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FINITE QUOTIENTS OF THE ALGEBRAIC FUNDAMENTAL GROUP OF PROJECTIVE CURVES IN POSITIVE CHARACTERISTIC

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Let X be a smooth connected projective curve defined over an algebraically closed field k of characteristic $p > 0$. Let G be a finite group whose order is divisible by p. Suppose that G has a normal p-Sylow subgroup. We give a necessary and sufficient condition for G to be a quotient of the algebraic fundamental group $\pi_1(X)$ of X.

1. Introduction.

Let X be a smooth projective connected algebraic curve of genus g defined over an algebraically closed field k of characteristic $p > 0$. In this paper we study necessary and sufficient conditions for a finite group G to be a quotient of the algebraic fundamental group $\pi_1(X)$ of X. We denote by $\pi_A(X)$ the set of isomorphism classes of finite groups which are quotients of $\pi_1(X)$. Recall that a group $G \in \pi_A(X)$ will occur as a Galois group of an étale Galois cover $Z \to X$. In this paper we will call $Z \to X$ a Galois G-cover.

Let G be a finite group and suppose that its order is not divisible by p. In [**Groth71**, Corollary 2.12] Grothendieck showed that $G \in \pi_A(X)$ if and only if G is a quotient of the topological fundamental group Γ_g of a compact Riemann surface of genus g.

We consider next a finite p-group G. Denote by $\Phi(G) = [G,G]G^p$ its Frattini subgroup and let $\mathcal{G} = G/\Phi(G)$. This group is an elementary p-abelian group. The p-torsion subgroup $J_X[p]$ of the Jacobian variety J_X of X is an $\mathbb{F}_p$-vector space whose dimension γ_X is called the Hasse-Witt invariant of X. It follows from [**Ser56**, §11] that $\mathcal{G} \in \pi_A(X)$ if and only if $\mathcal{G}$ has p-rank at most γ_X. Suppose now that $G \in \pi_A(X)$, then $\mathcal{G} \in \pi_A(X)$, therefore the p-rank of G (the minimal number of generators of its maximal p-quotient) is at most γ_X. Actually, this condition is also sufficient. This follows from the fact that the p-cohomological dimension $\mathrm{cd}_p(\pi_1(X))$ of $\pi_1(X)$ is at most 1 (cf. end of proof of Theorem 1.3).

Now these two situations are understood, the next step to study is the case of a finite group G whose order is divisible by p. Consider the case where G has a normal p-Sylow subgroup P. Let $H = G/P$. The main theorem (Theorem 1.3) addresses the question of when a Galois P-cover $Z \to Y$ and a Galois H-cover $Y \to X$ can be composed to give a Galois G-cover $Z \to X$ (recall that in general the cover may not be Galois). Roughly speaking the theorem says that $G \in \pi_A(X)$ if and only if the action of H on P is compatible with the action of H on $J_Y[p]$. The result fits nicely with the fact (implied above) that the p-torsion of the Jacobian variety of Y regulates the Galois P-covers of Y. In order to state the main theorem precisely we need to introduce some notation.

1.1. Group theory. Let G be a finite group with normal p-Sylow subgroup P and quotient $H = G/P$. A theorem of Schur and Zassenhaus assures that G is isomorphic to the semi-direct product $P \rtimes H$ taken with respect to the the action $\eta : H \to \mathrm{Aut}(P)$ defined by conjugation. Let $\Phi(P) = [P,P]P^p$ be the Frattini subgroup of P. The quotient $\mathcal{P} = P/\Phi(P)$ is the maximal elementary abelian quotient of P, hence it is an $\mathbb{F}_p$-vector space. This action induces an $\mathbb{F}_p$-representation $\rho : H \to \mathrm{Aut}(\mathcal{P})$.

Let $Z(H)$ be the set of irreducible characters χ of H with values in the algebraically closed field k of characteristic $p > 0$. Let χ^0 be the trivial character of H and $\rho_\chi : H \to \mathrm{GL}(V_\chi)$ an irreducible k-representation of H of character χ of degree n_χ. The canonical decomposition of $\mathcal{P} \otimes_{\mathbb{F}_p} k$ as a $k[H]$-module is given by

$$\mathcal{P} \otimes_{\mathbb{F}_p} k = \bigoplus_{\chi \in Z(H)} V_\chi^{m_\chi}. \tag{1.1}$$

1.2. Generalized Hasse-Witt invariants. Let $Y \to X$ be a Galois cover with $\mathrm{Gal}(Y/X) \cong H$ and g_Y the genus of Y. Let J_Y be the Jacobian variety of Y and $J_Y[p]$ its p-torsion subgroup. Suppose that $\mathbb{F}_q = \mathbb{F}_{p^m}$ is a finite field large enough to contain the $|H|$-th roots of unity. Let $e_\chi = \frac{\chi(1)}{|H|}\sum_{h\in H} \chi\left(h^{-1}\right) h \in k[H]$ be the idempotent corresponding to χ. Denote by

$$J_Y[p] \otimes_{\mathbb{F}_p} \mathbb{F}_q = \bigoplus_{\chi \in Z(H)} J_Y[p]_\chi \tag{1.2}$$

the canonical decomposition of $J_Y[p] \otimes_{\mathbb{F}_p} \mathbb{F}_q$, where $J_Y[p]_\chi = e_\chi \cdot (J_Y[p] \otimes_{\mathbb{F}_p} \mathbb{F}_q)$.

Definition 1.1. The generalized Hasse-Witt invariant $\gamma_{Y,\chi}$ of Y with respect to χ is defined as the dimension of $J_Y[p]_\chi$ as an $\mathbb{F}_q$-vector space (cf. [**Ruc86**, §2]). A surjection $\phi : \pi_1(X) \twoheadrightarrow H$ corresponds to a Galois H-cover $Y \to X$, and in the case that the cover $Y \to X$ is not named, we will use $\gamma_{\phi,\chi}$ to denote the generalized Hasse-Witt invariant $\gamma_{Y,\chi}$ of Y with respect to χ.

The notation $\gamma_{\phi,\chi}$ has the advantage that it emphasizes that the generalized Hasse-Witt invariants are invariants of the *cover* $Y \to X$ (corresponding to the surjection $\phi : \pi_1(X) \twoheadrightarrow H$) rather than of the curve Y alone. Also, the main result of this paper is phrased in terms of embedding problems involving ϕ. Thus, the notation $\gamma_{\phi,\chi}$ eases the exposition in that case. However, in the literature the notation $\gamma_{Y,\chi}$ is standard. Moreover, in this paper when we are dealing directly with the cover $Y \to X$, as opposed to the surjection ϕ, we use the notation $\gamma_{Y,\chi}$.

A consequence of (1.2) is

$$\gamma_Y = \sum_{\chi \in Z(H)} \gamma_{Y,\chi}. \tag{1.3}$$

1.3. Embedding problems.

Definition 1.2 ([**Har95**, p. 366])**.** An embedding problem for a profinite group Λ is a pair of surjective profinite group homomorphisms ($\alpha : \Lambda \to \mathcal{K}_2$, $\delta : E \to \mathcal{K}_2$). The embedding problem is finite, if E is a finite group, and trivial, if δ is an isomorphism. A weak, respectively proper, solution to the embedding problem is a homomorphism, respectively a surjective homomorphism, $\beta : \Lambda \to E$ such that $\alpha = \delta \circ \beta$.

1.4. Main theorem. Here we give the statement of the main result. Since a surjection $\phi : \pi_1(X) \twoheadrightarrow H$ corresponds to a unique Galois H-cover $Y \to X$, embedding problems ($\phi : \pi_1(X) \twoheadrightarrow H, G \twoheadrightarrow H$) relate to Galois theory. Specifically, a proper solution to such an embedding problem corresponds to the existence of a Galois G-cover $Z \to X$ dominating the Galois H-cover $Y \to X$. Thus we use the language of embedding problems to state the main theorem.

Again we assume throughout this paper that all curves are smooth connected projective k-curves. For such a curve X we make the following notation.

Notation. Given a group G with normal p-Sylow subgroup P and quotient $H = G/P$, and given $\phi : \pi_1(X) \twoheadrightarrow H$, let m_χ, n_χ, and $\gamma_{\phi,\chi}$ be as in Sections 1.1 and 1.2. By *Condition A for the curve X* we will mean that for every $\chi \in Z(H)$ the following inequality holds: $m_\chi n_\chi \leq \gamma_{\phi,\chi}$.

Theorem 1.3. *Let G be a finite group having a normal p-Sylow subgroup P. Let $H = G/P$. An embedding problem ($\phi : \pi_1(X) \twoheadrightarrow H, G \twoheadrightarrow H$) has a proper solution if and only if Condition A holds for the curve X.*

The necessity of Condition A for the curve X in Theorem 1.3 was previously obtained in [**Ste96a**, Proposition 3.4]. In this paper we show that it is also sufficient. Rephrasing this in terms of covers we obtain the following immediate corollary.

Corollary 1.4. *Let G be a finite group having a normal p-Sylow subgroup P. Let $H = G/P$. Then, $G \in \pi_A(X)$ if and only if there exists a Galois H-cover $Y \to X$ such that $m_\chi n_\chi \leq \gamma_{Y,\chi}$, for every $\chi \in Z(H)$.*

Remark 1.5. In the case that ϕ corresponds to a Galois H-cover $Y \to X$ where Y is an ordinary curve (namely, that the genus of Y is equal to γ_Y) we show (Theorem 7.1) that an embedding problem $(\phi : \pi_1(X) \twoheadrightarrow H, G \twoheadrightarrow H)$ has a proper solution if and only if $m_\chi \leq g$, when χ is the trivial character of H, and $m_\chi \leq (g-1)n_\chi$, otherwise. The advantage here is that we eliminate the generalized Hasse-Witt invariant notation from the condition. This result appears in Section 7 where we discuss it and other consequences of Theorem 1.3. The existence or non-existence of 'ordinary Galois H-covers' is a difficult and open problem. However, in the case that H is abelian and X is 'generic' (cf. Section 7) a great deal of progress has been made by Nakajima [**Nak83**] and Zhang [**Zha92**]. We use their theorems in Section 7 to obtain some interesting results and examples (cf. Theorem 7.4 and Example 7.11).

We start with some Preliminaries which allow us to compute the generalized Hasse-Witt invariants in terms of differentials and to estimate how big they are. Next, in Section 3, we determine when $\mathcal{P} \rtimes H \in \pi_A(X)$ and develop some elementary representation theory tools which will be used in Section 6 to prove Theorem 1.3. In Section 4, some useful results regarding solutions of embedding problems are given. In Section 5, we prove that the p-cohomological dimension of $\pi_1(X)$ is at most 1. The main result is proved in Section 6, and in Section 7 we discuss some consequences of the main theorem and make some comparisons to previous work of Nakajima [**Nak87**] and Stevenson [**Ste96a**].

2. Preliminaries.

Let Y be a smooth projective connected algebraic curve of genus g_Y defined over an algebraically closed field k of characteristic $p > 0$.

Definition 2.1. Let Ω^1_Y be the space of differentials of Y and $\Omega^1_Y(0) \subset \Omega^1_Y$ the subspace of regular differentials. Let L be the function field of Y and t a separating variable of L. Given $\omega = f\,dt \in \Omega^1_Y$, the Cartier operator is defined by $\mathcal{C}\,(\omega) = (-d^{p-1}f/dt^{p-1})^{1/p}\,dt$. This is a $1/p$-linear operator, i.e., $\mathcal{C}(a^p\omega) = a\mathcal{C}(\omega)$, for any $a \in K$. Moreover, $\mathcal{C}$ acts on $\Omega^1_Y(0)$ (cf. [**Ser56**, §10, p. 39]).

It also follows from [**Ser56**, §10, p. 39] that there exists an $\mathbb{F}_p$-isomorphism between $J_Y[p]$ and $\mathrm{Ker}(1 - \mathcal{C} \,|\, \Omega^1_Y(0))$ given by $\mathrm{class}(D) \mapsto df/f$, where $p.\mathrm{class}(D) = \mathrm{div}(f)$. In particular, $\gamma_Y \leq g_Y$.

Definition 2.2. The curve Y is called ordinary if $\gamma_Y = g_Y$.

In order to understand how big the generalized Hasse-Witt invariants are we recall that a theorem of Nakajima ([**Nak84**] Corollary, one can also follow the proof in characteristic 0 of Chevalley and Weil [**CheWei34**]) which says that if $Y \to X$ is étale and $\mathrm{Gal}(Y/X) \cong H$, then we have an isomorphism of $k[H]$-modules

$$\Omega^1_Y(0) \cong k \oplus k[H]^{g-1}. \tag{2.1}$$

Given $\chi \in Z(H)$, let $\Omega^1_Y(0)_\chi = e_\chi.\Omega^1_Y(0)$ and $g_\chi = \dim_k \Omega^1_Y(0)_\chi$. Note that (2.1) implies that $g_{\chi^0} = g$ and $g_\chi = (g-1)n_\chi^2$ for every $\chi \in Z(H)$, $\chi \neq \chi^0$. It is a result due to Rück [**Ruc86**, Proposition 2.3] that the $\mathbb{F}_q[H]$-modules $J_Y[p]_\chi$ and $\mathrm{Ker}(1 - \mathcal{C}^m \,|\, \Omega^1_Y(0)_\chi)$ are isomorphic (this generalizes the above result of Serre). Hence, for each $\chi \in Z(H)$ we have

$$\gamma_{Y,\chi} \leq g_\chi. \tag{2.2}$$

Remark 2.3. In particular, by (1.3), we conclude that Y is ordinary if and only if for each $\chi \in Z(H)$ we have

$$\gamma_{Y,\chi} = \begin{cases} g, & \text{if } \chi = \chi^0 \text{ and} \\ (g-1)\, n_\chi^2, & \text{if } \chi \neq \chi^0. \end{cases} \tag{2.3}$$

3. Unramified covers and Galois modules.

Let $Y \to X$ be a Galois cover with $\mathrm{Gal}(Y/X) \cong H$ and $Z \to Y$ an étale Galois cover with $\mathrm{Gal}(Z/Y) \cong (\mathbb{Z}/p\mathbb{Z})^r$, for some $1 \leq r \leq \gamma_Y$. In [**Pac95**, Propositions 2.4 and 2.5] the first author determined a necessary and sufficient condition for $Z \to X$ to be also Galois. We review these results and as a consequence we obtain a necessary and sufficient condition for $\mathcal{P} \rtimes H \in \pi_A(X)$.

Denote by $\mathcal{S}_1$ the set of all étale Galois covers $Z \to Y$ with $\mathrm{Gal}(Z/Y) \cong (\mathbb{Z}/p\mathbb{Z})^r$ for some $1 \leq r \leq \gamma_Y$. This set corresponds bijectively to the set $\mathcal{S}_2$ of $\mathbb{F}_p$-vector subspaces of $\mathrm{Hom}(\pi_1(Y), \mathbb{Z}/p\mathbb{Z})$ by $(Z \to Y) \mapsto \mathrm{Hom}(\mathrm{Gal}(Z/Y), \mathbb{Z}/p\mathbb{Z})$, where we identify $\mathrm{Hom}(\mathrm{Gal}(Z/Y), \mathbb{Z}/p\mathbb{Z})$ with the $\mathbb{F}_p$-vector space of $\psi \in \mathrm{Hom}(\pi_1(Y), \mathbb{Z}/p\mathbb{Z})$ such that $\pi_1(Z) \subset \mathrm{Ker}(\psi)$. Its inverse is equal to $V \mapsto (Z \to Y)$, where $\bigcap_{\psi \in V} (L^{\mathrm{un}})^{\mathrm{Ker}(\psi)}$ is the function field of Z, L is the function field of Y and L^{un} is the maximal unramified Galois extension of L. An element $(Z \to Y)$ of $\mathcal{S}_1$ is explicitly described as follows.

For each $Q \in Y$, let L_Q be the completion of L at Q, $U_L = \bigcap_{Q \in Y} (\wp(L_Q) \cap L)$, where $\wp$ denotes the operator $\wp(x) = x^p - x$. Let $W_L = U_L/\wp(L)$ and for each $a \in U_L - \wp(L)$, let $\langle a + \wp(L) \rangle$ be the cyclic subgroup of order p of W_L generated by $a + \wp(L)$. Denote by $\wp^{-1}(a)$ a solution of $\wp(T) = a$ in the algebraic closure of L.

Lemma 3.1 ([**Pac95**, Proposition 2.4])**.** *Let* $(Z \to Y) \in \mathcal{S}_1$ *with* $\mathrm{Gal}(Z/Y) \cong (\mathbb{Z}/p\mathbb{Z})^r$ *for some* $1 \leq r \leq \gamma_Y$. *There exist* $\mathbb{F}_p$*-linearly independent* $a_1 + \wp(L), \ldots, a_r + \wp(L) \in W_L$ *such that* $k(Z) = k(\wp^{-1}(a_1), \cdots, \wp^{-1}(a_r))$. *Moreover, the cover* $(Z \to Y)$ *is uniquely determined by the* $\mathbb{F}_p$*-vector subspace* $A_{Z/Y} = \bigoplus_{j=1}^r \langle a_j + \wp(L) \rangle$ *of* W_L *and* $\mathrm{Gal}(Z/Y) = \mathrm{Hom}(A_{Z/Y}, \mathbb{Z}/p\mathbb{Z})$.

Lemma 3.2 ([**Pac95**, Proposition 2.5])**.** *With hypothesis and notation as in Lemma 3.1, $Z \to X$ is Galois if and only if $A_{Z/Y}$ is an $\mathbb{F}_p[H]$-module. In this case,* $\mathrm{Gal}(Z/X) \cong \mathrm{Gal}(Z/Y) \rtimes H$ *and the action of H on $A_{Z/Y}$ is contragradient to the natural action of H on* $\mathrm{Gal}(Z/Y)$*.*

Our goal now is to describe the $\mathbb{F}_p[H]$-module structure of $\mathcal{P}$ and compare it with the $\mathbb{F}_p[H]$-module structure of $\mathrm{Ker}(1-\mathcal{C} \mid \Omega^1_Y(0))$. In order to do this we introduce some basic facts on representation theory.

Definition 3.3. Let $\chi \in Z(H)$ and denote by $\rho_\chi : H \to \mathrm{GL}(V_\chi)$ an irreducible representation with character χ. Given $h \in H$, let $(a_{ij}(h))$ be the matrix of $\rho_\chi(h)$ with respect to some fixed basis of V_χ. For each $m \geq 0$, let $\rho_{\chi^{p^m}} : H \to \mathrm{GL}(V_\chi)$ be the map defined by $\rho_{\chi^{p^m}}(h) = \rho_\chi(h)^{p^m}$.

Lemma 3.4 ([**Isa76**, p. 151])**.** *The map $\rho_{\chi^{p^m}}$ is an irreducible k-representation of H with character χ^{p^m} defined by $\chi^{p^m}(h) = \chi(h)^{p^m}$.*

Definition 3.5 ([**Isa76**, p. 152])**.** Denote by $\mathbb{F}_{p^{l_\chi}}$ the field $\mathbb{F}_p(\chi)$ generated by $\mathbb{F}_p$ and the character values $\{\chi(h)\,;\, h \in H\}$. Given $\chi, \psi \in Z(H)$, define $\chi \sim \psi$ if and only if there exists $0 \leq m < l_\chi$ such that $\psi = \chi^{p^m}$. Let $[\chi]$ be the class of χ in $\mathcal{Z}(H) = Z(H)/\sim$. Let $\mathcal{F}$ be the set of $\mathbb{F}_p$- irreducible representations $\rho : H \to \mathrm{GL}(U)$ of H.

Lemma 3.6 ([**Isa76**, Theorem 9.21])**.** *There is a bijection between the sets $\mathcal{F}$ and $\mathcal{Z}(H)$ given by $\rho \longmapsto [\chi]$, where $\rho \otimes_{\mathbb{F}_p} k : H \to \mathrm{GL}(U \otimes_{\mathbb{F}_p} k)$ is isomorphic to $\rho_{[\chi]} = \bigoplus_{j=0}^{l_\chi - 1} \rho_{\chi^{p^j}}$.*

The action $\eta : H \to \mathrm{Aut}(P)$ given by conjugation induces an $\mathbb{F}_p$-representation $\rho : H \to \mathrm{Aut}(\mathcal{P})$. By Lemma 3.6, $\rho \otimes_{\mathbb{F}_p} k$ is a sum of the representations $\rho_{[\chi]}$ with multiplicities m_χ (note that since ρ is defined over $\mathbb{F}_p$, $m_\psi = m_\chi$, for $\psi \sim \chi$). Denote $V_{[\chi]} = \bigoplus_{j=0}^{l_\chi - 1} V_{\chi^{p^j}}$. Hence,

$$\mathcal{P} \otimes_{\mathbb{F}_p} k \cong \bigoplus_{[\chi] \in \mathcal{Z}(H)} V_{[\chi]}^{m_\chi}. \tag{3.1}$$

Let $\mathcal{V}_{[\chi]}$ be the irreducible $\mathbb{F}_p[H]$-module such that $\mathcal{V}_{[\chi]} \otimes_{\mathbb{F}_p} k \cong V_{[\chi]}$. It follows from (3.1) that

$$\mathcal{P} \cong \bigoplus_{[\chi] \in \mathcal{Z}(H)} \mathcal{V}_{[\chi]}^{m_\chi}. \tag{3.2}$$

Let $n_\chi = \dim_k V_\chi$, $g_\chi = \dim_k \Omega^1_Y(0)_\chi$ and $\Omega^1_Y(0)_{[\chi]} = \bigoplus_{i=0}^{l_\chi - 1} \Omega^1_Y(0)_{\chi^{p^i}}$. The Cartier operator $\mathcal{C}$ induces a k-isomorphism between $\Omega^1_Y(0)_{\chi^p}$ and $\Omega^1_Y(0)_\chi$ given by $\omega \mapsto \mathcal{C}(\omega)$. In particular, $\Omega^1_Y(0)_{[\chi]} \cong V_{[\chi]}^{g_\chi/n_\chi}$. Clearly $\mathcal{C}$ acts on $\Omega^1_Y(0)_{[\chi]}$. Hence, $\mathrm{Ker}(1 - \mathcal{C} \mid \Omega^1_Y(0)_{[\chi]}) \cong \mathcal{V}_{[\chi]}^{t_\chi}$, for some $1 \leq t_\chi \leq g_\chi/n_\chi$.

The canonical decomposition of $\Omega^1_Y(0)$ into irreducible $k[H]$-modules is given by

$$\Omega^1_Y(0) = \bigoplus_{\chi\in Z(H)} \Omega^1_Y(0)_\chi = \bigoplus_{[\chi]\in\mathcal{Z}(H)} \Omega^1_Y(0)_{[\chi]}.$$

As a consequence we obtain the canonical decomposition

$$\begin{aligned}\text{(3.3)}\qquad \operatorname{Ker}(1-\mathcal{C}\,|\,\Omega^1_Y(0)) &= \bigoplus_{[\chi]\in\mathcal{Z}(H)} \operatorname{Ker}(1-\mathcal{C}\,|\,\Omega^1_Y(0)_{[\chi]}) \\ &\cong \bigoplus_{[\chi]\in\mathcal{Z}(H)} \mathcal{V}^{t_\chi}_{[\chi]}\end{aligned}$$

of $\operatorname{Ker}(1-\mathcal{C}\,|\,\Omega^1_Y(0))$ into irreducible $\mathbb{F}_p[H]$-modules.

4. Cohomological dimension and embedding problems.

In this section we describe one tool from Galois cohomology which we use to prove that if $\mathcal{P}\rtimes H\in\pi_A(X)$ and $\operatorname{cd}_p(\pi_1(X))\le 1$, then $G\in\pi_A(X)$. This result is expressed in terms of embedding problems (cf. Remark 4.4). This concept is also reviewed here.

Definition 4.1 ([**Ser86**, I-17])**.** A profinite group Λ has p-cohomological dimension at most $d\ge 1$, if for every Λ-module M and for every integer $e>d$ the p-primary component of $H^e(\Lambda,M)$ is trivial. The infimum $\operatorname{cd}_p(\Lambda)$ of all such d is called the p-cohomological dimension of Λ.

Definition 4.2 ([**Ser86**, I-23, 3.4])**.** Let

$$\text{(4.1)}\qquad 1\to\mathcal{K}_1\to E\xrightarrow{\delta}\mathcal{K}_2\to 1$$

be an extension of profinite groups. A profinite group Λ has the lifting property for this extension, if for every homomorphism $\alpha:\Lambda\to\mathcal{K}_2$ there exists a homomorphism $\beta:\Lambda\to E$ such that $\alpha=\delta\circ\beta$.

Proposition 4.3 ([**Ser86**, Proposition 16, I-23])**.** *The inequality* $\operatorname{cd}_p(\Lambda)\le 1$ *holds if and only if the extension* (4.1) *has the lifting property, when* $\mathcal{K}_1$ *is a pro-p group.*

Remark 4.4. In the case where $\operatorname{cd}_p(\Lambda)\le 1$, it follows from Proposition 4.3 and Definition 1.2 that there exists a weak solution to the embedding problem

$$(\delta: E\twoheadrightarrow\mathcal{K}_2, \Lambda\twoheadrightarrow\mathcal{K}_2).$$

Let G be a finite group having a normal p-Sylow subgroup P, $H=G/P$ and $\mathcal{P}=P/\Phi(P)$. Recall that $G\cong P\rtimes H$. Define $\delta_G: G\to\mathcal{P}\rtimes H$ by $\delta_G((a,b)) = (a \bmod \Phi(P), b)$. This function is a surjective group homomorphism and $\operatorname{Ker}(\delta_G)=\Phi(P)$.

In particular, if $\mathrm{cd}_p(\pi_1(X)) \leq 1$ and $\mathcal{P} \rtimes H \in \pi_A(X)$, then there exists a weak solution $\pi_1(X) \to G$ to the embedding problem

$$(\delta_G : G \twoheadrightarrow \mathcal{P} \rtimes H, \pi_1(X) \twoheadrightarrow \mathcal{P} \rtimes H).$$

Furthermore, this weak solution is indeed a proper one, because $\Phi(P) \subset \Phi(G)$ and the latter set is exactly the set of "non-generators" of G, thus $\pi_1(X) \to G$ must be surjective.

5. Cohomological dimension at most one.

In this section we prove that the p-cohomological dimension $\pi_1(X)$ is at most 1. The proof follows the argument sketched out by Serre in [**Ser90**, Proposition 1] where he proves a similar result for an affine curve U (sf. also [**Kat88**]).

Definition 5.1. Let X be a smooth projective connected curve defined over k. Denote by $\mathbf{FEt}/X$ the category of finite étale covers of X. Given a closed point $\overline{x}$ of X define the functor $\mathfrak{F} : \mathbf{FEt}/X \to \mathbf{Sets}$ by $Y \mapsto \mathrm{Hom}_X(\overline{x}, Y)$.

Remark 5.2. It follows from [**Mil80**, Chapter I, §5, p. 39] that $\mathfrak{F}$ is strictly pro-representable, i.e., there exists a projective system $(X_\nu, \phi_{\nu\mu})$ in $\mathbf{FEt}/X$ where the transition morphisms $\phi_{\nu\mu} : X_\nu \to X_\mu$ are epimorphisms for $\nu \geq \mu$ and the elements $f_\nu \in \mathrm{Hom}_X(\overline{x}, X_\nu)$ satisfy

1) $f_\nu = \phi_{\nu\mu} \circ f_\mu$; and
2) for any $Y \in \mathbf{FEt}/X$ the natural map $\varinjlim_\nu \mathrm{Hom}_X(X_\nu, Y) \to \mathrm{Hom}_X(\overline{x}, Y)$ is an isomorphism.

Notation. Given a morphism $Y \to X$ and $\mathcal{F}$ an étale sheaf on X (cf. [**Mil80**, Chapter II]), we denote by $\mathcal{F}_{|Y}$ the pullback of $\mathcal{F}$ to Y. For any $n \geq 0$ and $\alpha \in H^n_{\text{ét}}(X, \mathcal{F})$ denote by $\alpha_{|Y} \in H^n_{\text{ét}}(Y, \mathcal{F}_{|Y})$ the pullback of α to Y.

Definition 5.3 ([**Mil80**, p. 155 and 220])**.** An étale sheaf $\mathcal{F}$ on X is called finite if for every quasi-compact $U \subset X$, $\mathcal{F}(U)$ is finite. $\mathcal{F}$ has finite stalks if for every geometric point $\overline{x}$ of X, $\mathcal{F}_{\overline{x}}$ is finite. $\mathcal{F}$ is called locally constant if there exists a covering $(U_\xi \to X)_{\xi \in \Xi}$ such that for every $\xi \in \Xi$, $\mathcal{F}_{|U_\xi}$ is constant. $\mathcal{F}$ is called a p-torsion sheaf if for every quasi-compact $U \subset X$, $\mathcal{F}(U)$ is killed by a power of p.

Proposition 5.4 ([**Mil80**, Proposition 1.1, Remark 1.2 (b)])**.** *Each locally constant sheaf $\mathcal{F}$ on X with finite stalks is finite and represented by a group scheme $\widetilde{\mathcal{F}}$ that is finite and étale over X. Furthermore, there exists a finite étale morphism $X' \to X$ such that $\widetilde{\mathcal{F}} \times_X X'$ is a disjoint union of copies of X' and $\mathcal{F}_{|X'}$ is constant.*

Convention. From this point till the end of this section, unless otherwise stated, $\mathcal{F}$ will denote a p-torsion locally constant sheaf on X with finite stalks.

Remark 5.5. It follows from Definition 5.3 and Proposition 5.4 that

$$\mathcal{F}_{|X'} \cong \bigoplus_{i=1}^{r} (\mathbb{Z}/p^{n_i}\mathbb{Z})^{m_i}, \tag{5.1}$$

where the n_i's and m_i's are positive integers.

Proposition 5.6. *For each $Y \in \mathbf{FEt}/X$ and $\beta \in H^1_{\text{et}}(Y, \mathcal{F}_{|Y})$ there exists $Z \in \mathbf{FEt}/X$ such that Z factors through Y and $\beta_{|Z} \in H^1_{\text{et}}(Z, \mathcal{F}_{|Z})$ is trivial.*

Proof. We start with the case where $Y = X$. Given $\beta \in H^1_{\text{et}}(X, \mathcal{F})$, let x' be as in Prop. 5.4 and $\beta' = \beta_{|X'} \in H^1_{\text{et}}(X', \mathcal{F}_{|X'})$. By (5.1)

$$H^1_{\text{et}}(X', \mathcal{F}_{|X'}) \cong \bigoplus_{i=1}^{r} H^1_{\text{et}}(X', \mathbb{Z}/p^{n_i}\mathbb{Z})^{m_i}.$$

So, we denote $\beta' = (\beta_{1,1}, \dots, \beta_{1,m_1}, \dots, \beta_{r,1}, \dots, \beta_{r,m_r})$ with $\beta_{i,j} \in H^1_{\text{et}}(X', \mathbb{Z}/p^{n_i}\mathbb{Z})$. Let $\mathcal{W}_n$ be the sheaf of Witt vectors of length n on X [**Ser56**, §2], $F_{\text{abs}} : X \to X$ the absolute Frobenius morphism and $\wp$ the operator $\wp(x) = x^p - x$. The exact sequence

$$1 \to \mathbb{Z}/p^n\mathbb{Z} \to \mathcal{W}_n \xrightarrow{\wp} \mathcal{W}_n \to 1,$$

gives an isomorphism $H^1_{\text{et}}(X, \mathbb{Z}/p^n\mathbb{Z}) \cong H^1(X, \mathcal{W}_n)^{F_{\text{abs}}}$, as in the usual Artin-Schreier theory [**Mil80**, p. 127-128]. Hence, by [**Ser56**, Proposition 13], we conclude that $\beta_{i,j}$ parametrizes a cyclic étale cover $X_{i,j} \to X'$ of degree p^{n_i}. Given $\alpha \in H^1_{\text{et}}(X', \mathbb{Z}/p^n\mathbb{Z})$ and $V \to X'$ any finite étale cover, let $X'' \to X'$ be the cyclic étale cover of degree p^n defined by α and let $\alpha' = \alpha_{|V} \in H^1_{\text{et}}(V, \mathbb{Z}/p^n\mathbb{Z}_{|V})$. Thus α' parametrizes the covering $W = V \times_{X'} X'' \to V$. In the case where $\alpha = \beta_{i,j}$, the covering $X_{i,j} \to X'$ plays the role of both $V \to X'$ and $X'' \to X'$. Therefore $\beta_{i,j|X_{i,j}}$ is trivial. Let $Z \to X'$ be a finite étale cover such that for every $i \in \{1, \dots, r\}$ and $j \in \{1, \dots, m_i\}$. The cover $Z \to X'$ factors through $X_{i,j} \to X'$. Therefore $\beta_{|Z} = \beta'_{|Z} \in H^1_{\text{et}}(Z, \mathcal{F}_{|Z})$ is trivial.

In the case where $Y \neq X$, let $Y' = Y \times_X X'$. We have

$$\mathcal{F}_{|Y'} \cong \bigoplus_{i=1}^{r} (\mathbb{Z}/p^{n_i}\mathbb{Z})^{m_i}.$$

It follows from the above argument that there exists a finite étale cover $Z \to Y'$ such that $\beta_{|Z} = (\beta_{|Y'})_{|Z} \in H^1_{\text{et}}(Z, \mathcal{F}_{|Z})$ is trivial. □

Proposition 5.7. *For each $Y \in \mathbf{FEt}/X$ there exists $Z \in \mathbf{FEt}/X$ which factors through Y such that $H^0_{\text{et}}(Z, \mathcal{F}_{|Z}) \cong \bigoplus_{i=1}^{r} (\mathbb{Z}/p^{n_i}\mathbb{Z})^{m_i}$.*

Proof. As in the proof of Proposition 5.6 it suffices to take $Z = Y \times_X X'$. □

Remark 5.8 ([**Mil80**, Chapter I, 5.4])**.** Given $Y \in \mathbf{FEt}/X$ denote by $\mathrm{Aut}_X(Y)$ the set of X-automorphisms of Y. There exists $Z \in \mathbf{FEt}/X$ such that $Z \to X$ is Galois and $Z \to Y$ is an X-morphism. In this case $\mathrm{Hom}_X(\overline{x}, Z)$ is isomorphic to $\mathrm{Aut}_X(Z)$. In particular the elements of the projective system $(X_\nu, \phi_{\nu\mu})$ can be taken so that for each ν the cover $X_\nu \to X$ is Galois. Furthermore, $\pi_1(X, \overline{x}) = \varprojlim_\nu \mathrm{Aut}_X(X_\nu)$.

Remark 5.9. Since for each ν the map $X_\nu \to X$ is finite, hence affine, it follows from [**SGA 4**, VII, §5] that the projective limit of schemes $\widehat{X} = \varprojlim_\nu X_\nu$ exists. Moreover, by [**Mil80**, Chapter III, Lemma 1.16], for any étale sheaf $\mathcal{F}$ on X and for any integer $n \geq 0$ we have

$$H^n_{\mathrm{et}}(\widehat{X}, \mathcal{F}_{|\widehat{X}}) \cong \varinjlim_\nu H^n_{\mathrm{et}}(X_\nu, \mathcal{F}_{|X_\nu}).$$

Corollary 5.10. $H^1_{\mathrm{et}}(\widehat{X}, \mathcal{F}_{|\widehat{X}}) = 0$ *and* $H^0_{\mathrm{et}}(\widehat{X}, \mathcal{F}_{|\widehat{X}}) \cong \bigoplus_{i=1}^r (\mathbb{Z}/p^{n_i}\mathbb{Z})^{m_i}$.

Proof. This is an immediate consequence of Propositions 5.6 and 5.7 and Remark 5.9. □

Theorem 5.11. *Let X be a smooth projective connected algebraic curve defined over an algebraically closed field of characteristic $p > 0$. For any closed point $\overline{x}$ of X we have* $\mathrm{cd}_p(\pi_1(X, \overline{x})) \leq 1$.

Proof. It follows from [**Sha72**, p. 55, Theorem 11] that it suffices to show that $H^2(\pi_1(X, \overline{x}), F) = 0$ for any finite simple $\pi_1(X, \overline{x})$-module F of p-power order. By [**Mil80**, p. 155-156], any such F is associated uniquely to a p-torsion locally constant étale sheaf $\mathcal{F}$ with finite stalks. Proposition 5.4 shows that there exists $X' \in \mathbf{FEt}/X$ such that

$$F \cong \mathcal{F}_{|X'} \cong \bigoplus_{i=1}^{r} (\mathbb{Z}/p^{n_i}\mathbb{Z})^{m_i}. \tag{5.2}$$

Furthermore, by [**SGA 4**, X, Corollary 5.2], since X is a smooth projective connected algebraic curve defined over k, we conclude that

$$H^n_{\mathrm{et}}(X, \mathcal{F}) = 0 \text{ for any } n \geq 2. \tag{5.3}$$

For every ν we consider the Hochschild-Serre spectral sequence [**Mil80**, p. 105, Theorem 2.20] $E_\nu^{r,s} \Rightarrow E^{r+s}$, where $E_\nu^{r,s} = H^r(\mathrm{Aut}_X(X_\nu), H^s_{\mathrm{et}}(X_\nu, \mathcal{F}_{|X_\nu}))$ and $E^{r+s} = H^{r+s}_{\mathrm{et}}(X, \mathcal{F})$. Also, as in [**Mil80**, p. 106 (b)], taking the projective limit we obtain a spectral sequence $E_\infty^{r,s} \Rightarrow E^{r+s}$, where $E_\infty^{r,s} = H^r(\pi_1(X, \overline{x}), H^s_{\mathrm{et}}(\widehat{X}, \mathcal{F}_{|\widehat{X}}))$. Furthermore, it follows from [**Mil80**, p. 309, 1.8] that there exists an exact sequence

(5.4)
$$0 \to H^1(\pi_1(X,\overline{x}), H^0_{\mathrm{et}}(\widehat{X}, \mathcal{F}_{|\widehat{X}})) \to H^1_{\mathrm{et}}(X,\mathcal{F}) \to H^0(\pi_1(X,\overline{x}), H^1_{\mathrm{et}}(\widehat{X}, \mathcal{F}_{|\widehat{X}}))$$
$$\to H^2(\pi_1(X,\overline{x}), H^0_{\mathrm{et}}(\widehat{X}, \mathcal{F}_{|\widehat{X}})) \to H^2_{\mathrm{et}}(X,\mathcal{F}) \to H^1(\pi_1(X,\overline{x}), H^1_{\mathrm{et}}(\widehat{X}, \mathcal{F}_{|\widehat{X}})).$$

Finally, we conclude from Corollary 5.10, (5.2), (5.3) and (5.4) that $H^2(\pi_1(X,\overline{x}),F) = 0$. Thus, $\mathrm{cd}_p(\pi_1(X,\overline{x})) \leq 1$. □

In the next two corollaries we assume that X has genus $g \geq 2$. In this case it follows from [**Ray82**, Corollaire 4.3.2] that the p-Sylow subgroups of $\pi_1(X,\overline{x})$ are non-trivial.

Corollary 5.12. *For every finite simple p-power order $\pi_1(X,\overline{x})$-module F we have $H^1(\pi_1(X,\overline{x}),F) \cong H^1_{\mathrm{et}}(X,\mathcal{F})$.*

Proof. The result is a consequence of Corollary 5.10 and (5.4). □

Corollary 5.13. *The p-Sylow subgroups of $\pi_1(X,\overline{x})$ are non-trivial and pro-p-free.*

Proof. Recall that [**Ser86**, p. I-20, Proposition 14 (i)] implies $\mathrm{cd}_p(P) = \mathrm{cd}_p(\pi_1(X,\overline{x}))$, for any p-Sylow subgroup P of $\pi_1(X,\overline{x})$. Moreover, it follows from Theorem 5.11 that $\mathrm{cd}_p(\pi_1(X,\overline{x})) \leq 1$. But, for a pro-$p$-group P this is equivalent to P being pro-p-free. □

6. Galois covers.

Proof of Theorem 1.3. Let $\pi_1(X) \twoheadrightarrow G$ be a proper solution for the embedding problem $(\phi : \pi_1(X) \twoheadrightarrow H, G \twoheadrightarrow H)$. Let $Y \to X$ be the Galois H-cover corresponding to ϕ. Thus, $\gamma_{\phi,\chi} = \gamma_{Y,\chi}$. Recall that $\Phi(P)$ is the Frattini subgroup of P and $\mathcal{P} = P/\Phi(P)$. Observe that $\mathcal{P} \in \pi_A(Y)$. It follows from the correspondence described in the second paragraph of Section 3 that $\mathrm{Hom}(\mathcal{P}, \mathbb{Z}/p\mathbb{Z})$ is an $\mathbb{F}_p$-subspace of $\mathrm{Hom}(\pi_1(Y), \mathbb{Z}/p\mathbb{Z})$. This latter space is $\mathbb{F}_p$-isomorphic to $\mathrm{Hom}(\mathrm{Ker}(1-\mathcal{C} \mid \Omega^1_Y(0)), \mathbb{F}_p)$ by Serre's duality [**Ser56**, §9]. Therefore, (3.2) and (3.3) imply $m_\chi \leq t_\chi$, for every $\chi \in Z(H)$. Note that $\mathrm{Ker}(1-\mathcal{C} \mid \bigoplus_{j=0}^{l_\chi-1} \Omega^1_Y(0)_{\chi^{p^j}})$ and $\mathrm{Ker}(1-\mathcal{C}^{l_\chi} \mid \Omega^1_Y(0)_\chi)$ are $\mathbb{F}_p[H]$-isomorphic via $\omega = \sum_{j=0}^{l_\chi-1} \omega_j \mapsto \omega_0$ (cf. [**Pac95**, Lemma 2.14]). Moreover, $\dim_{\mathbb{F}_p} \mathrm{Ker}(1-\mathcal{C}^{l_\chi} \mid \Omega^1_Y(0)_\chi) = \gamma_{Y,\chi} l_\chi$, therefore $t_\chi = \gamma_{Y,\chi}/n_\chi$ (cf. [**Pac95**, Corollary 3.6]), hence $m_\chi n_\chi \leq \gamma_{Y,\chi}$ (cf. [**Ste96a**, Proposition 3.4]). Conversely, suppose that $m_\chi \leq \gamma_{Y,\chi}/n_\chi$, for every $\chi \in Z(H)$. Since $t_\chi = \gamma_{Y,\chi}/n_\chi$, it follows from (3.3) there exists an $\mathbb{F}_p[H]$-submodule $\mathcal{B}_\chi$ of $\mathrm{Ker}(1-\mathcal{C} \mid \bigoplus_{j=0}^{l_\chi-1} \Omega^1_Y(0)_{\chi^{p^j}})$ such that $\mathcal{B}_\chi \cong \mathcal{V}_{[\chi]}^{m_\chi}$. Let $\mathcal{B} = \bigoplus_{\chi \in Z(H)} \mathcal{B}_\chi$ and remark that there exists an $\mathbb{F}_p[H]$-isomorphism between $\mathcal{B}$ and $\mathcal{P}$. Once again by the the correspondence described in the second paragraph of Section 3, $\mathrm{Hom}(\mathcal{B}, \mathbb{F}_p)$ is $\mathbb{F}_p[H]$-isomorphic to $\mathrm{Hom}(\mathrm{Gal}(Z/Y), \mathbb{Z}/p\mathbb{Z})$ for some étale cover $Z \to Y$ and $\mathrm{Gal}(Z/Y) \cong \mathcal{P}$. Therefore, Lemma 3.2 implies that $Z \to X$ is Galois and $\mathrm{Gal}(Z/X) \cong \mathcal{P} \rtimes H$. Hence $\mathcal{P} \rtimes H \in \pi_A(X)$. It follows from Theorem 5.11 that $\mathrm{cd}_p(\pi_1(X, \overline{x})) \leq 1$ for any closed point $\overline{x}$ of X. Therefore, the argument of Remark 4.4 implies that $G \in \pi_A(X)$. □

7. A generic condition.

Theorem 1.3 tells us that if we are given a finite group G with a normal p-Sylow subgroup P and quotient H, then whether or not G lies in $\pi_A(X)$ depends not only on the size of P, but also on the specific action of H on P. The role that the action of H on P plays in this question was examined previously in the work of Nakajima [**Nak87**, Theorem A], Pacheco [**Pac95**, Propositions 2.4 and 2.5] and Stevenson [**Ste96a**, Proposition 3.5]. However, for the groups we are considering, Theorem 1.3 is stronger. In particular, it gives us a necessary and sufficient condition which is reasonably easy to compute. We begin this section with some consequences of Theorem 1.3. These involve situations where the generalized Hasse-Witt invariants can be most easily computed. At the end of this section we compute Condition A for the curve X_g (which represents the generic geometric point of the coarse moduli scheme $\mathcal{M}_g$ of curves of genus g) under the assumption that H is abelian. This situation is sufficient to demonstrate the strengths of our results while also distinguishing it from previous work.

As a preliminary step, we will prove the result mentioned in Remark 1.5, which deals with "ordinary Galois H-covers". The advantage in this case is that Condition A can be rephrased in a way that is independent of the H-cover. Given a finite group G having a normal p-Sylow subgroup P, recall that $H = G/P$, $Z(H)$ denotes the set of irreducible characters χ of H defined over the algebraically closed field k of characteristic $p > 0$ and χ^0 is the trivial character of H.

Theorem 7.1. *Let G be a finite group having a normal p-Sylow subgroup P. Let $H = G/P$. Suppose that $\phi : \pi_1(X) \twoheadrightarrow H$ corresponds to a Galois H-cover $Y \to X$ where Y is an ordinary curve. An embedding problem $(\phi : \pi_1(X) \twoheadrightarrow H, G \twoheadrightarrow H)$ has a proper solution if and only if $m_{\chi^0} \leq g$, and $m_\chi \leq (g-1)n_\chi$, for $\chi \neq \chi^0$.*

Proof. Notice that by Remark 2.3, the Galois H-cover Y is ordinary if and only if we have

$$\gamma_{Y,\chi} = \begin{cases} g, & \text{if } \chi = \chi^0 \text{ and} \\ (g-1)\, n_\chi^2, & \text{if } \chi \neq \chi^0. \end{cases} \tag{7.1}$$

Thus condition A is equivalent to the condition of Theorem 7.1. □

Let $g \geq 2$ be an integer and $\pi_A(g)$ the set of isomorphism classes of finite groups G such that $G \in \pi_A(X)$ for some smooth projective connected curve X of genus g.

Remark 7.2. Suppose that there exists some smooth projective connected curve X defined over k such that a finite group $G \in \pi_A(X)$. Denote by $x \in \mathcal{M}_g$ the point corresponding to X. In [**Ste96**, Proposition 4.2] Stevenson showed that in this case there exists an open subset U of $\mathcal{M}_g$ containing x such that for every $z \in U$ we have $G \in \pi_A(Z)$, where Z denotes the curve corresponding to z. In particular, $G \in \pi_A(X_g)$, therefore $\pi_A(X_g) = \pi_A(g)$.

Remark 7.3. It is an immediate consequence of the definition of $\pi_A(g)$ that a finite group G satisfying the hypothesis of Theorem 1.3 lies in $\pi_A(g)$ if and only if there exists a smooth projective connected curve X of genus g for which Condition A holds.

Notation. Let G be a finite group. Denote by $d(G)$ the minimum number of generators of G.

Now we can prove another consequence of Theorem 1.3.

Theorem 7.4. *Let G be a finite group having a normal p-Sylow subgroup P. Suppose that $H = G/P$ is abelian and $g \geq 2$. A necessary and sufficient condition for $G \in \pi_A(g)$ is $d(H) \leq 2g$, $m_{\chi^0} \leq g$ and $m_\chi \leq g-1$ for each $\chi \in Z(H)$ and $\chi \neq \chi^0$.*

Proof. Suppose that $G \in \pi_A(g)$. It follows from Remark 7.3 that there exists a smooth projective connected curve X and an étale Galois cover $Y \to X$ with $\mathrm{Gal}(Y/X) \cong H$ such that for every $\chi \in Z(H)$ we have $m_\chi \leq \gamma_{Y,\chi}$. By (2.2) we conclude that $\gamma_{Y,\chi^0} \leq g$ and $\gamma_{Y,\chi} \leq g-1$ for every $\chi \in Z(H)$, $\chi \neq \chi^0$. Moreover, since $H \in \pi_A(X)$, [**Groth71**, Corollary 2.12] implies that $d(H) \leq 2g$. In particular, the condition of Theorem 7.4 is satisfied. Conversely, suppose that $d(H) \leq 2g$, $m_{\chi^0} \leq g$ and $m_\chi \leq g-1$ for each $\chi \in Z(H)$ and $\chi \neq \chi^0$. Since H is abelian and $d(H) \leq 2g$, it follows from [**Groth71**, Corollary 2.12] that $H \in \pi_A(X_g)$, i.e., there exists an étale covering $Y_g \to X_g$ such that $\mathrm{Gal}(Y_g/X_g) \cong H$. It is a result due to Nakajima [**Nak83**, Theorem 2] that every étale cyclic covering $Z_g \to X_g$ of degree prime to p is ordinary. (It is essential here that X_g is generic.) This result was extended to all abelian prime to p groups by Zhang [**Zha92**, Théorème 3.1] (again for X_g). Hence Y_g is ordinary. So, by Theorem 7.1, $\gamma_{Y_g,\chi^0} = g$ and $\gamma_{Y_g,\chi} = g-1$ for every $\chi \in Z(H)$, $\chi \neq \chi^0$. Furthermore, by hypothesis, $m_{\chi^0} \leq g$ and $m_\chi \leq g-1$ for every $\chi \in Z(H)$, $\chi \neq \chi^0$. Therefore, Condition *A* holds for X_g and by Theorem 1.3, $G \in \pi_A(X_g)$. Finally, Remark 7.2 shows that this is equivalent to $G \in \pi_A(g)$. □

Another result in this direction is the following one from [**Ste96a**].

Theorem 7.5 ([**Ste96a**, Propositions 3.1 and 3.2])**.** *Let G be a finite group having a normal p-Sylow subgroup P and $H = G/P$. Suppose that $g \geq 2$ and $d(H) \leq g$. A necessary and sufficient condition for $G \in \pi_A(g)$ is $m_{\chi^0} \leq g$ and $m_\chi \leq (g-1)n_\chi$ for each $\chi \in Z(H)$ and $\chi \neq \chi^0$.*

Remark 7.6. Notice that for an abelian group H such that $d(H) \leq 2g$, Theorem 7.4 is stronger than Theorem 7.5 since the latter requires that $d(H) \leq g$. However, for arbitrary H with $d(H) \leq g$, Theorem 7.5 is stronger than Theorem 7.4.

Now we can compare these results to a result of Nakajima. Let G be a finite group, $I_G = \{\sum_{\sigma\in G} a_\sigma \sigma \in \mathbb{Z}[G]\,;\, \sum_{\sigma\in G} a_\sigma = 0\}$ its augmentation ideal and $t(G)$ the minimum number of generators of I_G. Suppose that there exists a smooth projective curve X of genus g such that $G \in \pi_A(X)$, i.e., $G \cong \mathrm{Gal}(Y/X)$ for some étale Galois cover $Y \to X$.

Theorem 7.7 (Nakajima, [**Nak84**, Theorem 4])**.** *There exists a short exact sequence of $k[G]$-modules*

$$1 \to \Omega^1_Y(0) \to k[G]^g \to I_G \to 1.$$

Corollary 7.8 (Nakajima, [**Nak87**, Theorem A])**.** $t(G) \leq g$.

Notation. We call Condition B the inequality of Corollary 7.8.

Remark 7.9. From the definition of $\pi_A(g)$, Theorem 7.7 and Corollary 7.8, we see that Condition B is necessary for $G \in \pi_A(g)$.

Corollary 7.10. *Let G be a finite group having a normal p-Sylow subgroup P, $H = G/P$. Suppose that either:* (a) *H is abelian and $d(H) \leq 2g$; or* (b) *$d(H) \leq g$. Under either hypothesis* (a) *or* (b) *Condition A is equivalent to Condition B.*

Proof. By [**Ste96a**, Proposition 3.5] Condition A implies Condition B without any restrictions on H. Conversely, by [**Ste96**, Proposition 3.1] Condition B implies that $m_{\chi^0} \leq g$ and $m_\chi \leq (g-1)n_\chi$ for each $\chi \in Z(H)$ and $\chi \neq \chi^0$. Under hypothesis (a) (resp. (b)) Theorem 7.4 (resp. 7.5) show that the latter condition implies that $G \in \pi_A(g)$. Now by Theorem 1.3 this implies Condition A. □

In order to obtain a converse in the case where H is a non-abelian finite quotient of Γ_g we need to generalize the Nakajima-Zhang result ([**Nak83**, Theorem 2] and [**Zha92**, Théorème 3.1]) to non-abelian Galois étale covers of degree prime to p of X_g. Another option is to show that there exists an ordinary Galois H-cover of a curve X of genus g and apply [**Ste96**] (cf. Remark 1.5). Very recently M. Raynaud has found a counter example to both these approaches.

Example 7.11. Theorem 7.4 gives a result which is not covered by [**Ste96a**, Theorem 3.2] in the case where H is abelian and $g < d(H) \le 2g$. Let $n \ge 1$ be an integer and let $g \ge 2$ be an integer. Let $H = (\mathbb{Z}/n\mathbb{Z})^{2g}$ and label the elements τ_j for $j = 1, \dots, n^{2g}$. For each $i = 1, 2, \dots, g-1$ let $P_i = (\mathbb{Z}/p\mathbb{Z})^{n^{2g}}$ and $P_g = \mathbb{Z}/p\mathbb{Z}$. Pick a basis $a_{i,\tau_1}, \dots, a_{i,\tau_{n^{2g}}}$ for P_i for $i = 1, \dots, g-1$ and let a_g be a basis of P_g. Then we define an action of H on each P_i for $i = 1, \dots, g-1$ as follows: $\rho_i : H \to \mathrm{Aut}(P_i)$ by $\rho_i(\tau_j)a_{i,\tau_l} = a_{i,\tau_j\tau_l}$. With this action each P_i is isomorphic to $\mathbb{F}_p[H]$, which is the H-module defined over $\mathbb{F}_p$ corresponding to the regular representation of H. Let H act on P_g trivially. Now let $P = \bigoplus_{i=1}^{g} P_i$ with the induced action of H on P. Then P is isomorphic to $(\mathbb{Z}/p\mathbb{Z})^{n^{2g}(g-1)+1}$ as a group and to $\mathbb{F}_p[H]^{g-1} \oplus \mathbb{F}_p$ as an $\mathbb{F}_p[H]$-module. Let G be defined as the semi-direct product $P \rtimes H$ with respect to this action.

By construction $P \otimes_{\mathbb{F}_p} k$ is isomorphic as a $k[H]$-module to $k[H]^{g-1} \oplus k$. Let $Z(H)$ be the set of irreducible characters of H defined over k and let χ^0 be the trivial character of H. Then using the notation of Section 1.1, $m_{\chi^0} = g$ and $m_\chi = g-1$ for $\chi \neq \chi^0$. Note that since H is abelian, by Zhang's theorem [**Zha92**, Théorème 3.1], any Galois H-cover Y_g of X_g is ordinary, thus $\gamma_{Y_g,\chi^0} = g$, and $\gamma_{Y_g,\chi} = g-1$ for $\chi \neq \chi^0$. In particular, Condition A is satisfied for the curve X_g, therefore $G \in \pi_A(X_g) = \pi_A(g)$.

Remark 7.12. In the set-up of Example 7.11, as the rank of H is greater than g, Theorem 7.5 does not apply. If we keep P the same but change the action of H on P in any way, then G will not lie in $\pi_A(g)$, because for some character $\chi \neq \chi^0$ we would have $m_\chi > g-1$ or $m_{\chi^0} > g$. Finally, if we replace P by any p-group Q with Frattini quotient isomorphic to P and extend the action of H in P of Example 7.11 to an action of H in Q, then we get (as in the end of the proof of Theorem 1.3) $Q \rtimes H \in \pi_A(X_g) = \pi_A(g)$.

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Universidade Federal do Rio de Janeiro
Rio de Janeiro RJ
Brasil
E-mail address: amilcar@impa.br

California Institute of Technology
Pasadena, CA 91125
E-mail address: kfs@vision.caltech.edu