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FINITE NON-NILPOTENT GENERALIZATIONS OF HAMILTONIAN GROUPS

Zhencai Shen, Wujie Shi, and Jinshan Zhang

Abstract. In J. Korean Math. Soc, Zhang, Xu and other authors investigated the following problem: what is the structure of finite groups which have many normal subgroups? In this paper, we shall study this question in a more general way. For a finite group G, we define the subgroup $\mathcal{A}(G)$ to be intersection of the normalizers of all non-cyclic subgroups of G. Set $\mathcal{A}_0 = 1$. Define $\mathcal{A}_{i+1}(G)/\mathcal{A}_i(G) = \mathcal{A}(G/\mathcal{A}_i(G))$ for $i \geq 1$. By $\mathcal{A}_\infty(G)$ denote the terminal term of the ascending series. It is proved that if $G = \mathcal{A}_\infty(G)$, then the derived subgroup G' is nilpotent. Furthermore, if all elements of prime order or order 4 of G are in $\mathcal{A}(G)$, then G' is also nilpotent.

1. Introduction

Let G be a finite group (all groups considered in this paper are finite). The notation and terminology used in this paper are standard, as in [14-16]. It is known that if G normalizes each subgroup of G, then G is a Dedekind group. We know that if G normalizes all cyclic subgroups of G, then G normalizes all subgroups of G. As a dual case, one can ask what can be said about the finite groups G satisfying the following condition: G normalizes all non-cyclic subgroups of G? Note that R. Baer and H. Wielandt in 1934 and 1958, respectively, introduced the following concepts: $N(G)$ denote the intersection of the normalizers of all subgroups of G and $\omega(G)$ denote the intersection of the normalizers of all subnormal subgroups of G. Those concepts were investigated by many authors, for example, see [1-4, 5, 10, 12, 24 and 26]. In fact, the generalization of the above problem had been considered by many authors, see [6-9, 17, 18, 20-22, 25 and 27]. In this paper, we shall study this question in a more general way. First of all, we give the following definition.

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Definition 1.1. Let $\mathcal{A}(G)$ be the intersection of the normalizers of all non-cyclic subgroups of G. That is,

$$\mathcal{A}(G) = \bigcap_{H \in \mathcal{S}(G)} N_G(H),$$

where $\mathcal{S}(G) = \{H | H \text{ is a non-cyclic subgroup of } G\}$.

Obviously, $\mathcal{A}(G)$ is a characteristic subgroup of G.

The case that $\mathcal{A}(G) = 1$ is possible for a solvable group G. For instance, the symmetric group S_4 on four letters satisfies $\mathcal{A}(S_4) = 1$.

Definition 1.2. For a group G, there exists a series of characteristic subgroups:

$$1 = \mathcal{A}_0(G) \leq \mathcal{A}_1(G) \leq \mathcal{A}_2(G) \leq \cdots \leq \mathcal{A}_n(G) \leq \cdots$$

satisfying $\mathcal{A}_{i+1}(G)/\mathcal{A}_i(G) = \mathcal{A}(G/\mathcal{A}_i(G))$ for $i = 0, 1, 2, \ldots$ and $\mathcal{A}_n(G) = \mathcal{A}_{n+1}(G)$ for some integer $n \geq 1$. Write $\mathcal{A}_\infty(G)$ for the terminal term of the ascending series.

Definition 1.3. A finite group G is called a $\mathcal{A}$-group if $G = \mathcal{A}(G)$, that is, all non-cyclic subgroups of G are normal.

Throughout the paper, we denote by $\mathcal{F}_{dn}$ the class of finite groups G with G' nilpotent. It is well-known that $\mathcal{F}_{dn}$ is a saturated formation containing all supersolvable groups. In addition, for a p-group P and p a prime, we denote $\Omega(P) = \Omega_1(P)$ if $p > 2$ and $\Omega(P) = \langle \Omega_1(P), \Omega_2(P) \rangle$ if $p = 2$, where $\Omega_i(P) = \{x | x^{p^i} = 1\}$. $\pi(G)$ denotes the set of primes dividing $|G|$; Z_n denotes the cyclic group of order n; Q_8 denotes the quaternion group of order 8; $[H]K$ means a split extension of a normal subgroup H by a complement subgroup K; G_p denotes a Sylow p-subgroup of G for $p \in \pi(G)$; $\Phi(G)$ is the Frattini subgroup of G; H_G is the normal core of the subgroup H in G; $l_p(G)$ is p-length; $F_p(G)$ is the largest normal p-nilpotent subgroup of G; $O_{p'}(G)$ is the largest normal p'-subgroup of G; $O_{p',p}(G)$ is the original image of $O_p(G/O_{p'}(G))$.

2. Preliminaries

First, we give two examples on $\mathcal{A}(G)$, which are useful in Sections 3 and 5. Example 2.1 indicates that the subgroup $\mathcal{A}(G)$ may be non-supersolvable. Example 2.2 shows that $1 < \mathcal{A}(G) < G$ is possible when G is a solvable group.

Example 2.1. (1) Assume that $G = A_4$. Then $\mathcal{A}(G) = G$;

(2) Assume that $G = [Q_8]Z_3 = SL(2, 3)$, where Z_3 is a cyclic subgroup of $\mathrm{Aut}(Q_8)$ of order 3. Then $\mathcal{A}(G) = G$ and $(\mathcal{A}(G))'(= G') = Q_8$ is non-abelian.

Proof. (1) Since the only non-cyclic subgroups of A_4 are A_4 and K_4 (Klein 4-group), $\mathcal{A}(G) = G$.

(2) The only non-cyclic subgroups of G are G and Q_8. Hence $\mathcal{A}(G) = G$, and G' is non-abelian. □

Example 2.2. As $\mathrm{Aut}(Q_8) \cong S_4$ and $D_8 \leq S_4$, we have the semidirect product $G = [Q_8]D_8$. Then G is a 2-group of order 2^6 and $1 < \mathcal{A}(G) < G$.

Proof. The derived subgroup A of D_8 acts faithfully on Q_8, so A is non-normal in G. Thus $1 < \mathcal{A}(G) < G$. □

The following basic properties of the subgroup $\mathcal{A}(G)$ are required in this paper.

Lemma 2.3. *Let $G = A \times B$ and $(|A|, |B|) = 1$. Then $\mathcal{A}(G) = \mathcal{A}(A) \times \mathcal{A}(B)$.*

Proof. Let H be any non-cyclic subgroup of G and let π be the set of primes dividing the order of A. Then A is a normal Hall π-subgroup of G and B is a normal Hall π'-subgroup of G. So $H \cap A$ is a normal Hall π-subgroup of H and $H \cap B$ is a normal Hall π'-subgroup of H. Therefore we have

$$H = (H \cap A) \times (H \cap B).$$

Thus

$$\begin{aligned} N_G(H) &= N_G((H \cap A)(H \cap B)) \\ &= N_G((H \cap A)) \cap N_G((H \cap B)) \\ &= (N_A(H \cap A)) \times B) \cap (A \times N_B(H \cap B)) \\ &= N_A((H \cap A)) \times N_B((H \cap B)). \end{aligned}$$

Now the result follows. □

Proposition 2.4. *If $M \leq G$, then $M \bigcap \mathcal{A}(G) \leq \mathcal{A}(M)$.*

Proof. Clearly, $M \bigcap \mathcal{A}(G) = M \bigcap_{H \in \mathcal{S}(G)} N_G(H) \leq M \bigcap_{H \in \mathcal{S}(M)} N_G(H) = \bigcap_{H \in \mathcal{S}(M)} N_M(H) = \mathcal{A}(M)$. □

Proposition 2.5. *Let $N \leq \mathcal{A}(G)$ and $N \trianglelefteq G$. Then $\mathcal{A}(G)/N \leq \mathcal{A}(G/N)$.*

Proof. It is clear by definition. □

3. $\mathscr{A}_\infty(G)$ and $\mathcal{F}_{dn}$-groups

Proposition 3.1. *For any finite group X, the subgroup $\mathcal{A}(X)$ is solvable.*

Proof. Write $G = \mathcal{A}(X)$. Then G has the property: All non-cyclic subgroups of G are normal in G. Consider a composition factor K of G. Then K is simple and $\mathcal{A}(K) = K$. If H is any proper subgroup of K, we have that H is cyclic. Now the theorem of Miller and Moreno [19] yields that K is solvable and hence abelian. Thus all composition factors of G are abelian, so G is solvable. □

Corollary 3.2. *For any finite group G, the subgroup $\mathcal{A}_\infty(G)$ is solvable.*

We can now characterize $\mathcal{F}_{dn}$-groups.

Proposition 3.3. *Let G be a finite group. Then the following statements are equivalent*:

(i) *G is an $\mathcal{F}_{dn}$-group*;

(ii) *$G/\mathcal{A}(G)$ is an $\mathcal{F}_{dn}$-group.*

Proof. (i) $\Rightarrow$ (ii): Clear. (ii)$\Rightarrow$ (i): We use induction on the order of G. If $\mathcal{A}(G) = 1$, then nothing needs to be shown. Suppose that $\mathcal{A}(G) > 1$. So we can find a minimal normal subgroup N of G such that $N \leq \mathcal{A}(G)$. By Proposition 3.1, $\mathcal{A}(G)$ is solvable, so N is an elementary abelian p-group for some prime p.

Firstly let $N \leq \Phi(G)$. By Proposition 2.5, $\mathcal{A}(G)/N \leq \mathcal{A}(G/N)$. It follows that $(G/N)/\mathcal{A}(G/N)$ is in $\mathcal{F}_{dn}$ because $G/\mathcal{A}(G) \in \mathcal{F}_{dn}$. We thus have that G/N satisfies the condition of the theorem. By induction, $(G/N)' = G'N/N$ is nilpotent. As $N \leq \Phi(G)$, it follows by [15, III, Satz 3.5] that $G'N$ and hence G' is nilpotent, which gives $G \in \mathcal{F}_{dn}$, as desired.

Next, let $N \not\subseteq \Phi(G)$. Then there is a maximal subgroup M of G such that $G = NM$ with $N \bigcap M = 1$. By Proposition 2.4, $M \bigcap \mathcal{A}(G) \leq \mathcal{A}(M)$. Thus, by the hypothesis that $G/\mathcal{A}(G) \in \mathcal{F}_{dn}$, and as $G/\mathcal{A}(G) \cong M/(\mathcal{A}(G) \cap M)$, we have $M/\mathcal{A}(M) \in \mathcal{F}_{dn}$. Hence M satisfies the condition. By induction, M' is nilpotent. Now $N \leq \mathcal{A}(G)$ and $\mathcal{A}(G)$ normalizes all non-cyclic subgroups of G. If M is cyclic, then $G' \leq N$ is a p-group and hence $G \in \mathcal{F}_{dn}$. Suppose that M is non-cyclic, and so N normalizes M. Thus M is normal in G and it follows that $G' = N' \times M'$. Since M' is nilpotent, we conclude that G' is nilpotent, as desired. $\square$

Proposition 3.4. *Let G be a finite group and $G = \mathcal{A}_\infty(G)$. Then $G \in \mathcal{F}_{dn}$.*

Proof. As $\mathcal{A}_\infty(G/\mathcal{A}(G)) = \mathcal{A}_\infty(G)/\mathcal{A}(G)$, by induction, $G/\mathcal{A}(G) \in \mathcal{F}_{dn}$. It follows from Proposition 3.3 that $G \in \mathcal{F}_{dn}$. $\square$

Theorem 3.5. *Let G be a finite group and $Z(G) > 1$. Then the following statements are equivalent*:

(i) $G \in \mathcal{F}_{dn}$;

(ii) $G = \mathcal{A}_\infty(G)$.

Proof. We only need to prove (i) $\Rightarrow$ (ii). Since $Z(G) > 1$, $\mathcal{A}(G) > 1$. It is clear that $G/\mathcal{A}(G) \in \mathcal{F}_{dn}$. Thus, by induction, we have $G/\mathcal{A}(G) = \mathcal{A}_\infty(G/\mathcal{A}(G))$. Moreover, $\mathcal{A}_\infty(G/\mathcal{A}(G)) = \mathcal{A}_\infty(G)/\mathcal{A}(G)$ gives that $G = \mathcal{A}_\infty(G)$. $\square$

4. $\mathscr{A}$-groups

The following facts are clear from Definition 1.3:

Proposition 4.1. (i) *The subgroups of a $\mathcal{A}$-group are $\mathcal{A}$-groups*;

(ii) *The quotient groups of a $\mathcal{A}$-group are $\mathcal{A}$-groups.*

By Proposition 2.3, we have:

Theorem 4.2. *If G is a finite nilpotent group, then G is a $\mathcal{A}$-group if and only if all Sylow subgroups of G are $\mathcal{A}$-groups.*

Remark. Finite nilpotent $\mathcal{A}$-groups were considered by Passman, Bozikov, Janko, Song and Qu, see [8, 22 and 25]. In addition, finite non-nilpotent $\mathcal{A}$-groups were considered by Zhang, Guo, Qu and Xu, see [27].

For convenience of the readers, we give the following:

Theorem 4.3. *G is a finite non-nilpotent $\mathcal{A}$-group if and only if G is one of the following groups*:
(i) $[Z_p]Z_n$, *where p is a prime and Z_n is not normal in G*;
(ii) $[Z_p^2]Z_n$, *where p is a prime, $(p,n)=1$ and Z_n acts irreducibly on Z_p^2*;
(iii) $([Q_8]Z_{3^m})\times Z_n$, *where $(6,n)=1$ and Q_8 is not normal in $[Q_8]Z_{3^m}$.*

5. Applications

Gaschütz and Itô proved that if all minimal subgroups of a group G are normal (in which case G is called a PN-group), then G is solvable and the Fitting length of G is at most 3 ([15, p. 436, Theorem 5.7 or 11, Theorem 1]). In this section, the following dual problem is considered: study the finite group G all of whose minimal subgroups normalize every non-cyclic subgroup of G.

Theorem 5.1. *Let G be a p-solvable group. Suppose that all elements of G of order p are in $\mathcal{A}(G)$. If $p=2$, suppose in addition that all elements of G of order 4 are in $\mathcal{A}(G)$. Then $l_p(G)\le 1$.*

Proof. We use induction on $|G|$. Clearly, $G/O_{p'}(G)$ satisfies the hypothesis and $l_p(G/O_{p'}(G))=l_p(G)$. So we may assume that $O_{p'}(G)=1$.

Let P be a Sylow p-subgroup of $\mathcal{A}(G)$. By Proposition 3.3, $\mathcal{A}(G)'$ is nilpotent. Thus $O_{p'}(G)=1$ implies $\mathcal{A}(G)'$ is a p-group, and hence P is normal in G. Also, $F_p(G)=O_{p',p}(G)=O_p(G)$. As G is p-solvable, by [23, p. 269, Theorem 9.3.1], we know

$$C_G(O_p(G))\le O_p(G).$$

We now claim that G is q-nilpotent for any prime $q\ne p$. Otherwise, there exists a prime q such that G is non-q-nilpotent. Then there exists a subgroup K with the following properties: K is non-q-nilpotent but all proper subgroups of K are q-nilpotent. By a theorem of Itô [23, p. 296, Theorem 10.3.3], K has a normal Sylow q-subgroup Q and $\exp(Q)=q$ or 4. By above, $\Omega(G_p)\le P\le O_p(G)$, so $\Omega(G_p)=\Omega(O_p(G))$. Since K is non-cyclic, by hypothesis, $\Omega(O_p(G))$ normalizes K. On the other hand, since $Q\trianglelefteq K$, it follows that $\Omega(O_p(G))$ normalizes Q and $[Q,\Omega(O_p(G))]=1$. By [15, p. 437, 5.12], we get $[Q,O_p(G)]=1$. Thus $Q\le C_G(O_p(G))$. As $C_G(O_p(G))\le O_p(G)$ and Q is a p'-group, Q must be 1, a contradiction.

Now let $G_{q'}$ denote the normal q-complement of G for every prime $q\ne p$. Then $G_p\le G_{q'}$ and G_p is the intersection of all $G_{q'}$, hence $G_p\trianglelefteq G$, of course, $l_p(G)=1$. The proof is now complete. □

Theorem 5.2. *Let G be a finite group. If all elements of G of prime order are in $\mathcal{A}(G)$, then G is solvable.*

Proof. Assume that the theorem is false and let G be a counterexample of minimal order. If M is a proper subgroup of G, by Proposition 2.4 we have $M \cap \mathcal{A}(G) \leq \mathcal{A}(M)$. Thus all subgroups of M of prime order are in $\mathcal{A}(M)$. So M satisfies the condition. By the choice of G, M is solvable. Consequently, G is a non-solvable group in which all proper subgroups are solvable, by [13, Theorem 4.1], $G/\Phi(G)$ is a minimal simple group. As $\mathcal{A}(G)$ is normal in G and solvable, it follows that $\mathcal{A}(G) \leq \Phi(G)$.

Let p be an odd prime dividing the order of G. We claim:

(1) $\Omega_1(G_p) \trianglelefteq G$.

It is well known that $\Phi(G)$ is nilpotent, so all Sylow subgroups of $\Phi(G)$ are normal in G. Let P be the Sylow p-subgroup of $\Phi(G)$. By the hypothesis, all subgroups of G of order p are in $\mathcal{A}(G)$ and hence in P, so $\Omega_1(G_p) = \Omega_1(P)$. Thus $\Omega_1(G_p)$ char $P \trianglelefteq G$, (1) follows.

(2) $C_G(\Omega_1(G_p)) \leq \Phi(G)$.

By (1), $\Omega_1(G_p)$ is normal in G, so it follows that $C_G(\Omega_1(G_p))$ is normal in G. Thus $G/\Phi(G)$ contains a normal subgroup $C_G(\Omega_1(G_p))\Phi(G)/\Phi(G)$. As $G/\Phi(G)$ has no non-trivial normal subgroups, we have $C_G(\Omega_1(G_p))\Phi(G) = \Phi(G)$ or $C_G(\Omega_1(G_p))\Phi(G) = G$. Suppose that the second case happens. Then we have $C_G(\Omega_1(G_p)) = G$, i.e., $\Omega_1(G_p) \leq Z(G)$. Thus all elements of G of order p are in $Z(G)$. Noting that p is an odd prime, we can apply the Itô lemma [15, p. 435, Theorem 5.5] to see that G is p-nilpotent. Because the quotient groups of a p-nilpotent group are also p-nilpotent, we see that $G/\Phi(G)$ would be p-nilpotent. But $G/\Phi(G)$ has no non-trivial normal subgroup, which implies that $G/\Phi(G)$ is a p'-group. However, by [15, III, Theorem 3.8], $p \mid |G/\Phi(G)|$ holds whenever $p \mid |\Phi(G)|$. This is a contradiction. We thus conclude that only the first case is true, which implies (2).

Fix an odd prime p as above. Consider the subgroup

$$N = N_G(G_p).$$

By the Schur-Zassenhaus theorem [23, p. 253, Theorem 9.1.2], N possesses a Hall p'-subgroup H such that $N = [G_p]H$. By the condition, $\Omega_1(G_p) \leq \mathcal{A}(G)$.

(3) $H' \leq \Phi(G)$ and $N' = G_p' \times H'$.

Case 1. If H is cyclic, then $N' \leq G_p'$. Moreover, as G_p is a subgroup of N, it follows that $G_p' \leq N'$. Thus $G_p' = N'$.

Case 2. If H is non-cyclic, then by the hypotheses $\Omega_1(G_p)$ normalizes H. On the other hand, by (1), we have $\Omega_1(G_p) \trianglelefteq N$. Thus $[\Omega_1(G_p), H] \leq \Omega_1(G_p) \cap H = 1$ and H acts trivially on $\Omega_1(G_p)$ by conjugation. Hence $H \leq C_G(\Omega_1(G_p)) \leq \Phi(G)$ and of course, $H' \leq \Phi(G)$. Moreover, by [15, p. 437, 5.12], H acts trivially on G_p. That is, $G_pH = G_p \times H$, so $N = G_p \times H$ and

$$N' = G_p' \times H'.$$

(4) $G_p' \leq \Phi(G)$, in particular, $N' \leq \Phi(G)$.

As G is non-solvable, there exists another odd prime q dividing the order G such that $q \neq p$. Let G_q be a Sylow q-subgroup of G. By (1), we have $\Omega_1(G_q) \trianglelefteq G$. Also, by the hypothesis, $\Omega_1(G_q) \leq \mathcal{A}(G)$. If G_p is cyclic, then $G_p' = 1 \leq \Phi(G)$. If G_p is non-cyclic, then $\Omega_1(G_q)$ normalizes G_p. Thus $\Omega_1(G_q)G_p = \Omega_1(G_q) \times G_p$, and hence $C_G(\Omega_1(G_q)) \geq G_p$. Applying (2), we see that $G_p \leq \Phi(G)$. Therefore, $G_p' \leq \Phi(G)$, as desired.

(5) The final contradiction.

Let $\overline{G} = G/\Phi(G)$. Then $\overline{G}_p = G_p\Phi(G)/\Phi(G)$ is a Sylow p-subgroup of $G/\Phi(G)$. Write $N_{\overline{G}}(\overline{G}_p) = M/\Phi(G)$. Then $G_p\Phi(G) \trianglelefteq M$ and G_p is a Sylow p-subgroup of $G_p\Phi(G)$. By the Frattini argument $M = N_M(G_p)\Phi(G)$. Therefore $N_{\overline{G}}(\overline{G}_p) = N_G(G_p)\Phi(G)/\Phi(G)$. Now, $N_{\overline{G}}(\overline{G}_p) \cong N_G(G_p)/(N_G(G_p) \cap \Phi(G))$, and, by (4), $N_G(G_p)' = N' \leq \Phi(G)$, so $N_G(G_p)/(N_G(G_p) \cap \Phi(G))$ is abelian. Consequently, $N_{\overline{G}}(\overline{G}_p)$ is abelian. By a theorem of Burnside [15, IV, Theorem 2.6], $\overline{G}$ is p-nilpotent. This is impossible because $\overline{G}$ is a minimal simple group. The proof now is complete. □

Theorem 5.3. *Let G be a finite group. If all elements of G of prime order or order 4 are in $\mathcal{A}(G)$, then G' is nilpotent.*

Proof. Let p be any prime dividing $|G|$ and let P be a Sylow p-subgroup of G. As G is solvable, it is p-solvable. According to Theorem 5.1, we have $F_p(G) = O_{p',p}(G) = O_{p'}(G)P$, the maximal normal p-nilpotent subgroup of G. Next, by the Frattini argument $G = N_G(P)O_{p'}(G)$. On the other hand, by the Schur-Zassensaus theorem [23, p. 253, Theorem 9.1.2], $N_G(P) = [P]M$, where M is a Hall p'-subgroup of $N_G(P)$ and hence $G = F_p(G)M$.

If M is cyclic, then $G' \leq F_p(G)M' = F_p(G)$ and G' is p-nilpotent.

If M is non-cyclic, by the hypothesis, $\Omega_1(P)$ and $\Omega_2(P)$ normalize M. Hence M centralizes $\Omega_1(P)$ and $\Omega_2(P)$, and thus centralizes P. Since $C_G(P) \leq F_p(G)$ by [23, p. 269, Theorem 9.3.1],

$$M \leq F_p(G).$$

Now $G = F_p(G)M$, so it follows that $G = F_p(G)$. Of course, G' is p-nilpotent. Hence G' is nilpotent. □

Theorem 5.4. *Let G be a finite group. If all elements of G of prime order or order 4 are in $\mathcal{A}(G)$, then*

(i) *G is solvable*;

(ii) *$l_p(G) \leq 1$ for every prime p, and*

(iii) *the Fitting length of G is bounded by 2.*

Proof. This follows from Theorems 5.1, 5.2 and 5.3. □

Let us compare Theorem 5.4 with the following well-known result: If all the cyclic subgroups of a group G of prime order or order 4 are normal, then G is supersolvable [11]. The previous Example 2.1 shows that the supersolvable conclusion cannot be expected under the condition of Theorem 5.4.

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Zhencai Shen
LMAM and School of Mathematical Sciences
Peking University
Beijing, 100871, P. R. China
E-mail address: `zhencai688@sina.com`

Wujie Shi
School of Mathematics and Statics
Chongqing University of Arts and Sciences
Chongqing, 402160, P. R. China
E-mail address: `wjshi@suda.edu.cn`

Jinshan Zhang
School of Science
Sichuan University of Science and Engineering
Zigong, 643000, P. R. China
E-mail address: `zjscdut@163.com`