicity, we will use only the notations $\mathcal{T}, \langle \cdot \rangle$, independently of acting on $L^2(\mathrm{d}x\mathrm{d}v)$ or $L^2\left(M^{-1}\mathrm{d}x\mathrm{d}v\right)$.

3. Average and first-order differential operators. We intend to average transport operators; see (1.8). Moreover, in order to handle the Fokker-Planck kernel we will need to average second-order differential operators. For doing that it is convenient to identify derivations which leave invariant $\ker \mathcal{T}$. It turns out that these derivations are those along the invariants

$$\psi_1 = x_1 + \frac{v_2}{\omega_c}, \quad \psi_2 = x_2 - \frac{v_1}{\omega_c}, \quad \psi_3 = x_3, \quad \psi_4 = \sqrt{(v_1)^2 + (v_2)^2}, \quad \psi_5 = v_3.$$

We also introduce $\psi_0 = -\frac{\alpha}{\omega_c}$, with $\overline{v} = |\overline{v}|e^{i\alpha}$, $\alpha \in [0, 2\pi[$. Notice that ψ_0 has a jump of $\frac{2\pi}{\omega_c}$ across $\overline{v} \in \mathbb{R}_+ \times \{0\}$ but not its gradient with respect to $\overline{v}$:

$$\nabla_{\overline{v}}\alpha = -\frac{^\perp\overline{v}}{|\overline{v}|^2}, \quad \nabla_{\overline{v}}\psi_0 = \frac{^\perp\overline{v}}{\omega_c|\overline{v}|^2}, \quad \mathcal{T}\psi_0 = 1.$$

The idea is to consider the fields $(b^i)_{0 \leq i \leq 5}$ such that

$$b^i \cdot \nabla_{x,v}\psi_j = \delta^i_j, \quad 0 \leq i, j \leq 5.$$

Indeed, the map $(x,v) \to (\psi_i(x,v))_{0 \leq i \leq 5}$ defines a change of coordinates

$$x_1 = \psi_1 + \frac{\psi_4}{\omega_c}\sin(\omega_c\psi_0), \quad x_2 = \psi_2 + \frac{\psi_4}{\omega_c}\cos(\omega_c\psi_0), \quad x_3 = \psi_3$$

$$v_1 = \psi_4\cos(\omega_c\psi_0), \quad v_2 = -\psi_4\sin(\omega_c\psi_0), \quad v_3 = \psi_5.$$

Therefore any function $u = u(x, v)$ can be written $u(x, v) = U(\psi(x, v))$, $\psi = (\psi_i)_{0 \le i \le 5}$ and thus, for any $i \in \{0, 1, ..., 5\}$ we have

$$b^i \cdot \nabla_{x,v} u = b^i \cdot \sum_{j=0}^{5} \frac{\partial U}{\partial \psi_j}(\psi(x, v)) \nabla_{x,v} \psi_j = \frac{\partial U}{\partial \psi_i}(\psi(x, v)).$$

In other words, the derivations $b^i \cdot \nabla_{x,v}$ act like ∂_{ψ_i}, $0 \le i \le 5$. In particular, if $u \in \ker \mathcal{T}$, meaning that U does not depend on ψ_0, then $b^i \cdot \nabla_{x,v} u = \partial_{\psi_i} U(\psi(x, v))$ does not depend on ψ_0, which is the same as saying that $\ker \mathcal{T}$ is left invariant by $b^i \cdot \nabla_{x,v}$, $0 \le i \le 5$. The following result comes by direct computation and is left to the reader. For any smooth vector fields ξ, η on $\mathbb{R}^6$, the notation $[\xi, \eta]$ stands for their Poisson bracket, *i.e.*,

$$[\xi, \eta] = (\xi \cdot \nabla_{x,v})\eta - (\eta \cdot \nabla_{x,v})\xi.$$

PROPOSITION 3.1. The fields $(b^i)_{0 \le i \le 5}$ satisfying $b^i \cdot \nabla_{x,v} \psi_j = \delta_j^i$, $0 \le i, j \le 5$ are given by

$$b^0 \cdot \nabla_{x,v} = \overline{v} \cdot \nabla_{\overline{x}} + \omega_c {}^\perp\overline{v} \cdot \nabla_{\overline{v}}, \quad b^1 \cdot \nabla_{x,v} = \partial_{x_1}, \quad b^2 \cdot \nabla_{x,v} = \partial_{x_2}, \quad b^3 \cdot \nabla_{x,v} = \partial_{x_3}$$

$$b^4 \cdot \nabla_{x,v} = -\frac{{}^\perp\overline{v}}{\omega_c |\overline{v}|} \cdot \nabla_{\overline{x}} + \frac{\overline{v}}{|\overline{v}|} \cdot \nabla_{\overline{v}}, \quad b^5 \cdot \nabla_{x,v} = \partial_{v_3}.$$

Moreover, the Poisson brackets between $(b^i)_{0 \le i \le 5}$ vanishes or, equivalently, the derivations $b^i \cdot \nabla_{x,v}$, $0 \le i \le 5$ are commuting.

REMARK 3.1. Notice that $(b^i)_{i \ne 4}$ are divergence-free and $\mathrm{div}_{x,v} b^4 = \frac{1}{|\overline{v}|}$.

We claim that the operators $u \to \mathrm{div}_{x,v}(ub^i)$, with domain

$$\mathrm{D}(\mathrm{div}_{x,v}(\cdot \, b^i)) = \{u \in L^2(\mathbb{R}^3 \times \mathbb{R}^3) \, : \, \mathrm{div}_{x,v}(ub^i) \in L^2(\mathbb{R}^3 \times \mathbb{R}^3)\}, \quad 0 \le i \le 5,$$

are commuting with the average operator. More generally, we establish the following result.

PROPOSITION 3.2. Assume that the field $c \cdot \nabla_{x,v}$ is in involution with $b \cdot \nabla_{x,v} = \overline{v} \cdot \nabla_{\overline{x}} + \omega_c {}^\perp\overline{v} \cdot \nabla_{\overline{v}}$, *i.e.*, $[c, b] = 0$. Then the operator $\mathrm{div}_{x,v}(\cdot \, c)$ is commuting with the average operator associated to the flow of $b \cdot \nabla_{x,v}$; that is, for any function $u \in \mathrm{D}(\mathrm{div}_{x,v}(\cdot \, c))$, its average $\langle u \rangle$ belongs to $\mathrm{D}(\mathrm{div}_{x,v}(\cdot \, c))$ and

$$\mathrm{div}_{x,v}(\langle u \rangle \, c) = \langle \mathrm{div}_{x,v}(uc) \rangle.$$

Proof. Let us consider $u \in \mathrm{D}(\mathrm{div}_{x,v}(\cdot \, c))$. For any $\varphi \in C_c^1(\mathbb{R}^3 \times \mathbb{R}^3) \cap \ker \mathcal{T}$ we have

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \langle \mathrm{div}_{x,v}(uc) \rangle \varphi \, \mathrm{d}v \mathrm{d}x = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \mathrm{div}_{x,v}(uc) \varphi \, \mathrm{d}v \mathrm{d}x = -\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} uc \cdot \nabla_{x,v} \varphi \, \mathrm{d}v \mathrm{d}x. \quad (3.1)$$

But $\mathcal{T}(c \cdot \nabla_{x,v} \varphi) = c \cdot \nabla_{x,v}(\mathcal{T}\varphi) = 0$, saying that $c \cdot \nabla_{x,v} \varphi \in \ker \mathcal{T}$, and thus

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} uc \cdot \nabla_{x,v} \varphi \, \mathrm{d}v \mathrm{d}x = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \langle u \rangle \, c \cdot \nabla_{x,v} \varphi \, \mathrm{d}v \mathrm{d}x. \quad (3.2)$$

Combining (3.1), (3.2) we obtain for any $\varphi \in C_c^1(\mathbb{R}^3 \times \mathbb{R}^3) \cap \ker \mathcal{T}$

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \langle \mathrm{div}_{x,v}(uc) \rangle \varphi \, \mathrm{d}v \mathrm{d}x = -\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \langle u \rangle \, c \cdot \nabla_{x,v} \varphi \, \mathrm{d}v \mathrm{d}x. \quad (3.3)$$

Actually, the previous equality also holds true for smooth functions $\varphi \in \ker \langle \cdot \rangle$. Indeed, by Proposition 2.2, for any smooth function $\varphi \in \ker \langle \cdot \rangle$ there is $\psi \in D(\mathcal{T}) \cap \ker \langle \cdot \rangle$ such that $\mathcal{T}\psi = \varphi$ and thus $c \cdot \nabla_{x,v}\varphi = c \cdot \nabla_{x,v}(\mathcal{T}\psi) = \mathcal{T}(c \cdot \nabla_{x,v}\psi) \in \text{Range } \mathcal{T} = \ker \langle \cdot \rangle$. Using now the orthogonality between $\ker \mathcal{T}$ and $\ker \langle \cdot \rangle$, we deduce that

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \langle \text{div}_{x,v}(uc) \rangle \, \varphi \, \mathrm{d}v\mathrm{d}x = 0 = -\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \langle u \rangle \, c \cdot \nabla_{x,v}\varphi \, \mathrm{d}v\mathrm{d}x, \quad \varphi \in C_c^1(\mathbb{R}^3 \times \mathbb{R}^3) \cap \ker \langle \cdot \rangle.$$

Finally (3.3) is verified for any smooth φ, implying that

$$\langle u \rangle \in D(\text{div}_{x,v}(\cdot \, c)) \text{ and } \text{div}_{x,v}(\langle u \rangle \, c) = \langle \text{div}_{x,v}(uc) \rangle.$$

$\square$

We want to average transport operators, which are written in conservative forms. In order to obtain an averaged model still written in conservative form, it is worthwhile to establish the following commutation formula between average and divergence. For the sake of simplicity we discard all difficulties related to the required minimal smoothness.

PROPOSITION 3.3. For any smooth field $\xi = (\xi_x, \xi_v) \in \mathbb{R}^6$ we have the equality

$$
\begin{aligned}
\langle \text{div}_{x,v}\xi \rangle &= \text{div}_{\overline{x}} \left\{ \left\langle \xi_{\overline{x}} + \frac{^\perp\xi_{\overline{v}}}{\omega_c} \right\rangle + \left\langle \xi_{\overline{v}} \cdot \frac{^\perp\overline{v}}{|\overline{v}|} \right\rangle \frac{\overline{v}}{\omega_c|\overline{v}|} - \left\langle \xi_{\overline{v}} \cdot \frac{\overline{v}}{|\overline{v}|} \right\rangle \frac{^\perp\overline{v}}{\omega_c|\overline{v}|} \right\} + \partial_{x_3} \langle \xi_{x_3} \rangle \\
&+ \text{div}_{\overline{v}} \left\{ \left\langle \xi_{\overline{v}} \cdot \frac{^\perp\overline{v}}{|\overline{v}|} \right\rangle \frac{^\perp\overline{v}}{|\overline{v}|} + \left\langle \xi_{\overline{v}} \cdot \frac{\overline{v}}{|\overline{v}|} \right\rangle \frac{\overline{v}}{|\overline{v}|} \right\} + \partial_{v_3} \langle \xi_{v_3} \rangle.
\end{aligned}
$$

In particular, we have for any smooth field $\xi_x \in \mathbb{R}^3$

$$\langle \text{div}_x \xi_x \rangle = \text{div}_x \langle \xi_x \rangle$$

and for any smooth field $\xi_v \in \mathbb{R}^3$

$$
\begin{aligned}
\langle \text{div}_v \xi_v \rangle &= \text{div}_{\overline{x}} \left\{ \left\langle \frac{^\perp\xi_{\overline{v}}}{\omega_c} \right\rangle + \left\langle \xi_{\overline{v}} \cdot \frac{^\perp\overline{v}}{|\overline{v}|} \right\rangle \frac{\overline{v}}{\omega_c|\overline{v}|} - \left\langle \xi_{\overline{v}} \cdot \frac{\overline{v}}{|\overline{v}|} \right\rangle \frac{^\perp\overline{v}}{\omega_c|\overline{v}|} \right\} \\
&+ \text{div}_{\overline{v}} \left\{ \left\langle \xi_{\overline{v}} \cdot \frac{^\perp\overline{v}}{|\overline{v}|} \right\rangle \frac{^\perp\overline{v}}{|\overline{v}|} + \left\langle \xi_{\overline{v}} \cdot \frac{\overline{v}}{|\overline{v}|} \right\rangle \frac{\overline{v}}{|\overline{v}|} \right\} + \partial_{v_3} \langle \xi_{v_3} \rangle.
\end{aligned}
$$

Proof. By construction we have $\sum_{i=0}^5 b^i \otimes \nabla_{x,v}\psi_i = I$, and thus

$$\xi = \sum_{i=0}^5 (\xi \cdot \nabla_{x,v}\psi_i)b^i.$$

The main statement follows thanks to Proposition 3.2, since we have

$$\langle \text{div}_{x,v}\xi \rangle = \left\langle \sum_{i=0}^5 \text{div}_{x,v}\{(\xi \cdot \nabla_{x,v}\psi_i)b^i\} \right\rangle = \text{div}_{x,v} \left\{ \sum_{i=0}^5 \langle \xi \cdot \nabla_{x,v}\psi_i \rangle \, b^i \right\}.$$

The other statements come by considering the fields $(\xi_x, 0)$ and $(0, \xi_v)$. $\square$
A direct consequence of Proposition 3.3 is the computation of the average for the transport operator in (1.6).

PROPOSITION 3.4. Assume that the electric field derives from a smooth potential, *i.e.*, $E = -\nabla_x \phi$. Then, for any $f \in C_c^1(\mathbb{R}^3 \times \mathbb{R}^3) \cap \ker \mathcal{T}$, we have

$$\left\langle \partial_t f + v_3 \partial_{x_3} f + \frac{q}{m} E \cdot \nabla_v f + \mathcal{T} f^1 \right\rangle = \partial_t f + \frac{\left\langle {}^\perp \overline{E} \right\rangle}{B} \cdot \nabla_{\overline{x}} f + v_3 \partial_{x_3} f + \frac{q}{m} \left\langle E_3 \right\rangle \partial_{v_3} f.$$

Proof. We can write

$$\left\langle \partial_t f + v_3 \partial_{x_3} f + \frac{q}{m} E \cdot \nabla_v f + \mathcal{T} f^1 \right\rangle = \partial_t f + \left\langle v_3 \partial_{x_3} f \right\rangle + \frac{q}{m} \left\langle E \cdot \nabla_v f \right\rangle$$

since $\left\langle \mathcal{T} f^1 \right\rangle = 0$ and $\left\langle \partial_t f \right\rangle = \partial_t \left\langle f \right\rangle = \partial_t f$. The average of $v_3 \partial_{x_3} f$ comes easily thanks to Proposition 3.2:

$$\left\langle v_3 \partial_{x_3} f \right\rangle = \left\langle \operatorname{div}_{x,v} \{ f v_3 b^3 \} \right\rangle = \operatorname{div}_{x,v} \{ \left\langle f v_3 \right\rangle b^3 \} = \operatorname{div}_{x,v} \{ f v_3 b^3 \} = v_3 \partial_{x_3} f.$$

Observe that $\mathcal{T}(f\phi) = f\, \overline{v} \cdot \nabla_{\overline{x}} \phi = -f\, \overline{v} \cdot \overline{E}$ and thus $\left\langle f\, \overline{v} \cdot \overline{E} \right\rangle = 0$. Thanks to Proposition 3.3, one gets

$$\left\langle \operatorname{div}_v \{ f E \} \right\rangle = \operatorname{div}_{\overline{x}} \left\langle f \frac{{}^\perp \overline{E}}{\omega_c} \right\rangle + \mathcal{T} \left\langle f \frac{{}^\perp \overline{v} \cdot \overline{E}}{\omega_c |\overline{v}|^2} \right\rangle + \partial_{v_3} \left\langle f E_3 \right\rangle$$

$$= \operatorname{div}_{\overline{x}} \left\{ f \left\langle \frac{{}^\perp \overline{E}}{\omega_c} \right\rangle \right\} + \partial_{v_3} \{ f \left\langle E_3 \right\rangle \},$$

implying that

$$\frac{q}{m} \left\langle \operatorname{div}_v \{ f E \} \right\rangle = \operatorname{div}_{\overline{x}} \left\{ f \frac{\left\langle {}^\perp \overline{E} \right\rangle}{B} \right\} + \frac{q}{m} \partial_{v_3} \{ f \left\langle E_3 \right\rangle \}.$$

Using Proposition 3.2 again, notice that

$$\partial_{v_3} \left\langle E_3 \right\rangle = \operatorname{div}_{x,v} \{ \left\langle E_3 \right\rangle b^5 \} = \left\langle \operatorname{div}_{x,v} \{ E_3 b^5 \} \right\rangle = \left\langle \partial_{v_3} E_3 \right\rangle = 0$$

and

$$\operatorname{div}_{\overline{x}} \left\langle {}^\perp \overline{E} \right\rangle = \left\langle \operatorname{div}_{\overline{x}} {}^\perp \overline{E} \right\rangle = 0,$$

and our statement follows. $\qquad \square$

REMARK 3.2. We have proved that averaging the transport operator $a \cdot \nabla_{x,v} := v_3 \partial_{x_3} + \frac{q}{m} E \cdot \nabla_v$ leads to $A \cdot \nabla_{x,v} := \frac{\left\langle {}^\perp \overline{E} \right\rangle}{B} \cdot \nabla_{\overline{x}} + v_3 \partial_{x_3} + \frac{q}{m} \left\langle E_3 \right\rangle \partial_{v_3}$, which verifies

$$\left\langle a \cdot \nabla_{x,v} f \right\rangle = A \cdot \nabla_{x,v} f, \quad f \in C_c^1(\mathbb{R}^3 \times \mathbb{R}^3) \cap \ker \mathcal{T}.$$

By construction, the operator $A \cdot \nabla_{x,v}$ leaves invariant the subspace of smooth functions of $\ker \mathcal{T}$. By antisymmetry (since $\operatorname{div}_{x,v} A = 0$) it is easily seen that $A \cdot \nabla_{x,v}$ also leaves invariant the subspace of smooth functions in $\ker \left\langle \cdot \right\rangle$. Indeed, consider h a zero average smooth function and let us prove that $\left\langle A \cdot \nabla_{x,v} h \right\rangle = 0$. For any smooth f in $\ker \mathcal{T}$ we have

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} A \cdot \nabla_{x,v} h\, f\, \mathrm{d}v \mathrm{d}x = -\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} h A \cdot \nabla_{x,v} f\, \mathrm{d}v \mathrm{d}x = 0$$

by the orthogonality between $\ker \left\langle \cdot \right\rangle$ and $\ker \mathcal{T}$, and thus $\left\langle A \cdot \nabla_{x,v} h \right\rangle = 0$. Finally $A \cdot \nabla_{x,v}$ is commuting with the average operator $\left\langle A \cdot \nabla_{x,v} f \right\rangle = A \cdot \nabla_{x,v} \left\langle f \right\rangle$ for any smooth f.

4. The relaxation collision operator. In this section we analyze the linear Boltzmann collision kernel [17, 18]

$$Q_B(f)(x,v) = \frac{1}{\tau} \int_{\mathbb{R}^3} s(v,v')\{M(v)f(x,v') - M(v')f(x,v)\} \, \mathrm{d}v' \tag{4.1}$$

where the scattering cross section satisfies

$$s(v,v') = s(v',v), \;\; 0 < s_0 \le s(v,v') \le S_0 < +\infty, \;\; v,v' \in \mathbb{R}^3. \tag{4.2}$$

We recall the standard properties of this operator. Here $Q_B^\pm$ denote the gain/loss relaxation collision operators

$$Q_B^+(f)(v) = \frac{1}{\tau} \int_{\mathbb{R}^3} s(v,v')M(v)f(v') \, \mathrm{d}v', \;\; Q_B^-(f)(v) = \frac{1}{\tau} \int_{\mathbb{R}^3} s(v,v')M(v')f(v) \, \mathrm{d}v'.$$

PROPOSITION 4.1. *Assume that the scattering cross section satisfies* (4.2). *Then*

1. *The gain/loss collision operators* $Q_B^\pm$ *are linear bounded operators on* $L^2(M^{-1}\mathrm{d}v)$, *with* $\|Q_B^\pm\| \le S_0/\tau$, *and symmetric with respect to the scalar product of* $L^2(M^{-1}\mathrm{d}v)$.

2. *For any* $f \in L^2(M^{-1}\mathrm{d}v)$ *we have*

$$\int_{\mathbb{R}^3} Q_B(f)(v)f(v) \, \frac{\mathrm{d}v}{M} = -\frac{1}{2\tau} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} s(v,v')M(v)M(v') \left[\frac{f(v)}{M(v)} - \frac{f(v')}{M(v')} \right]^2 \mathrm{d}v' \, \mathrm{d}v \le 0.$$

We want to average $Q_B(f)$ for functions f which satisfy the constraint (1.5). In this section the operators $\mathcal{T}$ and $\langle \cdot \rangle$ should be understood in the $L^2\left(M^{-1}\mathrm{d}x\mathrm{d}v\right)$ framework. We need to compute the average of functions like $\int_{\mathbb{R}^3} C(v,v')f(x,v') \, \mathrm{d}v'$ where $C(v,v')$ is a given function. The corresponding result in the bidimensional framework has been announced in [5]. We will see that we only need to consider functions C which are left invariant by any rotation around $e_3 = (0,0,1)$. Therefore, we assume that for any orthogonal matrix $\mathcal{O} \in \mathcal{M}_3(\mathbb{R})$ such that $\mathcal{O}e_3 = e_3$, we have

$$C({}^t\mathcal{O}v, {}^t\mathcal{O}v') = C(v,v'), \;\; v,v' \in \mathbb{R}^3. \tag{4.3}$$

These functions are precisely those depending only on $|\overline{v}|, v_3, |\overline{v'}|, v_3'$ and the angle between $\overline{v}$ and $\overline{v'}$:

$$C(v,v') = \tilde{C}(|\overline{v}|, v_3, |\overline{v'}|, v_3', \varphi), \;\; \varphi = \arg \overline{v'} - \arg \overline{v}.$$

PROPOSITION 4.2. *Assume that the function* $C(v,v')$ *satisfies* (4.3) *and belongs to the space* $L^2(M^{-1}(v)M(v')\mathrm{d}v\mathrm{d}v')$. *Then for any function* $f \in \ker \mathcal{T}$ *we have*

$$\left\langle \int_{\mathbb{R}^3} C(v,v')f(x,v') \, \mathrm{d}v' \right\rangle (x,v) = \omega_c^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \mathcal{C}(|\overline{v}|, v_3, |\overline{v'}|, v_3', z)f(\overline{x'}, x_3, v') \, \mathrm{d}v'\mathrm{d}x_1'\mathrm{d}x_2' \tag{4.4}$$

where $z = \omega_c\overline{x} + {}^\perp\overline{v} - (\omega_c\overline{x'} + {}^\perp\overline{v'})$,

$$\mathcal{C}(r, v_3, r', v_3', z) = \frac{\tilde{C}(r, v_3, r', v_3', \varphi) + \tilde{C}(r, v_3, r', v_3', -\varphi)}{2\pi^2\sqrt{|z|^2 - (r-r')^2}\sqrt{(r+r')^2 - |z|^2}} \, \mathbf{1}_{\{|r-r'|<|z|<r+r'\}},$$

and for any $|z| \in (|r-r'|, r+r')$, $\varphi \in (0,\pi)$ *is the unique angle such that*

$$|z|^2 = r^2 + (r')^2 - 2rr'\cos\varphi.$$

Proof. For $(x, v) \in \mathbb{R}^3 \times \mathbb{R}^3$ we have

$$\left(\int_{\mathbb{R}^3} C(v, v') f(x, v') \, \mathrm{d}v' \right)^2 \leq \int_{\mathbb{R}^3} (C(v, v'))^2 M(v') \, \mathrm{d}v' \int_{\mathbb{R}^3} \frac{(f(x, v'))^2}{M(v')} \, \mathrm{d}v',$$

implying that

$$\left\| \int_{\mathbb{R}^3} C(v, v') f(x, v') \, \mathrm{d}v' \right\|_{L^2(M^{-1} \mathrm{d}x \mathrm{d}v)} \leq \|C\|_{L^2(M^{-1}(v) M(v') \mathrm{d}v \mathrm{d}v')} \|f\|_{L^2(M^{-1} \mathrm{d}x \mathrm{d}v)}.$$

Therefore the function $(x, v) \to \int_{\mathbb{R}^3} C(v, v') f(x, v') \, \mathrm{d}v'$ belongs to $L^2 \left(M^{-1} \mathrm{d}x \mathrm{d}v \right)$ and can be averaged in $L^2 \left(M^{-1} \mathrm{d}x \mathrm{d}v \right)$. Consider $(x, v) \in \mathbb{R}^3 \times \mathbb{R}^3$. By formula (2.1) we have

$$I := \left\langle \int_{\mathbb{R}^3} C(v, v') f(x, v') \, \mathrm{d}v' \right\rangle (x, v)$$
$$= \frac{1}{2\pi} \int_0^{2\pi} \int_{\mathbb{R}^3} C(|\overline{v}|e^{i\alpha}, v_3, v') f \left(\overline{x} + \frac{^\perp \overline{v}}{\omega_c} - \frac{^\perp \{|\overline{v}|e^{i\alpha}\}}{\omega_c}, x_3, v' \right) \, \mathrm{d}v' \mathrm{d}\alpha.$$

For any fixed $\alpha \in [0, 2\pi)$, we use the cylindrical coordinates

$$v' = (r' e^{i(\varphi + \alpha)}, v_3'), \quad r' \in \mathbb{R}_+, \quad \varphi \in [-\pi, \pi)$$

and therefore

$$I = \frac{1}{2\pi} \int_0^{2\pi} \int_{\mathbb{R}} \int_{-\pi}^{\pi} \int_{\mathbb{R}_+} C(|\overline{v}|e^{i\alpha}, v_3, r' e^{i(\varphi+\alpha)}, v_3')$$
$$\times f \left(\overline{x} + \frac{^\perp \overline{v}}{\omega_c} - \frac{^\perp \{|\overline{v}|e^{i\alpha}\}}{\omega_c}, x_3, r' e^{i(\varphi+\alpha)}, v_3' \right) r' \mathrm{d}r' \mathrm{d}\varphi \mathrm{d}v_3' \mathrm{d}\alpha. \tag{4.5}$$

But $f \in \ker \mathcal{T}$, and thus there is a g such that

$$f(x, v) = g \left(\overline{x} + \frac{^\perp \overline{v}}{\omega_c}, x_3, |\overline{v}|, v_3 \right), \tag{4.6}$$

implying that

$$f \left(\overline{x} + \frac{^\perp \overline{v}}{\omega_c} - \frac{^\perp \{|\overline{v}|e^{i\alpha}\}}{\omega_c}, x_3, r' e^{i(\varphi+\alpha)}, v_3' \right)$$
$$= g \left(\overline{x} + \frac{^\perp \overline{v}}{\omega_c} - \frac{^\perp \{|\overline{v}|e^{i\alpha}\}}{\omega_c} + \frac{^\perp \{r' e^{i(\varphi+\alpha)}\}}{\omega_c}, x_3, r', v_3' \right).$$

On one hand, notice that $r' e^{i(\varphi+\alpha)} - |\overline{v}|e^{i\alpha} = l e^{i(\psi+\alpha)}$ where $l^2 = r^2 + (r')^2 - 2rr' \cos\varphi$, $r = |\overline{v}|$ and ψ depends on r, r', φ but not on α. On the other hand, since C is invariant by rotation around e_3, we deduce that

$$C(r e^{i\alpha}, v_3, r' e^{i(\varphi+\alpha)}, v_3') = \tilde{C}(r, v_3, r', v_3', \varphi), \quad \varphi = \arg \overline{v'} - \arg \overline{v}.$$

The map $\varphi \to l(\varphi) = \sqrt{r^2 + (r')^2 - 2rr' \cos\varphi}$ defines a coordinate change between $\varphi \in (0, \pi)$ and $l \in (|r - r'|, r + r')$, and

$$\mathrm{d}\varphi = \frac{2l \mathrm{d}l}{\sqrt{l^2 - (r - r')^2} \sqrt{(r + r')^2 - l^2}}.$$

By the Fubini theorem, one gets

$$
\begin{aligned}
I = {} & \frac{1}{2\pi} \int_{\mathbb{R}} \int_{-\pi}^{\pi} \int_{\mathbb{R}_+} \int_0^{2\pi} \tilde{C}(r, v_3, r', v_3', \varphi) g\left(\overline{x} + \frac{^{\perp}\overline{v}}{\omega_c} + \frac{^{\perp}\{le^{i(\psi+\alpha)}\}}{\omega_c}, x_3, r', v_3'\right) \\
& \hspace{7cm} \cdot r' \, \mathrm{d}\alpha \mathrm{d}r' \mathrm{d}\varphi \mathrm{d}v_3' \\
= {} & \frac{1}{2\pi} \int_{\mathbb{R}} \int_{-\pi}^{\pi} \int_{\mathbb{R}_+} \int_0^{2\pi} \tilde{C}(r, v_3, r', v_3', \varphi) g\left(\overline{x} + \frac{^{\perp}\overline{v}}{\omega_c} + \frac{^{\perp}\{le^{i\alpha}\}}{\omega_c}, x_3, r', v_3'\right) r' \, \mathrm{d}\alpha \mathrm{d}r' \mathrm{d}\varphi \mathrm{d}v_3' \\
= {} & \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}_+} \int_0^{2\pi} \int_0^{\pi} \{\tilde{C}(r, v_3, r', v_3', \varphi) + \tilde{C}(r, v_3, r', v_3', -\varphi)\} \\
& \times g\left(\overline{x} + \frac{^{\perp}\overline{v}}{\omega_c} + \frac{^{\perp}\{l(\varphi)e^{i\alpha}\}}{\omega_c}, x_3, r', v_3'\right) r' \, \mathrm{d}\varphi \mathrm{d}\alpha \mathrm{d}r' \mathrm{d}v_3' \\
= {} & \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}_+} \int_0^{2\pi} \int_{|r-r'|}^{r+r'} \{\tilde{C}(r, v_3, r', v_3', \varphi(l)) + \tilde{C}(r, v_3, r', v_3', -\varphi(l))\} \\
& \times g\left(\overline{x} + \frac{^{\perp}\overline{v}}{\omega_c} + \frac{^{\perp}\{le^{i\alpha}\}}{\omega_c}, x_3, r', v_3'\right) \frac{2lr' \, \mathrm{d}l \mathrm{d}\alpha \mathrm{d}r' \mathrm{d}v_3'}{\sqrt{l^2 - (r-r')^2}\sqrt{(r+r')^2 - l^2}}.
\end{aligned}
$$

For any $\alpha' \in [0, 2\pi)$ we have

$$
g\left(\overline{x} + \frac{^{\perp}\overline{v}}{\omega_c} + \frac{^{\perp}\{le^{i\alpha}\}}{\omega_c}, x_3, r', v_3'\right) = f\left(\overline{x} + \frac{^{\perp}\overline{v}}{\omega_c} + \frac{^{\perp}\{le^{i\alpha}\}}{\omega_c} - \frac{^{\perp}\{r'e^{i\alpha'}\}}{\omega_c}, x_3, r'e^{i\alpha'}, v_3'\right),
$$

and performing the change of coordinates $v' = (r'e^{i\alpha'}, v_3')$ leads to

$$
\begin{aligned}
I = {} & \frac{1}{2\pi^2} \int_0^{2\pi} \int_{\mathbb{R}_+} \int_{\mathbb{R}} \int_0^{2\pi} \int_{\mathbb{R}_+} \{\tilde{C}(r, v_3, r', v_3', \varphi(l)) + \tilde{C}(r, v_3, r', v_3', -\varphi(l))\} \\
& \times f\left(\overline{x} + \frac{^{\perp}\overline{v}}{\omega_c} + \frac{^{\perp}\{le^{i\alpha}\}}{\omega_c} - \frac{^{\perp}\{r'e^{i\alpha'}\}}{\omega_c}, x_3, r'e^{i\alpha'}, v_3'\right) \frac{\mathbf{1}_{\{|r-r'|<l<r+r'\}} r' \mathrm{d}r' \mathrm{d}\alpha' \mathrm{d}v_3' l \mathrm{d}l \mathrm{d}\alpha}{\sqrt{l^2 - (r-r')^2}\sqrt{(r+r')^2 - l^2}} \\
= {} & \frac{1}{2\pi^2} \int_0^{2\pi} \int_{\mathbb{R}_+} \int_{\mathbb{R}^3} \{\tilde{C}(r, v_3, |\overline{v'}|, v_3', \varphi(l)) + \tilde{C}(r, v_3, |\overline{v'}|, v_3', -\varphi(l))\} \\
& \times f\left(\overline{x} + \frac{^{\perp}\overline{v}}{\omega_c} + \frac{^{\perp}\{le^{i\alpha}\}}{\omega_c} - \frac{^{\perp}\overline{v'}}{\omega_c}, x_3, v'\right) \frac{\mathbf{1}_{\{|\,|\overline{v}|-|\overline{v'}|\,|<l<|\overline{v}|+|\overline{v'}|\}} \mathrm{d}v' l \mathrm{d}l \mathrm{d}\alpha}{\sqrt{l^2 - (|\overline{v}| - |\overline{v'}|)^2}\sqrt{(|\overline{v}| + |\overline{v'}|)^2 - l^2}}.
\end{aligned}
$$

Using the notation

$$
\mathcal{C}(r, v_3, r', v_3', z) = \frac{\tilde{C}(r, v_3, r', v_3', \varphi) + \tilde{C}(r, v_3, r', v_3', -\varphi)}{2\pi^2 \sqrt{|z|^2 - (r-r')^2}\sqrt{(r+r')^2 - |z|^2}} \mathbf{1}_{\{|r-r'|<|z|<r+r'\}}
$$

where for any $|z| \in (|r - r'|, r + r')$, $\varphi \in (0, \pi)$ is the unique angle such that

$$
|z|^2 = r^2 + (r')^2 - 2rr' \cos\varphi,
$$

one gets

$$
I = \int_{\mathbb{R}^3} \int_0^{2\pi} \int_{\mathbb{R}_+} \mathcal{C}(|\overline{v}|, v_3, |\overline{v'}|, v_3', -^{\perp}\{le^{i\alpha}\}) f\left(\overline{x} + \frac{^{\perp}\overline{v}}{\omega_c} + \frac{^{\perp}\{le^{i\alpha}\}}{\omega_c} - \frac{^{\perp}\overline{v'}}{\omega_c}, x_3, v'\right) \, l \mathrm{d}l \mathrm{d}\alpha \mathrm{d}v'.
$$

We take as new coordinates

$$\overline{x'} = \overline{x} + \frac{{}^\perp\overline{v}}{\omega_c} + \frac{{}^\perp\{le^{i\alpha}\}}{\omega_c} - \frac{{}^\perp\overline{v'}}{\omega_c}.$$

Observing that $\det \frac{\partial(x_1', x_2')}{\partial(l,\alpha)} = \frac{l}{\omega_c^2}$, we deduce that

$$I = \omega_c^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \mathcal{C}(|\overline{v}|, v_3, |\overline{v'}|, v_3', (\omega_c\overline{x} + {}^\perp\overline{v}) - (\omega_c\overline{x'} + {}^\perp\overline{v'})) f(x_1', x_2', x_3, v') \, \mathrm{d}v'\mathrm{d}x_1'\mathrm{d}x_2'.$$

$\square$

REMARK 4.1. The constraint $\mathcal{T}f = 0$ allows us to reduce the right-hand side of (4.4) to a four-dimensional integral. Indeed, thanks to (4.6) we obtain, using the notation $\overline{y} = \overline{x} + \frac{{}^\perp\overline{v}}{\omega_c}$,

$$I =$$

$$\omega_c^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \mathcal{C}(|\overline{v}|, v_3, |\overline{v'}|, v_3', (\omega_c\overline{x} + {}^\perp\overline{v}) - (\omega_c\overline{x'} + {}^\perp\overline{v'})) \, g\left(\overline{x'} + \frac{{}^\perp\overline{v'}}{\omega_c}, x_3, |\overline{v'}|, v_3'\right) \, \mathrm{d}v'\mathrm{d}x_1'\mathrm{d}x_2'$$

$$= \omega_c^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \mathcal{C}(|\overline{v}|, v_3, |\overline{v'}|, v_3', \omega_c(\overline{y} - \overline{y'})) \, g(\overline{y'}, x_3, |\overline{v'}|, v_3') \, \mathrm{d}v'\mathrm{d}y_1'\mathrm{d}y_2'$$

$$= 2\pi\omega_c^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}} \int_{\mathbb{R}_+} \mathcal{C}(|\overline{v}|, v_3, r', v_3', \omega_c(\overline{y} - \overline{y'})) \, g(\overline{y'}, x_3, r', v_3') \, r'\mathrm{d}r'\mathrm{d}v_3'\mathrm{d}\overline{y'}.$$

We prefer to keep the five-dimensional integral representation since, in the sequel, we will introduce similar integral terms, but with densities f not satisfying the constraint $\mathcal{T}f = 0$.

REMARK 4.2. If the function $\tilde{C}(r, v_3, r', v_3', \varphi)$ is odd with respect to φ, then $\mathcal{C} = 0$ and

$$\left\langle \int_{\mathbb{R}^3} C(v, v') f(x, v') \, \mathrm{d}v' \right\rangle = 0, \ \ f \in \ker \mathcal{T}.$$

REMARK 4.3. Let χ be the function

$$\chi(r, r', z) = \frac{\mathbf{1}_{\{|r-r'|<|z|<r+r'\}}}{\pi^2 \sqrt{|z|^2 - (r-r')^2} \sqrt{(r+r')^2 - |z|^2}}$$

for any $r, r' \in \mathbb{R}_+, z \in \mathbb{R}^2$. Then for every $r, r' \in \mathbb{R}_+$, $\chi(r, r', z)\mathrm{d}z$ is a probability measure on $\mathbb{R}^2$:

$$\int_{\mathbb{R}^2} \chi(r, r', z) \, \mathrm{d}z = 1, \ \ r, r' \in \mathbb{R}_+,$$

and $\left\langle \int_{\mathbb{R}^3} C(v, v') f(x, v') \, \mathrm{d}v' \right\rangle$ appears as a convolution with respect to the invariants $\omega_c\overline{x} + {}^\perp\overline{v}$. Indeed, using the formula

$$f(\overline{x'}, x_3, v') = g\left(\overline{x'} + \frac{{}^\perp\overline{v'}}{\omega_c}, x_3, |\overline{v'}|, v_3'\right),$$

we obtain

$$\left\langle \int_{\mathbb{R}^3} C(v, v') f(x, v') \, \mathrm{d}v' \right\rangle(x, v) = \int_{\mathbb{R}^3} \int_{\mathbb{R}^2} \mathcal{C}(|\overline{v}|, v_3, |\overline{v'}|, v_3', (\omega_c\overline{x} + {}^\perp\overline{v}) - (\omega_c\overline{x'} + {}^\perp\overline{v'}))$$

$$\times g\left(\overline{x'} + \frac{{}^\perp\overline{v'}}{\omega_c}, x_3, |\overline{v'}|, v_3'\right) \, \mathrm{d}(\omega_c\overline{x'} + {}^\perp\overline{v'})\mathrm{d}v'.$$

REMARK 4.4. The conclusion of Proposition 4.2 also holds true for bounded functions f which are constant along the flow (1.7), provided that $C(v, v') \in L^\infty(dv; L^1(dv'))$ and satisfies (4.3). Indeed, in this case, $f \to \int_{\mathbb{R}^3} C(v, v') f(x, v') \, dv'$ is bounded on $L^\infty(dxdv)$

$$\left\| \int_{\mathbb{R}^3} C(v, v') f(x, v') \, dv' \right\|_{L^\infty(dxdv)} \leq \|C\|_{L^\infty(dv; L^1(dv'))} \|f\|_{L^\infty(dxdv)}$$

and, using the L^∞ version of the average operator, the same computations as those in the proof of Proposition 4.2 show that

$$\left\langle \int_{\mathbb{R}^3} C(v, v') f(x, v') \, dv' \right\rangle(x, v) = \omega_c^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \mathcal{C}(|\overline{v}|, v_3, |\overline{v'}|, v_3', z) f(\overline{x'}, x_3, v') \, dv' dx_1' dx_2'.$$

COROLLARY 4.1. Assume that the scattering cross section satisfies (4.2) and

$$s(v, v') = \sigma(|v - v'|), \quad v, v' \in \mathbb{R}^3 \tag{4.7}$$

for some function $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$. Then for any $f \in \ker \mathcal{T}$, we have

$$\langle Q_B f \rangle(x, v) = \frac{\omega_c^2}{\tau} \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \mathcal{S}(|\overline{v}|, v_3, |\overline{v'}|, v_3', z)\{M(v)f(\overline{x'}, x_3, v') - M(v')f(x, v)\} \, dv' dx_1' dx_2'$$

with $z = \omega_c \overline{x} + {}^\perp \overline{v} - (\omega_c \overline{x'} + {}^\perp \overline{v'})$ and

$$\mathcal{S}(r, v_3, r', v_3', z) = \sigma(\sqrt{|z|^2 + (v_3 - v_3')^2}) \chi(r, r', z).$$

Proof. Clearly the function $C(v, v') = \sigma(|v - v'|) M(v)$ satisfies (4.3), belongs to $L^2(M^{-1}(v) M(v') dv dv')$ and we have

$$\tilde{s}(r, v_3, r', v_3', \varphi) = \sigma(\sqrt{r^2 + (r')^2 - 2rr' \cos\varphi + (v_3 - v_3')^2})$$

$$\mathcal{S}(r, v_3, r', v_3', z) = \sigma(\sqrt{|z|^2 + (v_3 - v_3')^2}) \chi(r, r', z).$$

Thanks to Proposition 4.2 we obtain, with $z = (\omega_c \overline{x} + {}^\perp \overline{v}) - (\omega_c \overline{x'} + {}^\perp \overline{v'})$,

$$\left\langle \int_{\mathbb{R}^3} s(v, v') M(v) f(x, v') \, dv' \right\rangle = \omega_c^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \mathcal{S}(|\overline{v}|, v_3, |\overline{v'}|, v_3', z) M(v) f(\overline{x'}, x_3, v') \, dv' dx_1' dx_2'.$$

Since f belongs to $L^2(M^{-1} dxdv)$ and remains constant along the flow (1.7), we have

$$\left\langle \int_{\mathbb{R}^3} s(v, v') M(v') f(x, v) \, dv' \right\rangle = f(x, v) \left\langle \int_{\mathbb{R}^3} s(v, v') M(v') \, dv' \right\rangle$$

where the first average operator should be understood in the $L^2(M^{-1} dxdv)$ setting and the second one in the $L^\infty(dxdv)$ setting. Remark 4.4 applied with $C(v, v') = s(v, v') M(v') \in L^\infty(dv; L^1(dv'))$ and the constant function $1 \in L^\infty(dxdv)$ yields

$$\left\langle \int_{\mathbb{R}^3} s(v, v') M(v') \, dv' \right\rangle = \omega_c^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \mathcal{S}(|\overline{v}|, v_3, |\overline{v'}|, v_3', z) M(v') \, dv' dx_1' dx_2'.$$

Therefore we obtain

$$\left\langle \int_{\mathbb{R}^3} s(v, v') M(v') f(x, v) \, dv' \right\rangle = \omega_c^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \mathcal{S}(|\overline{v}|, v_3, |\overline{v'}|, v_3', z) M(v') f(x, v) \, dv' dx_1' dx_2',$$

and our statement follows immediately. $\square$

We intend to extend the previous averaged collision operator to all densities f, not only those satisfying the constraint $\mathcal{T}f = 0$. Think that, when numerically simulating these models, the particle density may not exactly satisfy $\mathcal{T}f = 0$, and thus we need to construct such a extension. One possibility consists of appealing to the decomposition $f = \langle f \rangle + (f - \langle f \rangle)$ and neglecting the fluctuation $f - \langle f \rangle$, leading to the operator

$$f \to \tilde{Q}_B f := \langle Q_B \langle f \rangle \rangle$$

$$= \frac{\omega_c^2}{\tau} \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \mathcal{S}(|\overline{v}|, v_3, |\overline{v'}|, v_3', z)\{M(v) \langle f \rangle (\overline{x'}, x_3, v') - M(v') \langle f \rangle (x, v)\} \, \mathrm{d}v' \mathrm{d}x_1' \mathrm{d}x_2'$$

for any $f \in L^2\left(M^{-1}\mathrm{d}x\mathrm{d}v\right)$. Clearly $\tilde{Q}_B$ coincides with $\langle Q_B f \rangle$ for any $f \in \ker \mathcal{T}$. Notice that for any $(x, v), (x_3', v_3')$, the function

$$(\overline{x'}, \overline{v'}) \to \mathcal{S}(|\overline{v}|, v_3, |\overline{v'}|, v_3', \omega_c \overline{x} + {}^{\perp}\overline{v} - (\omega_c \overline{x'} + {}^{\perp}\overline{v'}))M(v)$$

depends only on $\omega_c \overline{x'} + {}^{\perp}\overline{v'}, |\overline{v'}|$ and therefore, thanks to Remark 2.1, we obtain a simpler form

$$\tilde{Q}_B f = \frac{\omega_c^2}{\tau} \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \mathcal{S}(|\overline{v}|, v_3, |\overline{v'}|, v_3', z)\{M(v)f(\overline{x'}, x_3, v') - M(v') \langle f \rangle (x, v)\} \, \mathrm{d}v' \mathrm{d}x_1' \mathrm{d}x_2'.$$

Nevertheless, notice that it is not possible to remove the average in the loss part of the previous operator. Since Q_B and $\langle \cdot \rangle$ are linear bounded operators on $L^2\left(M^{-1}\mathrm{d}x\mathrm{d}v\right)$, we deduce that $\tilde{Q}_B$ is linear bounded on $L^2\left(M^{-1}\mathrm{d}x\mathrm{d}v\right)$. The properties of $\tilde{Q}_B$ come immediately from the properties of Q_B, cf. Proposition 4.1. For example, we have for any $f \in L^2\left(M^{-1}\mathrm{d}x\mathrm{d}v\right)$

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \tilde{Q}_B f \frac{f}{M} \, \mathrm{d}v\mathrm{d}x = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \langle Q_B \langle f \rangle \rangle \frac{f}{M} \, \mathrm{d}v\mathrm{d}x = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} Q_B \langle f \rangle \frac{\langle f \rangle}{M} \, \mathrm{d}v\mathrm{d}x$$

$$= -\frac{1}{2\tau} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} s(v, v')M(v)M(v') \left[\frac{\langle f \rangle (x, v)}{M(v)} - \frac{\langle f \rangle (x, v')}{M(v')} \right]^2 \, \mathrm{d}v'\mathrm{d}v\mathrm{d}x \le 0.$$

Another possible extension is given by

$$\langle Q_B \rangle f := \frac{\omega_c^2}{\tau} \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \mathcal{S}(|\overline{v}|, v_3, |\overline{v'}|, v_3', z)\{M(v)f(\overline{x'}, x_3, v') - M(v')f(x, v)\} \, \mathrm{d}v' \mathrm{d}x_1' \mathrm{d}x_2'$$

$$\tag{4.8}$$

for any $f \in L^2\left(M^{-1}\mathrm{d}x\mathrm{d}v\right)$, which is very similar to the operator Q_B in (4.1). We keep this operator as an extension to the operator in Corollary 4.1. The properties of $\langle Q_B \rangle$ are summarized below.

PROPOSITION 4.3. Assume that the scattering cross section satisfies (4.2), (4.7). Then
    1. The operator $\langle Q_B \rangle$ is linear bounded on $L^2\left(M^{-1}\mathrm{d}x\mathrm{d}v\right)$ and symmetric with respect to the scalar product of $L^2\left(M^{-1}\mathrm{d}x\mathrm{d}v\right)$.
    2. For any $f \in L^2\left(M^{-1}\mathrm{d}x\mathrm{d}v\right)$ we have

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \langle Q_B \rangle (f) \frac{f}{M} \, \mathrm{d}v\mathrm{d}x = -\frac{\omega_c^2}{2\tau} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \mathcal{S}(|\overline{v}|, v_3, |\overline{v'}|, v_3', z)M(v)M(v')$$

$$\times \left[\frac{f(x, v)}{M(v)} - \frac{f(\overline{x'}, x_3, v')}{M(v')} \right]^2 \, \mathrm{d}v'\mathrm{d}x_1'\mathrm{d}x_2' \, \mathrm{d}v\mathrm{d}x \le 0.$$

Proof. 1. The boundedness of the loss part follows easily since it is the multiplication by the bounded function (see Remark 4.3)

$$\frac{\omega_c^2}{\tau} \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \mathcal{S}(|\overline{v}|, v_3, |\overline{v'}|, v_3', z) M(v') \, \mathrm{d}v' \mathrm{d}x_1' \mathrm{d}x_2'$$

$$= \frac{1}{\tau} \int_{\mathbb{R}^3} \int_{\mathbb{R}^2} \mathcal{S}(|\overline{v}|, v_3, |\overline{v'}|, v_3', -z') M(v') \, \mathrm{d}z' \mathrm{d}v'$$

$$\leq \frac{S_0}{\tau}.$$

For the gain part, we use the inequalities

$$\omega_c^2 \left(\int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \mathcal{S}(|\overline{v}|, v_3, |\overline{v'}|, v_3', z) f(\overline{x'}, x_3, v') \, \mathrm{d}v' \mathrm{d}x_1' \mathrm{d}x_2' \right)^2$$

$$\leq \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \mathcal{S} \frac{f^2(\overline{x'}, x_3, v')}{M(v')} \, \mathrm{d}v' \mathrm{d}x_1' \mathrm{d}x_2' \, \omega_c^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \mathcal{S} M(v') \, \mathrm{d}v' \mathrm{d}x_1' \mathrm{d}x_2'$$

$$\leq S_0 \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \mathcal{S} \frac{f^2(\overline{x'}, x_3, v')}{M(v')} \, \mathrm{d}v' \mathrm{d}x_1' \mathrm{d}x_2'.$$

Thanks to Remark 4.3 we deduce, with $L_M^2 = L^2 \left(M^{-1} \mathrm{d}x \mathrm{d}v \right)$, that

$$\left\| \frac{\omega_c^2}{\tau} \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \mathcal{S} M(v) f(\overline{x'}, x_3, v') \, \mathrm{d}v' \mathrm{d}x_1' \mathrm{d}x_2' \right\|_{L_M^2}^2 \leq \frac{\omega_c^2}{\tau^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} M(v) S_0$$

$$\times \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \mathcal{S} \frac{f^2(\overline{x'}, x_3, v')}{M(v')} \, \mathrm{d}v' \mathrm{d}x_1' \mathrm{d}x_2' \, \mathrm{d}v \mathrm{d}x$$

$$= \frac{S_0}{\tau^2} \int_{\mathbb{R}^3} M(v) \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{f^2(\overline{x'}, x_3, v')}{M(v')} \, \omega_c^2 \int_{\mathbb{R}^2} \mathcal{S} \, \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}v' \mathrm{d}x_1' \mathrm{d}x_2' \mathrm{d}x_3 \, \mathrm{d}v$$

$$\leq \frac{S_0^2}{\tau^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{f^2(x_1', x_2', x_3, v')}{M(v')} \, \mathrm{d}v' \mathrm{d}x_1' \mathrm{d}x_2' \mathrm{d}x_3 = \frac{S_0^2}{\tau^2} \|f\|_{L_M^2}^2.$$

2. Interchanging $(\overline{x'}, v')$ with $(\overline{x}, v)$ and observing that this change leaves invariant $\mathcal{S}$, yield for any $f, g \in L^2 \left(M^{-1} \mathrm{d}x \mathrm{d}v \right)$

$$(\langle Q_B \rangle f, g)_{L_M^2}$$

$$= \frac{\omega_c^2}{\tau} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \mathcal{S} M(v) M(v') \left[\frac{f(\overline{x'}, x_3, v')}{M(v')} - \frac{f(x, v)}{M(v)} \right] \, \mathrm{d}v' \mathrm{d}x_1' \mathrm{d}x_2' \frac{g(x, v)}{M(v)} \, \mathrm{d}v \mathrm{d}x$$

$$= -\frac{\omega_c^2}{\tau} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \mathcal{S} M(v) M(v') \left[\frac{f(\overline{x'}, x_3, v')}{M(v')} - \frac{f(x, v)}{M(v)} \right] \, \mathrm{d}v \mathrm{d}x_1 \mathrm{d}x_2$$

$$\times \frac{g(\overline{x'}, x_3, v')}{M(v')} \mathrm{d}v' \mathrm{d}x_1' \mathrm{d}x_2' \mathrm{d}x_3$$

$$= -\frac{\omega_c^2}{2\tau} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \mathcal{S} M(v) M(v') \left[\frac{f(\overline{x'}, x_3, v')}{M(v')} - \frac{f(x, v)}{M(v)} \right] \left[\frac{g(\overline{x'}, x_3, v')}{M(v')} - \frac{g(x, v)}{M(v)} \right]$$

$$\mathrm{d}v' \mathrm{d}x_1' \mathrm{d}x_2' \, \mathrm{d}v \mathrm{d}x$$

which justifies the symmetry of $\langle Q_B \rangle$ and its negativity. $\square$

REMARK 4.5. Contrary to Q_B, the operator $\langle Q_B \rangle$ is nonlocal in space. The value of $\langle Q_B \rangle f$ at the point (x, v) depends on all the values of f in the set

$$A(x, v) = \{(x_1', x_2', x_3, v') \ : \ \mathcal{S}(|\overline{v}|, v_3, |\overline{v'}|, v_3', (\omega_c \overline{x} + {}^\perp \overline{v}) - (\omega_c \overline{x'} + {}^\perp \overline{v'})) > 0\}$$
$$= \{(x_1', x_2', x_3, v') \ : \ |\, |\overline{v}| - |\overline{v'}|\,| < |(\omega_c \overline{x} + {}^\perp \overline{v}) - (\omega_c \overline{x'} + {}^\perp \overline{v'})| < |\overline{v}| + |\overline{v'}|\}.$$

Observe that if we denote by $C_{x,v}$ the Larmor circle

$$C_{x,v} = \{(x_1', x_2', x_3) \ : \ |\, \omega_c \overline{x'} - (\omega_c \overline{x} + {}^\perp \overline{v})\,| = |\overline{v}| \},$$

then we have

$$C_{x,v} \times \{v' \ : \ v' \in \mathbb{R}^3\} \subset \overline{A(x,v)}$$

where $\overline{X}$ stands for the adherence of X in $\mathbb{R}^6$. In particular, $\{x\} \times \mathbb{R}^3 \subset \overline{A(x,v)}$.

REMARK 4.6. The gain/loss parts of $\langle Q_B \rangle$ are bounded on $L^1(\mathbb{R}^3 \times \mathbb{R}^3)$, and for any $f \in L^1(\mathbb{R}^3 \times \mathbb{R}^3)$ we have the global balance of the mass $\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \langle Q_B \rangle f \ \mathrm{d}v \mathrm{d}x = 0$. Indeed, we have

$$\| \langle Q_B \rangle^+ f \|_{L^1} \leq \frac{\omega_c^2}{\tau} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \mathcal{S}(|\overline{v}|, v_3, |\overline{v'}|, v_3', z) M(v) |f(\overline{x'}, x_3, v')| \ \mathrm{d}v' \mathrm{d}x_1' \mathrm{d}x_2' \ \mathrm{d}v \mathrm{d}x$$

$$\leq \frac{S_0}{\tau} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} M(v) |f(\overline{x'}, x_3, v')| \ \mathrm{d}v' \mathrm{d}x_1' \mathrm{d}x_2' \mathrm{d}x_3 \mathrm{d}v$$

$$= \frac{S_0}{\tau} \|f\|_{L^1}$$

and similarly

$$\| \langle Q_B \rangle^- f \|_{L^1} \leq \frac{S_0}{\tau} \|f\|_{L^1}.$$

The global balance follows by interchanging $(\overline{x'}, v')$ with $(\overline{x}, v)$.

Combining (1.6), Propositions 3.4, 4.1 and (4.8) yields the limit model in Theorem 1.1.

Proof of Theorem 1.1. Clearly $0 \leq f^\varepsilon \in L^\infty \left(\mathbb{R}_+, L^1(\mathbb{R}^3 \times \mathbb{R}^3)\right)$ and

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f^\varepsilon(t, x, v) \ \mathrm{d}v \mathrm{d}x = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f^{\mathrm{in}}(x, v) \ \mathrm{d}v \mathrm{d}x, \quad t \in \mathbb{R}_+, \varepsilon > 0.$$

We consider a sequence $(\varepsilon_k)_k \subset \mathbb{R}_+^\star$ converging to 0 such that $\lim_{k \to +\infty} f^{\varepsilon_k} = f$ weakly $\star$ in $L^\infty \left(\mathbb{R}_+, L^2 \left(M^{-1} \mathrm{d}x \mathrm{d}v\right)\right)$. Using the weak formulation of (1.3), (1.2) with test functions $\eta(t) \varphi(x, v)$, $\eta \in C_c^1(\mathbb{R}_+), \varphi \in C_c^1(\mathbb{R}^3 \times \mathbb{R}^3)$, we deduce, after multiplication by ε_k and letting $k \to \infty$, that the limit density satisfies the constraint

$$\mathcal{T} f(t) = 0, \quad t \in \mathbb{R}_+. \tag{4.9}$$

Considering test functions like $\eta(t) \varphi(x, v)$ with $\eta \in C_c^1(\mathbb{R}_+)$, $\varphi \in C_c^1(\mathbb{R}^3 \times \mathbb{R}^3) \cap \ker \mathcal{T}$, one gets

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f^{\varepsilon_k} \{\eta' \varphi + \eta v_3 \partial_{x_3} \varphi + \eta \frac{q}{m} E(x) \cdot \nabla_v \varphi\} \ \mathrm{d}v \mathrm{d}x \mathrm{d}t + \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f^{\mathrm{in}} \eta(0) \varphi \ \mathrm{d}v \mathrm{d}x$$

$$= -\int_{\mathbb{R}_+} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \eta Q_B(f^{\varepsilon_k}) \varphi \ \mathrm{d}v \mathrm{d}x \mathrm{d}t. \tag{4.10}$$

The symmetry of Q_B, cf. Proposition 4.1, allows us to write

$$
\begin{aligned}
\lim_{k\to+\infty} \int_{\mathbb{R}_+}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \eta Q_B(f^{\varepsilon_k})\varphi \, \mathrm{d}v\mathrm{d}x\mathrm{d}t &= \lim_{k\to+\infty}\int_{\mathbb{R}_+}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\eta f^{\varepsilon_k}Q_B(\varphi M)\,\frac{\mathrm{d}v\mathrm{d}x}{M}\mathrm{d}t \\
&= \int_{\mathbb{R}_+}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\eta f Q_B(\varphi M)\,\frac{\mathrm{d}v\mathrm{d}x}{M}\mathrm{d}t = \int_{\mathbb{R}_+}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\eta Q_B(f)\varphi \, \mathrm{d}v\mathrm{d}x\mathrm{d}t \\
&= \int_{\mathbb{R}_+}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\eta \left\langle Q_B(f)\right\rangle \varphi \, \mathrm{d}v\mathrm{d}x\mathrm{d}t = \int_{\mathbb{R}_+}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\eta \left\langle Q_B\right\rangle (f)\varphi \, \mathrm{d}v\mathrm{d}x\mathrm{d}t
\end{aligned}
$$

$$(4.11)$$

since $f(t)\in\ker\mathcal{T}, t\in\mathbb{R}_+$ and thus $\langle Q_B(f)\rangle = \langle Q_B\rangle (f)$. For the other terms in (4.10) we obtain, thanks to Proposition 3.4 and Remark 3.2,

$$
\begin{aligned}
\lim_{k\to+\infty}\int_{\mathbb{R}_+}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f^{\varepsilon_k}(\partial_t + a\cdot\nabla_{x,v})(\eta\varphi)\,\mathrm{d}v\mathrm{d}x\mathrm{d}t &= \int_{\mathbb{R}_+}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}f(\partial_t + a\cdot\nabla_{x,v})(\eta\varphi)\,\mathrm{d}v\mathrm{d}x\mathrm{d}t \\
&= \int_{\mathbb{R}_+}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}f(\partial_t + A\cdot\nabla_{x,v})(\eta\varphi)\,\mathrm{d}v\mathrm{d}x\mathrm{d}t
\end{aligned}
$$

$$(4.12)$$

and

$$
\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f^{\mathrm{in}}(x,v)\eta(0)\varphi(x,v)\,\mathrm{d}v\mathrm{d}x = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\eta(0)\left\langle f^{\mathrm{in}}\right\rangle (x,v)\varphi(x,v)\,\mathrm{d}v\mathrm{d}x. \qquad (4.13)
$$

Combining (4.10), (4.11), (4.12), (4.13) yields for any smooth test function of the form $\eta(t)\varphi(x,v)$, with $\varphi\in\ker\mathcal{T}$.

$$
\begin{aligned}
\int_{\mathbb{R}_+}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f(\partial_t + A\cdot\nabla_{x,v})(\eta\varphi)\,\mathrm{d}v\mathrm{d}x\mathrm{d}t &+ \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\left\langle f^{\mathrm{in}}\right\rangle\eta(0)\varphi(x,v)\,\mathrm{d}v\mathrm{d}x \\
&= -\int_{\mathbb{R}_+}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\left\langle Q_B\right\rangle (f)\eta\varphi\,\mathrm{d}v\mathrm{d}x\mathrm{d}t. \qquad (4.14)
\end{aligned}
$$

By Remark 3.2 we know that $A\cdot\nabla_{x,v}$ leaves invariant the subspace of zero average functions, and therefore it is easily seen that (4.14) is trivially satisfied for any test function $\eta(t)\psi(x,v)$, with $\psi\in C^1_c(\mathbb{R}^3\times\mathbb{R}^3)\cap\ker\langle\cdot\rangle$. Finally, (4.14) holds true for any smooth test function, which amounts to saying that f solves (1.9), (1.10). We are done if we prove the uniqueness for the solution of (1.9), (1.10) (and in this case the entire family $(f^\varepsilon)_\varepsilon$ will converge toward f, weakly $\star$ in $L^\infty\left(\mathbb{R}_+, L^2\left(M^{-1}\mathrm{d}x\mathrm{d}v\right)\right)$). Assume that f solves (1.9) with zero initial condition. By standard arguments one gets

$$
\partial_t |f| + \frac{\left\langle {}^\perp\overline{E}\right\rangle}{B}\cdot\nabla_{\overline{x}}|f| + v_3\partial_{x_3}|f| + \frac{q}{m}\left\langle E_3\right\rangle \partial_{v_3}|f| = \left\langle Q_B\right\rangle (f)\,\mathrm{sgn}f,
$$

implying that

$$
\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|f(t,x,v)|\,\mathrm{d}v\mathrm{d}x = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\left\langle Q_B\right\rangle (f)\,\mathrm{sgn}f(t,x,v)\,\mathrm{d}v\mathrm{d}x,\ \ t\in\mathbb{R}_+.
$$

Our conclusion comes by observing that

$$
\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \langle Q_B \rangle (f)\, \mathrm{sgn} f \; \mathrm{d}v\mathrm{d}x
$$

$$
= \frac{\omega_c^2}{\tau} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \mathcal{S}\{M(v)f(t,\overline{x'},x_3,v') - M(v')f(t,x,v)\}\mathrm{sgn} f(t,x,v)\; \mathrm{d}v'\mathrm{d}x_1'\mathrm{d}x_2'\,\mathrm{d}v\mathrm{d}x
$$

$$
= \frac{\omega_c^2}{\tau} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \mathcal{S}M(v')\{f(t,x,v)\mathrm{sgn} f(t,\overline{x'},x_3,v') - |f(t,x,v)|\}\; \mathrm{d}v'\mathrm{d}x_1'\mathrm{d}x_2'\,\mathrm{d}v\mathrm{d}x \leq 0.
$$

$$\square$$

REMARK 4.7. It is easily seen that the integro-differential operator in (1.9) propagates the constraint $\mathcal{T}f = 0$. We are done if we prove that $f_s = f$ for any $s \in \mathbb{R}$, where $f_s(t,x,v) = f(t,X(s;x,v),V(s;x,v))$ and (X,V) is the characteristic flow (1.7). A direct computation shows that $\mathcal{T}$ and $A \cdot \nabla_{x,v} = \frac{\langle {}^{\perp}\overline{E} \rangle}{B} \cdot \nabla_{\overline{x}} + v_3\partial_{x_3} + \frac{q}{m}\langle E_3 \rangle \partial_{v_3}$ commute, implying that

$$
A \cdot \nabla_{x,v} f_s = (A \cdot \nabla_{x,v} f)_s.
$$

Observe also that

$$
\langle Q_B \rangle^+ f_s = \langle Q_B \rangle^+ f = (\langle Q_B \rangle^+ f)_s, \;\; \langle Q_B \rangle^- f_s = (\langle Q_B \rangle^- f)_s
$$

and therefore $\langle Q_B \rangle f_s = (\langle Q_B \rangle f)_s$. Finally both f, f_s satisfy (1.9), (1.10), and our statement follows by the uniqueness that we have established before.

Clearly, the transport equation (1.9) can be written in the reduced phase space ($\overline{y} = \overline{x} + \frac{{}^{\perp}\overline{v}}{\omega_c}, x_3, r = |\overline{v}|, v_3$) since, by the constraint $\mathcal{T}f = 0$, we know that $f(t,x,v) = g(t,\overline{y},x_3,r,v_3)$. We obtain

$$
\partial_t g + \frac{\langle {}^{\perp}\overline{E} \rangle}{B} \cdot \nabla_{\overline{y}} g + v_3\partial_{x_3}g + \frac{q}{m}\langle E_3 \rangle \partial_{v_3}g = 2\pi\frac{\omega_c^2}{\tau}\int_{\mathbb{R}^2}\int_{\mathbb{R}}\int_{\mathbb{R}_+} \mathcal{S}(r,v_3,r',v_3',\omega_c(\overline{y}-\overline{y'}))
$$

$$
\times \{M\,g(t,\overline{y'},x_3,r',v_3') - M'\,g(t,\overline{y},x_3,r,v_3)\}\, r'\mathrm{d}r'\mathrm{d}v_3'\mathrm{d}\overline{y'}
$$

where

$$
M = \frac{1}{(2\pi\theta/m)^{3/2}}e^{-\frac{m}{2\theta}(r^2+(v_3)^2)}, \;\; M' = \frac{1}{(2\pi\theta/m)^{3/2}}e^{-\frac{m}{2\theta}((r')^2+(v_3')^2)}.
$$

REMARK 4.8. The family $(f^\varepsilon)_{0<\varepsilon\leq 1}$ remains bounded in $L^\infty\left(\mathbb{R}_+, L^2\left(M^{-1}\mathrm{d}x\mathrm{d}v\right)\right)$ for potentials of the form $\phi(x) = \overline{\phi}(\overline{x}) + \phi_3(x_3)$. Indeed, in this case, observe that the energy function $W^\varepsilon(x,v) := \frac{m|v|^2}{2} + q(\varepsilon\,\overline{\phi}(\overline{x}) + \phi_3(x_3))$ satisfies

$$
\partial_t W^\varepsilon + \frac{1}{\varepsilon}(\overline{v}\cdot\nabla_{\overline{x}} + \omega_c\,{}^{\perp}\overline{v}\cdot\nabla_{\overline{v}})W^\varepsilon + v_3\partial_{x_3}W^\varepsilon + \frac{q}{m}E(x)\cdot\nabla_v W^\varepsilon = 0,
$$

and therefore one gets

$$
\left\{\partial_t + \frac{1}{\varepsilon}(\overline{v}\cdot\nabla_{\overline{x}} + \omega_c\,{}^{\perp}\overline{v}\cdot\nabla_{\overline{v}}) + v_3\partial_{x_3} + \frac{q}{m}E(x)\cdot\nabla_v\right\}\left(\frac{(f^\varepsilon)^2}{2M(v)\exp(-\frac{q}{\theta}[\varepsilon\,\overline{\phi}(\overline{x}) + \phi_3(x_3)])}\right)
$$

$$
= \frac{Q_B(f^\varepsilon)f^\varepsilon}{M(v)\exp(-\frac{q}{\theta}[\varepsilon\,\overline{\phi}(\overline{x}) + \phi_3(x_3)])}.
$$

Integrating with respect to (x, v) yields the bound

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(f^\varepsilon(t, x, v))^2}{M(v) \exp(-\frac{q}{\theta}[\varepsilon \overline{\phi}(\overline{x}) + \phi_3(x_3)])} \, \mathrm{d}v\mathrm{d}x$$

$$\leq \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(f^{\mathrm{in}}(x, v))^2}{M(v) \exp(-\frac{q}{\theta}[\varepsilon \overline{\phi}(\overline{x}) + \phi_3(x_3)])} \, \mathrm{d}v\mathrm{d}x$$

implying the uniform estimate

$$\|f^\varepsilon\|_{L^\infty(\mathbb{R}_+, L^2(M^{-1}\mathrm{d}x\mathrm{d}v))} \leq \exp\left(\frac{|q| \, (\|\overline{\phi}\|_{L^\infty} + \|\phi_3\|_{L^\infty})}{\theta}\right) \|f^{\mathrm{in}}\|_{L^2(M^{-1}\mathrm{d}x\mathrm{d}v)}, \ 0 < \varepsilon \leq 1.$$

Appendix A. Proof of Theorem 1.2.

Proof. With the Fokker-Planck kernel being a second-order differential operator, we appeal twice to Proposition 3.3. For any $f \in \ker \mathcal{T}$, taking $\xi_v = M\nabla_v(f/M)$ yields

$$\left\langle \mathrm{div}_v\left(M\nabla_v\left(\frac{f}{M}\right)\right)\right\rangle \tag{A.1}$$

$$= \mathrm{div}_{\omega_c \overline{x}}\left\{\left\langle M \, {}^\perp\nabla_{\overline{v}}\left(\frac{f}{M}\right)\right\rangle + \left\langle M \, \nabla_{\overline{v}}\left(\frac{f}{M}\right) \cdot \frac{{}^\perp\overline{v}}{|\overline{v}|}\right\rangle \frac{\overline{v}}{|\overline{v}|} - \left\langle M \, \nabla_{\overline{v}}\left(\frac{f}{M}\right) \cdot \frac{\overline{v}}{|\overline{v}|}\right\rangle \frac{{}^\perp\overline{v}}{|\overline{v}|}\right\}$$

$$+ \mathrm{div}_{\overline{v}}\left\{\left\langle M \, \nabla_{\overline{v}}\left(\frac{f}{M}\right) \cdot \frac{{}^\perp\overline{v}}{|\overline{v}|}\right\rangle \frac{{}^\perp\overline{v}}{|\overline{v}|} + \left\langle M \, \nabla_{\overline{v}}\left(\frac{f}{M}\right) \cdot \frac{\overline{v}}{|\overline{v}|}\right\rangle \frac{\overline{v}}{|\overline{v}|}\right\} + \partial_{v_3}\left\langle M\partial_{v_3}\left(\frac{f}{M}\right)\right\rangle.$$

Since ∂_{v_3} commutes with $\langle \cdot \rangle$ (cf. Proposition 3.2), we deduce that

$$\partial_{v_3}\left\langle M\partial_{v_3}\left(\frac{f}{M}\right)\right\rangle = \partial_{v_3}\left\{M\left\langle \partial_{v_3}\left(\frac{f}{M}\right)\right\rangle\right\}$$

$$= \partial_{v_3}\left\{M\partial_{v_3}\left\langle\frac{f}{M}\right\rangle\right\} = \partial_{v_3}\left\{M\partial_{v_3}\left(\frac{f}{M}\right)\right\}.$$

It remains to compute the averages

$$\left\langle M \, {}^\perp\nabla_{\overline{v}}\left(\frac{f}{M}\right)\right\rangle, \ \left\langle M \, \nabla_{\overline{v}}\left(\frac{f}{M}\right) \cdot \frac{{}^\perp\overline{v}}{|\overline{v}|}\right\rangle, \ \left\langle M \, \nabla_{\overline{v}}\left(\frac{f}{M}\right) \cdot \frac{\overline{v}}{|\overline{v}|}\right\rangle.$$

These averages come easily, thanks to Proposition 3.3, observing that

$$\partial_{v_1}\left(\frac{f}{M}\right) = \mathrm{div}_v\left(\frac{f}{M}, 0, 0\right), \ \partial_{v_2}\left(\frac{f}{M}\right) = \mathrm{div}_v\left(0, \frac{f}{M}, 0\right),$$

$$\nabla_{\overline{v}}\left(\frac{f}{M}\right) \cdot {}^\perp\overline{v} = \mathrm{div}_{\overline{v}}\left(\frac{f}{M} \, {}^\perp\overline{v}\right), \ \nabla_{\overline{v}}\left(\frac{f}{M}\right) \cdot \overline{v} = \mathrm{div}_{\overline{v}}\left(\frac{f}{M}\overline{v}\right) - 2\frac{f}{M}.$$

We obtain

$$\left\langle M \, {}^\perp\nabla_{\overline{v}}\left(\frac{f}{M}\right)\right\rangle = M \, \nabla_{\omega_c\overline{x}}\left(\frac{f}{M}\right) \tag{A.2}$$

$$\left\langle M \, \nabla_{\overline{v}}\left(\frac{f}{M}\right) \cdot \frac{{}^\perp\overline{v}}{|\overline{v}|}\right\rangle = M \, \left(\frac{(\overline{v}, 0)}{|\overline{v}|}, \frac{({}^\perp\overline{v}, 0)}{|\overline{v}|}\right) \cdot \nabla_{\omega_c x, v}\left(\frac{f}{M}\right) \tag{A.3}$$

$$\left\langle M \, \nabla_{\overline{v}}\left(\frac{f}{M}\right) \cdot \frac{\overline{v}}{|\overline{v}|}\right\rangle = -M \, \left(\frac{({}^\perp\overline{v}, 0)}{|\overline{v}|}, -\frac{(\overline{v}, 0)}{|\overline{v}|}\right) \cdot \nabla_{\omega_c x, v}\left(\frac{f}{M}\right). \tag{A.4}$$

Our conclusion follows by combining (A.1), (A.2), (A.3), (A.4). The diffusion matrix $\mathcal{L}$ is positive, and for any $\xi = (\xi_x, \xi_v) \in \mathbb{R}^6$, we have

$$\mathcal{L}\xi \cdot \xi = |\xi_{\overline{x}}|^2 + |\xi_{\overline{x}} - {}^\perp\xi_{\overline{v}}|^2 + (\xi_{v_3})^2 \geq 0$$

with equality iff $\xi_{\overline{x}} = \xi_{\overline{v}} = (0,0)$ and $\xi_{v_3} = 0$. $\square$

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